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DP AND OTHER MINIMALITIES

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Abstract. A first-order expansion of $(\mathbb{R}, +, <)$ is dp-minimal if and only if it is o-minimal. We prove analogous results for algebraic closures of finite fields, p-adic fields, ordered abelian groups with only finitely many convex subgroups (in particular archimedean ordered abelian groups), and abelian groups equipped with archimedean cyclic group orders. The latter allows us to describe unary definable sets in dp-minimal expansions of $(\mathbb{Z}, +, S)$, where S is a cyclic group order. Along the way we describe unary definable sets in dp-minimal expansions of ordered abelian groups. In the last section we give a canonical correspondence between dp-minimal expansions of $(\mathbb{Q}, +, <)$ and o-minimal expansions $\mathcal{R}$ of $(\mathbb{R}, +, <)$ such that $(\mathcal{R}, \mathbb{Q})$ is a "dense pair.".

In model theory one typically approaches a first-order structure $\mathcal{M}$ by showing $\mathcal{M}$ satisfies some form of model completeness which yields a description of definable sets highly specific to that structure. This description allows one to understand the abstract classification-theoretic properties of $\mathcal{M}$ and situate $\mathcal{M}$ in the landscape of first-order structures. Converse implications, situations in which abstract classification-theoretic properties yield precise descriptions of definable sets, are rare. One is Fact 0.1, a consequence of [40, Corollary 3.7].

FACT 0.1. *An expansion of $(\mathbb{R}, +, <)$ is dp-minimal if and only if it is o-minimal.*

Another is Fact 0.2 [2, Proposition 6.6].

FACT 0.2. *An expansion $\mathcal{Z}$ of $(\mathbb{Z}, +, <)$ is dp-minimal if and only if every $\mathcal{Z}$-definable subset of every $\mathbb{Z}^n$ is already $(\mathbb{Z}, +, <)$-definable.*

The initial motivation for this work was to prove an analogous result over $(\mathbb{Q}, +, <)$. We prove several results of this form and give a general framework for such results.

§1. Dp-rank. Our reference is [41, Chapter 4]. Let κ be a cardinal and T be a complete theory. A family $(I_t : t \in X)$ of sequences of elements of a highly saturated $\mathcal{M} \models T$ is *mutually indiscernible* over a small set A of parameters if each I_t is indiscernible over $A \cup (I_s : s \in X \setminus \{t\})$. The dp-rank $\mathrm{dp-rk}(T)$ of T is less than κ if for every highly saturated $\mathcal{M} \models T$, small set A of parameters, family $(I_t : t < \kappa)$ of mutually indiscernible sequences over A, and $b \in \mathcal{M}$ there is $\lambda < \kappa$ such that I_λ is indiscernible over Ab. The dp-rank of T is κ if $\mathrm{dp-rk}(T) < \kappa^+$ but $\mathrm{dp-rk}(T)$

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is not less than κ. The dp-rank of T is ∞ if we do not have $\mathrm{dp-rk}(T) < \kappa$ for any cardinal κ. It is known that $\mathrm{dp-rk}(T) < \infty$ if and only if $\mathrm{dp-rk}(T) < |T|^+$ if and only if T is NIP. It is easy to see $\mathrm{dp-rk}(T) = 0$ if and only if $\mathcal{M}$ is finite. We say T is *dp-minimal* when $\mathrm{dp-rk}(T) \leq 1$. A structure is dp-minimal if its theory is dp-minimal. See [1, 13, 40] for examples of dp-minimal structures. Dp-minimal fields are classfied in [22].

§2. Set up.

2.1. Conventions. All languages and structures are first order and "definable" without modification means "first order definable, possibly with parameters." Given languages $L \subseteq L'$ and an L'-structure $\mathcal{M}$ we let $\mathcal{M}|L$ be the L-reduct of $\mathcal{M}$. We let $|x| = n$ be the length of a tuple $(x_1, \ldots, x_n)$ of variables. Throughout n is a natural number and m, k, i, j are integers.

2.2. External definibility. Suppose $\mathcal{M}$ is a structure and $\mathcal{N}$ is an $|M|^+$-saturated elementary extension of $\mathcal{M}$. A subset of M^k is *externally definable* if is of the form $X \cap M^k$ for $\mathcal{N}$-definable $X \subseteq N^k$. The *Shelah expansion* $\mathcal{M}^{\mathrm{Sh}}$ of $\mathcal{M}$ is the expansion of $\mathcal{M}$ by a k-ary predicate defining $X \cap M^k$ for every $\mathcal{N}$-definable $X \subseteq N^k$. Up to interdefinibility, this construction does not depend on choice of $\mathcal{N}$. (Two structures on a common domain M are *interdefinable* if they define the same subsets of all M^n). Observe that if $\mathcal{M}$ expands a linear order then every convex set is externally definable. The following theorem of Shelah is deep [39]. (See [9] for another proof.)

FACT 2.1. *If $\mathcal{M}$ is* NIP *then every $\mathcal{M}^{\mathrm{Sh}}$-definable subset of every M^k is externally definable. It follows that $\mathrm{dp-rk}(\mathcal{M}^{\mathrm{Sh}}) = \mathrm{dp-rk}(\mathcal{M})$ for any $\mathcal{M}$ and consequently $\mathcal{M}^{\mathrm{Sh}}$ is* NIP *when $\mathcal{M}$ is* NIP *and $\mathcal{M}^{\mathrm{Sh}}$ is dp-minimal when $\mathcal{M}$ is dp-minimal. It follows that if $\mathcal{M}$ expands a linear order then any expansion of $\mathcal{M}$ by any family of convex sets is* NIP *when $\mathcal{M}$ is* NIP *and dp-minimal when $\mathcal{M}$ is dp-minimal.*

2.3. Weak minimality. Let $L \subseteq L^\diamondsuit$ be languages, T be a complete consistent L-theory, $T^\diamondsuit$ be a complete consistent $L^\diamondsuit$-theory extending T, $\mathcal{M}^\diamondsuit \models T^\diamondsuit$, and $\mathcal{M} = \mathcal{M}^\diamondsuit|L$. We say that $\mathcal{M}^\diamondsuit$ is $\boldsymbol{\mathcal{M}}$*-minimal* if every $\mathcal{M}^\diamondsuit$-definable subset of M is $\mathcal{M}$-definable. Furthermore $T^\diamondsuit$ is $\boldsymbol{T}$*-minimal* if for every $L^\diamondsuit$-formula $\varphi(x, y)$, $|x| = 1$ there is an L-formula $\psi(x, z)$ such that for every $\mathcal{O}^\diamondsuit \models T^\diamondsuit$ and $b \in O^{|y|}$ there is $c \in O^{|z|}$ such that

$$\{a \in O : \mathcal{O}^\diamondsuit \models \varphi(a, b)\} = \{a \in O : \mathcal{O}^\diamondsuit \models \psi(a, c)\}.$$

We say that $\mathcal{M}^\diamondsuit$ is *weakly* $\boldsymbol{\mathcal{M}}$*-minimal* if every $\mathcal{M}^\diamondsuit$-definable subset of M is externally definable in $\mathcal{M}$. We say that $T^\diamondsuit$ is *weakly* $\boldsymbol{T}$*-minimal* if for every $L^\diamondsuit$-formula $\varphi(x, y)$ with $|x| = 1$, there is an L-formula $\psi(x, z)$ such that for every $\mathcal{O}^\diamondsuit \models T^\diamondsuit$, sufficiently saturated $\mathcal{P} \succ \mathcal{O}^\diamondsuit|L$, and $a \in O^{|y|}$ there is a $b \in P^{|z|}$ such that

$$\{c \in O : \mathcal{O}^\diamondsuit \models \varphi(c, a)\} = \{c \in O : \mathcal{P} \models \psi(c, b)\}.$$

If L is the empty language then $\mathcal{M}^\diamondsuit$ is $\mathcal{M}$-minimal if and only if $\mathcal{M}$ is minimal and $T^\diamondsuit$ is T-minimal if and only if $T^\diamondsuit$ is strongly minimal. If T is the theory of a dense linear order then $\mathcal{M}^\diamondsuit$ is $\mathcal{M}$-minimal if and only if $\mathcal{M}^\diamondsuit$ is o-minimal. If T is the

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theory of a C-relation, then $T^\diamond$ is T-minimal if and only if $T^\diamond$ is C-minimal. If T is the theory of $\mathbb{Q}_p$ then $T^\diamond$ is T-minimal if and only if $T^\diamond$ is P-minimal. (See [18, 19] for an account of C-, P-minimality, respectively.) If $\mathcal{M}$ is a dense linear order then $\mathcal{M}^\diamond$ is weakly $\mathcal{M}$-minimal if and only if $\mathcal{M}^\diamond$ is weakly o-minimal and $T^\diamond$ is weakly T-minimal if and only if $T^\diamond$ is weakly o-minimal. (An expansion of a dense linear order is *weakly o-minimal* if every definable unary set is a finite union of convex sets and a theory is weakly o-minimal if every model is.)

It is easy to see that Proposition 2.2 is true. We leave this to the reader.

PROPOSITION 2.2. *Suppose $T^\diamond$ is weakly T-minimal. Then $T^\diamond$ is* NIP *when T is* NIP *and $T^\diamond$ is dp-minimal when T is dp-minimal.*

If every externally definable subset of M is already definable in $\mathcal{M}$ then weak $\mathcal{M}$-minimality is equivalent to $\mathcal{M}$-minimality. This condition is satisfied in the following situations: $\mathcal{M}$ is stable, $\mathcal{M}$ is an o-minimal expansion of $(\mathbb{R}, <)$, $\mathcal{M}$ is $\mathbb{Q}_p$, or $\mathcal{M}$ is $(\mathbb{Z}, +, <)$. In each of the preceding examples $\mathcal{M}^{\mathrm{Sh}}$ is interdefinable with $\mathcal{M}$. The first case holds as stability of a theory T is equivalent to the assertion that all externally definable sets in all models of T are definable. The second is a consequence of the Marker–Steinhorn theorem [30], the third is a result of Delon [12], and the fourth follows by Facts 2.1 and 0.2.

We seek dp-minimal $\mathcal{M}$ which satisfy the following for all expansions $\mathcal{M}^\diamond$ of $\mathcal{M}$:

If $\mathcal{M}^\diamond$ is dp-minimal then $\mathcal{M}^\diamond$ is weakly $\mathcal{M}$-minimal,

or, stronger,

$$\mathcal{M}^\diamond \textit{ is dp-minimal if and only if } \mathrm{Th}(\mathcal{M}^\diamond) \textit{ is weakly } T\textit{-minimal.} \qquad (\bigstar)$$

Weak $\mathcal{M}$-minimality is the strictest condition we can impose on definable unary sets as $\mathcal{M}^{\mathrm{Sh}}$ is dp-minimal whenever $\mathcal{M}$ is dp-minimal. Outside of very special cases it does not seem possible to replace "dp-minimal" with a weaker NIP-theoretic property in $(\bigstar)$. For example if $\mathcal{M}$ is a strongly dependent structure and A is a dense (in the sense of [7]) acl-independent subset of $\mathcal{M}$, then $(\mathcal{M}, A)$ is strongly dependent by [7] and almost never weakly $\mathcal{M}^{\mathrm{Sh}}$-minimal. It also seems that $(\bigstar)$ requires $\mathcal{M}$ to have a certain amount of algebraic structure. If $\mathcal{M}$ is a linear order, more generally a tree, then the expansion of $\mathcal{M}$ by *all* subsets of M is dp-minimal [40, Proposition 4.7] and clearly not $\mathcal{M}^{\mathrm{Sh}}$-minimal.

Theorem 2.3 summarizes most of the results of this paper. We emphasize that most examples of ordered abelian groups which arise away from "valuational" settings have only finitely many convex subgroups.

THEOREM 2.3. *Suppose $\mathcal{M}$ is one of the following*:

(1) *the algebraic closure of a finite field,*
(2) *a non-singular ordered abelian group with only finitely many convex subgroups* (*in particular*:
   (a) *a non-singular archimedean ordered abelian group,*
   (b) *a non-singular subgroup of $(\mathbb{R}^n, +)$ with the lexicographic order, or*
   (c) *a finite rank ordered abelian group*), *or*
(3) *a non-singular subgroup of $\mathbb{R}/\mathbb{Z}$ with the induced cyclic order.*

Then $\mathcal{M}^\diamond$ is dp-minimal if and only if $\mathrm{Th}(\mathcal{M}^\diamond)$ *is weakly* $\mathrm{Th}(\mathcal{M})$*-minimal.*

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An abelian group A is *non-singular* if $|A/nA| < \aleph_0$ for all $n \geq 1$. An ordered abelian group is dp-minimal if and only if it is non-singular, so the assumption of non-singularity simply ensures that weak $\mathrm{Th}(\mathcal{M})$-minimality implies dp-minimality.

Item (3) is a corollary to (2a). Our interest in (3) is an application to the classification problem for dp-minimal expansions of $(\mathbb{Z}, +)$. Every known proper dp-minimal expansion of $(\mathbb{Z}, +)$ defines either $<$, a p-adic valuation [1], or a dense cyclic group order. Walsberg and Tran [44] show that $(\mathbb{Z}, +, C)$ is dp-minimal for any dense cyclic group order C. The proof of (3) characterizes dp-minimal expansions of such $(\mathbb{Z}, +, C)$ in terms of unary definable sets.

Fix a prime p. We also prove a weaker p-adic result. We let $\mathbb{K}$ be a finite extension of $\mathbb{Q}_p$ throughout. Recall that the p-adic valuation on $\mathbb{K}$ is $\mathbb{K}$-definable [4, Lemma 1.5], so we do not distinguish between $\mathbb{K}$ as a field or as a valued field. We also make use of the fact that the p-adic valuation $\mathbb{K}$ is discrete. The field $\mathbb{Q}_p$ is shown to be dp-minimal in [2, Corollary 7.8] and, as the authors observe in the first paragraph of [2, Section 7.2], their argument shows that $\mathbb{K}$ is dp-minimal. (Hence all characteristic zero local fields are dp-minimal.)

THEOREM 2.4. *Any dp-minimal expansion of $\mathbb{K}$ is $\mathbb{K}$-minimal.*

Combining with Fact 0.1 we see that if K is a characteristic zero local field other than $\mathbb{C}$ then any dp-minimal expansion of K is K-minimal. This is sharp, if v is a nontrivial valuation on $\mathbb{C}$ then $(\mathbb{C}, v)$ is dp-minimal and not strongly minimal, hence not $\mathbb{C}$-minimal. We do not know if the theory of every dp-minimal expansion of $\mathbb{Q}_p$ is $\mathrm{Th}(\mathbb{Q}_p)$-minimal. Theorem 2.3.2.C shows that if R is an archimedean real closed field then any dp-minimal expansion of R is weakly o-minimal. We prove the p-adic version of this result. We first recall Fact 2.5, which characterizes the relevant fields.

FACT 2.5. *Let K be a field of characteristic zero. Then the following are equivalent*:

(1) *K admits a discrete Henselian valuation v with finite residue field $\mathbb{F}$.*
(2) *K is isomorphic to an elementary subfield of a finite extension of $\mathbb{Q}_p$, where p is the characteristic of $\mathbb{F}$.*

If (1) *is satisfied then v is K-definable.*

PROOF. It is easy to see that (2) implies (1). Suppose (1) and let v be such a valuation on K. Then the completion of (K, v) is a complete discretely valued field of characteristic zero with residue field $\mathbb{F}$. Hence the completion is isomorphic to a finite extension $\mathbb{K}$ of $\mathbb{Q}_p$. We therefore assume that K is a subfield of $\mathbb{K}$ and v is the induced valuation. By [37, Theorem 3.1] K is p-adically closed. Model completeness for p-adically closed fields [37, Theorem 5.1] shows that K is an elementary subfield of $\mathbb{K}$. ⊣

With this in place we state Theorem 2.6.

THEOREM 2.6. *Suppose that K is discretely Henselian valued field of characteristic zero with finite residue field. Then any dp-minimal expansion of K is weakly K-minimal.*

Ideally, results such as Theorems 2.3, 2.4, and 2.6 should be corollaries to more general results. For example Fact 0.1 is a corollary to the theorem of Simon that an expansion of an ordered abelian group is o-minimal if and only if it is definably connected and dp-minimal. Fact 0.2 generalizes to Theorem 5.27 below.

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Theorem 2.3.1 is a corollary to the theorem of Johnson that a non-strongly minimal dp-minimal field admits a definable type V field topology. The first claim of Theorem 2.4 is a corollary to a more general result on dp-minimal expansions of discretely valued fields, Theorem 4.1 below. Theorem 2.3.2 is a special case of Theorem 2.7.

THEOREM 2.7. *Suppose $(M, +, <)$ is an ordered abelian group and $\mathcal{M}$ is an expansion of $(M, +, <)$ which defines only finitely many convex subgroups. Then $\mathcal{M}$ is dp-minimal if and only if every unary definable set in every elementary extension $\mathcal{N}$ of $\mathcal{M}$ is a finite union of sets of the form $C \cap (a + nN)$ for convex C and $a \in N$.*

Theorem 2.7 follows from Theorem 2.8.

THEOREM 2.8. *Suppose $(M, +, <)$ is an ordered abelian group, $\mathcal{M}$ is a dp-minimal expansion of $(M, +, <)$, and X is an $\mathcal{M}$-definable subset of M. Then there is n and an $\mathcal{M}$-definable convex equivalence relation E on M such that E has only finitely many finite classes and for every $a \in M$ there is a finite $B \subseteq M$ such that $E_a \cap X = E_a \cap [\bigcup_{b \in B} b + nM]$, where E_a is the E-class of a.*

Theorem 2.7 shows in particular that a dp-minimal expansion of $(\mathbb{Q}, +, <)$ is weakly o-minimal. The reader might wonder if there are any proper weakly o-minimal expansions of $(\mathbb{Q}, +, <)$. The o-minimal version of this question was at one point a subject of intense interest. Wilkie [50] showed that there are proper o-minimal expansions of $(\mathbb{Q}, +, <)$. In Section 7 we describe a correspondence between weakly o-minimal expansions of $(\mathbb{Q}, +, <)$ and o-minimal expansions $\mathcal{R}$ of $(\mathbb{R}, +, <)$ such that $(\mathcal{R}, \mathbb{Q})$ is a "dense pair."

§3. Algebraic closures of finite fields. Suppose $\mathbb{F}$ is the algebraic closure of a finite field and $\mathcal{F}$ is an expansion of $\mathbb{F}$. Johnson [23] has shown that a dp-minimal expansion of a field is either strongly minimal or admits a definable non-trivial type V field topology. A type V field topology is induced by an absolute value or valuation, see [15, Appendix B]. It is well-known that $\mathbb{F}$ does not admit a non-trivial absolute value or valuation. We recall the proof. Fix $a \in \mathbb{F}^\times$. Then $a^n = 1$ for some n. Suppose v is a valuation on $\mathbb{F}$. As $a^n = 1$ we must have $v(a) = 0$. So v is a trivial valuation. Suppose $\|, \|$ is an absolute value on $\mathbb{F}$. Then $\|a^n\| = \|1\| = 1$ and $\|a^n\| = \|a\|^n$, so we must have $\|a\| = 1$. So $\| \ \|$ is a trivial absolute value. Strong minimality implies dp-minimality, hence:

COROLLARY 3.1. *The expansion $\mathcal{F}$ is dp-minimal if and only if $\mathcal{F}$ is strongly minimal.*

So $\mathcal{F}$ is dp-minimal if and only if $\mathrm{Th}(\mathcal{F})$ is $\mathrm{Th}(\mathbb{F})$-minimal. By applying a recent results of Johnson we also obtain a somewhat weaker result for expansions of finite dp-rank. We say that a structure is *dp-finite* if it has finite dp-rank. Johnson has shown that an unstable dp-finite expansion of a field admits a definable V topology [25] and a stable dp-finite expansion of a field has finite Morely rank [24]. Finite Morley rank implies finite dp-rank, so we obtain:

COROLLARY 3.2. *The structure $\mathcal{F}$ is dp-finite if and only if $\mathcal{F}$ has finite Morley rank.*

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We are not aware of any proper expansions of $\mathbb{F}$ of finite Morley rank. For example it is still an open question if there is a nontrivial multiplicative subgroup G of $\mathbb{F}$ such that $(\mathbb{F}, G)$ has finite Morley rank. If there are infinitely many $\mathrm{Char}(\mathbb{F})$-Mersenne primes then there is no such subgroup [46, Theorem 4].

§4. Discretely valued fields. We first show that dp-minimal expansions of discretely valued fields are "locally P-minimal." In this section K is a field, $v : K^\times \to \mathbb{Z}$ is a non-trivial discrete valuation on K, and $\mathcal{K}$ is an expansion of (K, v). We let $B(a, k)$ be the ball $\{b \in K : v(a - b) \geq k\}$, $B(k) := B(0, k)$, and $P_n := \{a^n : a \in K^\times\}$ for all n. Dp-minimal valued fields are Henselian [21], so we suppose that (K, v) is Henselian.

THEOREM 4.1. *Suppose $\mathcal{K}$ is dp-minimal, X is a $\mathcal{K}$-definable subset of X, and $a \in K$. Then there is k such that $B(a, k) \cap X = B(a, k) \cap Y$ where Y is a finite union of sets of the form $a + bP_n$ for some $b \in K$ and n.*

Furthermore if L is a dp-minimal field then the valued field $L((t))$ of Laurent series is dp-minimal [22]. In particular Theorem 4.1 applies to dp-minimal expansions of $\mathbb{K}((t))$ for $\mathbb{K}$ a characteristic zero local field. We gather some basic lemmas.

FACT 4.2. *Suppose $\mathcal{K}$ is dp-minimal and X is a $\mathcal{K}$-definable subset of K. Then X is the union of a definable open set and a finite set. Furthermore the closure of X in K is the union of X together with a finite set.*

The first claim of Fact 4.2 is [21, Corollary 4.7]. The second claim is an easy consequence of the first.

FACT 4.3. *Let $(M, +)$ be a non-singular abelian group. Then any finite index subgroup of $(M, +)$ is a finite union of cosets of nM for some n.*

PROOF. Suppose H is an index m subgroup of $(M, +)$. We show that $(m!)M$ is a subgroup of H and it follows by non-singularity that H is a finite union of cosets of $(m!)H$. Fix $a \in M$. There are $1 \leq i < j \leq m + 1$ such that ia, ja lie in the same coset of H. So $(j - i)a$ is in H. As $j - i$ divides $m!$ we have $(m!)a \in H$. ⊣

Fact 4.4 is a special case of [21, Lemma 3.5]. We say that $X, X' \subseteq K$ have the same germ at $a \in K$ if $B(a, k) \cap X = B(a, k) \cap X'$ for some k. A family $\mathcal{X}$ of subsets of X has only finitely many germs at a if there are $X_1, \ldots, X_n \in \mathcal{X}$ such that every $X \in \mathcal{X}$ has the same germ at a as some X_i.

FACT 4.4. *Suppose $\mathcal{K}$ is dp-minimal. Then any $\mathcal{K}$-definable family of subsets of K has only finitely many germs at any $a \in K$.*

LEMMA 4.5. *Suppose K is dp-minimal. Then P_n is finite index in $K^\times$ for any $n \geq 1$ and any finite index subgroup of $K^\times$ is a finite union of cosets of some P_n.*

PROOF. The second claim follows from the first claim and Fact 4.3. We prove the first claim. Fix $n \geq 1$. It is easy to see that 0 is an accumulation point of P_n, so 0 is an accumulation point of aP_n for each $a \in K^\times$. Now apply Fact 4.4. ⊣

We now prove Theorem 4.1.

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PROOF. After possibly translating we suppose $a = 0$. If 0 is not an accumulation point of X then we take $Y = \emptyset$. Suppose 0 is an accumulation point of X. Let G be the set of $a \in K^\times$ such that $B(k) \cap aX = B(k) \cap X$ for some k. Then G is a $\mathcal{K}$-definable subgroup of $K^\times$, the local multiplicative stabilizer of X at 0. By Fact 4.4 the family $(aX : a \in K^\times)$ has finitely many germs at 0. It follows that $|K^\times/G| < \aleph_0$. Applying Lemma 4.5 we get n such that $P_n \subseteq G$.

Fix $t \in K$ such that $v(t) = 1$, so $v(t^n) = n$. Fix $r \in \mathbb{Z}$ such that $B(r) \cap t^n X = B(r) \cap X$. Fix representatives $\beta_1, \ldots, \beta_l \in X$ of the cosets of P_n such that 0 is an accumulation point of each $\beta_i P_n \cap X$ and set $m_i := v(\beta_i) > r$ for all i. We show that

$$B(m_i) \cap \beta_i P_n \cap X = B(m_i) \cap \beta_i P_n \quad \text{for all} \quad 1 \leq i \leq l.$$

It follows that X and $\beta_1 P_n \cup \cdots \cup \beta_l P_n$ have the same germ at 0.

Fix $1 \leq i \leq l$ and set $m := m_i$ and $\beta := \beta_i$. Now observe that multiplication by t^n maps βP_n to βP_n bijectively and maps $v^{-1}(j) \cap X$ to $v^{-1}(j+n) \cap X$ bijectively for all $j \geq m$. So it suffices to show that $v^{-1}(m) \cap \beta P_n \cap X = v^{-1}(m) \cap \beta P_n$.

We have $v^{-1}(m) \cap \beta P_n = \{a\beta : a \in P_n, v(a) = 0\}$. We fix $a \in P_n$ such that $v(a) = 0$ and show that $a\beta \in X$. As $a \in P_n$ and $v(a) = 0$ multiplication by a maps $v^{-1}(k) \cap X$ to itself bijectively for all sufficiently large k. Fix such a $k \geq m$ and such that $k \in n\mathbb{Z} + m$. Let $j = k - m$ and observe that t^j is a power of t^n. So multiplication by p^j gives a bijection between $v^{-1}(m) \cap \beta P_n \cap X$ and $v^{-1}(k) \cap \beta P_n \cap X$. It follows that multiplication by $t^{-j} a t^j = a$ gives a bijection from $v^{-1}(m) \cap \beta P_n \cap X$ to itself. So $a\beta$ is an element of X. ⊣

THEOREM 4.6. *Let $\mathbb{K}$ be a finite extension of $\mathbb{Q}_p$ for some prime p. Then any dp-minimal expansion of $\mathbb{K}$ is $\mathbb{K}$-minimal.*

PROOF. Let V be the valuation ring of the p-adic valuation on $\mathbb{K}$, recall that V is compact. Suppose that $X \subseteq \mathbb{K}$ is definable in a dp-minimal expansion of $\mathbb{K}$. As $\mathbb{K} = V \cup V^{-1}$ we may suppose that $X \subseteq V$. Theorem 4.1 shows that for every $a \in V$ there is k_a such that $B(a, k_a) \cap X$ is $\mathbb{K}$-definable. As V is compact there is finite $A \subseteq V$ such that $\{B(a, k_a) : a \in A\}$ covers V. So $X = \bigcup_{a \in A}(B(a, k_a) \cap X)$ is $\mathbb{K}$-definable. ⊣

The proof of Theorem 4.6 applies compactness of the valuation ring in a crucial way and so does not extend to elementary extensions of $\mathcal{K}$. So we cannot show that $\mathcal{K}$ is P-minimal.

QUESTION 1. *Is every dp-minimal expansion of $\mathbb{Q}_p$ P-minimal? Equivalently: is the theory of every dp-minimal expansion of $\mathbb{Q}_p$ $\mathrm{Th}(\mathbb{Q}_p)$-minimal?*

It is easy to see that Question 1 is equivalent to: Suppose $\mathcal{Q}$ is a dp-minimal expansion of $\mathbb{Q}_p$, $\mathcal{Q}'$ is an elementary extension of $\mathcal{Q}$, and X is a $\mathcal{Q}'$-definable clopen subset of the valuation ring of $\mathcal{Q}'$. Must X be a finite union of balls?

Recall our standing assumption that (K, v) is a discrete Henselian valued field.

THEOREM 4.7. *Suppose that K is characteristic zero with finite residue field and $\mathcal{K}$ is a dp-minimal expansion of K. Then $\mathcal{K}$ is weakly K-minimal.*

Let $\mathbb{Q}_p^{\mathrm{alg}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{Q}_p$. Then $\mathbb{Q}_p^{\mathrm{alg}}$ is p-adically closed and is hence an elementary subfield of $\mathbb{Q}_p$. So any dp-minimal expansion of $\mathbb{Q}_p^{\mathrm{alg}}$ is weakly $\mathbb{Q}_p^{\mathrm{alg}}$-minimal. We now prove Theorem 4.7.

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PROOF. By Fact 2.5 we may suppose that K is an elementary subfield of a finite extension $\mathbb{K}$ of $\mathbb{Q}_p$, where p is the residue characteristic of (K, v). Let $\mathcal{F}$ be a sufficiently saturated elementary extension of $\mathcal{K}$ with underlying field F. By saturation we may suppose that $\mathbb{K}$ is an elementary subfield of F. We first realize $\mathbb{K}$ as an F^{Sh}-definable set of imaginaries. Let $u : F^\times \to \Gamma$ be the p-adic valuation on F, so u is F-definable. Note that $\mathbb{Z}$ is the minimal non-trivial convex subgroup of Γ. Hence $\mathbb{Z}$ is F^{Sh}-definable. Let $w : F^\times \to \Gamma/\mathbb{Z}$ be the composition of u with the quotient $\Gamma \to \Gamma/\mathbb{Z}$. We equip $\Gamma/\mathbb{Z}$ with a group order by declaring $a + \mathbb{Z} \leq b + \mathbb{Z}$ when $a \leq b$, so w is an externally definable valuation on K. Let W be the valuation ring of w and $\mathfrak{m}$ be the maximal ideal of W. Then W is the set of $\alpha \in K$ such that $u(\alpha) \geq m$ for some $m \in \mathbb{Z}$ and $\mathfrak{m}$ is the set of $\alpha \in K$ such that $u(\alpha) > m$ for all $m \in \mathbb{Z}$. It is easy to see that for every $\alpha \in W$ there is a unique $\alpha^* \in \mathbb{K}$ such that $\alpha - \alpha^* \in \mathfrak{m}$. We identify $W/\mathfrak{m}$ with $\mathbb{K}$ so that the residue map $\mathrm{st} : W \to \mathbb{K}$ is the usual standard part map. Then $\mathbb{K}$ is an F^{Sh}-definable set of imaginaries. Let $\boldsymbol{\mathcal{K}}$ be the structure induced on $\mathbb{K}$ by $\mathcal{F}^{\mathrm{Sh}}$. By Fact 2.1 $\mathcal{F}^{\mathrm{Sh}}$ is dp-minimal, it follows that $\boldsymbol{\mathcal{K}}$ is dp-minimal. By Theorem 4.6 $\boldsymbol{\mathcal{K}}$ is $\mathbb{K}$-minimal.

Suppose X is a $\mathcal{K}$-definable set. We show that X is externally definable in K. By Fact 4.2 we may suppose that X is closed in K. Let X' be the subset of L defined by any formula defining X and let $Y = \mathrm{st}(X' \cap V)$. An easy saturation argument shows that Y is the closure of X in $\mathbb{K}$, so $X = Y \cap K$. Note that Y is $\boldsymbol{\mathcal{K}}$-definable, hence Y is $\mathbb{K}$-definable. So X is externally definable in K. ⊣

We discuss the relationship between $\mathcal{K}$ and $\boldsymbol{\mathcal{K}}$ further in Section 7.6.

§5. Ordered abelian groups with finitely many convex subgroups. Throughout this section $(M, +, <)$ is an ordered abelian group and $\mathcal{M}$ is an expansion of $(M, +, <)$. Section 5 is devoted to the proof of Theorem 5.1.

THEOREM 5.1. *Suppose that $\mathcal{M}$ defines only finitely many convex subgroups of $(M, +, <)$. Then $\mathcal{M}$ is dp-minimal if and only if every unary definable subset of every elementary extension $\mathcal{N}$ of $\mathcal{M}$ is a finite union of sets of the form $C \cap (a + nN)$ for convex $C \subseteq N$, n, and $a \in M$.*

In particular if $(M, +)$ is divisible and $\mathcal{M}$ defines only finitely many convex subgroups then $\mathcal{M}$ is weakly o-minimal if and only if $\mathcal{M}$ is dp-minimal. Theorem 5.1 and Fact 5.3 below show that if $(M, +)$ is non-singular and $(M, +, <)$ has only finitely many convex subgroups then $\mathcal{M}$ is dp-minimal if and only if $\mathrm{Th}(\mathcal{M})$ is weakly $\mathrm{Th}(M, +, <)$-minimal.

There are dp-minimal expansions of divisible ordered abelian groups which are not weakly o-minimal and hence fail the hypothesis of Theorem 5.1. In particular there are dp-minimal ordered fields which are not weakly o-minimal. A weakly o-minimal field is real closed by [29], the ordered field $\mathbb{R}((\Gamma))$ of Hahn series with exponents in a non-singular abelian group Γ is dp-minimal [23], and $\mathbb{R}((\Gamma))$ is real closed if and only if Γ is divisible, see [15, Theorem 4.3.7]. In particular the field $\mathbb{R}((t))$ of Laurent series is dp-minimal and not weakly o-minimal.

We first gather some results and definitions concerning ordered abelian groups.

5.1. Preliminaries on ordered abelian groups. We make frequent use of the elementary facts that $(M, +, <)$ is either dense or discrete, if $(M, +, <)$ is discrete

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then $(M,+,<)$ has a minimal positive element which we denote by 1_M, and $\{m1_M : m \in \mathbb{Z}\}$ is the minimal convex subgroup of $(M,+,<)$ which is not $\{0\}$. A subset X of M is *convex* if whenever $a, a' \in X$ and $a < b < a'$ then $b \in X$. The convex hull of $X \subseteq M$ is the set of $b \in M$ such that $a \leq b \leq a'$ for some $a, a' \in X$. A subgroup of $(M,+)$ is non-trivial if it is not $\{0\}$ or M. We will make use of the fact that the convex subgroups of $(M,+,<)$ form a chain under inclusion.

For our purposes a *cut* in M is a downwards closed subset of M which either does not have a supremum or is of the form $(\infty, a]$ for some $a \in M$. We let $\mathcal{C}(M)$ be the set of cuts in M, order $\mathcal{C}(M)$ under inclusion, and equip $\mathcal{C}(M)$ with the resulting order topology. We identify each $a \in M$ with the cut $(\infty, a]$ so $\mathcal{C}(M)$ is the order-completion of $(M,<)$. A cut C *lies in* a convex $D \subseteq M$ if D intersects both C and $M \setminus C$. A family of cuts is nowhere dense if it is nowhere dense in $\mathcal{C}(M)$.

A *convex equivalence relation* E on M is an equivalence relation with convex equivalence classes. We let E_b be the E-class of $b \in M$. There is a canonical linear order on M/E given by declaring $E_a < E_b$ when every element of E_a is strictly less than every element of E_b. If H is a convex subgroup of $(M,+,<)$ then equivalence modulo H is a convex equivalence relation and the linear order on M/H is a group order. So we always regard the quotient of $(M,+,<)$ by a convex subgroup as an ordered abelian group.

A *cnc subset* of M is a set of the form $C \cap X$ for **c**onvex $C \subseteq M$ and a **c**oset X of nM for some n. Observe that every cnc subset of M is definable in $(M,+,<)^{\mathrm{Sh}}$. We make use of the following fact, whose verification we leave to the reader.

FACT 5.2. *Suppose $(M,+)$ is non-singular. Then the collection of finite unions of cnc subsets of M forms a boolean algebra.*

See [21] for the first claim of Fact 5.3. The second claim follows from the first claim and Fact 5.2

FACT 5.3. *The ordered abelian group $(M,+,<)$ is dp-minimal if and only if $(M,+)$ is non-singular. So if $(M,+,<)$ is dp-minimal then the collection of finite unions of cnc subsets of M is a boolean algebra.*

We say that $(M,+,<)$ is **n**-*regular* if every interval of cardinality at least n contains an n-divisible element and $(M,+,<)$ is regular if it is n-regular for all n. If $(M,+,<)$ is dense then $(M,+,<)$ is n-regular if and only if nM is dense in M. It is a theorem of Robinson and Zakon [38] that $(M,+,<)$ is regular if and only if $(M,+,<)$ is elementarily equivalent to an archimedean ordered abelian group. For each n we let R_n be the set of $a \in M$ such that every interval in $[0,|a|]$ with cardinality at least n contains an n-divisible element. Note that each R_n is $(M,+,<)$-definable. Fact 5.4 shows that R_n is the maximal n-regular convex subgroup of $(M,+,<)$. See e.g., [5, Theorem 3.1] for a proof of Fact 5.4.

FACT 5.4. *Each R_n is a convex subgroup of $(M,+,<)$.*

In general R_n could be $\{0\}$. Lemma 5.5 rules out this situation in our case.

LEMMA 5.5. *If $(M,+,<)$ is discrete or $|M/nM| < \aleph_0$ then $R_n \neq \{0\}$.*

PROOF. If $(M,+,<)$ is discrete then $\{k1_M : k \in \mathbb{Z}\}$ is contained in R_n. Now suppose that $(M,+,<)$ is dense and $|M/nM| < \aleph_0$. Then M is a finite union of

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cosets of nM so nM is somewhere dense. Suppose I is a nonempty open interval in which nM is dense. Fix $a \in I \cap nM$. Let δ be a positive element of M such that $J := (a-\delta, a+\delta)$ is a subset of I. Then $(J \cap nM) - a$ is dense in $(-\delta, \delta)$ and $(J \cap nM) - a \subseteq nM$. So nM is dense in $(-\delta, \delta)$, so $(-\delta, \delta)$ is a subset of R_n. ⊣

We leave Lemma 5.6 to the reader.

LEMMA 5.6. *Suppose I is an interval in R_n which contains at least n elements. Then I intersects $a + nR_n$ for every $a \in R_n$.*

5.2. Right to left. We first prove the right to left implication of Theorem 5.1.

PROPOSITION 5.7. *Suppose that every unary definable set in every elementary extension of $\mathcal{M}$ is a finite union of cnc sets. Then $\mathcal{M}$ is dp-minimal.*

PROOF. By [36, Theorem 2.2] $(M, +)$ is non-singular. So $(M, +, <)$ is dp-minimal by Fact 5.3. An application of compactness shows that for every $\mathcal{M}$-definable family $(X_a)_{a \in M^k}$ of subsets of M there is n such that every X_a is a union of no more than n sets of the form $C \cap (a + nM)$ for convex $C \subseteq M$ and $a \in M$. It is now easy to see that $\mathrm{Th}(\mathcal{M})$ is weakly $\mathrm{Th}(M, +, <)$-minimal. So $\mathcal{M}$ is dp-minimal by Proposition 2.2. ⊣

5.3. Unary definable sets in dp-minimal expansions of ordered abelian groups. In this section we prove Theorem 2.8.

LEMMA 5.8. *Suppose G is a group, $\mathcal{G}$ is a first-order expansion of G, H is an infinite definable subgroup of G, and X is an infinite definable subset of G such that $X \cap aH$ has at most one element for every $a \in G$. Then $\mathcal{G}$ is not dp-minimal.*

We construct an array witnessing failure of dp-minimality, see [41, Definition 4.21] for an overview of such arrays.

PROOF. Let $(a_i : i \in \mathbb{N}), (b_j : j \in \mathbb{N})$ be sequences of distinct elements of X, H, respectively. Let $\phi_1(x, y) := (x^{-1}y \in H)$ and $\phi_2(x, y) := (yx^{-1} \in X)$. Fix $k, l \in \mathbb{N}$. Then $\mathcal{G} \models \phi_1(a_i, a_k b_l)$ if and only if $i = k$ and $\mathcal{G} \models \phi_2(b_j, a_k b_l)$ if and only if $j = l$. So $(\phi_1(a_i, y) : i \in \mathbb{N})$ and $(\phi_2(b_j, y) : j \in \mathbb{N})$ form an array witnessing $\text{dp-rk}(\mathcal{G}) \geq 2$. ⊣

LEMMA 5.9. *Let H be an infinite $\mathcal{M}$-definable subgroup of $(M, +)$, $X \subseteq M$ be $\mathcal{M}$-definable, and $|X \cap (a + H)| < \aleph_0$ for all $a \in M$. If $\text{dp-rk}(\mathcal{M}) = 1$ then X is finite.*

PROOF. Let Y be the set of $a \in X$ such that a is maximal in $X \cap (a + H)$. Then $|Y \cap (a + H)| \leq 1$ for all $a \in M$. Apply Lemma 5.8. ⊣

The first claim of Fact 5.10 is [40, Lemma 3.2]. Note that the convex hull of a subgroup of an ordered abelian group is a subgroup.

FACT 5.10. *If $\mathcal{M}$ is dp-minimal then every definable subgroup of $(M, +)$ has finite index in its convex hull.*

Lemma 5.11 shows that every definable subgroup is a finite union of cnc sets.

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LEMMA 5.11. *Suppose $\mathcal{M}$ is dp-minimal. Let H be an $\mathcal{M}$-definable subgroup of $(M,+)$. Then H is of the form $H' \cap C$ where C is the convex hull of H and H' is a finite union of cosets of nM for some n.*

PROOF. Let C be the convex hull of H in M. So C is a subgroup of $(M,+)$. By Fact 5.10 H has finite index in C. By Fact 4.3 H is a finite union of cosets of nC for some n. Finally, it is easy to see that $nC + a = (nM + a) \cap C$ for any $a \in C$. ⊣

LEMMA 5.12. *Suppose $\mathcal{M}$ is dp-minimal and one of the following is satisfied*:

(1) *$(M,+,<)$ is dense, or*

(2) *$(M,+,<)$ is discrete and $\mathcal{M}$ is sufficiently saturated.*

Let $X \subseteq M$ be $\mathcal{M}$-definable. There is n such that every infinite interval I contains an infinite interval J such that $J \cap X = J \cap [\bigcup_{a \in A} a + nM]$ for finite $A \subseteq M$.

The proof of Lemma 5.12 splits into the dense and discrete cases. In the discrete case we apply Facts 0.2 and 2.1. In the dense case we apply results of Goodrick and Simon. Fact 5.13 is [16, Lemma 3.3].

FACT 5.13. *If $(M,+,<)$ is dense and $\mathcal{M}$ is dp-minimal then every nowhere dense definable subset of M is finite.*

Fact 5.14 is proven in the proof of [40, Theorem 3.6]. Note that while [40, Theorem 3.6] is stated for expansions of divisible ordered abelian groups the proof only uses this assumption in the final step.

FACT 5.14. *Suppose that $(M,+,<)$ is dense, $\mathcal{M}$ is dp-minimal, and X is an infinite definable subset of M. Then there is $a \in X$ and an interval I containing 0 such that*

(1) *$X^* = (X - a) \cap I$ is dense in I,*

(2) *If $g, h \in X^*$ and $g + h \in I$ then $g + h, -g \in X^*$.*

We now prove Lemma 5.12.

PROOF. Applying non-singularity and either compactness (in the dense case) or saturation (in the discrete case) we see that it suffices to show that every infinite interval I has an infinite subinterval J such that $J \cap X = J \cap Y$ for some finite union Y of cosets of some nM.

Let I be an infinite open interval. The case when $I \cap X$ is finite is trivial, so suppose that $I \cap X$ is infinite. After replacing X with $I \cap X$ if necessary we suppose $X \subseteq I$. It now suffices to produce $p \in X$ and an infinite subinterval J of I containing p such that $J \cap X = J \cap Y$ where Y is a finite union of cosets of some nM.

Suppose $(M,+,<)$ is dense. In this case every nonempty open interval is infinite. After passing to an elementary extension of $\mathcal{M}$ if necessary we suppose $\mathcal{M}$ is sufficiently saturated. By Fact 5.14 we suppose, after translating and shrinking I if necessary, that I contains 0, $-I = I$, X is dense in I, and if $g, h \in X$ satisfy $g + h \in I$ then $g + h, -g \in X$. Saturation yields a convex subgroup C contained in I. Then $X \cap C$ is a subgroup of $(M,+)$ and X is dense and hence cofinal in C. Fact 2.1 and Lemma 5.11 together show that $X \cap C$ is equal to $H \cap C$ where H is a finite union of cosets of nM for some n. Now take $J \subseteq C$ to be any open interval containing 0.

Now suppose (2) holds. We identify 1_M with 1 and $1_M\mathbb{Z}$ with $\mathbb{Z}$, so $\mathbb{Z}$ is a convex subgroup of $(M,+,<)$. Saturation implies $\mathbb{Z}$ is non-trivial. Fact 2.1 shows

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that $(\mathcal{M}, \mathbb{Z})$ is dp-minimal. Lemma 5.9 shows that some coset of $\mathbb{Z}$ has infinite intersection with X. After possibly translating and reflecting we suppose $\mathbb{N} \cap X$ is infinite. This implies that $\mathbb{N} \subseteq I$. Fact 0.2 yields $b \in \mathbb{Z}$, $n \geq 1$, and a finite union Y of cosets of nM such that

$$\{a \in \mathbb{Z} : a > b\} \cap X = \{a \in \mathbb{Z} : a > b\} \cap Y.$$

An application of saturation yields $b' \in I, b' > \mathbb{N}$ such that $(b, b') \cap X = (b, b') \cap Y$. Take $J = (b, b')$ and note that J is infinite. ⊣

LEMMA 5.15. *Suppose $\mathcal{M}$ is dp-minimal and one of the following holds*:

(1) $(M, +, <)$ *is dense, or*
(2) $(M, +, <)$ *is discrete and $\mathcal{M}$ is sufficiently saturated.*

Let E be a convex equivalence relation such that for every infinite interval I we have $|I \cap E_b| \geq \aleph_0$ for some $b \in M$. Then there are only finitely many finite E-classes.

PROOF. Suppose $(M, +, <)$ is dense. So an E-class is finite if and only if it is a singleton. Let Y be the set of $b \in M$ such that $E_b = \{b\}$. We show that Y is finite. By Fact 5.13 it suffices to show that any nonempty open interval I contains a nonempty interval J such that $J \cap Y = \emptyset$. Fix $a \in I$ such that $I \cap E_a$ is infinite. As $I \cap E_a$ is convex and infinite there is a nonempty open interval $J \subseteq I \cap E_a$, so $J \cap Y = \emptyset$.

Suppose (2) holds. We identify 1_M with 1 and $1_M\mathbb{Z}$ with $\mathbb{Z}$. The E-class of $a \in M$ is infinite if and only if it is either contains an initial or final segment of $a + \mathbb{Z}$. So $\{a \in M : |E_a| < \aleph_0\}$ is $(\mathcal{M}, \mathbb{Z})$-definable. Let Y be the $(\mathcal{M}, \mathbb{Z})$-definable set of minimal elements of finite E-classes. Suppose toward a contradiction that there are infinitely many finite E-classes.

First suppose there is $a \in M$ such that $a + \mathbb{Z}$ contains infinitely many finite E-classes. After translating and reflecting if necessary suppose that $\mathbb{N}$ contains infinitely many finite E-classes. Then $Y \cap \mathbb{N}$ is infinite. Dp-minimality of $(\mathcal{M}, \mathbb{Z})$ and Fact 0.2 together yield $n \geq 1$, $b \in \mathbb{N}$, and a finite union Z of cosets of nM such that

$$\{a \in \mathbb{Z} : a > b\} \cap Y = \{a \in \mathbb{Z} : a > b\} \cap Z.$$

Applying saturation we obtain $b' \in M, b' > \mathbb{N}$ such that $(b, b') \cap Y = (b, b') \cap Z$. So the E-class of each element of (b, b') is finite, contradiction.

Now suppose that each coset of $\mathbb{Z}$ contains only finitely many finite E-classes. Then Y is infinite and $|Y \cap (a + \mathbb{Z})| < \aleph_0$ for all $a \in M$. This contradicts Lemma 5.9. ⊣

THEOREM 5.16. *Suppose $\mathcal{M}$ is dp-minimal and X is an $\mathcal{M}$-definable subset of M. Then there is n and an $\mathcal{M}$-definable convex equivalence relation E such that*

(1) *there are only finitely many finite E-classes,*
(2) *every infinite E-class is open, and*
(3) *for every $a \in M$ we have $E_a \cap X = E_a \cap Y$ where Y is a (necessarily finite) union of cosets of nM.*

PROOF. Let $\mathcal{M} \prec \mathcal{N}$ be sufficiently saturated and X' be the subset of N defined by same formula as X. Lemma 5.12 yields n such that every infinite interval $I \subseteq N$ has an infinite subinterval J such that $J \cap X' = J' \cap Y$ for some union Y of cosets of nN. Let R_n be as above. Recall that R_n is $(M, +, <)$-definable. We define a binary

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relation E on M. We declare $(a, a) \in E$ for all a, and if $a < b$ then a, b are E-related if and only if there are $a' < a$ and $b' > b$ such that:

(1) a', b' lie in the same coset of R_n,
(2) (a', a) and (b, b') both have at least n elements, and
(3) $(a', b') \cap X = (a', b') \cap Y$ for some union Y of cosets of nM.

Finally, if $a > b$ then $(a, b) \in E$ if and only if $(b, a) \in E$. Note that E is $\mathcal{M}$-definable. We now show that E is an equivalence relation. It suffices to show that E is transitive. Fix distinct $a, b, c \in M$ such that a, b and b, c are both E-equivalent. There are several cases to consider. Without loss of generality we suppose $a < b$, the case when $a > b$ is handled in the same way. The cases when $a < c < b$ and $c < a < b$ are easy.

We treat the case when $a < b < c$. Fix $a' < a$, $b_0 < b < b_1$, $c < c'$ and finite unions Y_0, Y_1 of cosets of nM such that (a', a), (b_0, b), (b, b_1), and (c, c') all have at least n elements and $(b_0, c') \cap X = (b_0, c') \cap Y_0$, $(a', b_1) \cap X = (a', b_1) \cap Y_1$. If $Y_0 = Y_1$ then $(a', c') \cap X = (a', c') \cap Y_1$ and a, c are E-equivalent. We therefore show that $Y_0 = Y_1$. We first reduce to the case when a, b, c are in R_n. Note that $a', a, b_0, b, b_1, c, c'$ all lie in the same coset of R_n. After replacing X with X^* and E with E^* we suppose that $a', a, b_0, b, b_1, c, c'$ are all in R_n. The intersection of any coset of nM with R_n is a coset of nR_n so it is now enough to show that Y_0 and Y_1 contain the same cosets of nR_n.

We show that every coset of nR_n contained in Y_0 is contained in Y_1, the same argument yields the other direction. Fix $d \in M$ and suppose that $d + nR_n \subseteq Y_0$. As (b_0, b) has at least n elements Lemma 5.6 shows that $d + nR_n$ intersects (b_0, b). Hence $d + nR_n$ intersects Y_1 which implies $d + nR_n \subseteq Y_1$.

We now show that E is a convex equivalence relation. Suppose $a < b$ are E-equivalent and fix $a < c < b$. Fix $a' < a < b < b'$ such that a', a, b, b' all lie in the same coset of R_n, (a', a), (b, b') both have at least n elements, and $(a', b') \cap X = (a', b') \cap Y$ for some union Y of cosets of nM. Each coset of R_n is convex so c lies in the same coset of R_n as a', b'. Furthermore (c, b') has at least n elements and $(a', b') \cap X = (a', b') \cap Y$. So a, c are E-equivalent.

We show that every infinite E-class is open. This is trivially true when $(M, +, <)$ is discrete so suppose $(M, +, <)$ is dense. Then any nonempty open interval is infinite. Suppose E_a is infinite. After possibly reflecting there is $b > a$ such that $b \in E_a$. Fix $a' < a$ and $b' > b$ such that a', b' lie in the same coset of R_n and $(a', b') \cap X = (a', b') \cap Y$ for some union Y of cosets of nM. It is easy to see that if $a' < c < b'$ then $c \in E_a$ so $(a', b') \subseteq E_a$.

It remains to show that there are only finitely many finite E-classes. Let E' be the convex equivalence relation on N defined by the same formula as E. It suffices to show that E' has only finitely many finite classes. By Lemma 5.15 it suffices to fix an infinite interval $I \subseteq N$ and show that $E'_a \cap I$ is infinite for some $a \in I$. Applying Lemma 5.12 we obtain an infinite subinterval J of I such that $J \cap X' = J \cap Y$ for some union Y of cosets of nN. Applying saturation we obtain elements $a' < a < b < b'$ of J such that (a, b) is infinite, (a', a), (b, b') both have at least n elements, and a', b' lie in the same coset of R_n. So a, b are E-equivalent and E_a contains (a, b) and is therefore infinite. ⊣

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Theorem 5.17 is a corollary to Theorem 5.16. When $(M,+)$ is divisible Theorem 5.17 shows that any $\mathcal{M}$-definable subset of M is the union of an open set and a finite set, this is already proven in [40].

THEOREM 5.17. *Suppose $\mathcal{M}$ is dp-minimal, $(M,+,<)$ is dense, and X is an $\mathcal{M}$-definable subset of M. Then X is a finite union of sets of the form $(nM+a)\cap U$ where U is $\mathcal{M}$-definable and open.*

PROOF. When X is finite let $n=0$ and $U=M$. Suppose X is infinite. Let n and E be as in Theorem 5.16. Let A be a set of representatives of the cosets of nM. Let $F:=\{a\in M: E_a=\{a\}\}$. So F is the set of $a\in M$ with finite E-class and F is finite. For each $a\in A$ let U_a be the union of all $b\in M$ such that $b\notin F$ and $(a+nM)\cap X\subseteq E_b\cap X$. Each U_a is a union of infinite E-classes and is therefore open. We have

$$X=(F\cap X)\cup\bigcup_{a\in A}[U_a\cap(a+nM)].$$ ⊣

5.4. Proof of Theorem 5.1. We prove the left to right implication of Theorem 5.1. A cut C is *valuational* if $C+a=C$ for some positive $a\in C$, and *non-valuational* otherwise. An $\mathcal{M}$-definable convex subgroup of $(M,+,<)$ is valuational if it contains an $\mathcal{M}$-definable non-trivial convex subgroup, and non-valuational otherwise. We say that $\mathcal{M}$ is valuational if there is a non-trivial $\mathcal{M}$-definable convex subgroup, and non-valuational otherwise.

LEMMA 5.18. *Suppose H is an $\mathcal{M}$-definable convex subgroup of $(M,+,<)$. Then H is valuational if and only if there is an $\mathcal{M}$-definable valuational cut C which lies in some coset of H. In particular $\mathcal{M}$ is valuational if and only if there is an $\mathcal{M}$-definable valuational cut.*

PROOF. Suppose H is valuational. Let K be a non-trivial definable convex subgroup of H and let C be the downwards closure of K. Then $C+a=C$ for any positive element $a\in K$. So C is valuational. Observe that C lies in H.

Suppose C is a definable valuational cut which lies in $b+H$. It is easy to see that $\{a\in M: C+a=C=C-a\}$ is a non-trivial definable convex subgroup of H. ⊣

If $(M,+,<)$ is non-archimedean then there are $a,b>0$ such that $na<b$ for all n, in which case the convex hull of $a\mathbb{Z}$ is a nontrivial convex subgroup of $(M,+,<)$. Conversely, if K is a convex subgroup of $(M,+,<)$, and a,b are positive elements of M such that $a\in K, b\notin K$, then $na<b$ for all n. Hence $(M,+,<)$ is archimedean if and only if $(M,+,<)$ does not have a non-trivial convex subgroup, equivalently if every cut in M is valuational. Thus the property of being non-valuational is a definable analog of the archimedean property.

LEMMA 5.19. *Suppose $(M,+,<)$ is discrete and C is a cut in M. Then C is non-valuational if and only if C has a maximal element. Furthermore $\mathcal{M}$ is non-valuational if and only if every nonempty definable subset of M which is bounded above has a maximal element.*

PROOF. Suppose a is maximal in C. Then $a+b\notin C$ for any positive $b\in M$, so C is non-valuational. Suppose C is non-valuational. Then there is $a\in C$ such that $a+1_M\notin C$. Then a is maximal in C as $a+1_M$ is the successor of a.

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We now treat the second claim. Suppose C is a valuational cut in $\mathcal{M}$ and $a \in M_{>0}$ satisfies $b + a \in C$ for any $b \in C$. As $b < b + a$ for all $b \in C$, C does not have a maximum. Suppose $\mathcal{M}$ is non-valuational. Suppose X is a nonempty definable subset of M which is bounded above. Let C be the downwards closure of X. Then C has a maximal element, which is the maximal element of X. ⊣

PROPOSITION 5.20. *Suppose $\mathcal{M}$ is dp-minimal and H is an $\mathcal{M}$-definable convex subgroup. Then the following are equivalent*:

(1) *H is valuational*,
(2) *there is an $\mathcal{M}$-definable convex equivalence relation E such that every coset of H contains infinitely many infinite E-classes*,
(3) *there is an $\mathcal{M}$-definable convex equivalence relation E with infinitely many E-classes E_a such that $|E_a| \geq \aleph_0$ and E_a does not contain $a + H$.*

In particular $\mathcal{M}$ is valuational if and only if there is an $\mathcal{M}$-definable convex equivalence relation with infinitely many infinite equivalence classes.

Proposition 5.20 requires Lemma 5.21 and Fact 5.22. We leave the easy proof of Lemma 5.21 to the reader.

LEMMA 5.21. *Suppose $(M, +, <)$ is dense. Suppose $\mathcal{E}$ is a collection of pairwise disjoint infinite convex subsets of M which are bounded above. Let $\mathcal{C}$ be the collection of downwards closures of elements of $\mathcal{E}$. Then $\mathcal{C}$ is a nowhere dense family of cuts.*

We now recall a result of Goodrick [16, Lemma 3.3] which generalizes Fact 5.13.

FACT 5.22. *Suppose $\mathcal{M}$ is dp-minimal and $(M, +, <)$ is dense. Let $\mathcal{C}$ be an $\mathcal{M}$-definable family of cuts. If $\mathcal{C}$ has infinitely many non-valuational elements then $\mathcal{C}$ is somewhere dense.*

We now prove Proposition 5.20.

PROOF. If K is a non-trivial $\mathcal{M}$-definable convex subgroup of H then equivalence modulo K is a convex equivalence relation and every coset of H contains infinitely many classes. So (1) implies (2).

It is clear that (2) implies (3). We show that (3) implies (1). Suppose E satisfies the conditions of (3). If E_a does not contain $a + H$ then E_a is disjoint from either an initial or final segment of $a + H$. After reflecting if necessary we suppose that there are infinitely many E-classes E_a such that E_a is infinite and does not contain a final segment of $a + H$. Let $\mathcal{B}$ be the family of downwards closures of such E-classes. Note that every element of $\mathcal{B}$ lies in some coset of H.

Suppose $(M, +, <)$ is dense. Then an E-class is infinite if and only if it has at least two elements, so $\mathcal{B}$ is an $\mathcal{M}$-definable family of cuts. Lemma 5.21 shows that $\mathcal{B}$ is nowhere dense. So $\mathcal{B}$ contains a valuational cut by Fact 5.22. Lemma 5.18 now shows that H is valuational.

Now suppose $(M, +, <)$ is discrete. We identify the minimal positive element of M with 1 and consider $\mathbb{Z}$ as a convex subgroup of $(M, +, <)$. Then $(\mathcal{M}, \mathbb{Z})$ is dp-minimal by Fact 2.1. The downwards closure of E_a is in $\mathcal{B}$ if there is $b \in E_a$ such that E_a contains an initial segment of $b + \mathbb{Z}$ but does not contain $b + H$. So $\mathcal{B}$ is an $(\mathcal{M}, \mathbb{Z})$-definable family of cuts. Suppose H is non-valuational. Lemma 5.18 implies that each cut in $\mathcal{B}$ is non-valuational. Lemma 5.19 shows that each cut in $\mathcal{B}$

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has a maximal element. Let Y be the $(\mathcal{M}, \mathbb{Z})$-definable set of maximal elements of cuts in $\mathcal{B}$. Note that distinct elements of Y lie in distinct cosets of $\mathbb{Z}$. As Y is infinite Lemma 5.8 implies $(\mathcal{M}, \mathbb{Z})$ is not dp-minimal, contradiction. ⊣

Proposition 5.23 in particular shows that if $\mathcal{M}$ is non-valuational then every definable subset of M is a finite union of cnc sets.

PROPOSITION 5.23. *Suppose $\mathcal{M}$ is dp-minimal and H is an $\mathcal{M}$-definable non-valuational convex subgroup of $(M, +, <)$. Let $N := M/H$ and π be the quotient map $M \to N$. Then every $\mathcal{M}$-definable subset of M is a finite union of cnc sets and sets of the form $\pi^{-1}(Y) \cap (b + nM)$ for $\mathcal{M}$-definable $Y \subseteq N$, $b \in M$, and n.*

Taking $H = M$ we see that if $\mathcal{M}$ is non-valuational then every definable unary set is a finite union of cnc sets.

PROOF. Let X be an $\mathcal{M}$-definable subset of M. Let n and E be as obtained in Theorem 5.16. Let A be a set of representatives of the cosets of nM. There are only finitely many finite E-classes, so all but finitely many cosets of H intersect an infinite E-class. Proposition 5.20 implies that all but finitely many cosets of H are contained in some E-class. Let $F \subseteq N$ be finite such that if $c \in N \setminus F$ then $\pi^{-1}(c)$ is contained in an E-class. So if $c \in N \setminus F$ then

$$\pi^{-1}(c) \cap X = \pi^{-1}(c) \cap \bigcup_{a \in B} a + nM$$

for some $B \subseteq A$. For each $b \in A$ let Y_b be the $\mathcal{M}$-definable set of $c \in N \setminus F$ such that

$$\pi^{-1}(c) \cap [b + nM] \subseteq \pi^{-1}(c) \cap X.$$

Then

$$X \setminus \pi^{-1}(F) = \bigcup_{b \in A} \pi^{-1}(Y_b) \cap [b + nM].$$

So it is enough to show that $X \cap \pi^{-1}(c)$ is a finite union of cnc sets for each $c \in F$. Fix $c \in F$. As H is non-valuational and there are only finitely many finite E-classes Proposition 5.20 shows that only finitely many E-classes intersect $\pi^{-1}(c)$. For each $b \in A$ let $\mathcal{C}_b$ be the set of all E_a such that E_a intersects $\pi^{-1}(c)$ and $b + nM \subseteq E_a \cap X$. Then

$$\pi^{-1}(c) \cap X = \bigcup_{b \in A} \bigcup_{C \in \mathcal{C}_b} C \cap (b + nM).$$

So $\pi^{-1}(c) \cap X$ is a finite union of cnc sets. ⊣

We now finally prove Theorem 5.1.

PROOF. Suppose $\mathcal{M}$ defines exactly m non-trivial convex subgroups of $(M, +, <)$. We apply induction on m. If $m = 0$ then $\mathcal{M}$ is non-valuational so this case follows from Proposition 5.23. Suppose $m \geq 1$. Let H be the intersection of all non-trivial definable convex subgroups of $(M, +, <)$. Then H is the minimal non-trivial definable convex subgroup of $(M, +, <)$, so H is non-valuational. Let $N := M/H$.

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If K is a convex subgroup of N then $\pi^{-1}(K)$ is a convex subgroup of M containing H. Note that there are exactly $m-1$ non-trivial definable convex subgroups of N.

By Proposition 5.23 and Fact 5.2 it suffices to suppose that $Y \subseteq N$ is definable and show that $\pi^{-1}(Y)$ is a finite union of cnc sets. Induction yields a finite collection $\mathcal{Y}$ of cnc subsets of N such that $Y = \bigcup \mathcal{Y}$. As $\pi^{-1}(Y) = \bigcup_{Z\in\mathcal{Y}} \pi^{-1}(Z)$ we may suppose that Y is a cnc subset of N. Let $Y = C \cap (a + nN)$ for convex $C \subseteq N$. So $\pi^{-1}(Y) = \pi^{-1}(C) \cap \pi^{-1}(a + nN)$. As π is monotone $\pi^{-1}(C)$ is convex. Fix $a' \in M$ such that $\pi(a') = a$. Then $\pi^{-1}(a + nN) = a' + \pi^{-1}(nN)$. Then $\pi^{-1}(nN)$ is a subgroup of $(M, +)$ containing nM. By non-singularity $|M/nM| < \aleph_0$ so $\pi^{-1}(nN)$ is a finite union of cosets of nM. So $\pi^{-1}(a + nN)$ is a finite union of cosets of nM. ⊣

We now gather some corollaries to Theorem 5.1.

5.5. Concrete examples. We first discuss some examples of ordered abelian groups with only finitely many convex subgroups. Let $<_{\text{Lex}}$ be the lexicographic order on $\mathbb{R}^n$. It is easy to see that if M is a subgroup of $(\mathbb{R}^n, +)$ then every convex subgroup of $(M, +, <_{\text{Lex}})$ is of the form

$$\{(a_1, \dots, a_n) \in M : a_1 = a_2 = \cdots = a_k = 0\}$$

for $0 \leq k \leq n$. So in this case $(M, +, <_{\text{Lex}})$ has at most $n-1$ non-trivial convex subgroups. Corollary 5.24 follows from Theorem 5.1, Fact 5.3, and Proposition 2.2.

COROLLARY 5.24. *Suppose M is a subgroup of $(\mathbb{R}^n, +)$ and $\mathcal{M}$ expands $(M, +, <_{\text{Lex}})$. Then $\mathcal{M}$ is dp-minimal if and only if $(M, +)$ is non-singular and* $\text{Th}(\mathcal{M})$ *is weakly* $\text{Th}(M, +, <_{\text{Lex}})$*-minimal. If $(M, +)$ is divisible and $\mathcal{M}$ is dp-minimal then $\mathcal{M}$ is weakly o-minimal.*

Thus a dp-minimal expansion of $(\mathbb{R}^n, +, <_{\text{Lex}})$ or $(\mathbb{Q}^n, +, <_{\text{Lex}})$ is weakly o-minimal.

We let $\text{rk}(A)$ be the rank of an abelian group A, i.e., the dimension of the $\mathbb{Q}$-vector space $A \otimes \mathbb{Q}$. We apply the well-known fact that rank is additive across short exact sequences.

FACT 5.25. *If $\text{rk}(M) < \aleph_0$ then $(M, +, <)$ has only finitely many convex subgroups.*

PROOF. The case when $M = \{0\}$ is trivial, so we suppose that M is infinite. We show that if $(M, +, <)$ has at least n non-trivial convex subgroups then $\text{rk}(M) \geq n + 1$ by applying induction on n. As M is torsion free $\text{rk}(M) \geq 1$, so the case $n = 0$ is trivial. Suppose $n \geq 1$. Let H be a non-trivial convex subgroup of $(M, +, <)$ with at least $n-1$ convex subgroups. By induction $\text{rk}(H) \geq n$. As H is proper $M/H \neq \{0\}$. So M/H is torsion free, hence $\text{rk}(M/H) \geq 1$. We have an exact sequence

$$0 \longrightarrow H \longrightarrow M \longrightarrow M/H \longrightarrow 0.$$

So $\text{rk}(M) = \text{rk}(H) + \text{rk}(M/H) \geq n + 1$. ⊣

Corollary 5.26 follows from Theorem 5.1 and Fact 5.25.

COROLLARY 5.26. *Suppose $(M, +)$ has finite rank. Then $\mathcal{M}$ is dp-minimal if and only if $(M, +)$ is non-singular and* $\text{Th}(\mathcal{M})$ *is weakly* $\text{Th}(M, +, <)$*-minimal. In particular if $(M, +)$ is divisible and $\mathcal{M}$ is dp-minimal then $\mathcal{M}$ is weakly o-minimal.*

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5.6. The discrete non-valuational case. We prove Theorem 5.27, a generalization of Fact 0.2. An ordered abelian group is a $\mathbb{Z}$-group if it is elementarily equivalent to $(\mathbb{Z}, +, <)$. Model completeness of $(\mathbb{Z}, +, <)$ shows that if $(M, +, <)$ is a $\mathbb{Z}$-group then the embedding $\mathbb{Z} \to M$ given by $k \mapsto k1_M$ is elementary.

THEOREM 5.27. *Suppose $(M, +, <)$ is discrete. Then the following are equivalent*

(1) *$\mathcal{M}$ is dp-minimal and non-valuational,*
(2) *$(M, +, <)$ is a $\mathbb{Z}$-group and $\mathcal{M}$ is interdefinable with $(M, +, <)$.*

Theorem 5.27 requires several tools. Standard model-theoretic results on ordered abelian groups imply that a non-valuational discrete ordered abelian group is a $\mathbb{Z}$-group. We summarize these results in the proof of Fact 5.28.

FACT 5.28. *Suppose $(M, +, <)$ is discrete. Then $(M, +, <)$ is non-valuational if and only if $(M, +, <)$ is a $\mathbb{Z}$-group.*

PROOF. Any $\mathbb{Z}$-group is non-valuational as $(\mathbb{Z}, +, <)$ is non-valuational. A regular ordered abelian group is elementairly equivalent to an archimedean ordered abelian group by [38]. So if $(M, +, <)$ is regular then $(M, +, <)$ is a $\mathbb{Z}$-group. Suppose $(M, +, <)$ is not regular and fix n such that $(M, +, <)$ is not n-regular. Then R_n is a non-trivial $(M, +, <)$-definable convex subgroup. So $(M, +, <)$ is valuational. ⊣

Fact 5.29 is due to Cluckers [10, Theorem 5]. It is a generalization of the theorem of Micheaux and Villemaire [31].

FACT 5.29. *Suppose $(M, +, <)$ is a $\mathbb{Z}$-group and $\mathcal{M}$ is sufficiently saturated. If $\mathcal{M}$ is $(M, +, <)$-minimal then $\mathcal{M}$ is interdefinable with $(M, +, <)$.*

We now prove Theorem 5.27.

PROOF. It is easy to see that (2) implies (1). We show that (1) implies (2). Suppose (2). Then $(M, +, <)$ is a $\mathbb{Z}$-group by Fact 5.28. We may pass to an elementary extension of $\mathcal{M}$ without loss of generality, so suppose $\mathcal{M}$ is sufficiently saturated. By Fact 5.29 it suffices to show that $\mathcal{M}$ is $(M, +, <)$-minimal. By Proposition 5.23 it suffices to show that every definable cnc subset of M is $(M, +, <)$-definable. It suffices to show that every definable convex set is $(M, +, <)$-definable. As $\mathcal{M}$ is discrete and non-valuational Lemma 5.19 shows that every $\mathcal{M}$-definable convex subset of M which is bounded above has a maximum and every bounded below $\mathcal{M}$-definable convex subset of M has a minimum. So every $\mathcal{M}$-definable convex subset of M is an interval. ⊣

5.7. An application to dp-minimal expansions of fields. Recall that a valuation on an ordered field is convex if its valuation ring is convex.

COROLLARY 5.30. *Let $(K, +, \times, <)$ be an ordered field and $\mathcal{K}$ be an expansion of $(K, +, \times, <)$. If $\mathcal{K}$ is dp-minimal then $\mathcal{K}$ is either weakly o-minimal or defines a non-trivial convex valuation.*

If $(K, +, \times, <)$ is real closed then $(K, +, \times, <)^{\mathrm{Sh}}$ is weakly o-minimal by Fact 2.1 and $(K, +, \times, <)^{\mathrm{Sh}}$ defines every convex valuation on $(K, +, \times, <)$.

PROOF. If $\mathcal{K}$ is non-valuational then $\mathcal{K}$ is weakly o-minimal by Theorem 5.1. Suppose $\mathcal{K}$ is valuational and let H be a non-trivial convex subgroup of the additive

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group of K. It is easy to see that $\{a \in K : aH \subseteq H\}$ is a non-trivial convex subring of K, and any convex subring of K is a valuation ring. ⊣

Weakly o-minimal non-valuational structures are quite similar to o-minimal structures [26, 48, 49]. So Corollary 5.30 shows that a dp-minimal expansion of an ordered field is either "almost o-minimal" or admits a definable valuation.

Suppose that L is a field and $\mathcal{L}$ is a dp-minimal expansion of L. It is an open question whether one of the following necessarily holds:

(1) $\mathcal{L}$ is strongly minimal,
(2) $\mathcal{L}$ admits a definable field order, or
(3) $\mathcal{L}$ admits a non-trivial definable valuation.

Johnson [22] has shown that if L is not algebraically closed then (2) or (3) holds, so it suffices to show that a non strongly minimal dp-minimal expansion of an algebraically closed field admits a non-trivial definable valuation. If the trichotomy above holds, then any dp-minimal expansion of a field is either strongly minimal, "valuational," or "almost o-minimal."

§6. Cyclically ordered abelian groups. Throughout this section $(M, +)$ is an abelian group. A cyclic group order on $(M, +)$ is a ternary relation C such that for all $a, b, c \in M$:

(1) if $C(a, b, c)$, then $C(b, c, a)$,
(2) if $C(a, b, c)$, then $C(c, b, a)$ fails,
(3) if $C(a, b, c)$ and $C(a, c, d)$ then $C(a, b, d)$,
(4) if a, b, c are distinct, then either $C(a, b, c)$ or $C(c, b, a)$,
(5) C is invariant under the group operation.

We say that a subset J of M is convex if whenever $a, b \in J$ are distinct we either have $\{t : C(a, t, b)\} \subseteq J$ or $\{t : C(b, t, a)\} \subseteq J$. Given $a, c \in M$ we define the open interval with endpoints a, c to be the set of $b \in M$ such that $C(a, b, c)$. The collection of open intervals forms the basis for a group topology on $(M, +)$. We say that C is dense if the group topology is non-discrete.

Throughout this section C is a dense cyclic group order on $(M, +)$ and $\mathcal{M}$ is an expansion of $(M, +, C)$.

We equip $\mathbb{R}/\mathbb{Z}$ with the cyclic group order S such that whenever $a, b, c \in \mathbb{R}$ and $0 \leq a, b, c < 1$ then $S(a + \mathbb{Z}, b + \mathbb{Z}, c + \mathbb{Z})$ holds if and only if either $a < b < c$, $b < c < a$, or $c < a < b$. Given an injective character $\chi : M \to \mathbb{R}/\mathbb{Z}$ we let S_χ be the cyclic group order on M where $S_\chi(a, b, c)$ if and only if $S(\chi(a), \chi(b), \chi(c))$. We say that C is *archimedean* if it is of this form. One can show that C is archimedean if and only if there are no $a, b \in M$ such that $C(0, na, b)$ for all n, see [43].

This section is devoted to the Proof of Theorem 6.1.

THEOREM 6.1. *Suppose that $(M, +, C)$ is archimedean. Then $\mathcal{M}$ is dp-minimal if and only if $(M, +)$ is non-singular and* $\mathrm{Th}(\mathcal{M})$ *is weakly* $\mathrm{Th}(M, +, C)$*-minimal. Furthermore if $\mathcal{M}$ is dp-minimal and $\mathcal{M} \preceq \mathcal{N}$ then every $\mathcal{N}$-definable subset of N is a finite union of sets of the form $a + nJ$ for convex $J \subseteq N$.*

Note that every set of the form $a + nJ$, $J \subseteq N$ convex, is $\mathcal{N}^{\mathrm{Sh}}$-definable. We first use the universal cover described below to prove the right to left implication of Theorem 6.1.

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6.1. The universal cover. The universal cover largely reduces the theory of cyclically ordered abelian groups to that of linearly ordered abelian groups. A universal cover $(H, +, <, u, \pi)$ of $(M, +, C)$ consists of an ordered abelian group $(H, +, <)$, a distinguished positive element u of H such that $u\mathbb{Z}$ is cofinal in H, and a surjective group homomorphism $\pi : H \to M$ with kernel $u\mathbb{Z}$ such that for all $0 \leq a, b, c < u$ we have $C(\pi(a), \pi(b), \pi(c))$ if and only if one of the following holds: $a < b < c$, $b < c < a$, or $c < a < b$. We call π a covering map. It is easy to see that $(H, +, <, u, \pi)$ is unique up to unique isomorphism. Note π restricts to a bijection $[0, u) \to M$.

Note that $(\mathbb{R}, +, <, 1)$ is a universal cover of $(\mathbb{R}/\mathbb{Z}, +, S)$, and if $(M, +, C)$ is a subgroup of $(\mathbb{R}/\mathbb{Z}, +)$ with the induced cyclic group order, and H is the preimage of M under the quotient map $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ then $(H, +, <, 1)$ is a universal cover of $(M, +, C)$.

We describe the standard construction of a universal cover. Let $\prec$ be the binary relation on M where $a \prec b$ if either $C(0, a, b)$ or $a = 0$ and $b \neq 0$. It is easy to see that $\prec$ is a linear order on M ($\prec$ is typically not a group order). We let H be $\mathbb{Z} \times M$, let $<$ be the lexicographic product of the usual order on $\mathbb{Z}$ and $\prec$, let u be $(1, 0)$, let $+$ be given by

$$(k, a) + (k', a') = \begin{cases} (k + k', a + a') & \text{if } a = 0 \text{ or } a' = 0 \text{ or } C(0, a, a + a'), \\ (k + k' + 1, a + a') & \text{otherwise,} \end{cases}$$

and let $\pi : H \to M$ be the projection onto the second coordinate. Then $(H, +, <, u)$ is a universal cover of $(M, +, C)$ with covering map π.

We now observe that $(M, +, C)$ is definable in $(H, +, <, u)$. We define

$$a \tilde{+} b = \begin{cases} a + b & \text{if } a + b \in [0, u), \\ a + b - u & \text{otherwise} \end{cases} \quad \text{for all} \quad a, b \in [0, u).$$

We define $\tilde{C}$ by setting $\tilde{C}(a, b, c)$ for any $a, b, c \in [0, u)$ such that $a < b < c$ or $b < c < a$ or $c < a < b$. Then $\pi : H \to M$ induces an isomorphism $([0, u), \tilde{+}, \tilde{C}) \to (M, +, C)$.

We let $I := (-u, u)$ and let $+_u$ be the restriction of $+$ to I, i.e., the ternary relation on I where $a +_u b = c$ when $a, b, c \in I$ and $a + b = c$. So $(I, +_u, <)$ is a local topological group. We show that $(I, +_u, <)$ is definable in $(M, +, C)$. Let $M^{\geq}$ be a copy of M and $M^{-} = \{-1\} \times (M \setminus \{0\})$. We denote an element $(-1, a)$ of M^{-} as $-a$. We will identify $M^{\geq}$ with $[0, u)$ and M^{-} with $(-u, 0)$. We define an order $\lhd$ on $M^{-} \cup M^{\geq}$ in the following way

(1) if $a, b \in M^{\geq}$ and $a \prec b$ then $a \lhd b$,
(2) if $-a, -b \in M^{-}$ and $b \prec a$ then $-a \lhd -b$,
(3) if $-a \in M^{-}$ and $b \in M^{\geq}$ then $-a \lhd b$.

We define a ternary relation $\hat{+}$ on $M^{-} \cup M^{\geq}$ by declaring the following

(1) if $a, b, c \in M^{-} \cup M^{\geq}$ and $a \hat{+} b = c$ then $b \hat{+} a = c$,
(2) $a \hat{+} 0 = a$ for all $a \in M^{-} \cup M^{\geq}$,
(3) if $a, b, c \in M^{\geq}$ are non-zero then $a \hat{+} b = c$ if and only if $C(0, a, a + b)$ and $a + b = c$,

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(4) if $-a,-b,-c \in M^-$ then $-a\hat{+}(-b) = -c$ if and only if $C(0,a,a+b)$ and $a+b=c$,
(5) if $a \in M^{\geq}, -b \in M^-$, and $c \in M^{\geq}$ then $a\hat{+}(-b) = c$ if and only if $b\hat{+}c = a$,
(6) if $a \in M^{\geq}, a \neq 0, -b \in M^-$, and $-c \in M^-$ then $a\hat{+}(-b) = -c$ if and only if $a\hat{+}c = b$.

Let $\iota : M^- \cup M^{\geq} \to I$ be given by declaring $\iota(a) = b$ if either $a \in M^{\geq}$ and $b \in [0,u)$ satisfies $\pi(b) = a$ or $a \in M^-$ and $b \in (-u,0)$ satisfies $\pi(-b) = a$. It is easy to see that ι gives an isomorphism $(M^- \cup M^{\geq}, \hat{+}, \lhd) \to (I, +_u, <)$. We therefore regard $(I, +_u, <)$ as an $(M,+,C)$-definable local topological group.

We let $\equiv_n$ be the relation of equivalence modulo nM on I.

LEMMA 6.2. *For all n, $\equiv_n$ is $(I, +_u, <)$-definable. For all n and $a \in H$, $I \cap (nH + a)$ is $(I, +_u, <)$-definable.*

PROOF. Note that the second claim follows from the first. Observe that $a \in I$ is an element of nH if and only if there is a $b \in I$ such that $nb = a$. Therefore $nH \cap I$ is $(I, +_u, <)$-definable. If $a, b \in [0,u)$ then $a - b \in I$. So if $a, b \in I$ then $a \equiv_n b$ if and only if there is a $c \in nH \cap [0,u)$ such that $a +_u c = b$. If $a, b \in (-u, 0)$ then $a \equiv_n b$ if and only if $-a \equiv_n -b$. If $a \in [0,u)$ and $b \in (-u,0)$ then $a \equiv_n b$ if and only if $a \equiv_n -b$. ⊣

6.2. Proof of Theorem 6.1. We continue with the notation of Section 6.1. The right to left implication of Theorem 6.1 follows from Propositions 2.2 and 6.3.

PROPOSITION 6.3. *Suppose that $(M,+,C)$ is archimedean. Then $(M,+,C)$ is dp-minimal if and only if $(M,+)$ is non-singular.*

We will apply Lemma 6.4 to prove Proposition 6.3.

LEMMA 6.4. *Suppose $(H,+,<,u)$ is a universal cover of $(M,+,C)$ with covering map π. Then $(H,+)$ is non-singular if and only if $(M,+)$ is non-singular.*

Below $\otimes$ is the tensor product of abelian groups. We use some basic facts about $\otimes$.

PROOF. The left to right implication holds as a quotient of a non-singular abelian group is non-singular. We show that $|H/nH| \leq n|M/nM|$, the other implication follows. Let $\iota : \mathbb{Z} \to H$ be given by $\iota(k) := ku$. The sequence

$$\mathbb{Z} \xrightarrow{\iota} H \xrightarrow{\pi} M \to 0$$

is exact, so tensoring with $\mathbb{Z}/n\mathbb{Z}$ we get an exact sequence

$$\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \longrightarrow H \otimes \mathbb{Z}/n\mathbb{Z} \longrightarrow M \otimes \mathbb{Z}/n\mathbb{Z} \longrightarrow 0.$$

Recall that $A \otimes \mathbb{Z}/n\mathbb{Z}$ is isomorphic to A/nA for any abelian group A. So we get an exact sequence

$$\mathbb{Z}/n\mathbb{Z} \longrightarrow H/nH \longrightarrow M/nM \longrightarrow 0.$$

So $|H/nH| \leq n|M/nM|$. ⊣

We now prove Proposition 6.3.

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PROOF. Suppose that $(M,+)$ is non-singular. So $(H,+)$ is non-singular by Lemma 6.4 hence $(H,+,<)$ is dp-minimal by Fact 5.3. The manner in which $(M,+,C)$ is defined in $(H,+,<)$ above shows that $(M,+,C)$ is dp-minimal. Now suppose that $(M,+)$ is singular. Fix n such that $|M/nM| \geq \aleph_0$. As $(M,+,C)$ is archimedean and dense we see that nM is dense in M, so each coset of nM is also dense in M. We construct an array violating dp-minimality. Let $(a_i : i \in \mathbb{N})$ be a sequence of elements of M which lie in distinct cosets of nM. Let $(I_i : i \in \mathbb{N})$ be a sequence of pairwise disjoint nonempty open intervals in M. Then $I_i \cap (a_j + nM) \neq \emptyset$ for all i, j. It follows that $(a_i + nM : i \in \mathbb{N})$ and $(I_i : i \in \mathbb{N})$ form an array witnessing dp-rk$(M,+,C) \geq 2$. ⊣

We now prove the left to right implication of Theorem 6.1. Let $\mathfrak{I}$ be the structure induced on I by $\mathcal{M}$. Note that $\mathfrak{I}$ expands $(I,+_u,<)$. Therefore $\mathcal{M}$ and $\mathfrak{I}$ define isomorphic copies of each other and $\mathcal{M}$ is dp-minimal if and only if $\mathfrak{I}$ is dp-minimal.

LEMMA 6.5. *Suppose $\mathfrak{I}$ is dp-minimal and $X \subseteq I$ is $\mathfrak{I}$-definable and nonempty. Then there is a $p \in X$ and a nonempty open interval $J \subseteq I$ containing p such that $J \cap X = J \cap Y$ where Y is a finite union of cosets of nH for some n.*

PROOF. The lemma is trivial when X is empty, so we suppose X is nonempty. Fix $a \in X$ and let $L \subseteq I$ be an open interval containing a such that $L - a \subseteq I$. Note that the map $L \to L - a$ given by $x \mapsto x - a$ is then $(I,+_u,<)$-definable. After replacing X and L with $(X \cap L) - a$ and $L - a$ we suppose that $0 \in X$. After passing to an elementary extension if necessary suppose $\mathfrak{I}$ is sufficiently saturated. Applying saturation we obtain convex $C \subseteq I$ which contains a positive element and is closed under addition and additive inverse, so $(C,+_u,<)$ is an ordered abelian group. By Fact 2.1 the structure $\mathcal{C}$ induced on C by $\mathfrak{I}$ is dp-minimal. Applying Lemma 5.12 to $\mathcal{C}$ and $C \cap X$ we obtain a $p \in C \cap X$ and a nonempty open interval $J \subseteq C$ such that $p \in J$ and $J \cap X = J \cap Y$ where Y is a finite union of cosets of nC for some n. The lemma now follows easily. ⊣

Lemma 6.6 now follows in the same way as Theorem 5.17. We omit the proof.

LEMMA 6.6. *Suppose $\mathfrak{I}$ is dp-minimal and $X \subseteq I$ is $\mathfrak{I}$-definable. Then X is a finite union of sets of the form $U \cap (nH + a)$ where $U \subseteq I$ is definable and open and $a \in I$.*

We now describe $\mathfrak{I}$-definable sets.

LEMMA 6.7. *Suppose that $(M,+,C)$ is archimedean and $\mathfrak{I}$ is dp-minimal. Then every $\mathfrak{I}$-definable subset of I is a finite union of sets of the form $J \cap (a + nH)$ for convex $J \subseteq I$.*

PROOF. By Lemma 6.6 it suffices to let U be an $\mathfrak{I}$-definable open subset of I and show that U is a finite union of convex sets. We define a convex equivalence relation E on I by declaring a, b to be E-equivalent if and only if a, b lie in the same convex component of U. Let $\mathcal{C}$ be the family of downwards closures of E-classes. So $\mathcal{C}$ is an $\mathfrak{I}$-definable family of cuts. It suffices to show that $\mathcal{C}$ is finite. It is easy to see that $\mathcal{C}$ is nowhere dense. As $(H,+,<)$ is archimedean every $C \in \mathcal{C}$ is non-valuational. A straightforward modification of the proof of Fact 5.22 shows that $\mathcal{C}$ is finite. ⊣

We now prove the left to right implication of Theorem 6.1.

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PROOF. Suppose $\mathcal{M}$ is dp-minimal. Then $(M,+)$ is non-singular by Lemma 6.4. We suppose that X is an $\mathcal{M}$-definable subset of M and show that X is a finite union of sets of the form $a+nJ$ for convex $J \subseteq M$. The same proof goes through for elementary extensions of $\mathcal{M}$, it easy follows that $\mathrm{Th}(\mathcal{M})$ is weakly $\mathrm{Th}(M,+,C)$-minimal.

Let Y be the set of $a \in [0,u)$ such that $\pi(a) \in X$. As $\mathcal{U}$ is dp-minimal if follows by Lemma 6.6 that Y is a finite union of sets of the form $J \cap (a+nM)$ for convex $J \subseteq I$. So we may suppose that Y is of this form. As $X = \pi(Y)$ and π is a homomorphism we have $X = \pi(a) + n\pi(J)$. It is easy to see that $\pi(J)$ is convex, this is also shown carefully in [44, Lemma 3.1]. ⊣

6.3. An application to dp-minimal expansions of $(\mathbb{Z},+)$. Given irrational $\alpha \in \mathbb{R}$ let χ_α be the character $\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ given by $\chi_\alpha(k) = \alpha k + \mathbb{Z}$. Every injective character $\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is of this form. Let S_α be the cyclic group order induced on $\mathbb{Z}$ by χ_α. It is known that any cyclic group order on $(\mathbb{Z},+)$ is one of the following, see for example [44, Proposition 2.5].

(1) S_α for irrational $\alpha \in \mathbb{R}$,
(2) S^+ where $S^+(i,j,k)$ if and only if $i<j<k$ or $j<k<i$ or $k<i<j$,
(3) S^- where $S^-(i,j,k)$ if and only if $k<j<i$ or $j<i<k$ or $i<k<k$.

It is easy to see that $(\mathbb{Z},+,S^+)$ and $(\mathbb{Z},+,S^-)$ are both interdefinable with $(\mathbb{Z},+,<)$. So Corollary 6.8 follows from Theorem 6.1 and Fact 0.2.

COROLLARY 6.8. *Let C be a cyclic group order on $(\mathbb{Z},+)$ and $\mathcal{Z}$ expand $(\mathbb{Z},+,C)$. Then $\mathcal{Z}$ is dp-minimal if and only if* $\mathrm{Th}(\mathcal{Z})$ *is weakly* $\mathrm{Th}(\mathbb{Z},+,C)$*-minimal.*

The only other known examples of dp-minimal expansions of $(\mathbb{Z},+)$ are $(\mathbb{Z},+)$ and $(\mathbb{Z},+,\prec_p)$ where p is prime and $k \prec_p k'$ if and only if the p-adic valuation of k is strictly less than that of k'. It is a theorem of Conant and Pillay [11] that there are no proper stable dp-minimal expansions of $(\mathbb{Z},+)$. It is natural to conjecture, analogously with the remarks in Section 5.7, that an unstable dp-minimal expansion of $(\mathbb{Z},+)$ defines either a cyclic group order or a (perhaps generalized) valuation.

Let $\mathcal{Z}$ be an expansion of $(\mathbb{Z},+,\prec_p)$. It is natural to ask if $\mathcal{Z}$ is dp-minimal if and only if $\mathrm{Th}(\mathcal{Z})$ is weakly $\mathrm{Th}(\mathbb{Z},+,\prec_p)$-minimal. This fails. Let

$$\mathrm{Exp}(\alpha) := \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \quad \text{for all } \alpha \in p\mathbb{Z}_p$$

be the p-adic exponential and $\mathcal{P}$ be the expansion of $(\mathbb{Z},+)$ by all sets of the form $\{(k_1,\dots,k_n) \in \mathbb{Z}^n : (\mathrm{Exp}(pk_1),\dots,\mathrm{Exp}(pk_n)) \in X\}$ for $(\mathbb{Z}_p,+,\times)$-definable $X \subseteq \mathbb{Z}_p^n$. It will be shown in forthcoming work that $\mathcal{P}$ is a dp-minimal expansion of $(\mathbb{Z},+,\prec_p)$ and it is easy to see that $\mathcal{P}$ is not weakly $(\mathbb{Z},+,\prec_p)$-minimal.

§7. Dp-minimal expansions of divisible archimedean groups. Suppose that H is a dense divisible subgroup of $(\mathbb{R},+)$ and $\mathcal{H}$ expands $(H,+,<)$. In this section we show that if $\mathcal{H}$ is dp-minimal then there is a canonical o-minimal expansion $\mathcal{R}$ of $(\mathbb{R},+,<)$ such that $\mathcal{H}^{\mathrm{Sh}}$ is interdefinable with the structure induced on H by $\mathcal{R}$. We first describe $\mathcal{R}$ in the case when $\mathcal{H}$ is o-minimal. Suppose $\mathcal{H}$ is o-minimal. It is

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a theorem of Laskowski and Steinhorn [27] that $\mathcal{H}$ is an elementary substructure of a unique o-minimal expansion $\mathcal{R}$ of $(\mathbb{R}, +, <)$. Note that the structure induced on H by $\mathcal{R}$ is a reduct of $\mathcal{H}^{\mathrm{Sh}}$. We show that $\mathcal{H}^{\mathrm{Sh}}$ is a reduct of the induced structure. Suppose that X is an $\mathcal{H}^{\mathrm{Sh}}$-definable subset of H^n. Let $\mathcal{N}$ be a sufficiently saturated elementary extension of $\mathcal{R}$ and X^* be an $\mathcal{N}$-definable subset of N^n such that $X = X^* \cap H^n$. By the Marker–Steinhorn theorem $\mathcal{R}^{\mathrm{Sh}}$ is interdefinable with $\mathcal{R}$, hence $X^* \cap \mathbb{R}^n$ is $\mathcal{R}$-definable. Thus $X = (X^* \cap \mathbb{R}^n) \cap H^n$ is definable in the structure induced on H by $\mathcal{R}$.

We first gather some background. An NIP structure $\mathcal{M}$ is *Shelah complete* if $\mathcal{M}^{\mathrm{Sh}}$ is interdefinable with $\mathcal{M}$. It follows easily from Fact 2.1 that $\mathcal{M}^{\mathrm{Sh}}$ is Shelah complete when $\mathcal{M}$ is NIP, so an NIP structure is Shelah complete if and only if it is interdefinable with the Shelah expansion of some structure. Fact 7.1 is a special case of the Marker–Steinhorn theorem [30].

FACT 7.1. *Any o-minimal expansion of* $(\mathbb{R}, <)$ *is Shelah complete.*

The structure *induced* on $B \subseteq M^n$ by $\mathcal{M}$ is the structure on B with an m-ary predicate defining $B^m \cap X$ for every $\mathcal{M}$-definable $X \subseteq M^{mn}$. If $\mathcal{M}$ expands a linear order then the *open core* $\mathcal{M}^\circ$ of $\mathcal{M}$ is the reduct of $\mathcal{M}$ generated by all closed $\mathcal{M}$-definable sets. Suppose that $\mathcal{M}$ is an o-minimal expansion of an ordered abelian group $(M, +, <)$ and H is a dense divisible subgroup of $(M, +)$. We say that $(\mathcal{M}, H)$ is a *dense pair* if for all $(\mathcal{M}, H) \preceq (\mathcal{N}, K)$ we have

(A1) $(\mathcal{M}, H)$ is NIP,
(A2) $(\mathcal{N}, K)^\circ$ is interdefinable with $\mathcal{N}$, and
(A3) every $(\mathcal{N}, K)$-definable subset of K^n is of the form $K^n \cap X$ for an $\mathcal{N}$-definable subset X of N^n.

We also use the term "dense pair" in slight variations of the situation above. What we mean in these cases should be clear to the reader. By (A3) the theory of the structure induced on H by $(\mathcal{M}, H)$ is weakly o-minimal and therefore dp-minimal.

Suppose that H is a dense divisible subgroup of $(\mathbb{R}, +, <)$. In this section we describe a canonical correspondence between Shelah complete dp-minimal expansions of $(H, +, <)$ and dense pairs $(\mathcal{R}, H)$. We show that if $(\mathcal{R}, H)$ is a dense pair then the structure induced on H by $(\mathcal{R}, H)$ is Shelah complete and that for every Shelah complete dp-minimal expansion $\mathcal{H}$ of $(H, +, <)$ there is a canonical o-minimal expansion $\mathcal{R}$ of $(\mathbb{R}, +, <)$ such that $\mathcal{H}$ is interdefinable with the structure induced on H by $(\mathcal{R}, H)$.

7.1. Weakly o-minimal non-valuational structures. Suppose that $(M, +, <)$ is a divisible ordered abelian group and $\mathcal{M}$ is a weakly o-minimal non-valuational expansion of $(M, +, <)$. Let $\overline{M}$ be the collection of all nonempty $\mathcal{M}$-definable cuts in M which are bounded above. (Recall our convention that every cut in M either does not have a supremum or is of the form $(-\infty, a]$ for some $a \in M$.) Let $\boldsymbol{\mathcal{M}}$ be the structure on $\overline{M}$ with an n-ary predicate defining the closure in $\overline{M}^n$ of every $\mathcal{M}$-definable subset of M^n. The natural addition on cuts makes $\boldsymbol{\mathcal{M}}$ an expansion of an divisible ordered abelian group. The completion $\boldsymbol{\mathcal{M}}$ is studied in [26, 48, 49]. The expansion $(\boldsymbol{\mathcal{M}}, M)$ of $\boldsymbol{\mathcal{M}}$ by a unary predicate defining M is studied in [3].

FACT 7.2. *The completion* $\boldsymbol{\mathcal{M}}$ *is o-minimal,* $(\boldsymbol{\mathcal{M}}, M)$ *is a dense pair, and the structure induced on* M *by* $(\boldsymbol{\mathcal{M}}, M)$ *is interdefinable with* $\mathcal{M}$.

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The first claim of Fact 7.2 is shown in [26]. The other claims follow from [3]. It is not explicitly shown in [3] that $(\mathcal{M}, M)$ is NIP, but this follows immediately from the results of that paper and [9].

7.2. The correspondence. Suppose that H is a dense divisible subgroup of $(\mathbb{R}, +)$. We describe a canonical correspondence (up to interdefinability) between

(1) Shelah complete dp-minimal expansions of $(H, +, <)$, and
(2) o-minimal expansions $\mathcal{R}$ of $(\mathbb{R}, +, <)$ such that $(\mathcal{R}, H)$ is a dense pair.

Lemma 7.3 gives one direction of this correspondence.

LEMMA 7.3. *Suppose that $\mathcal{R}$ is an o-minimal expansion of $(\mathbb{R}, +, <)$ and $(\mathcal{R}, H)$ is a dense pair. Then the structure $\mathcal{H}$ induced on H by $(\mathcal{R}, H)$ is Shelah complete.*

PROOF. Let $(\mathcal{N}, K)$ be a highly saturated elementary extension of $(\mathcal{R}, H)$ and Y be an $(\mathcal{N}, K)$-definable subset of K^n. We show that $H^n \cap Y$ is $(\mathcal{R}, H)$-definable. By (A3) there is an $\mathcal{N}$-definable subset X of N^n such that $Y = K^n \cap X$. By Fact 7.1 $\mathbb{R}^n \cap X$ is $\mathcal{R}$-definable. So $H^n \cap X = H^n \cap (\mathbb{R}^n \cap X)$ is $(\mathcal{R}, H)$-definable. ⊣

We now describe the other direction of the correspondence. Let $\mathcal{H}$ be a Shelah complete dp-minimal expansion of $(H, +, <)$. By Corollary 5.24 $\mathcal{H}$ is weakly o-minimal so Fact 7.2 shows that $\boldsymbol{\mathcal{H}}$ is an o-minimal expansion of $(\mathbb{R}, +, <)$ and $(\boldsymbol{\mathcal{H}}, H)$ is a dense pair. By Fact 7.2 the structure induced on H by $(\boldsymbol{\mathcal{H}}, H)$ is interdefinable with $\mathcal{H}$. So we can recover $\mathcal{H}$ up to interdefinibility from $\boldsymbol{\mathcal{H}}$.

LEMMA 7.4. *Suppose that $\mathcal{R}$ is an o-minimal expansion of $(\mathbb{R}, +, <)$ and $(\mathcal{R}, H)$ is a dense pair. Let $\mathcal{H}$ be the structure induced on H by $(\mathcal{R}, H)$. Then $\boldsymbol{\mathcal{H}}$ is interdefinable with $\mathcal{R}$.*

Our proof follows that of [47, Proposition 3.4]. Recall that an open subset of a topological space is *regular* if it is the interior of its closure.

PROOF. By [47, Lemma 3.3] every $\mathcal{R}$-definable subset of $\mathbb{R}^n$ is a boolean combination of regular open $\mathcal{R}$-definable sets. So it suffices to suppose that U is a regular open $\mathcal{R}$-definable subset of $\mathbb{R}^n$ and show that U is $\boldsymbol{\mathcal{H}}$-definable. Note that $H^n \cap U$ is $\mathcal{H}$-definable, so the closure of $H^n \cap U$ in $\mathbb{R}^n$ is $\boldsymbol{\mathcal{H}}$-definable. Note that the closure of $H^n \cap U$ agrees with the closure of U as H^n is dense. So by regularity U is the interior of the closure of $H^n \cap U$ in $\mathbb{R}^n$. So U is definable in $\boldsymbol{\mathcal{H}}$. ⊣

7.3. Expansions of $(\mathbb{Q}, +, <)$. We are particularly interested in the case $H = \mathbb{Q}$.

QUESTION 2. *For which o-minimal expansions $\mathcal{R}$ of $(\mathbb{R}, +, <)$ is $(\mathcal{R}, \mathbb{Q})$ a dense pair*?

The classification of o-minimal expansions of $(\mathbb{R}, +, <)$ breaks Question 2 into several cases. Fact 7.5 is a special case of the Peterzil–Starchenko trichotomy theorem [35]. We let $\mathbb{R}_{\text{Vec}}$ be the ordered vector space $(\mathbb{R}, +, <, (x \mapsto \lambda x)_{\lambda \in \mathbb{R}})$.

FACT 7.5. *Suppose $\mathcal{R}$ is an o-minimal expansion of $(\mathbb{R}, +, <)$. Then exactly one of the following holds*:

(1) *$\mathcal{R}$ is a reduct of $\mathbb{R}_{\text{Vec}}$,*
(2) *there is a nonempty open interval I and $\mathcal{R}$-definable $\oplus, \otimes : I^2 \to I$ such that $(I, \oplus, \otimes, <)$ is isomorphic to $(\mathbb{R}, +, \times, <)$.*

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It is shown in [8] that $(\mathbb{R}_{\mathrm{Vec}}, \mathbb{Q})$ is a dense pair. So we consider the case when (2) above holds. Fact 7.6 is a special case of a theorem of Edmundo [14].

FACT 7.6. *Suppose $\mathcal{R}$ is an o-minimal expansion of $(\mathbb{R}, +, <)$. Then exactly one of the following holds*:

(1) *There is a subfield K of $(\mathbb{R}, +, \times)$ and a collection $\mathcal{B}$ of bounded subsets of Euclidean space such that $(\mathbb{R}, +, <, \mathcal{B})$ is o-minimal and $\mathcal{R}$ is interdefinable with $(\mathbb{R}, +, <, \mathcal{B}, (x \mapsto \lambda x)_{\lambda \in K})$,*
(2) *There are $\mathcal{R}$-definable $\oplus, \otimes : \mathbb{R}^2 \to \mathbb{R}$ such that $(\mathbb{R}, \oplus, \otimes, <)$ is isomorphic to $(\mathbb{R}, +, \times, <)$.*

7.4. Global field structure. We discuss case (2) of Fact 7.6. Let ι be the unique isomorphism $(\mathbb{R}, \oplus, \otimes, <) \to (\mathbb{R}, +, \times, <)$, $\mathcal{R}'$ be the pushforward of $\mathcal{R}$ by ι, $\boxplus$ be the pushforward of $+$ by ι, and $Q := \iota(\mathbb{Q})$. So $\mathcal{R}'$ is an expansion of $(\mathbb{R}, +, \times)$ and ι gives an isomorphism $(\mathcal{R}, \mathbb{Q}) \to (\mathcal{R}', Q)$.

The *o-minimal two group question* asks if there is necessarily either an $\mathcal{R}'$-definable isomorphism $(\mathbb{R}, \boxplus, <) \to (\mathbb{R}, +, <)$ or $(\mathbb{R}, \boxplus, <) \to (\mathbb{R}_{>0}, \times)$. This question is open, but it holds for all known o-minimal expansions of $(\mathbb{R}, +, \times)$, see [33].

Suppose there is an $\mathcal{R}'$-definable isomorphism $\tau : (\mathbb{R}, \boxplus, <) \to (\mathbb{R}, +, <)$. Then there is $t \in \mathbb{R}_{>0}$ such that $(\tau \circ \iota)(a) = ta$ for all $a \in \mathbb{R}$, hence $\tau(Q) = t\mathbb{Q}$. It follows that $(\mathcal{R}', Q)$ defines $\mathbb{Q}$ and is therefore totally wild.

Suppose there is an $\mathcal{R}'$-definable isomorphism $\tau : (\mathbb{R}, \boxplus, <) \to (\mathbb{R}_{>0}, \times)$. Then there is $t \in \mathbb{R}_{>0}$ such that $(\tau \circ \iota)(a) = t^a$ for all $a \in \mathbb{R}$, hence $\tau(Q) = t^{\mathbb{Q}} := \{t^q : q \in \mathbb{Q}\}$. This leads to Question 3.

QUESTION 3. *Fix $t \in \mathbb{R}_{>0}$. For which o-minimal expansions $\mathcal{S}$ of $(\mathbb{R}, +, \times)$ is $(\mathcal{S}, t^{\mathbb{Q}})$ a dense pair*?

It is known that $(\mathbb{R}, +, \times, t^{\mathbb{Q}})$ is a dense pair [17]. Hieronymi has shown that there is a co-countable subset Λ of $\mathbb{R} \setminus \mathbb{Q}$ such that if $r \in \Lambda$ and $t \in \mathbb{R}_{>0}$ lies in the algebraic closure of $\mathbb{Q}(r)$ then $(\mathbb{R}, +, \times, x^r, t^{\mathbb{Q}})$ is a dense pair [20].

If $\mathcal{S}$ defines the exponential function then $(\mathcal{S}, t^{\mathbb{Q}})$ defines $\mathbb{Q}$ and is therefore totally wild so by Miller's dichotomy [32] if $(\mathcal{S}, t^{\mathbb{Q}})$ is a dense pair then $\mathcal{S}$ is polynomially bounded. It follows from [45, Proposition 1] that if $\mathcal{S}$ defines the restriction of the exponential to some nonempty open interval than $(\mathcal{S}, t^{\mathbb{Q}})$ defines $\mathbb{Q}$. So for example $(\mathbb{R}_{\mathrm{an}}, t^{\mathbb{Q}})$ is totally wild.

Let I be a closed bounded interval and $C^\infty(I)$ be the space of smooth functions $I \to \mathbb{R}$ equipped with the topology induced by the seminorms $f \mapsto \max\{f^{(n)}(r) : r \in I\}$. So $C^\infty(I)$ is Polish. Le Gal has shown that there is a comeager subset Γ of $C^\infty(I)$ such that if $f \in \Gamma$ the $(\mathbb{R}, +, \times, f)$ is o-minimal and polynomially bounded [28]. Fix $t \in \mathbb{R}_{>0}$. There should be a comeager subset Γ_t of $C^\infty(I)$ such that if $f \in \Gamma_t$ then $(\mathbb{R}, +, \times, f)$ is o-minimal and $(\mathbb{R}, +, \times, f, t^{\mathbb{Q}})$ is a dense pair.

7.5. Local field structure. We now consider the case when there is a local field structure but no global field structure, i.e., when item (1) of Fact 7.5 and item (2) of Fact 7.6 both fail. In this case the field structure is on a bounded interval I and there is no definable injection $\mathbb{R} \to I$, so only bounded segments of $\mathbb{Q}$ show up inside the field structure. Suppose I is a bounded open interval, $\oplus, \otimes : I^2 \to I$ are $\mathcal{R}$-definable, and $\iota : (\mathbb{R}, +, \times, <) \to (I, \oplus, \otimes, <)$ is an isomorphism. Let $\mathcal{R}'$ be pullback by ι of

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the structure induced on I by $\mathcal{R}$. So $\mathcal{R}'$ is an o-minimal expansion of $(\mathbb{R}, +, \times)$. After rescaling and translating if necessary we suppose $[0, 1]$ is contained in I. Let

$$s\tilde{+}t = \begin{cases} s+t & s+t<1 \\ s+t-1 & \text{otherwise.} \end{cases} \quad \text{for all} \quad 0 \le s, t < 1.$$

Let $\mathbb{S}$ be the pullback of $([0,1), \tilde{+})$ by ι, so $\mathbb{S}$ is an $\mathcal{R}'$-definable group. Note $\mathbb{S}$ is isomorphic to $(\mathbb{R}/\mathbb{Z}, +)$ and $q \in [0,1)$ is rational if and only if q is a torsion point of $\mathbb{S}$. So if $(\mathcal{R}, \mathbb{Q})$ is a dense pair then the expansion of $\mathcal{R}'$ by the torsion points of $\mathbb{S}$ is a dense pair. This leads us to Question 4.

QUESTION 4. *If the expansion of $\mathcal{R}'$ by the torsion points of $\mathbb{S}$ is a dense pair then must $(\mathcal{R}, \mathbb{Q})$ be a dense pair? For which o-minimal expansions $\mathcal{S}$ of $(\mathbb{R}, +, \times)$ and compact, connected, one-dimensional $\mathcal{S}$-definable groups $\mathbf{T}$ is the expansion of $\mathcal{S}$ by a predicate defining the torsion points of $\mathbf{T}$ a dense pair?*

If the first part of Question 4 has a positive answer then Question 2 largely reduces to Question 3 and the second part of Question 4.

Question 4 is closely related to the following question of Peterzil [34, 11.2]: Suppose $\mathcal{R}$ is an o-minimal expansion of a real closed field and G is a definably compact definable group, what is the induced structure on the torsion points of G?

Consider the case when $\mathbf{T}$ is the unit circle equipped with complex multiplication. Then T is the set of roots of unity. The expansion of $(\mathbb{R}, +, \times)$ by a predicate defining the set T of roots of unity is a dense pair. This again follows by combining general results on preservation of NIP from [9] with specific tameness results for this structure given in [6]. The results of [6] rely on the "Lang property" for roots of unity: if V is a subvariety of $\mathbb{C}^n$ then $V \cap U^n$ is a boolean combination of sets of the form

$$\{(g_1, \dots, g_n) \in U^n : g_1^{m_1} \dots g_n^{m_n} = h\}$$

for integers $m_1, \dots, m_n$ and $h \in \mathbb{C}^\times$. Other positive instances of Question 4 may depend on other instances of the "Lang property."

7.6. The *p*-adic case. We finally give a partial p-adic analog of the results on dp-minimal expansions of divisible archimedean ordered abelian groups given above. We continue where we left off at the end of Section 4 and continue to work in the set up of the proof of Theorem 4.7. Let $\mathbb{K}$ be a finite extension of $\mathbb{Q}_p$, K be an elementary subfield of $\mathbb{K}$, $\mathcal{K}$ be a dp-minimal expansion of K, $\mathcal{F}$ be a highly saturated elementary extension of $\mathcal{K}$ such that $\mathbb{K}$ is an elementary subfield of the underlying field F of $\mathcal{F}$, and $\boldsymbol{\mathcal{K}}$ be the associated expansion of $\mathbb{K}$ defined as above.

PROPOSITION 7.7. *The structure induced on K by $\boldsymbol{\mathcal{K}}$ is interdefinable with $\mathcal{K}^{\mathrm{Sh}}$.*

Thus any Shelah complete dp-minimal expansion of K is the structure induced on K by some dp-minimal expansion of $\mathbb{K}$. We first recall the necessary facts. Fact 7.8 is immediate from the definitions and left to the reader.

FACT 7.8. *Suppose that $\mathcal{M} \prec \mathcal{N}$ are structures and $X \subseteq N^n$ is externally definable in $\mathcal{N}$. Then $X \cap M^n$ is externally definable in $\mathcal{M}$.*

Fact 7.9 is one way of phrasing existence of honest definitions, see [9].

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FACT 7.9. *Suppose that $\mathcal{M}$ is* NIP, *$\mathcal{M} \prec \mathcal{N}$ is sufficiently saturated, and $X \subseteq M^n$ is externally definable in $\mathcal{M}$. Then there is a $\mathcal{N}$-definable $D \subseteq N^n$ such that $X = D \cap M^n$ and $D \cap Z^* = \emptyset$ whenever Z is an $\mathcal{M}$-definable subset of M^n such that $X \cap Z = \emptyset$ and Z^* is the subset of N^n defined by the same formula as Z.*

Fact 7.10 is a special case of the results of [42].

FACT 7.10. *Suppose that (E, v) is a valued field and $\mathfrak{E}$ is a dp-minimal expansion of (E, v). Then every $\mathfrak{E}$-definable subset of E^n is a finite boolean combination of closed $\mathfrak{E}$-definable subsets of E^n.*

We now prove Proposition 7.7. We let $\mathrm{st} : W^n \to \mathbb{K}^n$ be the coordinate-wise standard part map given by $\mathrm{st}(x_1, \dots, x_n) = (\mathrm{st}(x_1), \dots, \mathrm{st}(x_n))$.

PROOF. We first show that the structure induced on K by $\boldsymbol{\mathcal{K}}$ is a reduct of $\mathcal{K}^{\mathrm{Sh}}$. It suffices to assume that $Y \subseteq \mathbb{K}^n$ is $\boldsymbol{\mathcal{K}}$-definable and show that $X = Y \cap K^n$ is $\mathcal{K}^{\mathrm{Sh}}$-definable. By dp-minimality of $\boldsymbol{\mathcal{K}}$ and Fact 7.10 we may suppose that Y is closed. Let $Y^* = \mathrm{st}^{-1}(Y)$. As Y is closed we have $X = Y^* \cap K^n$. Then Y^* is definable in $\mathcal{F}^{\mathrm{Sh}}$, hence externally definable in $\mathcal{F}$. Hence by Fact 7.8 Y is externally definable in $\mathcal{K}$.

We now suppose that $X \subseteq K^n$ is definable in $\mathcal{K}^{\mathrm{Sh}}$ and show that X is definable in the structure induced on K by $\boldsymbol{\mathcal{K}}$. By dp-minimality of $\mathcal{K}^{\mathrm{Sh}}$ and Fact 7.8 we may suppose that X is closed in K^n. Let X' be the closure of X in $\mathbb{K}^n$. As X is closed we have $X = X' \cap K^n$, so it is enough to show that X' is definable in $\boldsymbol{\mathcal{K}}$. We let D be the $\mathcal{F}$-definable set provided by Fact 7.9. We show that $\mathrm{st}(D \cap W^n) = X'$. An easy saturation argument shows that $\mathrm{st}(D \cap W^n)$ is closed, and X is contained in $D \cap W^n$, so X' is contained in $\mathrm{st}(D \cap W^n)$. We show that $\mathrm{st}(D \cap W^n)$ is contained in X'. Fix $p \notin X'$. As X' is closed there is a definable open neighbourhood U of p such that $U \cap X' = \emptyset$. We may suppose that $U = U_1 \times \cdots \times U_n$ for balls $U_1, \dots, U_n$ in $\mathbb{K}$. Let U^* be the subset of F^n defined by the same formula as U. As U is clopen we have $U^* = \mathrm{st}^{-1}(U)$. By Fact 7.9 D is disjoint from U^*. Hence $\mathrm{st}(D \cap W^n)$ does not intersect U, so $p \notin \mathrm{st}(D \cap W^n)$. ⊣

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