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Hyperbolicity versus non-hyperbolic ergodic measures inside homoclinic classes

CHENG CHENG†, SYLVAIN CROVISIER‡, SHAOBO GAN†, XIAODONG WANG§ and DAWEI YANG¶

† *School of Mathematical Sciences, Peking University, Beijing 100871, PR China*
(e-mail: chocolate-74@163.com, gansb@pku.edu.cn)
‡ *CNRS - Laboratoire de Mathématiques d'Orsay, Université Paris-Sud 11, Orsay 91405, France*
(e-mail: Sylvain.Crovisier@math.u-psud.fr)
§ *School of Mathematical Sciences, Shanghai Jiao Tong University, Shanghai 200240, PR China*
(e-mail: xdwang1987@sjtu.edu.cn, xdwang1987@gmail.com)
¶ *School of Mathematical Sciences, Soochow University, Suzhou 215006, PR China*
(e-mail: yangdw1981@gmail.com, yangdw@suda.edu.cn)

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Abstract. We prove that, for C^1-generic diffeomorphisms, if a homoclinic class is not hyperbolic, then there is a non-trivial non-hyperbolic ergodic measure supported on it. This proves a conjecture by Díaz and Gorodetski.

1. *Introduction*

1.1. *Characterization of non-hyperbolicity.* The study of dynamics beyond uniform hyperbolicity has been a major problem for dynamists since the 1960s when Abraham and Smale [**3**] found that the set of *hyperbolic diffeomorphisms* (i.e., satisfying axiom A and the no-cycle condition) is not dense in dynamical systems. The lack of hyperbolicity may be characterized in different ways. Let M be a compact smooth manifold without boundary of dimension d and denote by $\mathrm{Diff}^r(M)$ the space of C^r-diffeomorphisms of M for any $r \geq 1$.

- Thanks to the contributions of Aoki, Hayashi, Liao and Mañé [**4**, **30**, **32**, **33**] to the C^1-stability conjecture, it is known that any non-hyperbolic diffeomorphism can be perturbed as a diffeomorphism with a non-hyperbolic periodic orbit. In particular, a C^1-generic diffeomorphism which is not hyperbolic has arbitrarily weak periodic orbits.

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- Palis has proposed other obstructions to hyperbolicity, still related to periodic orbits: he conjectured [**34**] that any diffeomorphism can be C^r-approximated by one which is hyperbolic or by one which exhibits a homoclinic bifurcation. These bifurcations (homoclinic tangencies and heterodimensional cycles) have strong dynamical consequences and can sometimes be strengthened as robust obstructions to hyperbolicity; see [**13**]. Much progress has been made in the direction of this conjecture [**20**, **22**, **24**, **36**].
- Pesin's theory [**35**] weakens the notion of hyperbolicity (non-uniform hyperbolicity), and gives a possible approach to characterize non-hyperbolic behavior through the invariant measures. This is the goal of the present paper.

Recall that, for an ergodic measure μ of a diffeomorphism $f \in \mathrm{Diff}^1(M)$, there are d numbers $\chi_1 \le \chi_2 \le \cdots \le \chi_d$ such that, for μ-almost every point $x \in M$ and for any $v \in T_xM \setminus \{0\}$, $\lim_{n\to+\infty} (1/n) \log \|Df^n(v)\| = \chi_i$ for some $i \in \{1, 2, \ldots, d\}$. The d numbers χ_i are called the *Lyapunov exponents* of the measure μ. We call μ a *hyperbolic measure* if all its Lyapunov exponents are non-zero.

Clearly, if a diffeomorphism has a non-hyperbolic ergodic measure, then it cannot be hyperbolic. The converse is not true, in general (see [**5**, **18**]), but we may expect that it is the case for typical systems, as conjectured by Díaz and Gorodetski in [**25**].

CONJECTURE 1. (Díaz–Gorodetski) *There is an open dense subset $\mathcal{U} \subset \mathrm{Diff}^r(M)$, where $r \ge 1$, such that every diffeomorphism $f \in \mathcal{U}$ either is uniformly hyperbolic or has an ergodic non-hyperbolic invariant measure.*

The existence of an open set of diffeomorphisms having an ergodic non-hyperbolic invariant measure has been addressed in different papers [**7**, **28**, **31**].

In order to study the dynamics, one usually concentrates on the set of points that have some recurrence properties, for instance, the *chain-recurrent set* which splits into disjoint invariant compact sets called *chain-recurrence classes*. For C^1-generic diffeomorphisms (i.e., diffeomorphisms in a dense G_δ subset of $\mathrm{Diff}^1(M)$), periodic points are dense in the chain-recurrent set and any chain-recurrence class which contains a (hyperbolic) periodic point p, coincides with its *homoclinic class* $H(p)$, which is the closure of the set of transverse intersection points between the stable and unstable manifolds of $\mathrm{orb}(p)$ (see [**9**, **19**]). There may exist other chain-recurrence classes called *aperiodic classes*.

This viewpoint allows one to derive local versions of the previous problems: in [**38**], it is shown that any non-hyperbolic homoclinic class of a C^1-generic diffeomorphism contains arbitrarily weak periodic orbits. Versions of Palis's conjecture inside homoclinic classes have been partially addressed in the works mentioned above; in particular, in [**22**]. A local version of the above conjecture for homoclinic classes has been also stated in [**25**].

CONJECTURE 2. (Díaz–Gorodetski) *For generic $f \in \mathrm{Diff}^1(M)$, every homoclinic class either is uniformly hyperbolic or supports an ergodic non-hyperbolic invariant measure.*

We point out, here, that Conjecture 2 is not true without the genericity assumption. In [**18**, **37**], the authors construct a homoclinic class containing a homoclinic tangency inside it by destroying hyperbolic horseshoes in parameterized families of diffeomorphisms on the surface. Such a homoclinic class is not hyperbolic because of the existence of

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homoclinic tangencies, but it is uniformly hyperbolic in the measure sense. To be precise, all Lyapunov exponents of all invariant measures supported on the homoclinic class are uniformly bounded away from zero. However, a homoclinic tangency is not persistent under perturbations, and hence one expects a positive answer to Conjecture 2, which is what we prove in this paper. An invariant measure is called trivial if it is the probability atomic measure uniformly distributed on a periodic orbit.

MAIN THEOREM. *For generic $f \in \mathrm{Diff}^1(M)$, if a homoclinic class $H(p)$ is not hyperbolic, then there is a non-trivial non-hyperbolic ergodic measure μ, such that* $\mathrm{supp}(\mu) \subset H(p)$.

Let us emphasize that the same result does not hold for aperiodic classes: [**12**] builds an open set of diffeomorphisms whose C^1-generic elements have aperiodic classes which only support hyperbolic ergodic measures. We do not have an answer to Conjecture 1. A possible approach would be to answer the following problem; see [**1**, Conjecture 1], [**8**, Conjecture 1] or [**21**, §6.2.Ia].

CONJECTURE 3. *For any C^1-generic diffeomorphism which is not hyperbolic, there exists a non-hyperbolic homoclinic class.*

1.2. *Criteria for the existence of an ergodic non-hyperbolic measure.* Let us comment on the techniques for getting ergodic measures with one zero Lyapunov exponent.

In the special case when a homoclinic class $H(p)$ has a partially hyperbolic splitting $E^s \oplus E^c \oplus E^u$, $\dim(E^c) = 1$, the class is hyperbolic if and only if all its ergodic measures have a non-zero central Lyapunov exponent with the same sign; any hyperbolic measure is approximated by hyperbolic periodic orbits in the class with the same stable dimension (see [**20**] and Proposition 2.12 below). Hence, if $H(p)$ is not hyperbolic, either it contains a non-hyperbolic ergodic measure (as required), or it contains two hyperbolic periodic orbits with different stable dimensions. In this second case, it is possible to build a non-hyperbolic measure by mixing the two period orbits, using certain shadowing properties. This has been developed in many works, including [**7**, **15**, **25**, **28**]. Partial hyperbolicity on a subset of the class is enough for this argument, and it provides the following partial answer to Conjecture 2.

THEOREM (Díaz–Gorodetski) *For generic $f \in \mathrm{Diff}^1(M)$, if a homoclinic class contains periodic points with different stable dimensions, then there is a non-hyperbolic ergodic measure μ supported on it.*

The main difficulty in this paper is to obtain periodic points with different stable dimensions in a same non-hyperbolic homoclinic class. This is achieved under new settings, thanks to the recent result [**38**] (which produces weak periodic orbits) combined with [**11**] (in order to turn them to heterodimensional cycles). There are still cases where we do not manage to get such periodic points (see the discussion in §5), but the existence of non-hyperbolic measures is ensured then.

In the following, we call the stable dimension of a hyperbolic periodic point p the *index* of p, and denote it by $\mathrm{Ind}(p)$. Recall that, for a positive integer T, a *T-dominated*

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splitting over an invariant compact set K is a continuous Df-invariant splitting $T_K M = E \oplus F$, such that $\|Df^T|_{E(x)}\| \cdot \|Df^{-T}|_{F(f^T(x))}\| < \frac{1}{2}$ for any $x \in K$. We say that K has a *dominated splitting* if K has a T-dominated splitting for some T. Moreover, the dimension of the bundle E is called the *index* of the dominated splitting.

The main theorem follows immediately from the following two theorems, by considering whether or not the homoclinic class admits a dominated splitting corresponding to the index of p. The proofs of Theorems A and B are given in §§3 and 4, respectively.

THEOREM A. *For generic $f \in \mathrm{Diff}^1(M)$, if the homoclinic class $H(p)$ of a periodic point p of index i admits a dominated splitting $E \oplus F$ with $\dim E = i$, and if the bundle E is not uniformly contracted, then there exists a non-trivial ergodic measure supported on $H(p)$ whose ith Lyapunov exponent equals zero.*

THEOREM B. *For generic $f \in \mathrm{Diff}^1(M)$, if the homoclinic class $H(p)$ of a periodic point p of index i does not admit a dominated splitting $E \oplus F$ with $\dim E = i$, then there exists a non-trivial non-hyperbolic ergodic measure μ such that $\mathrm{supp}(\mu) = H(p)$. Moreover, if the ith and $(i+1)$th Lyapunov exponents χ_i, χ_{i+1} of p satisfy $\chi_i + \chi_{i+1} < 0$, then the $(i+1)$th Lyapunov exponent of μ vanishes.*

1.3. *Structure of the set of non-hyperbolic orbits.* In order to study non-hyperbolic homoclinic classes, one would like to understand the dynamics on the set of points having a vanishing Lyapunov exponent and to describe how large this set is. The first question deals with the support of the non-hyperbolic measures.

(1) *Does a non-hyperbolic homoclinic class coincide with the support of a non-hyperbolic ergodic measure?*

In the following statement, we give a partial answer. It generalizes a previous work [**15**] (where the existence of a dominated splitting is assumed). The proof is given in §4.

THEOREM C. *For generic $f \in \mathrm{Diff}^1(M)$, if a homoclinic class $H(p)$ contains periodic points with different indices, then there is a non-trivial non-hyperbolic ergodic measure μ with $\mathrm{supp}(\mu) = H(p)$.*

The size of the set of non-hyperbolic orbits may also be described by its entropy.

(2) *Is there a non-hyperbolic ergodic measure with positive entropy?*

This has been obtained in [**7**] for homoclinic classes containing periodic points with different indices.

(3) *What is the supremum of the entropy of the non-hyperbolic measures?*

In a locally maximal transitive hyperbolic set, the set of ergodic measures is dense in the space of invariant probabilities. Bonatti asked if the set of non-hyperbolic ergodic measures of a non-hyperbolic homoclinic class is 'convex', i.e., if its closure is convex.

(4) *Consider the closure of the set of ergodic measures whose ith Lyapunov exponent vanishes. Is it a convex set?*

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1.4. *Organization of the paper.* In §2, we will give some definitions and known results. The proof of Theorem A will be given in §3 and the proof of Theorems B and C will be given in §4. In §5, we will give some classification of non-hyperbolic homoclinic classes.

2. *Preliminary*

2.1. *Hyperbolicity and Lyapunov exponents.* For an f-invariant measure μ, we list all its Lyapunov exponents as $\chi_1(\mu, f) \le \chi_2(\mu, f) \le \cdots \le \chi_d(\mu, f)$. We denote these by $\chi_i(\mu)$ if there is no ambiguity. We define a function

$$L_i(\mu, f) = \liminf_{m\to+\infty} \int L_i^{(m)}(x, f)\, d\mu(x),$$

where

$$L_i^{(m)}(x, f) = \frac{1}{m} \log \|\wedge^i Df^m(x)\|.$$

Then $\chi_i(\mu, f) = L_{d-i+1}(\mu, f) - L_{d-i}(\mu, f)$. In particular, if μ is ergodic, then, for μ-a.e. $x \in M$,

$$L_i(\mu, f) = \lim_{m\to+\infty} L_i^{(m)}(x, f)$$

and

$$\chi_i(\mu, f) = \lim_{m\to+\infty} (L_{d-i+1}^{(m)}(x, f) - L_{d-i}^{(m)}(x, f)).$$

Now we recall the definition of hyperbolicity of an invariant compact set and give some notation.

Definition 2.1. Assume that $f \in \mathrm{Diff}^1(M)$. An invariant compact set K is *hyperbolic* if there is a continuous Df-invariant splitting $T_K M = E^s \oplus E^u$ such that E^s is contracted by Df and E^u is expanded by Df. To be precise, there are two constants, $C > 0$ and $\lambda \in (0, 1)$, such that, for any point $x \in K$ and any $n \ge 0$, the following two inequalities are satisfied.

$$\|Df^n|_{E(x)}\| < C\lambda^n \quad \text{and} \quad \|Df^{-n}|_{F(x)}\| < C\lambda^n.$$

In particular, a periodic point p is a *hyperbolic periodic point* if its orbit is a hyperbolic set, and the dimension of E^s is called the *index* of p and is denoted by $\mathrm{Ind}(p)$. A hyperbolic saddle p of index i is said to be *center-dissipative* if $\chi_i(p, f) + \chi_{i+1}(p, f) < 0$.

Remark 2.2. It is well known that a hyperbolic periodic point always has a *continuation*. More precisely, for a hyperbolic periodic point p of a diffeomorphism $f \in \mathrm{Diff}^1(M)$, there is a C^1-neighborhood $\mathcal{U}$ of f and a neighborhood U of $\mathrm{orb}(p)$ such that any diffeomorphism $g \in \mathcal{U}$ has a unique periodic orbit contained in U. Moreover, this periodic orbit is hyperbolic and has the same period as p. We denote by p_g the continuation of p for any $g \in \mathcal{U}$.

Sometimes, a non-hyperbolic set has some weaker hyperbolicity.

Definition 2.3. Let $f \in \mathrm{Diff}^1(M)$. An invariant compact set K is *partially hyperbolic* if there is a continuous splitting $T_K M = E^s \oplus E^c \oplus E^u$ such that $(E^s \oplus E^c) \oplus E^u$ and $E^s \oplus (E^c \oplus E^u)$ are dominated, the bundle E^s is contracted by Df, the bundle E^u is expanded by Df and at least one of E^s and E^u is non-trivial.

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2.2. *Sufficient conditions for existence of a non-hyperbolic ergodic measure.* We define a relationship called *multiple almost shadowing* between two periodic orbits, which was called good approximation in [**15**, **25**].

Definition 2.4. (Multiple almost shadowing) Consider two periodic orbits P and Q of a map $f : M \to M$ and two numbers $\gamma > 0$ and $0 < \varkappa \leq 1$. Denote by $\pi(P)$ the period of P. We say that P is $(\gamma, \varkappa)$-*multiple almost shadowed* by Q if there is a subset $\Gamma \subset Q$ and a map $\rho : \Gamma \to P$ that satisfy the following properties.

- $\#\Gamma/\#Q \geq \varkappa$.
- $\#\rho^{-1}(f^j(p))$ is constant, for any $p \in P$.
- $d(f^j(q),\ f^j(\rho(q))) < \gamma$, for any $q \in \Gamma$ and for any $j = 0, 1, \ldots, \pi(P) - 1$.

One says that a periodic point p has *simple spectrum* if the d Lyapunov exponents of $\mathrm{orb}(p)$ are mutually different. The following lemma is standard; see a similar statement in [**15**, Theorem 3.5].

LEMMA 2.5. *For generic $f \in \mathrm{Diff}^1(M)$, consider a periodic point p with simple spectrum whose homoclinic class $H(p)$ is non-trivial. Then, for any $\varepsilon, \gamma > 0$ and any $\varkappa \in (0, 1)$, there is a periodic point q with simple spectrum homoclinically related to p, such that the following properties are satisfied.*

(1) $\mathrm{orb}(q)$ *is ε-dense in $H(p)$.*

(2) $\mathrm{orb}(p)$ *is $(\gamma, \varkappa)$-multiple almost shadowed by* $\mathrm{orb}(q)$.

(3) $\chi_i(q,\ f)$ *is ε-close to $\chi_i(p,\ f)$, for any $i = 1, 2, \ldots, d$.*

In particular, if p is center-dissipative, then q can be chosen to be center-dissipative.

Sketch of the proof. The properties in the statement are persistent under C^1 perturbations. Hence we only show that one can obtain such a periodic point q by C^1-small perturbations, and, by a standard Baire argument, the statement holds for generic systems.

Since p has simple spectrum, there is a dominated splitting $E_1 \oplus E_2 \oplus \cdots \oplus E_d$ over $\mathrm{orb}(p)$ such that each E_i is the one-dimensional sub-bundle corresponding to the ith Lyapunov exponent of $\mathrm{orb}(p)$. Take a transverse homoclinic point $x \in W^s(p) \cap W^u(p)$, such that $\mathrm{orb}(x) \cup \mathrm{orb}(p)$ is $\varepsilon/2$-dense in $H(p)$. By a similar technique to the proof of [**20**, Proposition 1.10, p. 689], arbitrarily C^1-close to f in $\mathrm{Diff}^1(M)$, there is a diffeomorphism g, which coincides with f on $\mathrm{orb}(x)$ and outside a small neighborhood of the point x, such that the dominated splitting $E_1 \oplus E_2 \oplus \cdots \oplus E_d$ can be spread to the set $\mathrm{orb}(x) \cup \mathrm{orb}(p)$. Since $\mathrm{orb}(x) \cup \mathrm{orb}(p)$ is still a hyperbolic set with respect to g, by the shadowing lemma, there is a hyperbolic periodic point q homoclinically related to p, such that $\mathrm{orb}(q)$ spends an arbitrarily large portion of time close to $\mathrm{orb}(p)$ and $\varepsilon/2$-shadows $\mathrm{orb}(x) \cup \mathrm{orb}(p)$. Then the items (1), (2) are satisfied. Since the dominated splitting $E_1 \oplus E_2 \oplus \cdots \oplus E_d$ with each bundle of dimension one spreads to the set $\mathrm{orb}(x) \cup \mathrm{orb}(q)$, each Lyapunov exponent $\chi_i(q,\ g)$ can be presented as $(1/(\pi(q,\ g)))\ \log \|Dg^{\pi(q,g)}|_{E_i(q)}\|$. Then the item (3) can be obtained by the fact that $\mathrm{orb}(q)$ spends most of the time close to $\mathrm{orb}(p)$. □

The following lemma is proved in [**15**] to obtain ergodic measures, using the method developed in [**28**], by taking the weak-$*$-limit of atomic measures supported on periodic orbits.

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LEMMA 2.6. [**15**, Lemma 2.5] *Consider two sequences of numbers,* $(\gamma_n)_{n\geq 1}$ *in* $(0, +\infty)$ *and* $(\varkappa_n)_{n\geq 1}$ *in* $(0, 1]$*, such that* $\sum_{n=1}^{\infty} \gamma_n < \infty$ *and* $\prod_{n=1}^{\infty} \varkappa_n > 0$*. Assume that* $(p_n)_{n\geq 1}$ *is a sequence of periodic points of a map* $f : M \to M$ *with increasing periods* $\pi(p_n)$*, and denote by* μ_n *the probability atomic measure uniformly distributed on the orbit of* p_n*. If* $\mathrm{orb}(p_n)$ *is* $(\gamma_n, \varkappa_n)$*-multiple almost shadowed by* $\mathrm{orb}(p_{n+1})$ *for any* $n \geq 1$*, then the sequence of measures* (μ_n) *converges to an ergodic measure* μ *and* $\mathrm{supp}(\mu) = \bigcap_{n=1}^{\infty}(\overline{\bigcup_{k=n}^{\infty} \mathrm{orb}(p_k)})$.

2.3. *Perturbation lemmas about periodic cocycles.* Consider a family of linear maps $A_1, \ldots, A_n \in \mathrm{GL}(d, \mathbb{R})$. Define $B = A_n \circ \cdots \circ A_1$ and denote by $\lambda_1(B), \ldots, \lambda_d(B)$ the eigenvalues of B counted by multiplicity such that $|\lambda_1| \leq \cdots \leq |\lambda_d|$. Then the ith *Lyapunov exponent* of B is defined as $\chi_i(B) = (1/n) \log |\lambda_i(B)|$.

The following statement follows from [**6**], which allows one to modify only two consecutive Lyapunov exponents of a cocycle; see also [**11**, Lemma 4.4]. A similar result can be found in [**27**, Theorem 1.2].

LEMMA 2.7. [**6**, Theorem 4.1 and Proposition 3.1] *For any constants* $D > 1$*,* $\varepsilon > 0$ *and* $d \geq 2$*, there are two constants* T *and* n_0 *satisfying the following property.*

Consider a family of linear maps $A_1, \ldots, A_n \in \mathrm{GL}(d, \mathbb{R})$ *with* $n \geq n_0$*, such that* $\|A_i\|, \|A_i^{-1}\| \leq D$ *and the linear map* $B = A_n \circ \cdots \circ A_1$ *has no* T*-dominated splitting of index* i *for some* $i \in \{1, \ldots, d-1\}$*. Then there exist one-parameter families of linear maps* $(A_{m,t})_{t\in[0,1]}$ *for any* $1 \leq m \leq n$*, satisfying the following properties.*

(1) $A_{m,0} = A_m$*, for any* $1 \leq m \leq n$.
(2) $\|A_{m,t} - A_m\| < \varepsilon$ *and* $\|A_{m,t}^{-1} - A_m^{-1}\| < \varepsilon$*, for any* $t \in [0, 1]$ *and for any* $1 \leq m \leq n$.
(3) *Consider the linear map* $B_t = A_{n,t} \circ \cdots \circ A_{1,t}$*. Then the Lyapunov exponents of* B_t *satisfy the following properties.*
   - $\chi_j(B_t) = \chi_j(B)$ *for any* $j \neq i, i+1$.
   - $\chi_i(B_t) + \chi_{i+1}(B_t) = \chi_i(B) + \chi_{i+1}(B)$.
   - $\chi_i(B_t)$ *is non-decreasing and* $\chi_{i+1}(B_t)$ *is non-increasing, that is,*

$$\chi_i(B_t) \leq \chi_i(B_{t'}) \leq \chi_{i+1}(B_{t'}) \leq \chi_{i+1}(B_t) \quad \text{for any } t < t'.$$

   - $\chi_i(B_1) = \chi_{i+1}(B_1)$.

We state a generalized version of Franks's lemma [**26**], which is proved in [**29**].

LEMMA 2.8. [**29**] *Consider a diffeomorphism* $f \in \mathrm{Diff}^1(M)$*, a hyperbolic periodic point* q *of period* π *and a constant* $\varepsilon > 0$*. Assume that, for any* $n = 0, 1, \ldots, \pi - 1$*, there is a one-parameter family of linear maps* $(A_{n,t})_{t\in[0,1]}$ *in* $\mathrm{GL}(d, \mathbb{R})$*, for which the following properties are satisfied.*

- $A_{n,0} = Df(f^n(q))$.
- $\|Df(f^n(q)) - A_{n,t}\| < \varepsilon$ *and* $\|Df^{-1}(f^n(q)) - A_{n,t}^{-1}\| < \varepsilon$ *for any* $t \in [0, 1]$.
- $A_{\pi-1,t} \circ \cdots \circ A_{0,t}$ *is hyperbolic for any* $t \in [0, 1]$.

Then, for any neighborhood V *of* $\mathrm{orb}(q)$*, any constant* $\delta > 0$ *and any pair of compact sets* $K^s \subset W_\delta^s(q, f)$ *and* $K^u \subset W_\delta^u(q, f)$ *that are disjoint from* V*, there is a diffeomorphism* g *that is* ε*-close to* f *in* $\mathrm{Diff}^1(M)$ *and satisfies the following properties.*

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(1) *g coincides with f on $M\backslash V$ and* $\mathrm{orb}(q)$.
(2) *$K^s \subset W^s_\delta(q, g)$ and $K^u \subset W^u_\delta(q, g)$.*
(3) *$Dg(g^n(q)) = Df(f^n(q)) = A_{n,1}$ for all $n = 0, \ldots, \pi - 1$.*

2.4. *Generic properties.* Recall that a subset $\mathcal{R}$ of a topological Baire space X is called a *residual* set, if it contains a dense G_δ set of X. A property is a *generic* property of X if there is a residual set $\mathcal{R} \subset X$ such that each element contained in $\mathcal{R}$ satisfies the property. The following theorem summarizes some classical generic properties (see, for instance, [**2**, **9**, **14**, **16**, **17**, **25**]).

THEOREM 2.9. *There is a residual subset, $\mathcal{R} \subset \mathrm{Diff}^1(M)$, such that, for any diffeomorphism $f \in \mathcal{R}$, the following properties are satisfied.*

(1) *The diffeomorphism f is Kupka–Smale: all periodic points are hyperbolic and the stable and unstable manifolds of any two periodic orbits intersect transversely. Moreover, every periodic point is center-dissipative with respect to f or to f^{-1}.*
(2) *For any hyperbolic periodic point p, there is a C^1-neighborhood $\mathcal{U}$ of f such that every diffeomorphism $g \in \mathcal{U} \cap \mathcal{R}$ is a continuity point of the map $g \mapsto H(p_g, g)$ with respect to the Hausdorff topology on the set of non-empty compact sets of M.*
(3) *Any two homoclinic classes, $H(p_1, f)$ and $H(p_2, f)$, either coincide or are disjoint. Moreover, any two periodic orbits of the same index contained in a homoclinic class of f are homoclinically related.*
(4) *For any homoclinic class $H(p)$, there is an interval of natural numbers $[\alpha, \beta]$ and a C^1-neighborhood $\mathcal{V}$ of f such that, for any $g \in \mathcal{V}$, the set of indices of hyperbolic periodic points contained in $H(p_g, g)$ is exactly the interval $[\alpha, \beta]$.*
(5) *Consider a periodic point p. If, for any C^1-neighborhood $\mathcal{U}$ of f, there is a diffeomorphism $g \in \mathcal{U}$ having a heterodimensional cycle associated with p_g and some periodic point q, then the homoclinic class $H(p)$ contains a periodic point whose index equals* $\mathrm{Ind}(q)$.
(6) *Consider an invariant compact K which is a Hausdorff limit set of a sequence of periodic orbits O_n. If, for any neighborhood $\mathcal{U}$ of f and any $N > 0$, there is $g \in \mathcal{U}$ and $n > N$ such that O_n is a periodic orbit of g of index i, then K is the Hausdorff limit set of a sequence of periodic orbits of f of index i.*
(7) *Consider a non-trivial homoclinic class $H(p)$ of f. For any $\varepsilon > 0$, the set*

$$\{q \in \mathrm{Per}(f) : q \text{ has simple spectrum, and } |\chi_i(q) - \chi_i(p)| < \varepsilon, \text{for } \forall\, i = 1, 2, \ldots, d\}$$

is dense in $H(p)$.

2.5. *Previous results.* In this subsection, we state some previous results, some of which have already been mentioned in §1. First, we have the following result which is a combination of [**15**, **25**].

THEOREM 2.10. [**15**, **25**] *For generic $f \in \mathrm{Diff}^1(M)$, assume that p is a hyperbolic periodic point of index i. If the homoclinic class $H(p)$ contains a hyperbolic point q of index smaller than i, then there is an ergodic measure supported on $H(p)$ whose ith Lyapunov exponent is 0. Moreover, if $\mathrm{Ind}(q) = i - 1$ and there is a dominated*

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splitting $T_{H(p)}M = E \oplus F \oplus G$ *such that* $\dim(E) = i - 1$ *and* $\dim(F) = 1$, *then there is an ergodic measure* μ *such that* $\mathrm{supp}(\mu) = H(p)$ *and the* i*th Lyapunov exponent of* μ *is zero.*

The next result characterizes the non-hyperbolicity of a homoclinic class by the existence of weak periodic orbits contained in it for C^1-generic systems.

THEOREM 2.11. [**38**] *For generic* $f \in \mathrm{Diff}^1(M)$, *if a homoclinic class* $H(p)$ *is not hyperbolic, then there is a sequence of periodic orbits contained in* $H(p)$ *with a Lyapunov exponent converging to zero. Moreover, we have the following facts.*

(1) *Assume that* $H(p)$ *has a dominated splitting,* $T_{H(p)}M = E \oplus F$, *where* $\dim(E) = \mathrm{Ind}(p) = i$ *such that the bundle* E *is not contracted. Then, for any* $\varepsilon > 0$, *there is a periodic orbit* q *homoclinically related to* p *such that* $\chi_i(q) \in (-\varepsilon, 0)$.

(2) *Assume that* $H(p)$ *has a dominated splitting,* $T_{H(p)}M = E \oplus F$, *such that* $\dim(E)$ *is smaller than the smallest index of periodic orbits contained in* $H(p)$. *Then the bundle* E *is contracted. Symmetrically, if* $\dim(E)$ *is larger than the largest index of periodic orbits contained in* $H(p)$, *then the bundle* F *is expanded.*

Recall that, for a hyperbolic invariant measure μ, there is a full μ-measure set Λ such that there is a splitting, $T_\Lambda M = \mathcal{E}^s \oplus \mathcal{E}^u$, such that the bundles $\mathcal{E}^s$ and $\mathcal{E}^u$ are associated to the negative and positive Lyapunov exponents, respectively. We call this splitting the *non-uniform hyperbolic splitting* of μ and the dimension of $\mathcal{E}^s$ is called the *index* of μ. The following result claims that the support of a hyperbolic measure intersects a homoclinic class if the above splitting is a dominated splitting.

PROPOSITION 2.12. [**20**, Proposition 1.4] *Let* μ *be a hyperbolic measure of index* i. *If the non-uniform hyperbolic splitting of* μ *is a dominated splitting, then there is a hyperbolic periodic point* p *of index* i *such that* $\mathrm{supp}(\mu) \cap H(p) \neq \emptyset$. *Moreover, if* μ *is ergodic, then* $\mathrm{supp}(\mu) \subset H(p)$.

3. *The dominated case*

In this section, we complete the proof of Theorem A. The following result is a combination of [**11**, Proposition 7.1 and Proposition 8.1].

PROPOSITION 3.1. *Consider a diffeomorphism* $f \in \mathrm{Diff}^1(M)$ *and a hyperbolic periodic point* p *of index* i, *where* $2 \le i \le d - 1$. *Assume that the following properties are satisfied.*

- $H(p)$ *has no dominated splitting of index* $i - 1$.
- *For any* $\varepsilon > 0$, *there is a periodic point* p_ε, *homoclinically related to the orbit of* p, *such that* $\chi_i(p_\varepsilon) \in (-\varepsilon, 0)$.

Then for any C^1*-neighborhood* $\mathcal{U}$ *of* f, *there is a diffeomorphism* $g \in \mathcal{U}$ *having a heterodimensional cycle associated with* $\mathrm{orb}(p_g)$ *and a periodic point* q_g *of* g *of index* $i - 1$.

LEMMA 3.2. *For any diffeomorphism* $f \in \mathrm{Diff}^1(M)$, *consider a non-trivial homoclinic class* $H(p)$ *having a dominated splitting* $T_{H(p)}M = E \oplus F$ *such that* $\dim(E) = \mathrm{Ind}(p) = 1$. *If the bundle* E *is not contracted, then there is an ergodic measure* μ *supported on* $H(p)$ *whose first Lyapunov exponent vanishes.*

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Proof. Since the bundle E is not contracted, by [**20**, Claim 1.7], there is an ergodic measure μ, supported on $H(p)$, such that $\chi_1(\mu) \geq 0$, where $\chi_1(\mu)$ is the Lyapunov exponent of μ along the bundle E.

If $\chi_1(\mu) = 0$, then the ergodic measure μ which is supported on $H(p)$ is non-hyperbolic.

If $\chi_1(\mu) > 0$, then all Lyapunov exponents of μ are positive by the dominated splitting $E \oplus F$. Then μ is supported on a periodic source, which contradicts the fact that $\text{supp}(\mu) \subset H(p)$ and that $H(p)$ is non-trivial. This finishes the proof of Lemma 3.2. □

Now we will be able to prove Theorem A.

Proof of Theorem A. We consider a generic diffeomorphism $f \in \text{Diff}^1(M)$ which satisfies Theorem 2.9, a hyperbolic periodic point p of index i, where $1 \leq i \leq d-1$, and a dominated splitting $T_{H(p)}M = E \oplus F$, where $\dim(E) = i$ and E is not contracted. By Theorem 2.10, we can assume that all periodic points contained in $H(p)$ have index larger than or equal to i; otherwise, there is an ergodic measure supported on $H(p)$ whose ith Lyapunov exponent is zero and there is nothing we need to prove.

If $\dim(E) = 1$, then the conclusion can be obtained from Lemma 3.2. Hence we can assume that $\dim(E) \geq 2$. We consider two subcases that depend on whether the bundle E has a dominated splitting $E_1 \oplus E_2$ with $\dim(E_2) = 1$ or not. Equivalently, we distinguish between the cases when the homoclinic class $H(p)$ has a dominated splitting of index $i-1$ and when it does not.

Case 1. $H(p)$ has a dominated splitting of index $i-1$. In this case, the bundle E has a dominated splitting into two bundles $E = E^s \oplus E^c$ such that $\dim(E^c) = 1$. By Theorem 2.11, the bundle E^s is contracted by Df. The bundle E^c is not contracted, since the bundle E is not contracted. By [**20**, Claim 1.7], there is an ergodic measure μ, supported on $H(p)$, such that $\chi_i(\mu) \geq 0$, where $\chi_i(\mu)$ is the Lyapunov exponent of μ along the bundle E^c.

If $\chi_i(\mu) = 0$, then the conclusion of Theorem A holds.

If $\chi_i(\mu) > 0$, then μ is a hyperbolic measure because the bundle E^s is contracted by Df. Moreover, the non-uniform hyperbolic splitting of μ is a dominated spitting $E^s \oplus (E^c \oplus F)$. By Proposition 2.12, there is a hyperbolic periodic point q of index $i-1$ such that $\text{supp}(\mu) \subset H(q)$. By item (3) of Theorem 2.9, q belongs to $H(p)$, which contradicts the assumption that all periodic points contained in $H(p)$ have index larger than or equal to i.

Case 2. $H(p)$ has no dominated splitting of index $i-1$. By Theorem 2.11, since the bundle E is not contracted, for any $\varepsilon > 0$, there is a periodic point p_ε, homoclinically related to p, such that $\chi_i(p_\varepsilon) \in (-\varepsilon, 0)$. Since $H(p)$ has no dominated splitting of index $i-1$, by Proposition 3.1, there is an arbitrarily small perturbation g of f such that $H(p_g, g)$ has a heterodimensional cycle associated to $\text{orb}(p_g)$ and $\text{orb}(q_g)$ with $\text{Ind}(q_g) = i-1$. Then, by item (5) of Theorem 2.9, there is a periodic point $q \in H(p)$ whose index equals $i-1$, which contradicts the assumption that all periodic points contained in $H(p)$ have index larger than or equal to i.

The proof of Theorem A is now complete. □

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4. *The non-dominated case*

In this section, we prove Theorems B and C.

4.1. *Multiple almost shadowing of* orb(p) *with a weak Lyapunov exponent.*

The following proposition states that, for generic $f \in \mathrm{Diff}^1(M)$, if a homoclinic class $H(p)$ has no dominated splitting of index $\mathrm{Ind}(p)$, then, by a C^1-small perturbation, arbitrarily dense in $H(p)$, there is a periodic orbit that multiple almost shadows the orbit of p and has a Lyapunov exponent close to zero.

PROPOSITION 4.1. *For generic $f \in \mathrm{Diff}^1(M)$, consider a center-dissipative hyperbolic periodic saddle p of index i which has simple spectrum. Assume that the homoclinic class $H(p, f)$ has no dominated splitting of index i. Then, for any $\varepsilon, \gamma > 0$, for any $\varkappa \in (0, 1)$ and for any C^1-neighborhood $\mathcal{U}$ of f, there is a diffeomorphism $g \in \mathcal{U}$ and a hyperbolic saddle q of f such that:*

(1) *the saddle q is homoclinically related to p with respect to f and g;*
(2) *the orbit of p is $(\gamma, \varkappa)$-multiple almost shadowed by the orbit of q;*
(3) *the Hausdorff distance between* orb(q) *and $H(p, f)$ is less than ε;*
(4) *g coincides with f on* orb(q) *and outside a small neighborhood of* orb(q);
(5) *the saddle q has simple spectrum with respect to g; and*
(6) $\chi_{i+1}(q, g) \in (0, \varepsilon)$.

Proof. We assume that f satisfies the properties stated in Lemma 2.5 and Theorem 2.9. Without loss of generality, we can assume that $\varepsilon > 0$ is small such that any diffeomorphism h that is ε-close to f in $\mathrm{Diff}^1(M)$ is contained in $\mathcal{U}$ and such that $\varepsilon < |\chi_j|$, where $j = i, i+1$. Then there are two positive integers, T and τ, that satisfy the conclusion of Lemma 2.7 associated to the constant ε.

By the definition of dominated splitting, there is $\eta > 0$ such that any invariant compact set that is η-close to $H(p)$ in the Hausdorff distance has no T-dominated splitting of index i. Moreover, we can assume that $\eta < \varepsilon$.

By Lemma 2.5, there is a center-dissipative periodic saddle orb(q) with simple spectrum homoclinically related to p such that:

- the orbit of p is $(\gamma, \varkappa)$-multiple almost shadowed by the orbit of q;
- the Hausdorff distance between orb(q) and $H(p, f)$ is less than η;
- the Lyapunov exponents of orb(q) are close to those of orb(p).

Hence items (2) and (3) are satisfied. Moreover, the periodic point q can be chosen such that its period $\pi(q)$ is larger than τ.

Now we do a perturbation to get a diffeomorphism g that satisfies items (1), (4), (5) and (6).

Consider the hyperbolic periodic orbit $orb(q, f)$. By the choice of η, one can see that orb(q) has no T-dominated splitting of index i.

By Lemma 2.7, for each $n = 0, 1, \ldots, \pi(q) - 1$, there is a one-parameter family of matrices $(A_{n,t})_{t\in[0,1]}$ in $\mathrm{GL}(d, \mathbb{R})$ such that, denoting $B_t = A_{\pi(q)-1,t} \circ \cdots \circ A_{0,t}$ for $t \in [0, 1]$, the following properties are satisfied.

- $A_{n,0} = Df(f^n(q))$.
- $\|A_{n,t} - Df(f^n(q))\| < \varepsilon$ and $\|A_{n,t}^{-1} - Df^{-1}(f^n(q))\| < \varepsilon$ for any $t \in [0, 1]$.

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- $\chi_{i+1} \in (0, \varepsilon)$.
- $\chi_j(B_t) = \chi_j(B_0)$ for any $t \in [0, 1]$ and for any $j \in \{1, 2, \ldots, d\} \setminus \{i, i+1\}$.
- B_t is hyperbolic for any $t \in [0, 1]$.

Take a small constant, $\delta > 0$, such that the local manifolds $W^s_\delta(\mathrm{orb}(q), f)$ and $W^u_\delta(\mathrm{orb}(q), f)$ of size δ are two embedded sub-manifolds of dimension i and $d-i$, respectively. Then there are two transverse homoclinic points $z \in W^s_\delta(\mathrm{orb}(q), f) \pitchfork W^u(p, f)$ and $y \in W^u_\delta(\mathrm{orb}(q), f) \pitchfork W^s(p, f)$ since the periodic orbit $orb(q)$ is homoclinically related to p with respect to f. Consider the two small compact sets $\{z\}$ and $\{y\}$ as K^s and K^u. There is a small neighborhood V of $\mathrm{orb}(q)$ such that $\overline{V}$ is disjoint with $\mathrm{orb}^-(z, f)$, $\mathrm{orb}^+(y, f)$ and $\mathrm{orb}(p)$. By Lemma 2.8, there is a diffeomorphism g that is ε-close to f in $\mathrm{Diff}^1(M)$ and satisfies the following properties.

(a) g coincides with f on the orbit of $\mathrm{orb}(q)$ and outside V.

(b) $z \in W^s_\delta(\mathrm{orb}(q), g)$ and $y \in W^u_\delta(\mathrm{orb}(q), g)$.

(c) $Dg(g^n(q)) = Dg(f^n(q)) = A_{n,1}$ for all $n = 0, \ldots, \pi(q) - 1$.

Then $z \in W^s(\mathrm{orb}(q), g) \cap W^u(p, g)$ and $y \in W^u(\mathrm{orb}(q), g) \cap W^s(p, g)$, and by an arbitrarily small C^1-perturbation, if necessary, we can assume that the intersections are transverse. Hence $\mathrm{orb}(q)$ is still homoclinically related to p under g, which is item (1) in Proposition 4.1. Item (4) is automatically satisfied by item (a). Items (5) and (6) are satisfied by item (c) and the properties of the one-parameter families $(A_{n,t})_{t\in[0,1];n=0,1,\ldots,\pi(q)-1}$.

The proof of Proposition 4.1 is now complete. □

4.2. *Construction of sequences of weak periodic orbits.* The following proposition gives a sequence of periodic orbits that have some shadowing properties for C^1-generic diffeomorphisms.

PROPOSITION 4.2. *For generic $f \in \mathrm{Diff}^1(M)$, assume that p is a center-dissipative hyperbolic saddle of index i with simple spectrum whose homoclinic class $H(p)$ has no dominated splitting of index i. Then, there is a sequence of center-dissipative periodic points $(q_n)_{n\geq 1}$ with simple spectrum, together with a sequence of positive numbers $(\gamma_n)_{n\geq 1}$, such that, for any $n \geq 1$, the following properties are satisfied.*

(1) *q_n is homoclinically related to p.*

(2) *$\mathrm{orb}(q_n, f)$ is $1/4^n$-dense in $H(p)$.*

(3) *$\gamma_n < \frac{1}{2}\gamma_{n-1}$ and the orbit of q_{n-1} is $(\gamma_{n-1}, 1 - 1/2^{n-1})$-multiple almost shadowed by the orbit of q_n.*

(4) *There exists a positive integer $N_n > \pi(q_n)$ such that, for any point x contained in the $2\gamma_n$-neighborhood of $\mathrm{orb}(q_n)$, $L^{(N_n)}_{d-i}(x) - L^{(N_n)}_{d-i-1}(x) \in (0, 1/2^n)$.*

Proof. Since f is a C^1-generic diffeomorphism, by item (2) of Theorem 2.9, for a diffeomorphism g close to f in $\mathrm{Diff}^1(M)$, the homoclinic class $H(p_g, g)$ is close to $H(p, f)$ in the Hausdorff topology. Hence one can see that items (1), (2), (3), (5) and (6) in Proposition 4.1 are persistent under C^1-perturbations. Therefore, by a standard Baire argument, the following statement holds.

For generic $f \in \mathrm{Diff}^1(M)$, assume that p is a center-dissipative hyperbolic saddle of index i which has simple spectrum. If the homoclinic class $H(p)$ has no dominated

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splitting of index i, then, for any $\varepsilon, \gamma > 0$ and any $\varkappa \in (0, 1)$, there is a center-dissipative periodic saddle q with simple spectrum homoclinically related to p, such that:

- $\mathrm{orb}(p)$ *is* $(\gamma, \varkappa)$*-multiple almost shadowed by* $\mathrm{orb}(q)$*;*
- $\mathrm{orb}(q)$ *is* ε*-dense in* $H(p)$*; and*
- $\chi_{i+1}(q, f) \in (0, \varepsilon)$.

One may assume that the diffeomorphism f in the statement of Proposition 4.2 satisfies the property above and the properties of Lemma 2.5 and Theorem 2.9. Now we construct the sequence of periodic orbits. To make it complete, we take $q_0 = p$ and $\gamma_0 = 1$.

Assume that $\mathrm{orb}(q_n)$ and γ_n have been taken to satisfy the properties stated in the proposition for any $n \le k-1$. We construct $\mathrm{orb}(q_k)$, γ_k and N_k. We know that $H(q_{k-1}) = H(p)$. Consider the periodic point q_{k-1}. Since $H(q_{k-1})$ has no domination of index i and, by the choice of $\mathcal{R}$, there is a periodic point q_k with simple spectrum homoclinically related to q_{k-1} such that $\mathrm{orb}(q_{k-1})$ is $(\gamma_{k-1}, 1 - 1/2^{k-1})$-multiple almost shadowed by $\mathrm{orb}(q_k)$, $\mathrm{orb}(q_k)$ is $1/4^k$-dense in $H(p)$ and $\chi_{i+1}(q_k, f) \in (0, 1/4^k)$. Hence items (1), (2) and (3) are satisfied. By the fact that $\chi_{i+1}(\mu, f) = \lim_{m\to+\infty}(L^{(m)}_{d-i}(x, f) - L^{(m)}_{d-i-1}(x, f))$, there is $N_k > \pi(q_k)$ such that, for any $x \in \mathrm{orb}(q_k)$, $L^{(N_k)}_{d-i}(x) - L^{(N_k)}_{d-i-1}(x) \in (0, 1/2^k)$. Then there is a constant $\gamma_k \in (0, (\gamma_{k-1})/2)$ such that, for any x contained in the $2\gamma_k$-neighborhood of $\mathrm{orb}(q_k)$, $L^{(N_k)}_{d-i}(x) - L^{(N_k)}_{d-i-1}(x) \in (0, 1/2^k)$. Then item (4) is satisfied.

The proof of Proposition 4.2 is now complete. □

4.3. *End of the proof of Theorem B.* Now we can prove Theorem B. By items (1) and (7) of Theorem 2.9, we can assume that the periodic point p is center-dissipative and has simple spectrum. Then there is a sequence of center-dissipative hyperbolic periodic orbits (q_n) that satisfies the properties in Proposition 4.2.

By Lemma 2.6, denoting by μ_n the probability atomic measure uniformly distributed on the orbit $orb(q_n)$ for each n, the weak-$*$-limit of μ_n is an ergodic measure μ, whose support is

$$\mathrm{supp}(\mu) = \bigcap_{n=1}^{\infty} \left(\overline{\bigcup_{k=n}^{\infty} \mathrm{orb}(q_n)} \right) = H(p).$$

It only remains to show that μ is a non-hyperbolic measure, which comes from the following claim.

CLAIM 4.3. *For the ergodic measure μ, $\chi_{i+1}(\mu, f) = 0$.*

Proof. Since the orbit of q_n is a $(\gamma_n, 1 - 1/2^n)$-multiple almost shadowed by the orbit of q_{n+1}, there is a subset $\Gamma_n \subset \mathrm{orb}(q_n)$ and a map $\rho_n : \Gamma_n \mapsto \mathrm{orb}(q_{n-1})$ for each $n \ge 2$ from Definition 2.4. Take $Y_n = \rho_n^{-1} \circ \cdots \circ \rho_2^{-1}(\mathrm{orb}(q_1))$; we can see that Y_n is well defined. Take the upper topological limit

$$Y = \limsup_{n\to+\infty} Y_n.$$

Since Y is a compact set and μ is the limit measure of μ_n,

$$\mu(Y) \ge \limsup_{n\to+\infty} \mu_n(Y_n) \ge \prod_{k=1}^{n-1} \left(1 - \frac{1}{2^n}\right) > 0.$$

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By the fact that $\gamma_{n+1} < \frac{1}{2}\gamma_n$, we can see that the set Y is contained in the $2\gamma_n$-neighborhood of orb(q_n) for every $n \geq 1$. Then, for any $x \in Y$ and any $n \geq 1$, $L_{d-i}^{(N_n)}(x) - L_{d-i-1}^{(N_n)}(x) \in (0, 1/2^n)$ for the strictly increasing sequence $(N_n)_{n\geq 1}$. By the fact that μ is ergodic and that $\mu(Y) > 0$ for μ-a.e. $x \in Y$,

$$\begin{aligned}\chi_{i+1}(\mu, f) &= \lim_{m\to+\infty}(L_{d-i}^{(m)}(x, f) - L_{d-i-1}^{(m)}(x, f)) \\ &= \lim_{n\to+\infty}(L_{d-i}^{(N_n)}(x, f) - L_{d-i-1}^{(N_n)}(x, f)) = 0.\end{aligned}$$ □

4.4. *Proof of Theorem C.* By items (3) and (4) of Theorem 2.9, we can assume that there is a periodic point q such that $H(q) = H(p)$ and that p, q have indices $i, i+1$, respectively. We consider the following two cases.

If $H(p)$ has no dominated splitting of index i or of index $i+1$, then, by Theorem B, there is a non-hyperbolic ergodic measure μ such that $\text{supp}(\mu) = H(p)$.

If, otherwise, the homoclinic class $H(p)$ has both a dominated splitting of index i and a dominated splitting of index $i+1$, then, by Theorem 2.10, there is a non-hyperbolic ergodic measure μ such that $\text{supp}(\mu) = H(p)$.

5. *Classification of non-hyperbolic homoclinic classes*

Bonatti and Díaz [**14**] proposed a generalized Palis's conjecture for the C^1-topology: heterodimensional cycles (i.e., the existence of two hyperbolic periodic orbits with different indices linked by heteroclinic orbits) should appear densely in the interior of the set of non-hyperbolic C^1-diffeomorphisms. Here we state a local version of this conjecture for homoclinic classes.

CONJECTURE 4. [**8**, **11**, **13**, **21**] *For any generic $f \in \text{Diff}^1(M)$, if a homoclinic class $H(p)$ is not hyperbolic, then, arbitrarily C^1-close to f, there is a diffeomorphism g that exhibits a heterodimensional cycle associated to p_g and another periodic point, where p_g is the continuation of p.*

Note that, since on surfaces there is no heterodimensional cycle, these conjectures imply 'Smale's conjecture': hyperbolic systems are dense in the space of C^1 surface diffeomorphisms.

This conjecture, if satisfied, would generalize Theorem C to any non-hyperbolic homoclinic class. For this reason, we discuss partial results known in this direction. The next two statements are consequences of the results in [**11**] and [**38**].

PROPOSITION 5.1. *For any generic $f \in \text{Diff}^1(M)$, and for any non-hyperbolic homoclinic class $H(p)$ associated to a hyperbolic saddle p of index i, with $2 \leq i \leq d-2$, one of the following two possibilities holds. Either:*

- *$H(p)$ contains a periodic point with different index; or*
- *$H(p)$ has a partially hyperbolic splitting $E^s \oplus E^c$, $\dim(E^s) = i-1$, or $E^c \oplus E^u$, $\dim(E^c) = i+1$.*

Proof. We will assume that every periodic point contained in $H(p)$ has index i, since, otherwise, the first case in the statement holds. We consider the following two possibilities.

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Assume that there is a dominated splitting $T_{H(p)}M = E \oplus F$ with $\dim(E) = i - 1$. Then, by Theorem 2.11 and the fact that every periodic point contained in $H(p)$ has index i, the bundle E is contracted by Df, and hence $E \oplus F$ is a partially hyperbolic splitting $E^s \oplus E^c$. Similarly, if $\dim(E) = i + 1$, then F is expanded by Df and $E \oplus F$ is a partially hyperbolic splitting $E^c \oplus E^u$. The second case stated in Proposition 5.1 holds.

Assume, otherwise, that $H(p)$ admits neither domination of index $i - 1$ nor domination of index $i + 1$. Since $H(p)$ is not hyperbolic, by Theorem 2.11 and the fact that every periodic point contained in $H(p)$ has index i, for any $\varepsilon > 0$, there is a periodic point $q \in H(p)$ such that $\chi_i(q) \in (-\varepsilon, 0)$ or $\chi_{i+1}(q) \in (0, \varepsilon)$. Then, by Proposition 3.1, the diffeomorphism f can be C^1-approximated by diffeomorphisms with a heterodimensional cycle in the homoclinic class. By items (4) and (5) of Theorem 2.9, there is a periodic point of different index contained in $H(p)$, which contradicts the assumption. □

PROPOSITION 5.2. *Assume that* $\dim(M) = d \geq 3$. *For any generic* $f \in \mathrm{Diff}^1(M)$ *and for any non-hyperbolic homoclinic class* $H(p)$ *associated to a hyperbolic saddle* p *of index* $d - 1$, *one of the following three possibilities holds:*

- $H(p)$ *contains a periodic point with different index;*
- $H(p)$ *has a partially hyperbolic splitting* $E^s \oplus E^c$, $\dim(E^s) = d - 2$*; or*
- $H(p)$ *is the Hausdorff limit of periodic sinks.*

Proof. We will assume that all periodic points contained in $H(p)$ have the same index $d - 1$, otherwise the first case in the statement holds.

Assume that there is a dominated splitting $T_{H(p)}M = E \oplus F$ with $\dim(E) = d - 2$. Then, by Theorem 2.11 and the fact that every periodic point contained in $H(p)$ has index $d - 1$, the bundle E is contracted by Df and hence $E \oplus F$ is a partially hyperbolic splitting $E^s \oplus E^c$, which is the second case in the statement.

Assume, otherwise, that $H(p)$ admits no domination of index $d - 2$. Since $H(p)$ is non-hyperbolic, by Theorem 2.11 and the fact that every periodic point contained in $H(p)$ has index $d - 1$, there is a sequence of periodic points (q_n), contained in $H(p)$, such that $\chi_{d-1}(q_n) \to 0^-$ or $\chi_d(q_n) \to 0^+$ as $n \to +\infty$. Moreover, one can choose q_n to converge to $H(p)$ in the Hausdorff topology. If $\chi_{d-1}(q_n) \to 0^-$ occurs, then, by Proposition 3.1, one can get a heterodimensional cycle in the homoclinic class by arbitrarily C^1-small perturbation. By items (4) and (5) of Theorem 2.9, there is a periodic point of different index contained in $H(p)$, which contradicts the assumption. If, otherwise $\chi_d(q_n) \to 0^+$, then for any N and for any neighborhood, $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$, there is $g \in \mathcal{U}$ and $n > N$ such that q_n is a periodic sink of g. By item (6) of Theorem 2.9, there is a sequence of sinks converging to $H(p)$, which is the third case in the statement. □

This last property of Proposition 5.2 has to be compared with a similar statement on surfaces.

THEOREM (Pujals and Sambarino [**36**]) *For a generic surface diffeomorphism, any non-hyperbolic homoclinic class is the Hausdorff limit of periodic sinks or sources.*

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Based on the two propositions above and on the previous known results [**7**, **10**, **11**, **13**, **20**, **23**, **38**], one can list the different possibilities of a non-hyperbolic homoclinic class and discuss Conjecture 4 in each case.

5.1. *Non-hyperbolic homoclinic classes $H(p)$ for C^1-generic diffeomorphisms.*

Case (a). There exist two periodic points of different index.

This is the case satisfied by examples and which corresponds to the Conjecture 4.

Case (b). All periodic points have the same index i and the class $H(p)$ has a dominated splitting $T_{H(p)}M = E^s \oplus E^c_1 \oplus E^c_2 \oplus E^u$, with $\dim(E^c_j) \in \{0, 1\}$ and $i = \dim(E^s \oplus E^c_1)$. Maybe E^s and/or E^u is trivial.

This is exactly the case which occurs [**20**] when f is far from homoclinic tangencies and heterodimensional cycles, and hence it is the case in the spirit of Palis's conjecture.

Case (c). All periodic points have the same index i, the class $H(p)$ has a dominated splitting $T_{H(p)}M = E^s \oplus E^c \oplus E^u$, with $\dim(E^c) = 2$ and $\dim(E^s) = i - 1$, the bundle E^c does not split and there exist periodic points which contract and others which expand the volume along E^c. Maybe E^s and/or E^u is trivial.

Periodic points which expand the volume along E^c are dense in the class, and thus it is possible to turn them into points of index $i - 1$. Since the class has only points of index i (even after perturbation), the strong stable manifold of such a periodic point has to intersect $H(p)$ only at the periodic point itself. One may then expect that for any point x in the class $W^{ss}(x) \cap H(p) = \{x\}$. In this case, by [**10**], the class is contained in a submanifold tangential to $E^c \oplus E^u$. Arguing in a same way with periodic points which contract the volume along E^c, one deduces that $H(p)$ is contained in a locally invariant surface tangential to E^c. We are thus reduced to Smale's conjecture.

Case (d). All periodic points have the same index i, the class $H(p)$ has a dominated splitting $T_{H(p)}M = E \oplus E^u$, with $\dim(E) = i + 1$, there is no dominated splitting corresponding to index i and along any periodic orbit the volume of planes in E is contracted (sectional dissipation in E).

As in Case (c), one can expect that the class is contained in a locally invariant submanifold tangential to E. We are thus reduced to the case of a homoclinic class whose periodic points have one-dimensional unstable spaces and are sectionally dissipative and have no domination corresponding to index $\dim(M) - 1$. We are thus reduced to a generalized Smale's conjecture for higher dimension, as described in [**8**, Conjecture 8].

Case (d)′. This is similar to Case (d), but for f^{-1}.

Here, again, one may expect this to reduce to the generalized Smale's conjecture.

We prove that this classifies the dynamics on non-hyperbolic homoclinic classes.

THEOREM D. *For a generic diffeomorphism in $\mathrm{Diff}^1(M)$, any non-hyperbolic homoclinic class has to satisfy one of the cases above.*

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Proof. We consider a generic diffeomorphism $f \in \mathrm{Diff}^1(M)$ which satisfies Theorem 2.9, and a non-hyperbolic homoclinic class $H(p)$ associated to a hyperbolic periodic point p of index i. We assume that Case (a) does not occur, which means that all periodic points contained in $H(p)$ have the same index i. We consider the following possibilities.

Assume that there is a dominated splitting $T_{H(p)}M = E \oplus F$ with $\dim(E) = i$. If the bundle E is not uniformly contracted, then the bundle E has a dominated splitting $E^s \oplus E^c_1$ with $\dim(E^s) = i - 1$. Otherwise, using Proposition 3.1 and Theorem 2.11, one can get a heterodimensional cycle by arbitrarily C^1-small perturbation and, by items (4), (5) of Theorem 2.9, there is a periodic point of different index contained in $H(p)$, which contradicts the assumption. Moreover, by Theorem 2.11, the bundle E^s is uniformly contracted. Symmetrically, if the bundle F is not uniformly expanded, then it can be split as $E^c_2 \oplus E^u$, where $\dim(E^c_2) = 1$ and E^u is uniformly expanded. Hence, in this case, the homoclinic class has a partially hyperbolic splitting $T_{H(p)}M = E^s \oplus E^c_1 \oplus E^c_2 \oplus E^u$, with $\dim(E^c_j) \in \{0, 1\}$, and $i = \dim(E^s \oplus E^c_1)$. This is Case (b).

Assume, now, that the homoclinic class $H(p)$ admits no domination of index i. Consider the finest dominated splitting over $H(p)$. Combine all the contracted (center and expanded, respectively) bundles and denote it by $E^s \oplus E^c \oplus E^u$, such that E^s and E^u are uniformly contracted and expanded bundles, respectively, and E^c is the center bundle. By Propositions 5.1 and 5.2, $i = \dim(E^s) + 1$ or $i = \dim(E^s \oplus E^c) - 1$. We then consider the following two subcases.

If there exist both periodic points which contract and others which expand the volume along E^c, then there are both periodic orbits whose ith exponent is arbitrarily close to zero and those whose $(i + 1)$th exponent is arbitrarily close to zero. Then the homoclinic class $H(p)$ admits both a domination of index $i - 1$ and a domination of index $i + 1$. Otherwise, using Proposition 3.1, one can get a heterodimensional cycle by arbitrarily C^1-small perturbation and get a periodic point of different index in $H(p)$. Then $T_{H(p)}M = E^s \oplus E^c \oplus E^u$, with $\dim(E^s) = i - 1$ and $\dim(E^c) = 2$, and there is no finer dominated splitting along E^c. This is Case (c).

If there exist only periodic points which contract the volume along E^c, then, arguing as in the previous case, the homoclinic class $H(p)$ admits a domination $E \oplus E^u$, where $\dim(E) = i + 1$. This is Case (d). Symmetrically, if there exist only periodic points which expand the volume along E^c, then it is Case (d)′. □

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