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INTERNAL CATEGORICITY AND THE GENERIC MULTIVERSE

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Abstract. John Steel's theory, MV, of the generic multiverse provides a foundation for mathematics that aims to neutralize the effects of incompleteness brought on by forcing arguments. Jouko Väänänen's development of internal categoricity arguments provides opportunities to argue that the subject matter of some theory is, in some sense, determined. This paper investigates whether MV is internally categorical.

It is well known that second order ZFC is almost categorical. Given a pair of full models of second order ZFC, one of them will be isomorphic to a rank initial segment of the other and the rank of that initial segment will be an inaccessible cardinal. It is a pleasing uniqueness theorem that sits naturally beside the existence theorems delivered by consistency proofs. Moreover, the moths of philosophy have been drawn to its tantalizing flame since Zermelo [1976] and Kreisel [1969]. One might hope to use this result to conclusively demonstrate that a large chunk of the subject matter of set theory, including the continuum hypothesis, is definitively fixed. These hopes have largely been dashed on the rocks of circularity, however, more recently Jouko Väänänen has formulated an intriguing notion of *internal categoricity* that seems to avoid these problems and perhaps deliver opportunities for more promising philosophical arguments [Väänänen, 2021, Maddy and Väänänen, 2023]. Internal categoricity will be the focal concept of this paper.

As to this paper's target, we consider a theory that looks to be motivated by ideas that oppose the fixity of subject matter suggested by categoricity. Since the advent of Cohen's forcing technique and the realization that the continuum hypothesis is independent of ZFC, there has been a recurring thought that at least some of the incompleteness of ZFC is more of a feature than a bug [Mostowski, 1967]. This motivates the idea that there is a plurality of different universes wrought through forcing that might all be taken seriously as alternatives upon a somewhat level playing field. W. Hugh Woodin brought this idea to life, albeit in order to bury it, along with its provocative name: the *generic multiverse* [Woodin, 2012, 2011b, 2004]. This is a multiverse of universes linked together by sequences of generic extensions and refinements. More recently, John Steel gave this idea a more solid foundational footing in the form of a computably axiomatizable theory, MV [Steel, 2014]. MV will be the main target of this paper.

Putting these ideas together, we have an interesting tension. On one hand, we have an intriguing theorem template that might rigidify the subject matter of set theory; and on the other, we have a theory that appears to embrace a radically pluralistic vision of what set theory is about.[1] To bring

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[1]While I think it's fair to say Steel's theory is radically pluralist, the reader should note that it is much tamer than the wild multiverse proposed by [Hamkins, 2012].

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this contrast to a point, the categoricity arguments seem to tell us that the continuum hypothesis has a definitive answer, while the generic multiverse appears to embrace its indeterminacy. In spite of all these appearances, this paper has a simple, but – I think – surprising goal. We are going to prove that MV is internally categorical!

Before we get too carried away, I'll say something even stranger and then I'll try to clarify the underlying picture. So first, We are also going to prove that MV is *not* internally categorical. Now it sounds like we have a contradiction, but the devil is in the details; and in particular, the axioms we use. Just as we have with PA and ZFC, the internal categoricity of MV can be made to hold or not hold depending on the axioms we use to generate that theory.[2] Where does this leave the philosophical preambling above? We'll defer some discussion of that until then end, but I will say that these results nicely illustrate that internal categoricity is a subtle matter. Moreover, this subtlety warrants further investigation and MV presents an intriguing case study for that project.

The paper is organized as follows. In Section 1, we provide a brief overview of internal categoricity and some of the philosophical arguments it is entangled with. Section 2 provides an overview of Steel's MV that includes a rough classification of MV's models. In Section 3, we prove that MV is not internally categorical and then in Section 4, we prove that an alternative axiomatization of MV is internally categorical. The paper then closes in Section 5 with some discussion of these results.

1. What is internal categoricity?

We know from Dedekind that second order PA is categorical and from Zermelo and Shepherdson that second order ZFC is quasi-categorical. More formally,

Theorem 1. *(1) [Dedekind, 1963] If $\mathcal{M}$ and $\mathcal{N}$ are full models of second order PA, then $\mathcal{M} \cong \mathcal{N}$.*
(2) [Zermelo, 1976, Shepherdson, 1951] If $\mathcal{M}$ and $\mathcal{N}$ are full models of second order ZFC, then either: $\mathcal{M}$ is isomorphic to $V_\kappa^{\mathcal{N}}$ where κ is inaccessible in $\mathcal{N}$; or $\mathcal{N}$ is isomorphic to $V_\lambda^{\mathcal{M}}$ where λ is inaccessible in $\mathcal{M}$.

With regard to (1), we have an intriguing uniqueness result. To put some context around this, let's note that for most us, it's obviously desirable that, despite the weaknesses of first order logic in pinning down structures, there should be just one *intended* model of arithmetic (up to isomorphism).[3] It seems natural then to wonder if Dedekind's result *provides evidence* that this is the case. There is a lot of philosophy to unpack in such a conjecture, but fortunately this has been thoroughly discussed before [Kreisel, 1969, Parsons, 1990, Button and Walsh, 2016, Maddy and Väänänen, 2023]. As such, we'll satisfy ourselves by briefly considering just one compelling objection, which I'll motivate with a question: "If I were worried about whether there was a unique intended model of arithmetic, then surely I should be even more worried about full second order models of PA which presuppose that it makes sense to talk about the unique powerset of the natural numbers?" It has been suggested many times that the conjecture above makes use of a kind of circular reasoning [Weston, 1976, Putnam, 1980, Field, 2001, Hamkins, 2012, Meadows, 2013, Koellner, 2013, Maddy and Väänänen, 2023]. In order

[2]See, for example, Theorems 47 and 52 in [Enayat and Łełyk, 2024].
[3]As we progress further in this paper, I'll frequently omit the phrase "up to isomorphism" in the hopes that this will keep the words flowing smoothly and that the reader will charitably discern my true intentions.

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justify our thought that there is a unique model of arithmetic, we appeal to the uniqueness of a much more complex structure. As such, the idea that Dedekind's result provides evidence for our intuitions about arithmetic, looks to be on shaky ground. None of this is to say that there isn't a unique model of arithmetic, just that Dedekind's result doesn't provide compelling evidence for that claim.

With regard to (2), we get closer to the target of this paper and analogous remarks apply. Of course, we don't always get a genuine isomorphism between full models of second order ZFC. So there is less temptation to think we have evidence that the subject matter of set theory has been pinned down. But any pair of such models still have a great deal in common. In particular and saliently, any two full models of second order ZFC must agree on their evaluation of the continuum hypothesis. Thus, we might be tempted to think that Zermelo's result provides us with some evidence that the continuum hypothesis has a definitive answer. As with arguments making use of Dedekind's theorem, worries about circularity also emerge in this arena. Once again, the problem comes from the use of full second order models and their free wheeling use of the powerset operation.

Much more recently, Väänänen has offered a new perspective on categoricity theorems that makes no use of second order logic and thus, defuses the circularity worries discussed above. He calls this *internal categoricity* [Väänänen, 2021, 2019].[4] We start by considering an expansion of the ordinary language of set theory where we let $\mathcal{L}(\in_0, \in_1)$ be the language with a pair of 2-place relation symbols. We then let $ZFC(\in_0, \in_1)$ be the theory that has: an $\in_0$ and $\in_1$ version of every axiom of ZFC; and has $\in_0$ and $\in_1$ versions of the Replacement and Separation Schemata where the formulae used may make use of both $\in_0$ and $\in_1$.[5] We then obtain the following.

Theorem 2. *[Väänänen, 2019] If $\mathcal{M} = \langle M, \in_0^{\mathcal{M}}, \in_1^{\mathcal{M}} \rangle$ is a model of $ZFC(\in_0, \in_1)$, then there is an isomorphism*

$$\sigma : \langle M, \in_0^{\mathcal{M}} \rangle \cong \langle M, \in_1^{\mathcal{M}} \rangle$$

where σ is uniformly definable over $ZFC(\in_0, \in_1)$.

Some remarks about the relationship with Zermelo's theorem are warranted. With Zermelo, we learn something about a pair of models, while with Väänänen we are just considering a single model. The interesting thing is that this model is – in a sense – two models in disguise. Moreover, those two models are deeply entangled through the mashup theory $ZFC(\in_0, \in_1)$. In particular, they are allowed

[4]I should note that this attribution is a little vexed. The term, *internal categoricity*, was first introduced by Walmsley [2002] although the underlying idea arguably emerges with Parsons [1990]. More recently, it has been thoroughly investigated by Button and Walsh [2018, 2016] with some critical responses from Maddy and Väänänen [2023] and Picollo and Waxman [2023]. However, the notion of internal categoricity explored in this paper is importantly distinct from that work in that it is articulated in first order logic rather than second order logic. This approach was, I believe, first introduced by Väänänen [2019] with the theorem below. The proof might be understood as an optimization of a famous categoricity theorem for $ZFCU$ provided by McGee [1997]. Both approaches and their relationships are discussed by Maddy and Väänänen [2023]. More recently again, internal categoricity has been investigated by researchers in the relative interpretation literature [Enayat and Łełyk, 2024]. We shall be exclusively concerned with the first order version in this paper. Less philosophical attention seems to have been paid to the first order version so far. However, we shall discuss some its attractions and shortcomings below. I'm grateful to a referee for drawing my attention to some of the sources above and pressing me to be clearer about what I mean by internal categoricity.

[5]My notation is cosmetically different from that of Väänänen [2021]. I've done this so that I can notate things a little more compactly on the page, but do refer to Väänänen [2021, 2019], Maddy and Väänänen [2023] for the official notation. Moreover, Enayat and Łełyk [2024] provide an excellent notation system for generalizing these ideas to other theories. Given that our focus is quite narrow in this paper, we'll avoid this and just remind the reader of our meaning when required.

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to make us of each other's languages in their respective Replacement and Separation Schemata. Thus, and helpfully for the proof of the theorem above, each model can deliver the transitive collapse of the other.

But why should we allow this sharing of vocabulary between their Replacement and Separation Schemata? From a mathematical point of view, there is nothing stopping us. But from a more philosophical perspective, there is an intriguing argument in support of this move. If we consider the intention behind our use of a schema to articulate the idea behind Replacement, we might say that it is intended to be a first order approximation of the second order idea that the pointwise image of *any* class function through a set should itself be a set of outputs.[6] Thus, if we expand our language in such a way that could allow us to define new class functions, these class functions ought to also be subject to the closure conditions required by Replacement.

What should we make of this? To motivate the milieu behind such results, we might consider a thought experiment based on ideas from interpretability that is similar in spirit if not letter to Parsons [1990]. Suppose that you use a theory $DLO(<)$ of dense linear orders that is based on a strict $<$ relation, and I use a theory $DLO(\leq)$ based on non-strict relation $\leq$. There is, of course, no substantive debate to be had between us. But how might we demonstrate this? Given that you are working in $\mathcal{L}(<)$, you might move to an expansion $\mathcal{L}(<, \leq)$ that adds my non-strict $\leq$. You might then accommodate my thoughts about the $\leq$ by extending your theory to be $DLO(<) \cup DLO(\leq)$. This puts us in a position to model how you might think about my $\leq$ in terms of your $<$. At this point, we know nothing about that, but you might be suddenly struck by inspiration and observe that you can add an axiom to $DLO(<) \cup DLO(\leq)$ that defines $\leq$ in terms of $<$ by saying that $x \leq y$ iff $x < y$ or $x = y$. You might then prove that the resulting theory remains consistent. Moreover, I might then do the analogous thing but in the other direction. If we both succeed then it can be seen that our initial theories were very closely related; in particular $DLO(<)$ and $DLO(\leq)$ are definitionally equivalent.[7] This is a simple illustration of a pivotal concept based on the theory of interpretability. Results like this are a cornerstone of contemporary set theory.

I think there is a analogy to be had between the example above and a way of applying Väänänen's theorem. Suppose you and I are both users of ZFC. In this situation, we have no worries about whether our theories are definitionally equivalent since they are identical. However, we might wonder if your version of $\in$ has the same meaning as mine. To explore this question, we might tag our membership relations to differentiate them and thus, use $\in_0$ and $\in_1$. We then might say that you are working $ZFC(\in_0)$ articulated in $\mathcal{L}(\in_0)$ and I am working in $ZFC(\in_1)$ articulated in $\mathcal{L}(\in_1)$. In order to explore the relationship between our theories you might then expand your language from $\mathcal{L}(\in_0)$ to $\mathcal{L}(\in_0, \in_1)$ and accommodate my ideas about set theory by working in the theory $ZFC(\in_0) \cup ZFC(\in_1)$. This puts us in a good position to model how you might think about my $\in_1$ in terms of your $\in_0$. But at this point, our setup has nothing to say on this matter. Then, as before, you might be struck by inspiration

[6]See Parsons [1990] and Field [1999] for thorough arguments for this attitude toward schemata in the context of arithmetic. See Section 4.3 of [Maddy and Väänänen, 2023] for a nuanced discussion of the relationship Parson's thought experiment and internal categoricity. One might worry that this talk of approximating a second order concept with a schema doesn't take us far enough away from second order logic. See page 30 of [Maddy and Väänänen, 2023] for further discussion of this point. We shall raise this again in the Discussion section.

[7]See [Lefever and Székely, 2019] for an excellent overview of different characterizations of definitional equivalence.

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and adopt a *Parsonian* attitude to the Replacement and Separation Schemata by permitting formulae of the total language in both versions of those schemata. This leaves us in $ZFC(\in_0, \in_1)$ where we may finally use Väänänen's theorem to define an isomorphism between my version of the membership relation and yours. This might give you some reason to think that what I mean by $\in_1$ is very closely related to you mean by $\in_0$. Moreover, it might give you some reason to think that whatever you think about the truth value of the continuum hypothesis, I should think the same. The analogy with the dense linear orders is certainly not perfect, but I think it nicely illustrates the fact that we are dealing with a common or, at least, overlapping methodology. If we want to compare a pair of theories, then it makes good sense to consider ways of combining them. Finally, we note that second order logic has disappeared from view and so the circularity worries we had above no longer have a target. Putting all this together, it could seem that we have good reason to think that you and I should agree on the truth value of the continuum hypothesis using a methodology common in set theory that does not fall prey to worries about circularity.

I don't plan to precisely formulate arguments about the speculative reasoning above. Nor do I plan to evaluate their force. I have two reasons for this. First, I think there is value in merely illustrating the intuitive pull that surrounds Väänänen's theorem in an effort to flag it for more intense philosophical scrutiny. Second, I think the results that follow below place a sufficiently large question mark over such argument sketches that the effort required to understand what is going on will be substantial and well beyond the scope of this paper.

2. What is MV?

The incompleteness of ZFC as evidenced by forcing arguments is sometimes thought of as particularly deep. By contrast, shallow forms of incompleteness can be resolved by adopting a stronger theory that we think is true. For example, although ZFC cannot prove that it is consistent, if we adopt an axiom saying that there are inaccessible cardinals then the consistency of ZFC follows almost immediately. But over at the deep end, the continuum hypothesis can be both forced to hold and forced to fail. Moreover, no large cardinal axiom can come to the rescue and decide the matter. The generic multiverse was introduced as a means of taking such deep incompleteness seriously as something we might just have to live with. In its simplest form, a generic multiverse $\mathbb{V}_M$ is formed by taking a countable transitive model M of ZFC and then considering all the models that can be obtained from M by finite sequences of generic extensions and refinements [Woodin, 2012]. We might then live with the incompleteness of ZFC by only taking seriously those statements that are true in every world N in $\mathbb{V}_M$. In particular, the continuum hypothesis is relegated to the liminal space between always true and always false. Woodin describes the motivation behind the generic multiverse particularly clearly:

> Is the generic-multiverse position a reasonable one? The refinements of Cohen's method of *forcing* in the decades since his initial discovery of the method and the resulting plethora of problems shown to be unsolvable, have in a practical sense almost compelled one to adopt the generic-multiverse position.[8] [Woodin, 2012]

[8] I should mention at this point that although Woodin clearly describes why embracing the generic multiverse might be attractive and understandable, his goal in that paper is to argue against its adoption.

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If our goal is to take deep incompleteness seriously, then Woodin's $\mathbb{V}_M$ has some limitations. $\mathbb{V}_M$ is a model that we describe, so to speak, from the outside rather than a foundational framework like ZFC than we can use, so to speak, from the inside. It's still a valuable tool for understanding how the generic multiverse position works, but it doesn't offer us a way of truly inhabiting its multiverse. To address this, Steel introduced a computably axiomatizable theory of the generic multiverse that is designed to be an autonomous foundation for mathematics, which is on a par with ZFC, and within which we can live with deep incompleteness [Steel, 2014].

Steel's theory has received a healthy dose of high quality philosophical attention since its inception [Maddy and Meadows, 2020, Meadows, 2021, Bagaria and Ternullo, 2023, Blue, 2024]. My goals in this section and the rest of this paper are more logical in spirit. I want to give an overview of: what MV looks like from the outside, through a rough classification of its models; and what MV feels like on the inside, by highlighting some significant consequences of the theory. My thought is that by delivering a more comprehensive overview of MV, the internal categoricity results below will make more sense to the reader and perhaps over-optimistically that these results might make MV more accessible to a wider class of readers.

2.1. **Notation.** Before we launch into the theory, let's first make some remarks about notation. In general, I am aiming to follow the notation conventions used by contemporary set theorists as used in, for example [Jech, 2003, Kunen, 2011] and [Foreman and Kanamori, 2009]. However, I'll highlight a few local idiosyncrasies that I've taken up in order to make things look a little smoother on the page.

First and contrary to the usual conventions in mathematical logic, a theory will just be a set of sentences that we usually call axioms. We shall not demand, unless stated otherwise, that a theory is closed under logical consequence. Given that internal categoricity is sensitive to the axioms used to generate a theory, this is particularly important.[9]

We shall, below, be frequently concerned with models of $\mathcal{L}_\in = \{\in\}$. Most often these will be transitive models and for those, we adopt the usual convention of denoting the model $\langle M, \in\rangle$ by its domain M. However, when we come to denote models that have a non-standard membership relation we shall use the calligraphic font and write $\mathcal{M} = \langle M, \in_{\mathcal{M}}\rangle$.

In almost all of the many simple forcing arguments that follow we shall make great use of Lévy's collapse forcing. Recall that $Col(\omega, X)$ for $X \subseteq Ord$ is the partial order whose conditions are finite partial functions $p : \omega \times X \rightharpoonup X$ where for all $n \in \omega$ and $\alpha \in X$, $p(n, \alpha) < \alpha$ if it is defined. We then order $Col(\omega, X)$ be reverse inclusion. Rather than write $Col(\omega, X)$ again and again, we shall instead write $\mathbb{C}_X$ where the $\mathbb{C}$ is intended to be mnemonic for collapse. We shall also use interval notation for sets of ordinals so, for example $[\alpha, \beta) = \{\gamma \in \beta \mid \gamma \geq \alpha\}$. If H is $\mathbb{C}_\beta$-generic over V for some $\beta \in Ord \cup \{Ord\}$ and $\alpha < \beta$, we write $H \restriction \alpha$ to denote $\{p \in \mathbb{C}_\alpha \mid p \in H\}$.

The following theorem of Laver and Woodin plays a pivotal role in the theory of the generic multiverse. We shall also use the terms "generic refinement" and "ground" as synonyms below. U is a generic refinement of W if there is some U-generic G such that $W = U[G]$.

[9]We should note that this approach to theories is quite common in the theory of interpretability. See, for example, [Lindström, 2003] and [Enayat and Łełyk, 2024].

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Fact 3. *(Laver, Woodin) Every generic refinement U of V is uniformly definable in V using a parameter from U. That is, there is a formula $\psi(x, y)$ of $\mathcal{L}_\in$ such that if $V = U[G]$ is a forcing extension of U by a U-generic filter $G \subseteq \mathbb{P} \in U$, then there is some $r \in U$ such that*[10]

$$U = \{x \mid \psi(x, r)\}.$$

Let us fix a formula $\psi(y, x)$ as in the statement of the fact above. We shall write $x \in \mathsf{W}_r$ to mean that $\psi(x, r)$ holds. This is usually just written as W_r rather than W_r. However, when talking about the generic multiverse and using the theory, MV, upper case variables like W are in high demand. Thus, we use the sans serif font W to avoid confusion.

2.2. **The MV theory.** We are now ready to describe MV. The theory is articulated in the language $\mathcal{L}_{MV}$ which is a two sorted language extending $\mathcal{L}_\in$ with a new sort for *worlds* beyond the existing sort for *sets*. Sets can be members of both sets and worlds, but worlds can be members of neither sets nor worlds.[11] The axioms of MV can then be stated somewhat formally as follows.[12] We shall write $gen(\mathbb{P}, W)$ to denote the set of $\mathbb{P}$-generic filters of a world W.

(World-extensionality) $\forall x(x \in W \leftrightarrow x \in U) \rightarrow W = U$
(World-transitivity) $x \in W \rightarrow x \subseteq W$
(World-ordinals) $\alpha \in Ord^W \leftrightarrow \alpha \in Ord^U$
(ZFC-worlds) φ^W where φ is an axiom of ZFC.
(Set-capture) $\exists W\ x \in W$
(Extension) $\mathbb{P} \in W \exists U \exists g \in U\ (g \in gen(\mathbb{P}, W) \wedge U = W[g])^U$
(Refinement) $r \in W \rightarrow \exists U\ (U = (\mathsf{W}_r)^W$
(Amalgamation)

$$\exists \mathbb{P}_0 \in W_0 \exists \mathbb{P}_1 \in W_1 \exists U \exists g_0, g_1 \in U$$
$$(g_0 \in gen(\mathbb{P}_0, W_0) \wedge g_1 \in gen(\mathbb{P}_1, W_1) \wedge$$
$$W_0[g_0] = U = W_1[g_1])^U.$$

The first five axioms essentially say that worlds are inner models of ZFC and that every set x is captured by a world W in the sense that $x \in W$. The final three axioms breathe life into the *generic* aspect of this multiverse. Extension tells us that any generic extension of a world is itself a world. Refinement tells us that any generic refinement of a world is a world. Amalgamation tells us that any pair of worlds have a common generic extension.

Extension and Refinement are straightforwardly motivated. They tell us that the multiverse is closed under forcing and its inverse. Amalgamation is a little more complicated.[13] One quite simple-minded motivation for including it is that it ensures that our multiverse doesn't contain, so to speak, isolated galaxies since any pair of worlds are subsumed into a larger world. However, we shall see very soon

[10] See [Reitz, 2007] for a gentle proof and [Woodin, 2012] for something more general.
[11] Strictly we should have two membership relations: one for when sets are members of sets; and the other for when sets are members of worlds. However, using just one symbol lightens our notation load in a helpful way so we adopt this abuse of notation.
[12] I'm adopting the convention of omitting initial universal quantifiers below. Thus, for example, our articulation of Refinement is really just an abbreviation of $\forall W \forall r\ r \in W \exists U\ (U = (\mathsf{W}_r)^W$
[13] See Maddy and Meadows [2020] for some extensive discussion of this.

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that Amalgamation also allows us to obtain particularly elegant models of MV that have a structure which is very familiar to set theorists.

It's easy enough to offer these informal explanations but they only get us so far. In the next section, we'll start thinking more seriously about models of MV, but before we do this, let's ask a natural question about the consequences of this theory:

> Given that MV is a theory of sets and worlds, we can consider its reduction to the set domain, which gives us a theory in $\mathcal{L}_{\in}$. What is that theory like?

It turns out that it has a very nice theory indeed. Informally, we might say that it satisfies countable set theory. More precisely, let ZFC^{-} be ZFC articulated using the Collection schema (rather than Replacement) and a Set Induction schema (rather than Foundation) where we remove the Powerset axiom.[14] Let ZFC^{-}_{count} be ZFC^{-} with the following additional axiom: $\forall x\ |x| \leq \omega$. This is the natural theory of the hereditarily countable sets and it is implied by MV.[15]

Theorem 4. $MV \vdash ZFC^{-}_{count}$.

Before we remark on the proof of this theorem, this is a good moment to return our attention to the continuum hypothesis and consider how it should be understood in the context of MV. We remarked above that MV was developed with an eye to taking the independence of the continuum hypothesis seriously. But how is this achieved? It turns out that there are at least two ways of thinking about this. On the first way of understanding things, we treat each world in the multiverse as a genuine alternative and we see then that there will be worlds where the continuum hypothesis is true and worlds where it is false. This is a theorem of MV that is essentially a consequence of Cohen's initial work on forcing [Cohen, 1963]. We might call this the *local perspective*. From this point of view, the cardinality of the continuum is not determined since it has many possible values across the multiverse. The most prominent expositer of this perspective is Woodin. Although he argues that the position is misguided, he also acknowledges that it is an understandable one to take up given the radical incompleteness of ZFC and its extensions that has been revealed by forcing arguments since the 1960s [Woodin, 2012, 2011a, 2004].[16] By contrast, the *global perspective* takes up the point of view considered in Theorem 4. Rather than worrying about whether the continuum hypothesis is true in one world or false in another, we ask whether it is true in the underlying, global, universe of sets that sits behind each of the worlds in the multiverse. The theorem above then tells us that every set is countable. Unless we wish to contradict Cantor's famous theorem, this means that there cannot be a set of all subsets of the natural numbers: $\mathcal{P}(\omega)$ does not exist. Thus, from the global perspective the continuum hypothesis is simply inexpressible and thus, arguably enjoys an even greater degree of indeterminacy than is exhibited locally. This global perspective is closely related to philosophers of set theory interested in countabilism: the thesis that there are no uncountable infinities [Meadows, 2015, Pruss, 2020, Scambler, 2021, Builes and Wilson, 2022, Barton, 2025]. I do not wish to put forward

[14]Nothing important swings on using Set Induction rather than Foundation. Like Collection, it just behaves better in weakenings of ZFC.
[15]Note that Choice is already implied by $\forall x\ |x| \leq \omega$ but we retain the C just to emphasize that Choice is available.
[16]For some critical discussion of Woodin's arguments, see [Meadows, 2021].

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a claim that one perspective is better than the other or that a choice must be made.[17] After all, they both fit the formal theory quite well. Regardless, it is clear that – either way – the continuum hypothesis has no definitive answer in MV.

2.3. **The models of MV.** Let return to Theorem 4 and its proof. It shouldn't be too surprising to see that MV implies every set is countable: Extension ensures that for any cardinal, there will be a world where it has been collapsed. Since Cantor's theorem tells us that any set and its powerset have different cardinalities, we also see that the Powerset Axiom must fail. To show that ZFC^- is satisfied, it will be helpful to consider some existing results about models of MV. Let's begin with some technical preliminaries.

A model of MV is of the form $\mathcal{W} = \langle \mathbb{W}^{\mathcal{W}}, \mathbb{D}^{\mathcal{W}}, \in^{\mathcal{W}} \rangle$ where $\mathbb{W}^{\mathcal{W}}$ is the set domain, $\mathbb{D}^{\mathcal{W}}$ is the world domain and $\in^{\mathcal{W}}$ is the membership relation. In [Steel, 2014] and [Maddy and Meadows, 2020], we are given a particularly nice class of models for MV that we might call its *regular models*. Let $\mathcal{M} = \langle M, \in^{\mathcal{M}} \rangle$ be a model of ZFC and suppose that G is $(\mathbb{C}_{Ord})^{\mathcal{M}}$-generic over $\mathcal{M}$. Then let $\mathcal{M}[G] \supseteq \mathcal{M}$ be the (class) generic extension of $\mathcal{M}$ via G.[18] Note that since $(\mathbb{C}_{Ord})^{\mathcal{M}}$ is a pretame forcing over $\mathcal{M}$, we see that $\mathcal{M}[G]$ satisfies ZFC^-. A regular model $\mathcal{M}^G$ is then formed from the set $\mathbb{W}^{\mathcal{M}^G}$ of generic refinements of $\mathcal{M}[G \restriction \alpha]$ for each $\alpha \in Ord^{\mathcal{M}}$.[19] Thus, $\mathcal{M}^G = \langle \mathbb{W}^{\mathcal{M}^G}, \mathcal{M}[G] \rangle \in^{\mathcal{M}[G]}$. We then recall that:

Fact 5. *(Steel, Theorem 26 and 27 in [Maddy and Meadows, 2020]) (1) If $\mathcal{M}$ is a model of ZFC and G is $(\mathbb{C}_{Ord})^{\mathcal{M}}$-generic over $\mathcal{M}$, then $\mathcal{M}^G \models MV$.*

(2) If $\mathcal{W}$ is a countable model of MV and $\mathcal{M}$ is a world in $\mathcal{W}$, then for some G that is $(\mathbb{C}_{Ord})^{\mathcal{M}}$-generic over $\mathcal{M}$, we have

$$\mathcal{W} = \mathcal{M}^G.$$

Informally speaking, given any model $\mathcal{M}$ of ZFC and a generic to collapse its ordinals, we may use them to obtain a regular model of MV. And in the other direction, we see that if $\mathcal{W}$ is a *countable* model of MV then it must be a regular model. We can use this to prove the theorem about the set part of MV.

Proof. (of Theorem 4). Let $\mathcal{W}$ be a model of MV. We'd like to apply Fact 5(2), but for that we need $\mathcal{W}$ to be countable. But if $\mathcal{W}$ is not countable, we can simply take a countable Skolem hull. Now let

[17] We should also note that Steel himself offers a different perspective on MV. In particular, he thinks of $\mathcal{L}_{MV}$ as a sub-language of $\mathcal{L}_{\in}$ given by the range of the translation function t that translates sentences of $\mathcal{L}_{MV}$ into $\mathcal{L}_{\in}$. He remarks then that "$\mathcal{L}_{MV}$ is meant to be an artificial language, with no meaning except that given by the translation t" [Steel, 2024b]. This perspective is dependent on a theory of meaning in mathematics that requires further development and which has been criticized by [Maddy and Meadows, 2020]. Theorem 12 of [Maddy and Meadows, 2020] establishes that the range of t is (up to ZFC equivalence) exactly the generic invariant sentences of $\mathcal{L}_{\in}$; i.e., those that are either true at every world or false at every world. This is compatible with the local perspective described above and is closely related to Woodin's notion of a *multiverse truth* [Woodin, 2012]. However, if we extend ZFC with the Ground Axiom (see Definition 1.1 in [Reitz, 2007]), Theorem 17 of [Maddy and Meadows, 2020] it turns out that t becomes a surjection (up to to equivalence in this extension of ZFC). Discussion of these points and responses to them can be found in [Steel, 2024a]. At present, I think it is fair to say that more work will be required to resolve these issues.

[18] If $\mathcal{M}$ isn't transitive, it's a little fiddly to ensure that $\mathcal{M} \subseteq \mathcal{M}[G]$. A way of doing this is described in Appendix A.1 of [Maddy and Meadows, 2020], where it is denoted as $\mathcal{M}_{ult}[G]$, however, we'll gloss over that subtlety here.

[19] Or a little more precisely, $\mathcal{N} = \langle N, \in^{\mathcal{N}} \rangle$ is a world in $\mathcal{M}^G$ is there is some $\alpha \in Ord^{\mathcal{M}}$ and $r \in \mathcal{M}[G \restriction \alpha]$ such that $x \in N$ iff $\langle \mathcal{M}[G], \mathcal{M}, \in^{\mathcal{M}[G]} \rangle \models x \in (\mathbb{W}_r)^{\mathcal{M}[G \restriction \alpha]}$.

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$\mathcal{M}$ be an arbitrary world in $\mathcal{W}$. Fact 5(2) then tells us that we may fix some $(\mathbb{C}_{Ord})^{\mathcal{M}}$-generic over $\mathcal{M}$ such that $\mathcal{W} = \mathcal{M}^G$. Then since $(\mathbb{C}_{Ord})^{\mathcal{M}}$ is pretame, the set domain $\mathcal{M}[G]$ of $\mathcal{M}^G$ satisfies ZFC^-. Moreover, Extension ensures that every set is countable. □

The Fact above is stated quite generally in that it accommodates the worlds that are nonstandard in the sense that their membership relation is not well-founded. But most (although not all) of the time, we'll be dealing with models in which every world is transitive. In those cases, we know that $\in^{\mathcal{W}}$ is just the *real* $\in$ restricted to $\mathbb{D}^{\mathcal{W}}$. Moreover, in these case we can, without loss of generality, think of $\mathbb{W}^{\mathcal{W}}$ as literally being the set of those transitive sets. If we do this then we see that $\mathbb{D}^{\mathcal{W}} = \bigcup \mathcal{W}$.[20] Thus, when we deal with transitive models of MV, we see that $\mathbb{D}^{\mathcal{W}}$ and $\in^{\mathcal{W}}$ are redundant and so in such cases, we'll often abuse notation and let $\mathcal{W}$ just be its world domain $\mathbb{W}^{\mathcal{W}}$. Now if we restrict our attention countable transitive models we get a particularly nice model-theoretic representation MV.

Corollary 6. *For countable $\mathcal{L}_{MV}$-structures $\mathcal{W}$ containing transitive worlds, the following are equivalent:*

(1) $\mathcal{W} \models MV$*; and*
(2) $\mathcal{W} = M^G$ *for some G that is $Col(\omega, < Ord^M)$-generic over M.*

For a helpful way to think about regular models of MV, it is helpful to recall the proof behind Solovay's famous model where the reals are Lebesgue measurable, have the property of Baire and the perfect set property [Solovay, 1970]. We begin that proof by supposing that there is an inaccessible cardinal κ and then consider a model where all the ordinals below κ have been collapsed using a generic G for $\mathbb{C}_\kappa$. This was the first of many similar models obtained by collapsing large cardinals. It is a familiar playground for set theorists and – as we'll see many of the techniques for working there translate quite directly to models of MV. The main difference is that to make a regular model of MV we don't just collapse some of the ordinals, we collapse all of them. As we've seen this gives us our set domain. We might then regard the worlds of the multiverse are recording the history of all possible forcings along the road to this great collapse.

Regular models arguably provide the most natural tool for understanding MV and how it is connected to ZFC. Moreover, this bridge provides a helpful and well understood toolbox for proving theorems about MV. However, not all models of MV are regular. Given that we concerned in this paper with questions around categoricity, it will be helpful to survey the landscape beyond the realm of regular models. Of course, MV is just a two-sorted theory in first order logic and so, by the upward Löwenheim-Skolem theorem, it is obvious that there are models of MV of every possible cardinality. But if we place some natural constraints on the models and the worlds they contain a more interesting picture emerges. For example, given Corollary 6 we might wonder if every model $\mathcal{W}$ of MV containing countable transitive models must itself be countable. This is not the case.

Proposition 7. *If $MA(\kappa)$ holds for $\kappa \geq \omega$, then there are models $\mathcal{W}$ of MV with $|\mathcal{W}| = \kappa^+$ where every world M in $\mathcal{W}$ is countable and transitive (if there is a countable transitive model of ZFC). Moreover, any such $\mathcal{W}$ will be non-regular.*

[20] We'll see later that the converse does not hold: we cannot recover the world domain from the set domain.

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Proof. We start by observing that such models cannot be regular. To see this, suppose M is a world in $\mathcal{W}$ and G is $\mathbb{C}^M_{Ord}$-generic over M. Then for all $\alpha \in Ord^M < \omega_1$, $M[G \restriction \alpha]$ is countable and so it has countably many generic refinements, $\mathsf{W}_r^{M[G\restriction\alpha]}$, indexed by elements $r \in M[G \restriction \alpha]$. Thus, the worlds of M^G are a countable union of countable sets, while $\mathcal{W}$ is uncountable.

Now we prove that such models exist. Let M be a countable transitive model of ZFC. Before we build our target model, first observe that if G_0 and G_1 are $\mathbb{C}_{Ord^M}$-generic over M, then we can easily find G_2 that is $\mathbb{C}_{Ord^M}$-generic over M where $M[G_2] = M[G_0 \times G_1]$. Thus, we obtain a regular model M^{G_2} extending both M^{G_0} and M^{G_1}. Clearly, this can be generalized to any finite set of $\mathbb{C}_{Ord^M}$-generics. We might say that any finite collection regular models based on M can be amalgamated into another regular model. Similar remarks apply to countable collections of regular models, but as we've seen no amalgamation fact can hold for uncountable collections of regular models. We address this problem by inductively defining a sequence $\langle G_\alpha\rangle_{\alpha<\kappa^+}$ of filters that are mutually $\mathbb{C}_{Ord^M}$-generic over M. Let $\alpha < \kappa^+$ and suppose that we've already defined $\langle G_\beta\rangle_{\beta<\alpha}$ such that for all finite $X \subseteq \alpha$, $\prod_{\gamma\in X} G_\gamma$ is $\prod_{\gamma\in X} \mathbb{C}_{Ord^M}$-generic over M.[21] We wish to define G_α so that $\langle G_\beta\rangle_{\beta<\alpha+1}$ has the analogous property at $\alpha+1$. For finite $X \subseteq \alpha$, let

$$\mathcal{D}_X = \{D \in M[\prod_{\gamma\in X} G_\gamma] \mid D \text{ is dense in } \mathbb{C}_{Ord^M}\}.$$

Then let $\mathcal{D} = \bigcup_{X\in[\alpha]^{<\omega}} \mathcal{D}_X$. Since $\alpha < \kappa^+$, we see $|\mathcal{D}| \leq \kappa$ and since $\mathbb{C}_{Ord^M}$ has the ccc (in V), we may use $MA(\kappa)$ we may fix a filter G_α where $G_\alpha \cap D \neq \emptyset$ for all $D \in \mathcal{D}$. This ensures that for all finite $X \subseteq \alpha$, G_α is $\mathbb{C}_{Ord^M}$-generic over $M[\prod_{\gamma\in X} G_\gamma]$ and application of the product lemma completes our inductive argument. Finally, we then let $\mathcal{W}$ be the set of generic refinements of models of the form $M[\prod_{\gamma\in X}(G_\gamma \restriction \alpha)]$ where $X \in [\kappa^+]^{<\omega}$ and $\alpha \in Ord^M$. □

Thus, we see that in contexts where some form of Martin's axiom holds, there will be large non-regular models of MV in which every world is countable and transitive. Moreover, since $MA(\omega)$ is a theorem of ZFC, we see that such non-regular models are unavoidable.

Corollary 8. *There are models $\mathcal{W}$ of MV with $|\mathcal{W}| = \aleph_1$ where every world $\mathcal{M}$ in $\mathcal{W}$ is countable and transitive.*

Given that the delivery of generic sets for models of ZFC is often dependent on those models being countable, we might also wonder whether regular models of MV can only contain countable models. This is not the case.

Proposition 9. *If $0^\#$ exists, there there are regular models $\mathcal{W}$ of MV that contain uncountable transitive worlds.*

Proof. Since $0^\#$ exists, we know that L_{ω_1} satisfies ZFC. To deliver a regular model of MV as desired, it will suffice to show that there is some G that is $\mathbb{C}_{\omega_1}$-generic over L_{ω_1}. Since $\mathbb{C}_{\omega_1}$ is not countable, the usual Baire category argument will not work. We address this by defining the required generic inductively.[22]

[21] Note that $\prod_{\gamma\in X} \mathbb{C}_{Ord^M}$ is just $(\mathbb{C}_{Ord^M})^{|X|}$.

[22] A version of this argument can be found in the first claim within the proof of Theorem 3.5 in [Hjorth, 1995]. I learned it from a MathOverflow response by Farmer Schlutzenberg.

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First, we observe that for all $\alpha < \omega_1$, $\mathcal{P}(\mathbb{C}_\alpha)^L$ is countable and so we may obtain a $\mathbb{C}_\alpha$-generic filter over L_{ω_1} for any such α. To set up the induction, fix the club sequence $\langle \kappa_\alpha \rangle_{\alpha \leq \omega_1}$ of indiscernibles below ω_1 witnessing that $0^\#$ exists and where $\kappa_{\omega_1} = \omega_1$. We aim to deliver a sequence $\langle G_\alpha \rangle_{\alpha \leq \omega_1}$ such that:

(1) G_α is $\mathbb{C}_{\kappa_\alpha}$-generic over L; and
(2) $G_\alpha \subseteq G_\beta$ for all $\alpha \leq \beta \leq \omega_1$.

For successor ordinals, $\alpha + 1$, we just find H that is $\mathbb{C}_{[\kappa_\alpha, \kappa_{\alpha+1})}$-generic over $L[G_\alpha]$ and then $G_{\alpha+1}$ can be what is essentially the product of G_α and H. At limit ordinals β, we have to be a little more careful. Note first that each κ_β is inaccessible in L. Thus, L thinks that $\mathbb{C}_{\kappa_\beta}$ has the κ_β-cc property and so any maximal antichain A in $\mathbb{C}_{\kappa_\beta}$ will be a subset of $\mathbb{C}_{\kappa_\alpha}$ for some $\kappa_\alpha < \kappa_\beta$.[23] Given this, we can let $G_\beta = \bigcup_{\alpha<\beta} G_\alpha$. □

This result depends on a strong assumption beyond ZFC. Without it, it is possible to find models $\mathcal{W}$ of MV where every world is transitive, but $\mathcal{W}$ is not regular.

Proposition 10. *If $L_{\omega_1} \models ZFC$, then it is consistent that there is a model $\mathcal{W}$ of MV where every world M in $\mathcal{W}$ is uncountable and transitive, but $\mathcal{W}$ is not regular.*

Proof. Again, we are going to build a multiverse from L_{ω_1}. However, we know from Proposition 9 that this cannot be done within a model where $0^\#$ exists. More precisely, we need to find a model where there is a $\mathbb{C}_\alpha$-generic set for every $\alpha < \omega_1$, but no $\mathbb{C}_{\omega_1}$-generic set exists. To obtain such a model, we use Silver's collapse forcing. For $X \subseteq Ord$, we let $\mathbb{S}_X$ be the poset whose domain is formed from functions $p : s \times Y \to Ord$ where we have: the *first domain*, $s \in [\omega]^{<\omega}$; the *second domain*, $Y \in [X]^{\leq\omega}$; and for all $n \in s$ and $\alpha \in Y$, $p(n, \alpha) < \alpha$.[24] Silver's collapse forcing can be obtained from Lévy's collapse by weakening the constraint on the second domain so that it merely has to be countable rather than finite. This gives $\mathbb{S}_{\omega_1}$ quite different properties from $\mathbb{C}_{\omega_1}$. In particular, $\mathbb{S}_{\omega_1}$ has, so to speak, a certain amount of countable completeness in its second domain.

Now let H be $\mathbb{S}_{\omega_1}$-generic over L.[25] Since $\mathcal{P}(\mathbb{C}_\alpha)^L$ is countable for all $\alpha < \omega_1$, we see that there is a $\mathbb{C}_\alpha$ -generic over L for each such α. It will thus suffice to show that $L[H]$ contains no sets G that are $\mathbb{C}_{\omega_1}$-generic over L.[26] To demonstrate this, we prove a claim that suffices for the proposition.

Claim. (1) If $A \in [\omega_1]^{\omega_1} \cap V[H]$, then there is a countably infinite $X \subseteq A$ in L.

(2) If G is $\mathbb{C}_{\omega_1}$-generic over L, then for some $A \in [\omega_1]^{\omega_1} \cap L[G]$ there is no countably infinite $X \subseteq A$ in L.

[23] See the first lemma on page 15 of [Solovay, 1970] for the canonical proof of this fact.
[24] See Section 20 in Cummings' chapter in [Foreman and Kanamori, 2009] for more information on this and a more general definition.
[25] Strictly speaking, there is no reason to think such an H exists. However following usual conventions, we can think of ourselves as now working *virtually* within the context of $\Vdash_{\mathbb{S}_{\omega_1}}$ as calculated in L. This is sufficient for our consistency claim. Ordinarily, such metamathematical subtleties can be largely ignored. But given that this paper is so concerned with generic sets and whether they exist, it is worth taking the time to take a little care around this.
[26] Our proof is an elaboration of a sketch by Mohammad Golshani on MathOverflow. Nothing in the argument relies on us using L rather than V. It just lines up more smoothly with the story of the proof. I'm grateful to Jason Chen for drawing my attention to it.

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To see that this is sufficient, suppose toward a contradiction that $G \in V[H]$ is $\mathbb{C}_{\omega_1}$-generic over L. Then by (1) we may fix $A \in [\kappa]^\kappa \cap V[G]$ which has no countably infinite subset in L. But then we see that $A \in V[G] \subseteq V[H]$ and so by (2) A must have a countable infinite subset, which is impossible.

To prove (1), we make an argument that exploits the version of countable completeness possessed by the second domain of $\mathbb{S}_{\omega_1}$.[27] We suppose toward a contradiction that L contains no injection from ω into A. We continue by letting $\dot{A} \in L^{\mathbb{S}_{\omega_1}}$ be a name such that $\dot{A}_H = A$ and fixing $p \in H$ that forces the statement: there is is no injection $f : \omega \to \dot{A}$ with $f \in \check{V}$. Working in L, we then define sequences $\langle p_n \rangle_{n\in\omega}$ and $\langle \alpha_n \rangle_{n\in\omega}$ such that $p_0 \leq p$ and for all $n \in \omega$:

- $p_n \Vdash \alpha_n \in \dot{A}$;
- $\alpha_{n+1} > \alpha_n$;
- $p_{n+1} \leq p_n$; and
- $dom(p_n) = m \times Y$ where $m \in \omega$ is least among the second domains of conditions satisfying the first three bullet points.

We claim that there is some $k \in \omega$ such that for all $n \in \omega$, if m is the second domain of p_n, then $m \leq k$. To see this, we move back to V and observe that since $|A| = \kappa$ and there are only countably many possibilities for the first domains of conditions, there must be some $k \in \omega$ such that there are infinitely many $\alpha \in A$ with some $p \in H$ with $p \Vdash \alpha \in \dot{A}$ where the second domain of p is $< \kappa$. Since we have a bound on the first domain, it can be seen that $q = \bigcup_{n\in\omega} p_n \in \mathbb{S}_{\omega_1}$. Finally, let J be $\mathbb{S}_{\omega_1}$-generic over L with $q \in J$. Then we see that $\langle \alpha_n \rangle_{n\in\omega}$ is an injection from ω into A that is an element of L, which gives a contradiction completing our proof of (1).

To prove (2), we provide a $\mathbb{C}_{\omega_1}$-name $\dot{A}$ for a set A whose realization has no countably infinite subset in L. More specifically, we let

$$\dot{A} = \{\langle \check{\alpha}, p\rangle \in \check{\omega}_1 \times \mathbb{C}_{\omega_1} \mid p(0, \alpha) = 1\}$$

and $A = \dot{A}_G$. The see the idea behind this name, let $g = \bigcup G : \omega \times \omega_1 \to \omega_1$ and note that $\alpha \in \dot{A}_G$ iff $g(0, \alpha) = 1$. So we are restricting our attention to how conditions $p \in \mathbb{C}_{\omega_1}$ act on 0 in the first domain and we only care about whether the output is 1 or not. This gives us a name for an element of $[\omega_1]^{\omega_1} \cap L[G]$.

Next we claim that A is $Add(\omega_1, 1)$-generic over V. To see this, we recall a basic, abstract forcing fact. Let us say that $\sigma : \mathbb{P} \to \mathbb{Q}$ is *weakly dense* if σ is order preserving and for all $p \in \mathbb{P}$ and $q \in \mathbb{Q}$ with $q \leq \sigma(p)$, there is some $p^* \in \mathbb{P}$ such that $p \not\perp p^*$ and $\sigma^*(p) \leq q$. It is well-known then that if J is $\mathbb{P}$-generic over V, then the upward closure of σ"J is $\mathbb{Q}$-generic over V.[28] Now let us construe $Add(\omega_1, 1)$ as the poset whose conditions q are finite partial functions from ω_1 to 2; and let $\sigma : \mathbb{C}_{\omega_1} \to Add(\omega_1, 1)$ be such that for all $p \in \mathbb{C}_{\omega_1}$, $\sigma(p)(\alpha) = 1$ iff $p(0, \alpha) = 1$.[29] It can be seen that σ is a weakly dense embedding and so $J = \sigma$"G is $Add(\omega_1, 1)$-generic over L. Moreover, it is not difficult to see that $A = \bigcup J$. Finally, suppose toward a contradiction that $X \in L$ and $X \subseteq A$. Now let $\dot{J}$ be the canonical

[27]The argument is a simple generalization of the standard argument that forcings closed under ω-sequences don't add any new ω-sequences. See, for example, Theorem VII.6.14 in [Kunen, 2006].

[28]See Lemma 15.45 in [Jech, 2003] for a proof.

[29]This is not strictly correct. Since $p(0, 0)$ has no possible output and $p(0, 1)$ can only output 0, we can't just use α on both sides. Fixing this makes a simple idea ugly on the page, so I'll stick with the sloppiness. It is, however, easily addressed and we leave that to the reader.

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name for a $Add(\omega_1, 1)$-generic set.[30] Then we may fix $q \in J$ such that $q \Vdash X \subseteq \bigcup \dot{J}$. But this means that we have a finite condition $q \in Add(\omega_1, 1)$ determining infinitely many membership facts about $\bigcup \dot{J}$, which is impossible. □

Taking a little stock, we've seen that while regular models of MV are common, they are not ubiquitous. We should also note that in the models of MV above that contained uncountable transitive worlds, W, the ordinals of those worlds were identical to ω_1. We might then wonder if we can have models of MV containing worlds with a longer sequence of ordinals. This is not the case.[31] To see this, we start by noting that the "external" cardinality of any pair of sets in some $\mathcal{W}$ must be the same.

Proposition 11. *If $\mathcal{W}$ is a model of MV then for all x, y in the set domain of $\mathcal{W}$*

$$|\{z \in M \mid z \in^{\mathcal{W}} x\}| = |\{z \in M \mid z \in^{\mathcal{W}} y\}|.$$

Proof. First note that since MV implies ZFC^{-}_{count}, if x is any element of the set domain of $\mathcal{W}$, then there is some f_x, also in $\mathcal{W}$'s set domain, such that $\mathcal{W} \models f_x : \omega \cong x$. It is then clear that f can be pulled back into V to obtain a bijection

$$f^*_x : \omega^{\mathcal{W}} \cong \{z \in M \mid z \in^{\mathcal{W}} x\}.$$

Thus, $f^*_y \circ f^{*-1}_x$ witnesses the required claim. □

The nonstandardness of the worlds of such a model then follows easily.

Corollary 12. *If $\mathcal{W}$ is a model of MV in which there is a world $\mathcal{M}$ and some $x \in \mathcal{M}$ such that*

$$\{y \in M \mid y \in^{\mathcal{W}} x\}$$

is uncountable, then $\mathcal{W}$ is not an ω-model.

Proof. Let x be a member of the set domain of $\mathcal{W}$ be such that its $\in^{\mathcal{W}}$-members are uncountable. Then using the proof of Proposition 11, we may fix a bijection f^*_x between $\omega^{\mathcal{W}}$ and the $\in^{\mathcal{W}}$-members of x. This mean $\omega^{\mathcal{W}}$ is uncountable and thus $\mathcal{W}$ cannot be an ω-model. □

This concludes our rough classification of the models of MV.

3. When MV is not internally categorical

In this section, we show MV in its standard axiomatization is not internally categorical. The idea behind the proof is very simple. We are going to take a regular model $\mathcal{W}$ of MV and some world U from $\mathcal{W}$. We are then going to take a class generic extension $U[G]$ of $\mathcal{W}$ that cannot be accessed from U by set forcing but which is also a world in a multiverse with the same set domain as $\mathcal{W}$.

Theorem 13. *There exist distinct models $\mathcal{W}_0, \mathcal{W}_1$ of MV whose set domains are identical, assuming there is a countable transitive model of ZFC.*

[30] Here we mean that $\dot{J} = \{\langle \check{q}, q\rangle \mid q \in \mathbb{Q}\}$.
[31] This was observed by Usuba in the remarks following Definition 7.16 of [Usuba, 2017].

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Proof. Our goal is to describe a pair of models of MV whose set domains are identical. Let us start with a countable transitive model M of ZFC. Let $\mathbb{Q} \in M$ be M's version of the Easton support product of forcings that add a Cohen subset to every regular cardinal.[32] Now let H be $\mathbb{Q}$-generic over M and observe that this is a class generic extension of M for which there can be no set generic extension $M[g]$ of M with $M[g] = M[H]$. Next, we let J be $\mathbb{C}_{Ord^M}$-generic over $M[H]$. Thus, we have

$$M \subsetneq M[H] \subsetneq M[H \times J].$$

We plan to show that there is some G that is $\mathbb{C}_{Ord^M}$-generic over M with $M[G] = M[H \times J]$. To see that this will suffice, note that this will give us two regular models M^G and $M[H]^J$ of MV with the same set domain. However, their world domains cannot be the same since $M[H]$ is not a world in M^G.

To prove our target claim, we shall obtain the required set G from a generic filter on a class poset $\mathbb{P}^*$ over $M[H \times J]$ that will assemble G through a sequence of approximations. More specifically, we let $\mathbb{P}^*$ consist of sets $g \in M[H \times J]$ that are $\mathbb{C}_\alpha$-generic over M for some $\alpha \in Ord^M$, and we order $\mathbb{P}^*$ by reverse inclusion.[33] Now let G^* be $\mathbb{P}^*$-generic over $M[H \times J]$ and let $G = \bigcup G^*$. First, we claim that G is $\mathbb{C}_{Ord^M}$-generic over M. To see this let $A \subseteq \mathbb{C}_{Ord^M}$ be a maximal anti-chain that is definable in M. It can be seen that $\mathbb{C}_{Ord^M}$ is Ord-cc in M and so there must be some $\alpha \in Ord^M$ such that $A \subseteq \mathbb{C}_\alpha$.[34] Thus, it will suffice to show that there is some $g \in G^*$ where $g \cap \mathbb{C}_\alpha \neq \emptyset$. To see this, we exploit the genericity of G^* and make a density argument. In particular, we let

$$D = \{g \in \mathbb{P}^* \mid g \text{ is } \mathbb{C}_\alpha\text{-generic}/M\}$$

and show that D is pre-dense in $\mathbb{P}^*$. To see this, let $h \in \mathbb{P}^*$ and suppose h is $\mathbb{C}_\beta$-generic over M where $\beta \in Ord^M$. If $\beta \geq \alpha$, we would be done, so suppose $\beta < \alpha$. It will suffice to show that for some regular cardinal $\kappa > \alpha$ there is some $h^* \in M[H \times J]$ that is $\mathbb{C}_{[\beta,\kappa+1)}$-generic over $M[h]$. Let us start by fixing a $(\mathbb{Q} \times \mathbb{C}_{Ord^M})$-name, $\dot{h}$, such that $\dot{h}_{H\times J} = h$. Now since $\dot{h}$ is just a set it is clear that its transitive closure can only contain a set of conditions from $\mathbb{C} \times \mathbb{C}_{Ord^M}$. Thus, we may fix some $\gamma \in Ord^M$ such that $h = \dot{h}_{(H\times J)\restriction\gamma}$ and so $h \in M[(H \times J) \restriction \gamma]$. Now let κ be a regular M-cardinal $> \gamma, \alpha$. It is clear that $J_{\{\kappa\}}$ is $\mathbb{C}_{\{\kappa\}}$-generic over $M[(H \times J) \restriction \gamma]$.[35] Moreover, it can be seen that $\mathbb{C}_{\{\kappa\}}$ is forcing equivalent[36] to $\mathbb{C}_{[\beta,\kappa+1)}$ and so we may fix h^* that is $\mathbb{C}_{[\beta,\kappa+1)}$-generic over $M[(H \times J) \restriction \gamma] \supseteq M[h]$ as required.

Finally, we claim that $M[G] = M[H \times J]$. Clearly, $M[G] \subseteq M[H \times J]$ so we want to show that every $x \in M[H \times J]$ is in $M[G]$. To do this, we again exploit the genericity of G^* by showing that if $x \in M[H \times J]$ then

$$E = \{g \in \mathbb{P}^* \mid x \in M[g]\}$$

is dense in $\mathbb{P}^*$. To see this, let $\dot{x}$ be a $(\mathbb{Q} \times \mathbb{C}_{Ord^M})$-name in M such that $\dot{x}_{(H\times J)} = x$. Since $\dot{x}$ is just a set, we may fix, as in the previous claim, some $\alpha \in Ord^M$ such that $\dot{x}_{(H\times J)\restriction\alpha} = x$. Moreover, we may assume without loss of generality that $\alpha = \kappa + 1$ for some regular M-cardinal. Then it can be

[32] This is also the forcing used by Reitz to show that there is a model of ZFC that doesn't satisfy the Bedrock axiom. See Theorem 5.3 in [Reitz, 2007].

[33] A similar forcing is used in the proof of Theorem 3.1.5 in [Larson, 2004]. Indeed, it can be used to give a more elegant proof of Theorem 27 in [Maddy and Meadows, 2020]. I learned this from Gabe Goldberg.

[34] This can be established with a trivial generalization of the proof of first lemma on page 15 of [Solovay, 1970].

[35] We write $H \restriction \alpha$ to denote those $p \in H$ with $dom(p) \subseteq \alpha$.

[36] See Proposition 10.20 and and the proof of Proposition 10.21 in [Kanamori, 2003] for more details.

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seen that $\mathbb{C}_{\{\kappa\}}$, $\mathbb{C}_{\kappa+1}$ and $(\mathbb{Q} \times \mathbb{C}_{Ord^M}) \restriction (\kappa + 1)$ are all forcing equivalent to each other. Thus, there is some g that is $\mathbb{C}_{\kappa+1}$-generic over M with $x \in M[g]$. □

We shall now use this fact to establish that MV is not internally categorical. However, the underlying idea is quite obvious. The models given in the theorem above are not isomorphic, even though they share a set domain. Before we state the claim, let us first described our target theory. Let $\mathcal{L}_{MV}(\in_0, \in_1)$ be the language with a set sort and a world sort, but which has two membership relations that behave as in $\mathcal{L}_{MV}$. Let $MV(\in_0, \in_1)$ be the theory consisting of $\in_0$ and $\in_1$ versions of every axiom of MV where we allow formulae involving both $\in_0$ and $\in_1$ into both $\in_0$ and $\in_1$ versions of all schemata.

Corollary 14. *$MV(\in_0, \in_1)$ is not internally categorical, if there is a countable transitive model of ZFC.*

Proof. Using the proof of Theorem 13, let us start with a countable transitive model M of ZFC, but this time suppose that M satisfies $V = L$. We may then obtain regular models $\mathcal{W}_0 \neq \mathcal{W}_1$ of MV with the same world domain whose worlds are countable transitive models satisfying ZFC. In particular, $\mathcal{W}_1$ is generated from a world N that is obtained from M by adding a Cohen subset to every M-regular cardinal. It can be seen that no world in $\mathcal{W}_1$ satisfies the Ground Axiom.[37]

We now define a model $\langle \mathbb{W}, \mathbb{D}, \in_0, \in_1 \rangle$ of $MV(\in_0, \in_1)$. We let the set domain $\mathbb{D} = \bigcup \mathcal{W}_0 = \mathcal{W}_1$. It is clear that $\mathcal{W}_0$ and $\mathcal{W}_1$ are both countable, so we may fix a bijection $\sigma : \mathcal{W}_1 \to \mathcal{W}_0$. Let $\mathbb{W} = \mathcal{W}_0$. We then let $\in_0$ be $\in$ restricted to $\bigcup \mathcal{W}_0$. And for $\in_1$, we let: $x \in_1 y$ iff $x \in y$ where x and y are both sets; and we let $x \in_1 M$ iff $x \in \sigma(M)$ when x is a set and M is a world. Since $\in_0$ and $\in_1$ are the same on the set domain, it is easy to see that having both $\in_0$ and $\in_1$ available to the (relativized) Collection and Separation schemas, used in MV, makes no difference. Thus, $\langle \mathbb{W}, \mathbb{D}, \in_0, \in_1 \rangle$ satisfies $MV(\in_0, \in_1)$. Finally, to see that there is no definable isomorphism between $\mathcal{W}_0 = \langle \mathbb{W}, \mathbb{D}, \in_0 \rangle$ and $\mathcal{W}_1 = \langle \mathbb{W}, \mathbb{D}, \in_1 \rangle$ it suffices to observe that they are not even elementary equivalent. To see this we note that some world in $\mathcal{W}_0$ satisfies the Ground Axiom since $\mathcal{W}_0$ is generated from a model M satisfying $V = L$. But we've already seen that no world in $\mathcal{W}_1$ satisfies the Ground Axiom. □

Thus, we see that MV is not internally categorical, but it is natural to wonder what would happen if we augmented MV with axioms demanding that certain large cardinals exist at some, or perhaps, all worlds.[38] This can be quite attractive since, if every world contains a Woodin cardinal, then it can be seen that every world must contain a proper class of them. Moreover, if there is a proper class of Woodin cardinals in every world then the theory of $L(\mathbb{R})$ and analysis will be the same in every world.[39] Beyond the issue of whether it is interesting to add large cardinals to MV, it is also known that large cardinals can exert a powerful influence on the structure of the models of MV. In particular, Usuba [2017] has shown that if a world has an extendible cardinal, then the multiverse has a core, or bedrock; i.e., there is a world that is a submodel of every other world and every other world is a generic extension of it. This could give us reason to doubt that the failure internal categoricity will be preserved into strengthenings of MV. These doubts appear to be ill-founded although some work is required to show this.

[37] See Theorem 5.3 in [Reitz, 2007] for a proof.
[38] See Section 5 of [Steel, 2014] for some discussion of this.
[39] See Chapter 3 of [Larson, 2004] for a proof of this.

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When we argued that MV is not internally categorical, it sufficed to prove that we could have distinct models of MV with the same world domain. However, the proof of this relied on a class forcing argument. We started with a model of ZFC and added a class of Cohen sets. This forcing trivially preserves ZFC, but some remarks are warranted with regard to the preservation of large cardinal assumptions. This is not a small question, but rather overlaps an entire subfield of set theory concerned with the preservation and indestructibility of large cardinal axioms through forcing.[40] As such, we shall content ourselves with small sample of large cardinal axioms that are intended to cut across a wide swathe of consistency strengths.

Lemma 15. *Consider the theories obtained by expanding MV with each of the following axioms:*

(1) *There is a world with an inaccessible cardinal;*
(2) *There is a world with a measurable cardinal;*
(3) *There is a world with a supercompact cardinal; and*
(4) *There is a world with an extendible cardinal.*

Assuming sufficient large cardinals, each of these theories has a pair of distinct models $\mathcal{W}_0 \neq \mathcal{W}_1$ whose set domains are identical.

For the last two of these theories, we make use of the following facts about supercompact and extendible cardinals.

Fact 16. *[Reitz, 2007] If κ is a supercompact cardinal, then*[41] *there is a class forcing extension $V[G]$ in which κ is supercompact and if we class force over $V[G]$ to add a Cohen subset to every regular cardinal λ in $V[G]$ where $2^{<\lambda} = \lambda$, then κ remains supercompact in that extension.*

Fact 17. *[Bagaria and Poveda, 2023] Suppose Vopenka's principle holds. Then there is an extendible cardinal κ. Moreover,*[42] *if G is generic for the Easton support forcing adding a Cohen subset to every regular cardinal, then κ remains extendible in $V[G]$.*

Proof. (1) The argument used in Theorem 13 suffices since the class forcing adding Cohen subsets to each regular cardinal preserves inaccessible cardinals.

(2) The argument from Theorem 13 can be adapted by only adding Cohen subsets of regular cardinals greater than the least measurable cardinal of our initial model M.

(3) Suppose that $\kappa < \lambda$ are supercompact and inaccessible respectively. Then take the collapse of a countable Skolem hull to obtain a countable transitive model M that things some $\bar{\kappa}$ is supercompact. Now using Fact 16, we may adapt the argument of Theorem 13 to deal with a forcing that only adds Cohen sets to regular cardinals δ such that $2^{<\delta} = \delta$.

[40] See Cummings' chapter in [Foreman and Kanamori, 2009] for an introduction to this topic.
[41] This can be proved using two moves from [Reitz, 2007]. Starting in V with a supercompact cardinal, κ, we use the proof of Theorem 3.9 to class force to a model $V[G]$ where the continuum coding axiom holds and κ remains supercompact and is indestructible by $< \kappa$-directed forcing. Then we the proof of Theorem 5.5 to add a Cohen subset to every regular λ such that $2^{<\lambda} = \lambda$. It can then be seen via the indestructibility that the supercompactness of κ is preserved by this extension.
[42] This is essentially a suboptimal version Corollary 5.13 in [Bagaria and Poveda, 2023]. The assumptions of the Corollary follow from Theorems 6.1-3 as well as Theorem 2.4 and Corollary 2.6 in [Bagaria and Poveda, 2023].

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(4) Suppose that λ is a Vopenka cardinal. Then take the collapse of a countable Skolem hull of V_λ to obtain a countable transitive model M satisfying Vopenka's principle. Let $\kappa \in M$ be such that M thinks κ is extendible. Fact 17, then ensures that the class forcing used in Theorem 13 preserves the extendibility of κ. The rest of that argument remains the same.[43] □

Corollary 18. *None of the theories considered in Lemma 15 are internally categorical.*

Thus, in its standard axiomatization we have seen that MV and some its large cardinal variants are not internally categorical. Intuitively, this probably doesn't come as a surprise, however, there is value in thinking about how we proved this. In a nutshell, the trick was to find a pair of countable transitive models of ZFC that share a $\mathbb{C}_{Ord}$-collapse but where one is a (proper) class forcing extension of the other. Reflecting on this issue motivates a modified axiomatization of MV that avoids this problem.

4. When MV is internally categorical

We are now ready to prove the more surprising result. There is an alternative axiomatization MV^* of MV that *is* internally categorical. Let us begin by introducing the new axiom schemata.

(Global-ZFC^-_{count}) φ^W for every axiom of ZFC^-_{count}.
(World-domination) If N is a definable inner model then there is a world W, such that $N \subseteq W$.

We already saw above that Global-ZFC^-_{count} follows from MV. So – at least from a mathematical perspective – there is no harm in adding it to MV. World-domination is more complicated. Informally, it tells us that if you can define an inner model N of ZFC using $\mathcal{L}_{MV}$, then there will be a world W that dominates in the sense that $W \supseteq N$. Let us call the theory obtained by augmenting MV with the schemata above MV^*. We shall show first that MV^* is internally categorical and then we will show that MV^* is an alternative axiomatization of MV by demonstrating that MV^* follows from MV.

We begin by establishing that ZFC^-_{count} is internally categorical. This may be of independent interest to those with countabilist sympathies. However, it will also be helpful for our main proof since we have the Global-ZFC^-_{count} schema. We let $ZFC^-_{count}(\in_0, \in_1)$ be the theory articulated in $\mathcal{L}(\in_0, \in_1)$ that is obtained by adding $\in_0$ and $\in_1$ versions of every axiom of ZFC^-_{count} and by allowing formulae involving both $\in_0$ and $\in_1$ into both $\in_0$ and $\in_1$ versions of the Separation and Collection schemata.

Theorem 19. *$ZFC^-_{count}(\in_0, \in_1)$ is internally categorical; i.e., we can prove in $ZFC^-_{count}(\in_0, \in_1)$ that there is a definable isomorphism between $\in_0$ and $\in_1$.*

Proof. We work informally in $ZFC^-_{count}(\in_0, \in_1)$. Roughly speaking, our plan is to mutually collapse $\in_0$ and $\in_1$ within each other, then repeat the collapses and observe that there is no room in between for difference. Let us write $\mathcal{V}_0$ for $\langle V, \in_0\rangle$ and $\mathcal{V}_1$ to denote $\langle V, \in_1\rangle$ where V is the underlying universe.[44] Note that using the Set Induction schema, it is easy to see that $\mathcal{V}_0$ thinks $\in_1$ is well-founded and that $\mathcal{V}_1$ thinks that $\in_0$ is well-founded. Given this, we may fix collapse maps:

[43]This argument will also work for the theory where every world has an extendible cardinal and thus every world has a proper class of extendible cardinals.
[44]Strictly speaking, I cannot refer to order $\langle V, \in_0\rangle$ within the framework of $ZFC^-_{count}(\in_0, \in_1)$ since it is an ordered pair of proper classes. It is easy enough to eliminate this abuse of notation, but it makes things less pleasant on the page, so we shall continue to talk in this way.

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- $\pi_1 : \mathcal{V}_1 \cong \mathcal{M}_1 = \langle M_1, \in_0 \rangle$; and
- $\pi_0 : \mathcal{V}_0 \cong \mathcal{M}_0 = \langle M_0, \in_1 \rangle$.

Thus, $\mathcal{M}_1$ is the result of collapsing $\mathcal{V}_1$ in $\mathcal{V}_0$ and $\mathcal{M}_0$ is the result of collapsing $\mathcal{V}_0$ in $\mathcal{V}_1$. Since these are isomorphisms, we can then, so to speak, find a copy of $\mathcal{V}_1$ inside $\mathcal{M}_0$. More formally, we have:

- $\pi_1 \circ \pi_0 \text{“}\mathcal{V}_0 = \pi_1 \text{“}\mathcal{M}_0 \subseteq \mathcal{M}_1 \subseteq \mathcal{V}_0$; and
- $\pi_0 \circ \pi_1 \text{“}\mathcal{V}_1 = \pi_0 \text{“}\mathcal{M}_1 \subseteq \mathcal{M}_0 \subseteq \mathcal{V}_1$.

Now if things worked out particularly nicely, we'd see that $\pi_1 \circ \pi_0 \text{“}\mathcal{V}_0$ was the collapse of $\mathcal{V}_0$ and thus was $\mathcal{V}_0$ itself. This would essentially complete the argument. However, for all we know one of $\mathcal{V}_0$ and $\mathcal{V}_1$ is, so to speak, shorter than the other and this means that the collapse might not reach every element of the other model. We get around this by noting that, according to both $\mathcal{V}_0$ and $\mathcal{V}_1$, since every set is countable, every set can be coded by a set of natural numbers. It will, thus suffice to show that the $\pi_0 \circ \pi_1 \text{“}\mathcal{V}_0$ and $\mathcal{V}_0$ have the same powerset of the naturals.

To show this, we start by reducing our notational burden by letting $\mathcal{N}_0 = \pi_0 \circ \pi_1 \text{“}\mathcal{V}_0$. It will then suffice show that $\mathcal{P}(\omega)^{\mathcal{V}_0} \subseteq \mathcal{N}_0$ since the other direction is trivial. Let's start with some $x \in \mathcal{P}(\omega)^{\mathcal{V}_0}$.[45] It should be clear that $\langle trcl(\{x\})^{\mathcal{V}_0}, \in \rangle$ is set-like according to $\mathcal{V}_1$ and so $\pi_0(x)$ is defined. From here it is can be seen that $\pi_1 \circ \pi_0(x) = x \in \mathcal{N}_0$ as required. To close things out, note that – after unpacking the coding – this implies that $\mathcal{N}_0 = \mathcal{V}_0$ and so

$$\mathcal{V}_0 \subseteq \mathcal{M}_1 \subseteq \mathcal{V}_0$$

which means that $\mathcal{V}_0 = \mathcal{M}_1$ and thus, $\pi_1 : \mathcal{V}_1 \cong \mathcal{V}_0$ as required. □

4.1. MV^* **is internally categorical.** We are now ready for the main result. Let $MV^*(\in_0, \in_1)$ be the theory articulated in $\mathcal{L}_{MV}(\in_0, \in_1)$ that consists of $\in_0$ and $\in_1$ versions of every axiom of MV^* where we allow formulae involving both $\in_0$ and $\in_1$ into both $\in_0$ and $\in_1$ versions of all schemata.

Theorem 20. *$MV^*(\in_0, \in_1)$ is internally categorical; i.e., we can prove in $MV^*(\in_0, \in_1)$ that there is a definable bijection on the set domain σ witnessing an isomorphism between $\in_0$ and $\in_1$ where σ also determines an isomorphism on the world domain.*

Proof. We work in $MV^*(\in_0, \in_1)$. Let $\mathcal{V}_0 = \langle V, \in_0 \rangle$ and $\mathcal{V}_1 = \langle V, \in_1 \rangle$. By Fact 4, we see that both $\mathcal{V}_0$ and $\mathcal{V}_1$ satisfy ZFC^-_{count}. Thus, we may use Theorem 19, to obtain a definable isomorphism $\sigma : \mathcal{V}_0 \cong \mathcal{V}_1$ as we require. This leaves the world domain. For worlds $\in_0$-worlds, W, we let $\sigma_+(W) = \sigma \text{“}W$. It will suffice to show that σ_+ is a bijection. There are two things that might go wrong: (1) $\sigma_+(W)$ might not be a world with respect to $\in_1$; and (2) σ_+ might not be a surjection.

Regarding (1), we see using $(*)$ for $\in_1$ that there an $\in_1$-world U such that $\sigma_+(W) \subseteq U$. Moreover, using $(*)$ for $\in_0$, we also see that there is an $\in_0$-world $W^\dagger$ such that $U \subseteq \sigma_+(W^\dagger)$. By the intermediate submodel theorem, in $\sigma_+(W^\dagger)$, we see that $\sigma_+(W) \subseteq U \subseteq \sigma_+(W^\dagger)$ is a sequence of generic extensions. This means $\sigma_+(W)$ is a generic refinement of U and so $\sigma_+(W)$ is an $\in_1$-world as we required.

Regarding (2), we use a similar argument. Given an arbitrary $\in_1$-world U, it will suffices to show that $\sigma_+^{-1}(U)$ is an $\in_0$-world. To see this, we use $(*)$ for $\in_0$, to fix an $\in_0$-world W such that $\sigma_+^{-1}(U) \subseteq W$

[45] Note that it is not strictly correct to speak of $\mathcal{P}(\omega)^{\mathcal{V}_0}$ since $\mathcal{V}_0$ contains no set of all subsets of its ω. However, we can instead regard $\mathcal{P}(\omega)^{\mathcal{V}_0}$ as the definable class of subsets of $\mathcal{V}_0$'s ω.

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and so $U \subseteq \sigma_+(W)$. Then using $(*)$ for $\in_1$, we may fix an $\in_1$-world $U^\dagger$ such that $\sigma_+(W) \subseteq U^\dagger$. As in the previous case, this implies that $\sigma_+(W)$ is a generic extension of U and thus, that W is a generic extension of $\sigma_+^{-1}(U)$ and so $\sigma_+^{-1}(U)$ is an $\in_0$-world as we require. □

4.2. MV **implies** MV^***.** To complete our demonstration that an axiomatization of MV is internally categorical, we need to prove that MV implies MV^*. To do this, we need to show that:

(1) World-domination is expressible in $\mathcal{L}_{MV}$; and
(2) World-domination follows from MV.

4.2.1. *World-domination is expressible.* To motivate why there might even be a problem here, let us recall that the naive statement of the ordinary axioms of MV is not obviously expressible in $\mathcal{L}_{MV}$. For example, the Refinement axiom is often stated as follows: "If U is a world, and $U = W[G]$, where G is $\mathbb{P}$-generic over W, then W is a world" [Steel, 2014]. While this conveys the idea very clearly, it's not so clear where W came from. We appear to be quantifying over it like a world before we even know that it is a world. The problem is resolved by using Fact 3, which allows for the "official" formulation given in Section 2.2. As stated above, our formulation of World-domination has similar problems. It appears to quantify over inner models, but in fact, it is more correctly and much more formally stated as a schema as follows:

(World-domination) For all sets $y_0, ..., y_n$ and for all worlds $U_0, ..., U_m$, If

$$\varphi(x, y_0, ..., y_n, U_0, ..., U_m)$$

is a formula of $\mathcal{L}_{MV}$ that defines an *inner model* of ZFC, then there is some world W such that $\forall x(\varphi(x) \to x \in W)$.

By referring to formulae we are able to remove any quantification over inner models. However, we still need to know when a particular formula defines an inner model of ZFC. This can be addressed with a classic fact.

Fact 21. *(ZFC^-) A transitive class M is an inner model of ZF iff it is closed under the Gödel operations and is almost universal; i.e., for every subset $X \subseteq M$ there is some $Y \in M$ with $Y \supseteq X$.*[46]

It is easy to see that being almost universal and closed under the Gödel operations is expressible in $\mathcal{L}_\in$. Then using this and the fact that MV implies ZFC^-, we can reformulate World-domination by demanding that the class of x such that $\varphi(x, y_0, ..., y_n, U_0, ..., U_m)$ is closed under the Gödel operations and is almost universal. This is easily expressed in $\mathcal{L}_{MV}$ and so World-domination is expressible as a schema in $\mathcal{L}_{MV}$.

[46]For a proof see Theorem 13.9 in [Jech, 2003]. The proof there is conducted in ZFC rather than ZFC^-, however, the proof is the same. The only (cosmetic) modification occurs in establishing almost universality of an inner model, M of ZFC. We start with $X \subseteq M$, and then obtain that $X \subseteq V_\alpha^M \in M$ rather than $X \subseteq V_\alpha \cap M$ since, in general, we cannot guarantee that V_α exists in ZFC^-.

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4.2.2. *World-domination follows.* Now we move onto our final task by showing that World-domination follows from MV. Our strategy is to exploit the fact, explored earlier, that models of MV are closely related to models like Solovay's where an inaccessible cardinals is collapsed to be ω_1. The key idea of this proof was provided by Gabe Goldberg and I'm grateful to him for letting me use it here. We work our way gently toward our target by starting with a warmup lemma, which contains the main idea of the proof. We then use this and some lemmas from Steel and Goldberg to push the result into less familiar setting of the generic multiverse, which then gives us that result we require.

Lemma 22. *Suppose G is $\mathbb{C}_{Ord}$-generic over V and that $M \subseteq V[G]$ is an inner model of ZFC that is definable in $V[G]$ using V as a parameter and parameters from V. Then $M \subseteq V$.*

Proof. It will suffice to show that every set of ordinals in M is also in V. To show this, we let λ be an arbitrary ordinal and let $Z = \mathcal{P}(\lambda)^M$. Since λ is arbitrary, it will suffice to show that $Z \subseteq V$. Our plan is to exploit the homogeneity of $\mathbb{C}_{Ord}$ and chase a contradiction regarding the cardinality of Z. The proof closely follows the argument used by Solovay to establish that perfect set property in his well-known model. Suppose toward a contradiction that $Z \nsubseteq V$ and fix some $y \in V \backslash Z$ witnessing this.[47] By standard facts about collapse forcing, we may fix $\alpha < \kappa$ sufficiently large such that:

(1) $y \in Z \in V[G \restriction \alpha]$; and
(2) Z is countable in $V[G \restriction \alpha]$.

Using the homogeneity and absorption properties of $\mathbb{C}_{Ord}$ we may fix a formula $\varphi_Z(x)$ using λ and elements of V as parameters, which is such that for any H that is $\mathbb{C}_\alpha$-generic over V

$$x \in Z \;\Leftrightarrow\; V[H] \models \varphi_Z(x)$$

for all $x \in V[H]$. Next we let $\dot{y}$ be a $\mathbb{C}_\alpha$-name such that $\dot{y}_{G \restriction \alpha} = y$. Then we may put things together and fix $p \in \mathbb{C}_\alpha$ such that:

$$p \Vdash \varphi_Z(\dot{y}) \wedge \dot{y} \notin \check{V} \wedge |\{x \subseteq \lambda \mid \varphi_Z(x)\}| \leq \omega.$$

Using the first two conjuncts and noting that $y \subseteq \lambda \subseteq V$, it can be seen – using Solovay's argument – that in $V[G]$, Z contains a perfect set. But using the final conjunct, we see that Z is countable in $V[G]$, which gives us our desired contradiction. □

We now recall an important fact about Steel's generic multiverse.

Fact 23. *(Goldberg, Lemma 38 in [Maddy and Meadows, 2020]) There is a total computable function $e : \mathcal{L}_{MV} \to \mathcal{L}_\in(\dot{M})$ with the following property. Suppose $\mathcal{W} \models MV$, M is a world of $\mathcal{W}$, and N is the collection of sets of $\mathcal{W}$. Then*

$$\mathcal{W} \models \varphi(\bar{x}) \Leftrightarrow \langle N, M\rangle \models e(\varphi)(\bar{x})$$

where $\langle N, M\rangle$ denote a model of $\mathcal{L}_\in$ expanded by a one-place relation symbol.

We now use this fact to prove a special case of World-domination where only one world parameter is involved.

[47] A similar argument can be found in the proof of Theorem 1.1 in [Kanovie and Lyubetsky, 2017].

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Lemma 24. *MV proves the following. Suppose M is an inner model of ZFC that is definable using a single world parameter U and parameters from W. Then $M \subseteq U$.*

Proof. Let $\mathcal{W}$ be a model of MV and without loss of generality, we suppose that $\mathcal{W}$ is countable. We show that the claim holds in $\mathcal{W}$. We start by letting $\varphi_M(x, \bar{y}, U)$ be the formula defining M in $\mathcal{W}$ where U is a world parameter $\bar{y}$ is a finite sequence of set parameters from U. Now using Fact 5, we may fix a $\mathbb{C}_{Ord^U}$-generic G such that $U[G]$ is identical to the set domain of $\mathcal{W}$. Then using Fact 23, we see that

$$\mathcal{W} \models \varphi_M(x, \bar{y}, U) \;\Leftrightarrow\; \langle U[G], U\rangle \models e(\varphi)(x, \bar{y}).$$

Thus, we see that M is defined in $U[G]$ by a formula of $\mathcal{L}_\in$ using U as a parameter and parameters from U. Finally, we use Lemma 22 to see that $M \subseteq U$, as required. □

Next, we use Amalgamation to prove the general case where multiple world parameters are involved.

Theorem 25. *MV proves the following. Suppose M is an inner model of ZFC that is definable using set and world parameters. Then there is some world W such that $M \subseteq W$.*

Proof. Again, we show that the claim holds in arbitrary models of MV. We start by fixing a model $\mathcal{W}$ of MV and we shall now work inside $\mathcal{W}$. Our plan is to use the Amalgamation axiom to reduce our problem to that addressed in the previous lemma. We suppose that M is definable by the $\mathcal{L}_{MV}$-formula $\varphi(x, y_0, ..., y_m, W_0, ..., W_n,)$ where $y_0, ..., y_m$ are set parameters and $W_0, ..., W_n$ are world parameters. First, we let W_{y_i} be a world with $y_i \in W_{y_i}$ for all $i \leq m$,. Then by repeated applications of the Amalgamation axiom, we may fix a world W such that for all $j \leq n$, $W_j \subseteq W$ and for all $i \leq m$, $W_{y_i} \in W$. The observe by the Laver-Woodin theorem, each of these worlds is definable in W from a parameter in W. Thus, M is definable by a formula using W and elements of W. Thus by Lemma 24, we see that $M \subseteq W$ as required. □

Finally, our main claim is complete.

Corollary 26. *$MV \vdash$ World-domination.*

4.3. **A couple of applications in relative interpretablity.** Before we move on to discuss these results, we first take a slight detour into the world of interpretation. The reader with less interest in this material and who is eager to move onto the discussion section should feel free to do so. As has been noted by Enayat and Łełyk [2024], categoricity properties are often closely related to important equivalence relations between theories. We shall see that MV bears further witness to this close relationship. The following quite technical definition will allow us to set things up. For models $\mathcal{A}$ and $\mathcal{B}$ of some language $\mathcal{L}$ we shall write $\mathcal{A} \triangleright_{par} \mathcal{B}$ to indicate that $\mathcal{A}$ interprets $\mathcal{B}$ with parameters.[48]

Definition 27. Let T be a theory articulated in some language $\mathcal{L}$. We say:[49]

[48]See [Enayat and Łełyk, 2024] for a proper definition, but the essential idea is that the domain and relations on $\mathcal{B}$ are all definable in $\mathcal{A}$.

[49]I believe this terminology with the exception of (3) originates with [Enayat, 2016]. I include (3) as a weakening since it provides an interesting warmup for the main claims.

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(1) T is *solid* if whenever $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are models of T where $\mathcal{A} \rhd_{par} \mathcal{B} \rhd_{par} \mathcal{C}$ and $\sigma : \mathcal{A} \cong \mathcal{C}$ is definable in $\mathcal{A}$ from parameters, then there is $\tau : \mathcal{A} \cong \mathcal{B}$ that is also definable in $\mathcal{A}$ from parameters.
(2) T is *tight* if whenever T_0 and T_1 are bi-interpretable extensions of T, then T_0 and T_1 are the same theory in the sense that they have the same deductive closure.
(3) T is *weakly tight* if whenever T_0 and T_1 are definitionally equivalent extensions of T, then T_0 and T_1 are the same theory in the sense that they have the same deductive closure.

(1) implies (2) and (2) implies (3). [50] (1) is a technical, but very useful property. (2) and (3) seem to tell us something about the "stability" of a theory. Bi-interpretability and definitional equivalence are canonical equivalence relations that give us plausible accounts of when two theories might considered intuitively equivalent for some purposes.[51] (2) and (3) then tell us that – something like – any pair of intuitively equivalent theories are actually identical. For some examples, we've already seen above that both ZFC and PA are internally categorical. It can be seen using very similar arguments that they are also both solid. With regard to MV we can use what we already have above to show that it is weakly tight.

Corollary 28. *MV is weakly tight; i.e., If T and S are definitionally equivalent extensions of MV, then T and S are the same theory in the sense that $T \dashv\vdash S$.*

Proof. Suppose that T and S are extensions of MV that are definitionally equivalent. To make things simpler, suppose that T is articulated using $\in_0$ and S is articulated using $\in_1$. Then there are definitional expansions T^+ and S^+ that add definitions of $\in_1$ and $\in_0$ respectively to T and S such that T^+ and S^+ have the same deductive closure.[52] Call the deductive closure of these theories U. Since we can define $\in_0$ in terms of $\in_1$ and vice versa, this means that U extends $MV^*(\in_0, \in_1)$. Thus, by Theorem 20, there is a definable isomorphism between the multiverses associated with $\in_0$ and $\in_1$. This means that in any model $\mathcal{A}$ of U, the reducts $\mathcal{A} \restriction \in_0$ and $\mathcal{A} \restriction \in_1$ of $\mathcal{A}$ to $\in_0$ and $\in_1$ respectively are elementary equivalent. Thus, since T^+ and S^+ are conservative extensions of T and S, we see that they have the same theorems. □

In many interesting cases, tightness follows from weak-tightness. For some examples, Friedman and Visser [2025] have shown that for a wide class of theories, including ZFC, bi-interpretability implies definitional equivalence. More generally, Enayat and Łełyk [2024] have shown that for sequential theories that eliminate imaginaries, internal categoricity implies tightness.[53] ZFC is such a theory. When a theory eliminates imaginaries, this means (roughly) that it can deliver representatives for any definable equivalence relation. In ZFC this is easily achieved by using Scott's trick by taking the set of elements of minimal rank from each cell of the equivalence relation. This technique crucially

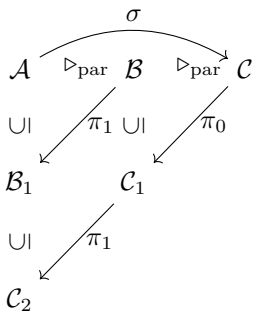
[50] See Remark 9 in [Enayat and Łełyk, 2024].
[51] [Visser, 2006] and [Button and Walsh, 2018] provide detailed discussion of these equivalence relations.
[52] Here we are using the characterization of definitional equivalence that Lefever and Székely [2019] call *definitional mergeability*. While this is not always the same as definitional equivalence, it is the same when the languages of the theories being compared are disjoint. Hence, our move to $\in_0$ and $\in_1$.
[53] See Proposition 50 in [Enayat and Łełyk, 2024]. See Section 4.4 of [Hodges, 1997] for a discussion of imaginary elimination. A sequential theory is defined in [Friedman and Visser, 2025], however, one might think of it as a particularly low bar for a foundational theory to meet and one that every theory in this paper traverses.

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relies on the powerset axiom to deliver the relevant V_α to intersect. A such, it is not available in ZFC^-_{count}. Nonetheless, it is possible to show that ZFC^-_{count} is solid as we now do. The original proof of this claim achieves is goal by drawing it as a corollary of the facts that: second order arithmetic is bi-interpretable with ZFC^-_{count}; second order arithmetic is tight; and tightness is preserved through bi-interpretations. The proof below is a little more direct and also harks back to our proof of Theorem m19. Thus, it seems worthwhile to include it.

Theorem 29. *[Enayat, 2016] ZFC^-_{count} is solid.*

Proof. Suppose that $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are models of ZFC^-_{count} such that $\mathcal{A} \rhd_{par} \mathcal{B} \rhd_{par} \mathcal{C}$ and there is some $\sigma : \mathcal{A} \cong \mathcal{C}$ that is definable in $\mathcal{A}$ from parameters. We claim that $\mathcal{A}$ can define with parameters some $\tau : \mathcal{A} \cong \mathcal{B}$. The argument is similar to our proof that ZFC^-_{count} is internally categorical. Since $\mathcal{C}$ is definable in $\mathcal{B}$ we can take the transitive collapse $\mathcal{C}_1$ of $\mathcal{B}$ in $\mathcal{C}$ via the possibly partial function $\pi_0 : \mathcal{C} \rightharpoonup \mathcal{C}_1$. At present, we don't know that $\mathcal{C}$ and $\mathcal{C}_1$ are isomorphic since we don't know that $\mathcal{C}$ is set-like in $\mathcal{B}$. Similarly, we can take the transitive collapse $\mathcal{B}_1$ of $\mathcal{B}$ in $\mathcal{A}$ by the possibly partial function $\pi_1 : \mathcal{B} \to \mathcal{B}_1$. Finally, wee may also use π_1 to collapse $\mathcal{C}_1$ in $\mathcal{A}$ to give us $\mathcal{C}_2$. The following diagram summarizes the situation.



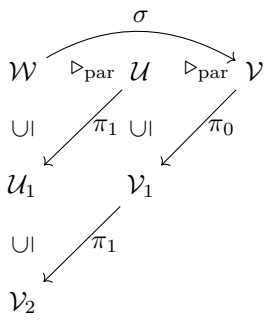
We claim that $\mathcal{A} = \mathcal{B}_1 = \mathcal{C}_2$. To see this, we show that it suffices – as in the proof of Theorem 19 – to observe that $\mathcal{A}$ and $\mathcal{C}_2$ have the same sets of natural numbers. We already can see that $\mathcal{C}_2 \subseteq \mathcal{A}$, so suppose that $\mathcal{A} \models x \subseteq \omega$. Then we see that $\mathcal{C} \models \sigma(x) \subseteq \omega$ since $\sigma(\omega^{\mathcal{A}}) = \omega^{\mathcal{C}}$. It is also clear that the transitive closure of $\{\sigma(x)\}$ in $\mathcal{C}$ is set-like in $\mathcal{C}$ and so we have $\mathcal{B} \models \pi_0 \circ \sigma(x) \subseteq \omega$. By similar reasoning we see that $\mathcal{A} \models \pi_1 \circ \pi_0 \circ \sigma(x) \subseteq \omega$. Moreover, it can be seen that $\pi_1 \circ \pi_0 \circ \sigma(z) = z$ for any z where the left hand side is defined. Thus, $x = \pi_1 \circ \pi_0 \circ \sigma(x) \in \mathcal{C}_2$ as required. Now since $\mathcal{A} = \mathcal{B}_1$, we see that $\pi_1 : \mathcal{B} \cong \mathcal{B}_1 = \mathcal{A}$ and so $\tau = \pi_1^{-1}$ is the isomorphism we've been looking for. □

As in Section 3, we can then piggyback on this argument to show that MV is also solid.

Theorem 30. *MV is solid.*

Proof. Suppose that $\mathcal{W}, \mathcal{U}$ and $\mathcal{V}$ are models of ZFC^-_{count} such that $\mathcal{W} \rhd_{par} \mathcal{U} \rhd_{par} \mathcal{V}$ and there is some $\sigma : \mathcal{W} \cong \mathcal{V}$ that is definable in $\mathcal{W}$ from parameters. We claim that $\mathcal{W}$ can define with parameters some $\tau : \mathcal{W} \cong \mathcal{U}$. As in the previous proof we take collapses of $\mathcal{V}$ in $\mathcal{U}$ and $\mathcal{U}$ in $\mathcal{V}$ to give us models $\mathcal{V}_1, \mathcal{U}_1$ and $\mathcal{V}_2$ as depicted in the following diagram.

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Given that MV implies ZFC^-_{count}, we see that $\pi_1 : \langle \mathbb{D}^{\mathcal{U}}, \in^{\mathcal{U}} \rangle \cong \langle \mathbb{D}^{\mathcal{W}}, \in^{\mathcal{W}} \rangle$ is an isomorphism on the set domains of $\mathcal{U}$ and $\mathcal{W}$ that is definable in $\mathcal{U}$. To address the worlds, it will suffices to show that $\mathcal{W}$ and $\mathcal{U}_1$ have the same worlds. Our argument is similar to the one we used to prove Theorem 20. First, suppose that W is a $\mathcal{W}$-world. Then since $\pi_1 \circ \pi_0 \circ \sigma$ is the identity function on sets in $\mathcal{W}$, we see that $\pi_1 \circ \pi_0 \circ \sigma$"$W = W \in \mathcal{V}_2 \subseteq \mathcal{U}_1$. Thus by World-domination in $\mathcal{U}_1$, we may fix a $\mathcal{U}_1$-world $U \supseteq W$. And then by World-domination in $\mathcal{W}$ we may fix a $\mathcal{W}$-world $W^* \supseteq U$. Thus, we have $W \subseteq U \subseteq W^*$ where $W^*$is a generic extension of W and so W is a generic refinement of a $\mathcal{U}_1$-world meaning that W is also a $\mathcal{U}_1$-world. A similar argument establishes that every $\mathcal{U}_1$ world is also a $\mathcal{W}$ world. □

With this in hand, we then see (roughly) that equivalent extensions of MV (and ZFC^-_{count}) are, in fact, identical.

Corollary 31. *ZFC^-_{count} and MV are both tight; i.e., any pair of theories extending MV (or ZFC^-_{count}) that are bi-interpretable have the same deductive closure.*

5. Discussion

I think the results above are sufficiently surprising, and perhaps confusing, that some discussion is warranted. However, I'll concede at the get-go that while I hope to clarify matters, I don't have anything definitive to say on the philosophy: just more questions. Nonetheless, from a logical point of view I think these results are of obvious interest and this justifies further investigation. We have here an example of a theory, MV, which is closely related to ZFC, whose canonical axiomatization fails to be internally categorical; but also has a modified axiomatization that is. Moreover, the internal categoricity of MV^* is quite unusual in that it is not established by a modification of a second order categoricity proof, because there isn't one. As such, I think MV^* will be a helpful example in the future study of internal categoricity and related properties like solidity and tightness.

Let us now move onto the philosophy. I'll start by considering how the internal categoricity result above might affect our interpretation of similar claims. Then I'll consider some reasons we might have for disregarding the significance of those considerations.

In the introduction to this paper we observed that traditional, external categoricity results for set theory rely on the use of full models of second order ZFC, which presuppose that it makes sense to talk of the "true" powerset operation in order to define those models. Given that large chunk of the burden in the categoricity claim is obtaining a unique powerset operation inside those models, we appear to be presupposing what we set out to obtain. This has been a classic reason to disregard

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the evidential value of these results when considering questions about the determinateness of the set theory's subject matter. However, internal categoricity makes no use of such second order resources. The theorem and the proof are all articulated in first order logic, within a theory $ZFC(\in_0, \in_1)$ that shares its membership relations evenly amongst its axiom schemata. Moreover loosely following Parsons [1990], we can justify the use of such a theory using a linguistically motivated thought experiment that is closely related to the way in which we use translation to interpret different extensions of ZFC. The theorem then shows that the universes associated with $\in_0$ and $\in_1$ are isomorphic and that this isomorphism is visible within the context of the thought experiment since it is definable. We might say this reveals that $\in_0$ and $\in_1$ are *structurally identical*. A little more concretely, if we then consider any statement articulated in $\mathcal{L}_\in$, like, say, the continuum hypothesis, we see that both the $\in_0$-user and the $\in_1$ -user must agree on its truth value (even if they are not in a position to decide it). Some have take this as reason to think that the continuum hypothesis has a determinate truth value. And pleasingly, this argument is not exposed to circularity worries and is based on a methodology common to the practice of working set theorists. Now this is hardly a watertight argument, but it does look like a promising sketch.

Unfortunately, I think the internal categoricity of MV^* suggests that this sketch cannot be filled in successfully. Or at least, any effort to fill it in will need to explain what is happening with MV^* and this will likely make the completed argument subtle and complex. This seems unattractive as one of the draws of using categoricity arguments for these purposes is the promise of extracting philosophy straightforwardly from the math. While this kind of strategy rarely works, when it does, the resulting claim is difficult to challenge when it is drawn from something incontrovertible: a theorem. But once the philosophy creeps in, the wiggle room quickly expands and the opportunities to disagree multiply. So why can't the argument sketch be filled in? The essential idea is that people using MV have access to an analogous argument that appears to lead to a conclusion that is in tension with the one drawn above. This suggests that something is wrong with this sketch. Let's try to step through this analogy patiently.

First let us observe that MV and ZFC are very closely related as alternative foundations for mathematics. It is important that we establish this closeness for our analogy since we want the failure of CH to be determinate in MV to have some bearing on its success in ZFC. We want a genuine clash.[54] We saw above that MV was introduced as an alternative to ZFC that takes the incompleteness wrought by forcing as a feature and not a bug. It was designed to be an alternative foundation for mathematics that is more conservative in the questions it permits us to ask. Regarding ZFC and its canonical extensions, Steel remarks, "the mathematical theory based on large cardinal hypotheses that we have produced to date can naturally be expressed in the multiverse language" [Steel, 2014]. We might then say that ZFC and MV are *functionally identical* as foundations for mathematics. To push this point a little harder, we observe that ZFC and MV are not only equiconsistent, but there are interpretations

[54]To illustrate its importance, note that ZFC^-_{count} deems CH to be globally inexpressible and it is also internally categorical. But ZFC^-_{count} and ZFC are very different theories. For example, it is clear that ZFC^-_{count} will be a poor foundation for any form of mathematics involving uncountable cardinals. Furthermore, ZFC obviously proves that $\mathbb{HC}$, the hereditarily countable sets, provides a natural model of ZFC^-_{count}. ZFC^-_{count} can thus be understood as providing a theory of a mere initial segment of what is described by ZFC. We see below that ZFC does not dominate MV in this way.

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witnessing this that display a degree of invertibility akin to bi-interpretability and definitional equivalence.[55] In particular, generalizing on work from Meadows [2021], Steel [2024b] shows that natural extensions of these theories are *weakly sententially equivalent*.[56] More specifically, we extend ZFC with the statement $V = \mathsf{C}$ which says that the universe is the core of its generic multiverse;[57] and we extend MV with the statement that some world W is the core: i.e., $(V = \mathsf{C})^W$ holds. The weak sentential equivalence of these theories then means that there are translations s and t from $\mathcal{L}_\in$ to $\mathcal{L}_{MV}$ and back such that:

(1) $MV + \exists W\ (W = \mathsf{C})^W \vdash \varphi \leftrightarrow s \circ t(\varphi)$, for all sentences φ of $\mathcal{L}_{MV}$; and
(2) $ZFC + V = \mathsf{C} \vdash \psi \leftrightarrow t \circ s(\psi)$ for all sentences ψ of $\mathcal{L}_\in$.

Informally speaking, this tells us that, from the perspective of either theory, if we translate some sentence into the other theory and back, we end up with a sentence that has what we might roughly gloss as the same meaning according that theory.[58] In this regard, Steel remarks, "I believe the proper conclusion is that $MV + \exists W\ (V = \mathsf{C})^W$ and $ZFC + V = \mathsf{C}$ are equivalent as foundations" [Steel, 2024b].[59] Despite this close relationship there are, of course, salient differences. For example, if we return to our stalking horse, we recall that in ZFC, the continuum hypothesis is naturally articulated as a simple statement about a putative bijection between two sets, but there is no corresponding formulation in MV. From the local perspective, it is true in some worlds and false in others. From the global perspective, it simply cannot be articulated. For these reasons, I claim that there is a sensible analogy to be had between ZFC and MV. While they differ in their treatment of deep incompleteness wrought by forcing, they remain functionally identical as foundations of mathematics.

For the next move in our analogy, recall that we motivated internal categoricity results for ZFC through worries about the fixedness of its subject matter. Users of MV may have similar worries: perhaps MV is an algebraic theory like group theory that has many very different models none of which could be thought of as its intended model. Without any indication that MV has a distinct intended subject matter, it may be difficult to interpret MV as a serious foundation for mathematics. This is a natural moment for internal categoricity to enter our story. We might follow something like the Parsonian thought experiment described above for ZFC, with MV^* in its place.[60] We might imagine two users of MV^* who wonder whether they are talking about the same thing. Following our template, we might then move to a combined language with two membership relations $\in_0$ and $\in_1$. Regarding the theory used to govern those relations, we might agree to use the combined language in each of the axiom schemata, thus, giving us $MV^*(\in_0, \in_1)$. The results above then tell us that there will be a definable isomorphism between the multiverse associated with $\in_0$ and that of $\in_1$. As above, we might then say that these relations are structurally identical. This appears to provide a helpful

[55]See [Visser, 2006, Button and Walsh, 2018, Barrett and Halvorson, 2016, Lefever and Székely, 2019] fore detailed information about definitional equivalence and bi-interpretability.
[56]This is one among a number of similar results, but it is the simplest to state, thus, most appropriate for our current purposes.
[57]This is also known as the Ground Axiom and it says that that there is no r such that $\mathsf{W}_r \subsetneq V$ [Reitz, 2007].
[58]See [Maddy and Meadows, 2020] and [Steel, 2024b] for discussion about the adequacy of this gloss.
[59]We should take care to note that these are extensions of MV and ZFC, so this doesn't give us an exact match. Nonetheless, these kinds of equivalence results hold rarely even for extensions. As such, I claim this does give us further evidence of their closeness as theories.
[60]The reader will note that I'm moving quite freely between MV and MV^* here. This will be discussed further below.

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response to qualms about the fixedness of MV's content. By contrast, it is obvious that group theory cannot be axiomatized in a way that is internally categorical.[61] So far so good for our analogy. But can we take it further?

We saw earlier that our (arguably straw) ZFC users took the solace drawn from the internal categoricity of ZFC and used that to sketch an argument toward a bold conclusion regarding the determinateness of the continuum hypothesis. Could incautious MV users do something similar? Given that we just have a coarse-grained sketch, I'm not at all sure that they can't. The MV users have a theory that is compatible with the idea that the continuum hypothesis is making something like a category error. Internal categoricity then appears to tell us something positive about the semantic stability of the perspective they have adopted. Any pair of MV^* users who commit to the combined theory $MV^*(\in_0, \in_1)$ will also be committed to the structural identity of their membership relations. And more particularly, given that the continuum hypothesis is indeterminate in this framework, they will continue to agree on its indeterminacy, but also that they are talking about essentially the same subject matter. Thus, when we follow through with our analogy, we appear to have landed, if not in contradiction, then at least in a zone of uncomfortable tension. We have two theories MV and ZFC that appear to be functionally identical as foundations for mathematics. Moreover, they are both internally categorical and, thus, both appear to display a fixedness of content that could give us confidence that they are both talking about a particular subject matter albeit in different ways. Nonetheless, it then seems that users of one theory should be committed to the determinacy of the continuum hypothesis while the users of the other should not, while at the same time, they both appear to be talking about much the same thing. Due to the inherent vagueness of both of these argument sketches, I find it difficult to know how to diagnose this situation. Something ought to give, but in the absence of a more thoroughgoing analysis, it seems difficult to conclude that either of these arguments lends much credence to the determinateness or indeterminateness of the continuum hypothesis. I think the bottom line, at least for now, is that we don't sufficiently understand how these formal tools interact with philosophical claims. Regardless, I think the internal categoricity of MV^* will provide a helpful foothold in developing such an understanding.

With that little argument out of the way, let's close by considering three concerns one might have about this line of reasoning. As we go, I'll also offer brief responses to each of them. For our first worry, the reader may have observed that I moved quite freely from talking about MV^* to MV in the previous paragraphs. These axiomatizations have an important difference. MV^* is internally categorical, while MV is not. Moreover, given that MV is the official axiomatization, we might wonder whether we should care about what MV^* and its peculiar properties. This might give us reason to doubt the robustness of the analogy described above. Our response is relatively straightforward. While ZFC is internally categorical, it also has an alternative axiomatization ZFC^* that is not internally categorical.[62] Thus, exactly the same problem emerges in the context of ZFC. So which axiomatization is the right one? This is an interesting question that has become more pressing in recent years.[63] But it is not so easy to answer. In contrast to the convention adopted in this paper, it is more common to regard a theory as

[61]Perhaps the easiest way to see this is to observe that group theory has, and is intended to have, distinct models with different cardinalities.

[62]See Theorem 52 in [Enayat and Łełyk, 2024] for a proof of this.

[63]See the conclusion of [Enayat and Łełyk, 2024] for some discussion of this.

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the deductive closure of some set of axioms rather than the axioms themselves. Moreover, this accords well with the ordinary practice of set theorists. We are generally just interested in what follows from ZFC regardless of how it is axiomatized. Evidence for this can be found in the fact that different textbooks frequently give quite different axiomatizations of ZFC. Some use Collection rather than Replacement. Some prefer Foundation over Set Induction.[64] But none of this changes the theorems of ZFC, so the mathematical practice of users of ZFC remains generally unaffected. Perhaps the right question to ask is not: is this axiomatization of theory T internally categorical; but rather, is there an axiomatization of theory T that is categorical?[65] If so, MV^*'s internal categoricity seems to be just as significant as that of ZFC. If not, then much more work needs to be done to understand when one axiomatization of a theory is preferable to another. Cards on the table, I think this is a fertile problem to explore that is likely to deliver significant foundational insights that could be pertinent to our current problem. But at present, this is a field in its infancy.

For our second concern, let's consider the World-domination schema. In $MV(\in_0, \in_1)$ both $\in_0$ and $\in_1$ are permitted into both the $\in_0$ and $\in_1$ versions of the World-domination schema. This is directly analogous to what we did with $ZFC(\in_0, \in_1)$. However, in the $ZFC(\in_0, \in_1)$ case we also spent some time discussing why we should allow the both membership relations into both versions of, say, the Replacement Schema. We are yet to do this for World-domination. And there is an important hitch. We motivated our use of the shared version of Replacement on the grounds that it provided a natural approximation of the full second order version of Replacement. No such motivation can be offered for World-domination. To see this recall that World-domination says that every *definable* inner model is covered by a world. What would it mean to say that this was an approximation of the full second order version of World-domination? To answer this question, we need to know what the full second order version of World-domination would look like. The obvious suggestion is the following: every inner model is covered by a world. With Replacement we moved from *definable* class functions to *arbitrary* class functions; here we move from *definable* inner models to *arbitrary* inner models. But second order World-domination is inconsistent.[66] To see this consider, the models $\mathcal{W}_0$ and $\mathcal{W}_1$ described in the proof of Theorem 13 (showing that MV is not internally categorical). We claim that $\mathcal{W}_0$ doesn't satisfy second order World-domination. To see this recall that while $\mathcal{W}_0$ is generated from a world M, $\mathcal{W}_1$ is generated from a class generic N extension of M that no world in $\mathcal{W}_0$ can cover. As such, we have

[64]For another classic difference, Kunen delivers a Pairing Axiom by saying that given any sets x and y there is a set z that contains x and y, while Jech says more by saying that there is a set w that contains *exactly* x and y [Kunen, 2006, Jech, 2003]. Of course, Separation and Extensionality wipe away any worries of inequivalence, but the point is just that there are many distinct axiomatizations of ZFC out there and we rarely have need to worry about this. I should also note that when we consider weakenings of ZFC then the question of axiomatization can become very important. For example, it is well known that in the absence of Extensionality or Powerset, Replacement behaves very oddly while Collection does not [Scott, 1961, Gitman et al., 2016].

[65]Of course, another option would be to consign the internal categoricity of both ZFC and MV^* to the flames. Perhaps the axiom sensitivity of internal categoricity makes it insignificant. I think it would be extremely premature to draw such a conclusion when so much remains unknown. Moreover, we've seen, with Corollary 28, that despite the axiom sensitivity, the internal categoricity of MV^* can still be used to draw significant conclusions about MV even when it is construed as a deductively closed theory.

[66]It is also worth bearing in mind that the other axiom schemata also face similar problems. For example, if we were to treat the Replacement Schemata relativized to worlds as an approximation of a second order version this also leads to inconsistency. To see this suppose W is a world and $W[c]$ is a generic extension of W by a Cohen real $c \subseteq \omega$. Now let $f : \omega \to \omega$ be the injection that enumerates c. This is a(n improper) class function over W, but a second order Replacement Axiom would tell us that $f``\omega = c \in W$, which is impossible.

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in N an uncovered inner model of $\mathcal{W}_0$. So how do we motivate World-domination and how does this affect the analogy above? Our response is to concede that World-domination is not an approximation of a second order axiom, but rather, it is about definability. Theorem 25 not only tells us that World-domination is a consequence of MV, its proof also gives us some explanation as to why it is true and some expectation of when it will continue to hold. In particular, we see that World-domination reveals deep limitations in what one can expect to define over a model of MV. In allowing both membership relations into the World-domination Schemata, we can be understood as merely flagging our expectation that these limitations will continue to hold. Of course, this is not as simple as saying that we are approximating a second order axiom, but this may be more of a blessing than a curse. When we motivated shared Replacement with the approximation story, we might wonder how far we've removed ourselves from second order logic and our worries about circularity. When motivating World-domination through definability rather than second order logic, there is no implicit appeal to the collection of all subclasses of the domain. Perhaps this is a benefit.[67] I think it's fair to say that more work is required here, and that this work will tell us much more about internal categoricity and foundational theories in mathematics. But even at just the beginning of this road, it is interesting to see an axiom schema in a foundational setting that cannot be explained as an approximation of a second order cousin.

For our third and final concern, we consider an important point of difference in the analogy between ZFC and MV.[68] For this we return again to our stalking horse, the continuum hypothesis. In the case of ZFC, we know from Gödel [1940] and Cohen [1963] that the continuum hypothesis cannot be decided. As such, we might have reason to doubt – as many have – that it has a determinate answer. However, in the case of MV, there is a sense in which the matter is already settled. MV doesn't prove or refute the continuum hypothesis, but with regard to both the global and local perspectives that we have entertained, we have taken it as indeterminate. We might summarize the situation as follows: ZFC, on its own, says nothing about the determinateness of the continuum hypothesis, but MV says, on its own, that it is not determinate. This makes a noticeable difference when internal categoricity enters the story. When we learn that $\in_0$ and $\in_1$ are structurally identical since ZFC is internally categorical, we might think we are learning that ZFC has a particular subject matter, but we also learn something new: $\in_0$ and $\in_1$ must agree on outcome the continuum hypothesis, be it true or false. In this situation, the internal categoricity of ZFC could be taken as indicating that the continuum hypothesis is determinate.[69] Thus, internal categoricity is being used to move from a position of ignorance about determinateness to a position where we claim that it holds. By contrast, the indeterminateness of the continuum hypothesis is built into MV from the start. The fact that MV^* is internally categorical doesn't change anything. This is the point of difference.

[67]To be fair, this is also a natural window for a Parsonian counter-move. We might take the failure of World Domination to be truly open-ended as showing us that the internal categoricity of MV^* is philosophically insignificant. This is an interesting way of applying some pressure to our discussion. There is, however, a standard response to which we alluded above. In particular, we might argue – as other have done – that adherence to fully open-ended schemata is tantamount to the adoption of full second order logic; and thus, reopens the problem of question-begging. This is discussed on page 30 of [Maddy and Väänänen, 2023], who direct the reader to proponents of this response including [Shapiro, 1991, Field, 2001] and [Button and Walsh, 2018]. Regardless, this is a point of contention in the literature and so this remains an interesting point to press upon. I'm grateful to a referee for pushing me on this.

[68]I'm particularly grateful to a referee for pressing me on this point.

[69]Of course, we have raised doubts about this kind of inference, but this is tangential to our current purposes.

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What should we say about it? First, I think we must concede that the difference is real and there is no pushing back on it. However, I don't think this presents a problem for our coarse-grained analogy. For our moment of tension to become palpable, it is not enough that MV deems the continuum hypothesis to be indeterminate. We needed more than that. In particular, we relied on a couple of crucial observations: (1) ZFC and MV have much the same subject matter, in the sense that they are functionally identical as foundations for mathematics; and (2) that MV can be understood as talking about a particular subject matter. For the first of these claims, we leaned on the folk set theory fact that MV can do anything that can be done with ZFC and its large cardinal extensions [Steel, 2014]; and results from relative interpretability regarding their equivalence [Meadows, 2021, Steel, 2024b]. For the second, we leaned on the internal categoricity of MV^*. In particular, and analogously to the situation with ZFC, we learned that in $MV^*(\in_0, \in_1)$, $\in_0$ is structurally identical to $\in_1$. Thus, we were tempted to conclude that MV is the kind of theory that should be about a particular subject matter. Putting these facts together with the fact that the indeterminacy of the continuum hypothesis was already baked into MV, we then end up in our standoff. So despite this salient point of difference, the underlying tension between the competing arguments is still manifest.

Conclusion. In this paper, we've shown that Steel's multiverse theory can be axiomatized in alternative manners such that it is both internally categorical and not internally categorical. This makes Steel's theory an interesting example to consider in the theory of relative interpretation. Given that internal categoricity is sometimes thought to have significance with regard to the determinateness of the continuum hypothesis and that MV is designed to, in a sense, ignore it, this also makes the internal categoricity of (a version of) MV interesting in the philosophy and foundations of mathematics. We have argued that this result suggests that arguments from internal categoricity to the fixedness of subject matter in a theory are unlikely to work or, at least, will be much more complicated than one might have expected.

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