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Pure point/continuous decomposition of translation-bounded measures and diffraction

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Abstract. In this work we consider translation-bounded measures over a locally compact Abelian group $\mathbb{G}$, with a particular interest in their so-called diffraction. Given such a measure ω, its diffraction $\widehat{\gamma}$ is another measure on the Pontryagin dual $\widehat{\mathbb{G}}$, whose decomposition into the sum $\widehat{\gamma} = \widehat{\gamma}_{\mathrm{p}} + \widehat{\gamma}_{\mathrm{c}}$ of its atomic and continuous parts is central in diffraction theory. The problem we address here is whether the above decomposition of $\widehat{\gamma}$ lifts to ω itself, that is to say, whether there exists a decomposition $\omega = \omega_{\mathrm{p}} + \omega_{\mathrm{c}}$, where ω_{p} and ω_{c} are translation-bounded measures having diffraction $\widehat{\gamma}_{\mathrm{p}}$ and $\widehat{\gamma}_{\mathrm{c}}$, respectively. Our main result here is the almost sure existence, in a sense to be made precise, of such a decomposition. It will also be proved that a certain uniqueness property holds for the above decomposition. Next, we will be interested in the situation where translation-bounded measures are weighted Meyer sets. In this context, it will be shown that the decomposition, whether it exists, also consists of weighted Meyer sets. We complete this work by discussing a natural generalization of the considered problem.

1. *Outline*

In this work, we consider certain measures over a second countable, σ-compact locally compact Abelian (LCA) group $\mathbb{G}$, for which we aim to establish a general result on structures that goes beyond the ones called *purely diffractive*. The motivation of this work comes from solid state physics, where materials with atomic-like kinematic diffraction have received much attention during the last 35 years since the discovery of *quasicrystal* by Shechtman *et al* [**55**], a work for which Schechtman was awarded the Nobel prize in 2011.

A material, or rather an infinite-sized idealization of a material, is mathematically described by a *translation-bounded measure* ω over the group $\mathbb{G}$ (thought as the ambient space, and usually taken as $\mathbb{R}^d$, $d = 1$, 2, or 3). Very often, a simpler picture is displayed by a weighted Dirac comb over a point set having specific geometric properties (lattice,

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Delone, finite local complexity (FLC), Meyer property and so on), which may be seen to model the position and nature of the constituents of the alloy. Now the mathematical diffraction theory as proposed by Hof in [**31**] (see also the recent review article [**9**] and the book [**10**]) assigns another measure γ on $\mathbb{G}$ to a general translation-bounded measure ω on a LCA group $\mathbb{G}$, given as a self-convolution product of ω passed to the infinite volume limit along a certain sequence $(\mathcal{A}_k)_k$ of sets in $\mathbb{G}$,

$$\gamma := \lim_{n\to\infty} \frac{1}{|\mathcal{A}_k|}\omega|_{\mathcal{A}_k} * \widetilde{\omega|_{\mathcal{A}_k}}.$$

The sequence $(\mathcal{A}_k)_k$ here is a van Hove sequence, which always exists in σ-compact LCA groups [**54**, §1]. If such a limit exists, and it always will after possibly extracting some sub-sequence, then it is called the *autocorrelation* of ω (with respect to $(\mathcal{A}_k)_k$). It is a positive-definite measure on $\mathbb{G}$ and thus admits a Fourier transform $\widehat{\gamma}$ called the *diffraction measure* (or simply diffraction) of ω. The latter is a positive translation-bounded measure on the Pontryagin dual $\widehat{\mathbb{G}}$, and is thought of as the outcome of a diffraction experiment set on an alloy modeled by ω, see again [**9**, **10**, **31**].

As any complex measure over $\widehat{\mathbb{G}}$, the diffraction measure splits as the sum

$$\widehat{\gamma} = \widehat{\gamma}_{\mathrm{p}} + \widehat{\gamma}_{\mathrm{c}}$$

of two mutually singular summands, namely a purely atomic (or *pure point*) part $\widehat{\gamma}_{\mathrm{p}}$ and a continuous part $\widehat{\gamma}_{\mathrm{c}}$. The summand $\widehat{\gamma}_{\mathrm{p}}$ is called the *Bragg spectrum* of the underlying translation-bounded measure ω, and any $\xi \in \widehat{\mathbb{G}}$ in its support, that is, with $\widehat{\gamma}(\{\xi\}) > 0$, is called a *Bragg peak position* for ω. The measure ω is then called *purely diffractive*, or said to have a pure point spectrum, whenever $\widehat{\gamma}$ is a pure point measure, that is, if its continuous part $\widehat{\gamma}_{\mathrm{c}}$ vanishes. Purely diffractive measures are, at least in the setting of weighted point sets, rather well understood, see the general characterizations made in [**13**, **17**, **29**, **37**]. It encompasses the (uniform Dirac combs over) lattices and regular model sets [**5**, **46**, **53**], and is illustrated by many explicit examples [**5**, **16**, **18**], see also the survey article [**9**] and the book [**10**]. The situation of a mixed diffraction spectrum is much less understood, although some explicit computations exist in this case [**6**, **11**].

In this work we are especially interested in the situation of a mixed diffraction spectrum. Namely, if ω is a translation-bounded measure with non-pure point diffraction, say $\widehat{\gamma}$, we wish to know if it is true that ω still admits a natural 'purely diffractive part' ω_{p} whose diffraction would exactly be the pure point part $\widehat{\gamma}_{\mathrm{p}}$, and such that, for the non-vanishing continuous part $\widehat{\gamma}_{\mathrm{c}}$, $\omega - \omega_{\mathrm{p}}$ would have diffraction. That is to say, we raise the following question.

Q. *Given a translation-bounded measure ω on $\mathbb{G}$ having diffraction $\widehat{\gamma} = \widehat{\gamma}_{\mathrm{p}} + \widehat{\gamma}_{\mathrm{c}}$, does there exist a decomposition of the form $\omega = \omega_{\mathrm{p}} + \omega_{\mathrm{c}}$, with ω_{p} and ω_{c} being translation-bounded measures having diffraction $\widehat{\gamma}_{\mathrm{p}}$ and $\widehat{\gamma}_{\mathrm{c}}$, respectively?*

As we will see, it is possible to give a rigorous answer to this question when one does not consider a translation-bounded measure ω as a deterministic object, but rather a *random* one. Physically this means that the precise structure of the underlying alloy is unknown, although one knows the frequency of any possible local configuration. On the mathematical side, this means that we do not consider a single translation-bounded

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measure but the whole collection $\mathcal{M}^\infty(\mathbb{G})$ of such measures, endowed with a translation-invariant (ergodic) probability distribution m on it. In other words, we shall deal here with an *ergodic system of translation-bounded measures* $(X, \mathbb{G}, m)$, $X \subset \mathcal{M}^\infty(\mathbb{G})$ being the support of m. In the setting of (uniform Dirac combs over) point sets one speaks of an (ergodic) stationary point process [**7**, **22**, **29**, **38**]. This approach by dynamical systems presents the great advantage of connecting the diffraction of a random translation-bounded measure modeled by a dynamical system $(X, \mathbb{G}, m)$, which is well-defined and unique, with the *dynamical spectrum* of $(X, \mathbb{G}, m)$ via the so-called Dworkin argument, see [**13**, **23**], also §3.4 in the main text.

1.1. *Overview on the content.* Our main result (see §4) is that, for an ergodic random translation-bounded measure, the decomposition almost surely exists, and is provided by two other ergodic random translation-bounded measures, which are naturally obtained from the former. In the vocabulary of dynamical systems, this can be stated as the following (Theorems 4.1 and 4.7 in the main text).

THEOREM 1. *Let $(X, \mathbb{G}, m)$ be an ergodic system of translation-bounded measures with diffraction $\widehat{\gamma} = \widehat{\gamma}_{\mathrm{p}} + \widehat{\gamma}_{\mathrm{c}}$.*

There exist two ergodic systems of translation-bounded measures:

- *$(X_{\mathrm{p}}, \mathbb{G}, m_{\mathrm{p}})$ having diffraction $\widehat{\gamma}_{\mathrm{p}}$;*
- *$(X_{\mathrm{c}}, \mathbb{G}, m_{\mathrm{c}})$ having diffraction $\widehat{\gamma}_{\mathrm{c}}$.*

Both of which are Borel factors of $(X, \mathbb{G}, m)$ under two Borel factor maps

$$X \ni \omega \longmapsto \omega_p \in X_p \qquad X \ni \omega \longmapsto \omega_c \in X_c$$

such that the equality $\omega = \omega_{\mathrm{p}} + \omega_{\mathrm{c}}$ occurs for m-almost every $\omega \in X$. These two ergodic systems are unique, as well as the involved Borel factor maps up to almost everywhere equality.

In this work we are particularly interested in the case where translation-bounded measures are *weighted Meyer sets* of $\mathbb{G}$. These are weighted Dirac combs supported on sets of a very special kind, the so-called Meyer sets. In this setting, the natural question is, given an ergodic system of weighted Meyer sets, to know whether translation-bounded measures in the spaces X_{p} and X_{c} resulting from Theorem 1 also are weighted Meyer sets. As we will see, this turns out to be correct (see §5).

In fact, much more is true: a particular property of weighted Meyer sets is that they are always supported on some point set of special feature, namely a *model set* [**2**, **9**, **10**, **14**, **17**, **18**, **26**, **31**, **32**, **36**, **39**, **42**, **45**–**47**, **53**, **54**, **58**]. Such point sets emerge if one not only considers the 'ambient' space $\mathbb{G}$ but a product $H \times \mathbb{G}$ of it with some locally compact Abelian group H, together with a lattice $\mathcal{L}$ in $H \times \mathbb{G}$ and a compact topologically regular subset W of H. Then denoting $z^{\parallel}$ and $z^{\perp}$ the projections of an element $z \in H \times \mathbb{G}$ along $\mathbb{G}$ and H respectively, one gets a point set, called here a 'closed' model set (the term 'closed' referring to the fact that W is closed in H), of the form

$$\mathcal{P}(W) := \{z^{\parallel} \in \mathbb{G} : z \in \mathcal{L},\, z^{\perp} \in W\}.$$

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Closed model sets are always Meyer sets, and for a partial converse any weighted Meyer set is supported on a point set of this form [**42**, **46**, **58**]. Considering this we have the following (see Theorem 5.10).

THEOREM 2. *Let $(X, \mathbb{G}, m)$ be an ergodic system of weighted Meyer sets. Then the following property holds m-almost everywhere: whenever ω is supported on a closed model set, then so are its summands ω_{p} and ω_{c}.*

This result asserts, in particular, that once X consists of weighted Meyer sets then so does X_{p} and X_{c}. It is in particular interesting (and surprising) to see the systematic emergence of weighted Meyer sets, namely those forming X_{c} (if this latter is non-trivial), having generically purely continuous diffraction. Only little is known about this latter type of weighted Meyer set, although some studies have been performed (on weighted lattices), for instance, in [**4**, **6**].

The remaining part (§6) of this work is about a generalization of our motivating question. Namely, in the decomposition $\widehat{\gamma} = \widehat{\gamma}_{\mathrm{p}} + \widehat{\gamma}_{\mathrm{c}}$ stated earlier the continuous part $\widehat{\gamma}_{\mathrm{c}}$ always splits into two mutually singular components

$$\widehat{\gamma}_{\mathrm{c}} = \widehat{\gamma}_{\mathrm{ac}} + \widehat{\gamma}_{\mathrm{sc}},$$

where $\widehat{\gamma}_{\mathrm{ac}}$ is its absolutely continuous part with respect to a fixed Haar measure on $\widehat{\mathbb{G}}$, and $\widehat{\gamma}_{\mathrm{sc}}$ the remaining part, its singular continuous part. It is then natural to ask whether a translation-bounded measure ω with diffraction $\widehat{\gamma}$ admits a decomposition of the form $\omega = \omega_{\mathrm{p}} + \omega_{\mathrm{ac}} + \omega_{\mathrm{sc}}$, with ω_{p}, ω_{ac}, and ω_{sc} being translation-bounded measures having diffraction $\widehat{\gamma}_{\mathrm{p}}$, $\widehat{\gamma}_{\mathrm{ac}}$, and $\widehat{\gamma}_{\mathrm{sc}}$, respectively. As before, we shall consider this question on a dynamical point of view and, given an ergodic system of translation-bounded measures $(X, \mathbb{G}, m)$, look for dynamical systems of translation-bounded measures $(X_{\mathrm{ac}}, \mathbb{G}, m_{\mathrm{ac}})$ and $(X_{\mathrm{sc}}, \mathbb{G}, m_{\mathrm{sc}})$ such that a generalized form of Theorem 1 holds.

We could not prove or disprove the existence of such systems. It is however possible to reformulate this problem, by noting that the two Borel factors we look for, whenever they exist, have diffraction being absolutely continuous with respect to $\widehat{\gamma}$, hence of the form $f.\widehat{\gamma}$ for functions f on $\widehat{\mathbb{G}}$ belonging in this case to $L^{\infty}(\widehat{\mathbb{G}}, \widehat{\gamma})$. Indeed this leads us to seek a criterion for the existence of a Borel factor of $(X, \mathbb{G}, m)$ whose diffraction has the form $f.\widehat{\gamma}$ where the density function f is prescribed. In this work we provide such a criterion, which involves certain operators acting on $L^2(X, m)$ called here *admissible* (see Definition 6.1), and that may be stated as follows (see Theorem 6.6).

THEOREM 3. *Let $(X, \mathbb{G}, m)$ be a dynamical system of translation-bounded measures with diffraction $\widehat{\gamma}$. There is a bijective correspondence between:*

(a) *Borel factor maps over some dynamical system of translation-bounded measures having diffraction $\widehat{\gamma}' \ll \widehat{\gamma}$, with Radon–Nikodym derivative $d\widehat{\gamma}'/d\widehat{\gamma} \in L^{\infty}(\widehat{\mathbb{G}}, \widehat{\gamma})$; and*

(b) *admissible operators Q on $L^2(X, m)$.*

Given a Borel subset $\mathcal{P} \subseteq \widehat{\mathbb{G}}$, one has $d\widehat{\gamma}'/d\widehat{\gamma} = \mathbb{I}_{\mathcal{P}}$ if and only if $|Q| = \mathsf{E}(\mathcal{P})P_{\Theta}$, with $\mathsf{E}(\mathcal{P})$ the spectral projector associated to $\mathcal{P}$ and P_{Θ} an appropriate projector on $L^2(X, m)$.

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Therefore, existence of our desired Borel factors rests on the existence of admissible operators on $L^2(X, m)$ whose absolute value is given by an appropriate spectral projector. It is worth noticing that this correspondence does not encompass many interesting factors, as in some cases the knowledge of the diffraction of all factors allows the dynamical spectrum to be completely determined, see the discussion in [**8**, **15**].

2. *Generalities on dynamical systems*

Throughout this work, $\mathbb{G}$ denotes a second countable LCA group. Such a group is automatically σ-compact, and admits a translation-invariant metric compatible with the topology, with all its closed spheres being compact [**56**]. We denote its group operation additively, and we fix a Haar measure whose valuation on a Borel set A will be denoted $|A|$. The integral with respect to the Haar measure will be noted $\int_{\mathbb{G}} \dots dt$ everywhere. Its Pontryagin dual $\widehat{\mathbb{G}}$ is also a second countable LCA group, whose group operation will be noted multiplicatively.

2.1. *Dynamical systems and Borel factors.*

By a *dynamical system* we mean a pair $(X, \mathbb{G})$ consisting of a compact metrizable space X together with a continuous $\mathbb{G}$-action (noted on the right)

$$X \times \mathbb{G} \ni (x, t) \longmapsto x.t \in X.$$

Whenever X is equipped with a $\mathbb{G}$-invariant (respectively ergodic) Borel probability measure m then the resulting triple $(X, \mathbb{G}, m)$ will be called a measured (respectively ergodic) dynamical system. The topological support $\mathrm{Supp}(m)$ of m is the smallest compact subset of X whose complementary set has measure zero, and $(X, \mathbb{G}, m)$ is said to have full support whenever $\mathrm{Supp}(m) = X$.

If $(X, \mathbb{G})$ is a dynamical system and $x \in X$ is chosen then we let X_x, the *hull* of x, to be the closure of the $\mathbb{G}$-orbit of x in X, giving rise to another dynamical system $(X_x, \mathbb{G})$ with restricted $\mathbb{G}$-action. A dynamical system $(X, \mathbb{G})$ is called minimal if $X_x = X$ for any $x \in X$. The following proposition is proved in [**25**, Ch. 9, Proposition 2.2.2].

PROPOSITION 2.1. *In any ergodic system $(X, \mathbb{G}, m)$ the set of point $x \in X$ such that $X_x = \mathrm{Supp}(m)$ is Borel of full measure in X.*

Let us now consider two measured dynamical systems $(X, \mathbb{G}, m)$ and $(X', \mathbb{G}, m')$. A Borel map $\pi : X \longrightarrow X'$ is called a Borel $\mathbb{G}$-map if for any fixed $t \in \mathbb{G}$ the set of points $x \in X$ such that $\pi(x.t) = \pi(x).t$ has full measure in X. It is called a Borel factor map if in addition m' is the push-forward of m by π, and in this case the system $(X', \mathbb{G}, m')$ is called a Borel factor of $(X, \mathbb{G}, m)$. Whenever π is a Borel $\mathbb{G}$-map (respectively a factor map) then any Borel function agreeing with π on a Borel set of full measure is again a Borel $\mathbb{G}$-map (respectively a factor map). By [**25**, Ch. 9, Proposition 2.2.1] any Borel $\mathbb{G}$-map $\pi : X \longrightarrow X'$ is almost everywhere equal to a Borel $\mathbb{G}$-map $\tilde{\pi} : X \longrightarrow X'$ admitting a $\mathbb{G}$-stable Borel subset $\tilde{X}$ of full measure in X such that $\tilde{\pi}(x.t) = \tilde{\pi}(x).t$ holds for any $x \in \tilde{X}$ and any $t \in \mathbb{G}$. This in particular implies that a Borel factor of an ergodic system is also ergodic.

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Let $(X, \mathbb{G}, m)$ be an ergodic dynamical system, and $(X', \mathbb{G}, m')$ some Borel factor with Borel factor map $\pi : X \longrightarrow X'$. Denote $\mathcal{L}^\infty(X)$ the space of bounded complex-valued Borel functions on X. Then the measure m disintegrates over the factor X' into a set of measures supported on fibers for the factor map π.

THEOREM 2.2. (Disintegration theorem) *Let $\pi : X \longrightarrow X'$ be a Borel map between compact metric spaces, m a Borel probability measure on X and m' its push-forward measure on X'. Then there is a Borel map valued in the probability measures over X,*

$$X' \ni x \longmapsto m_x \in \mathrm{Prob}(X)$$

satisfying the following two statements.
(D1) The push-forward measure $\pi(m_x) = \delta_x$ for m'-almost every $x \in X'$.
(D2) For each $f \in \mathcal{L}^\infty(X)$ one has $\int_X f\, dm = \int_{x\in X'} \left[\int_X f\, dm_x\right] dm'(x)$.
Moreover any two such maps coincide m'-almost everywhere on X'.

For a proof one can consult [**20**, Theorems 1 and 2(iii)]. The claim that a disintegration is a Borel map means that it is a Borel map when $\mathrm{Prob}(X)$ is endowed with its vague topology, which means exactly that for each $f \in \mathcal{L}^\infty(X)$ the mapping associating the value $\int_X f\, dm_x$ to $x \in X'$ is a Borel map. This also ensures that the right term of the equality in (D2) is well-defined. As a direct consequence of the almost everywhere uniqueness, if π is a Borel $\mathbb{G}$-map and m is $\mathbb{G}$-invariant then the disintegration is always a $\mathbb{G}$-map in the following sense: as $\mathbb{G}$ acts on X continuously it also acts on the space $\mathcal{M}^b(X)$ of bounded complex Radon measures of X via the formula $\mu.t(.) := \mu(. - t)$, and leaves stable the space $\mathrm{Prob}(X)$ of probability measures on X. Then the almost everywhere uniqueness ensures that for any $t \in \mathbb{G}$ one has $m_{x.t} = (m_x).t$ for m'-almost every $x \in X'$. Since any change of the disintegration map on a set of measure 0 in X' does not violate (D1) and (D2) one can choose the disintegration to also satisfy the following.
$(D3)$ One has $m_{x.t} = (m_x).t$ for all $t \in \mathbb{G}$ and all x in a full Borel subset $X' \subseteq X$.

2.2. *Unitary representation and dynamical spectrum.* A measured dynamical system $(X, \mathbb{G}, m)$ naturally gives rise to a strongly continuous unitary representation of $\mathbb{G}$ on the separable Hilbert space $L^2(X, m)$, where on a function $h \in L^2(X, m)$ the unitary operator associated with $t \in \mathbb{G}$ reads $U_t(h)(.) = h(. - t)$. From Stone's theorem, see [**41**, §36D] or [**24**, Vol. 2, §10.2], there is an associated *projection-valued measure* on the Borel sets of $\widehat{\mathbb{G}}$,

$$\mathsf{E} : \text{Borel sets of } \widehat{\mathbb{G}} \longrightarrow \text{Projectors of } L^2(X, m)$$

that is to say, a map such that $\mathsf{E}(\mathcal{P})$ is an orthogonal projector on $L^2(X, m)$ for each Borel subset $\mathcal{P} \subseteq \widehat{\mathbb{G}}$, with $\mathsf{E}(\emptyset)$ the trivial operator and $\mathsf{E}(\widehat{\mathbb{G}})$ the identity operator, and which is σ-additive on the Borel sets of $\widehat{\mathbb{G}}$. This projection-valued measure is uniquely characterized according to the following property: for any two $h, k \in L^2(X, m)$ the scalar product $\langle h, \mathsf{E}(.)k\rangle$ sets a bounded complex measure on $\widehat{\mathbb{G}}$, such that one has

$$\langle h, U_t k\rangle = \int_{\xi\in\widehat{\mathbb{G}}} \xi(t)\, d\langle h, \mathsf{E}(\xi)k\rangle. \tag{1}$$

In other words, $\langle h, \mathsf{E}(.)k\rangle$ is the unique bounded complex measure on $\widehat{\mathbb{G}}$ whose Fourier–Stieltjes transform is the continuous bounded function on $\mathbb{G}$ mapping t to $\langle h, U_t k\rangle$.

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A projector of the form $\mathsf{E}(\mathcal{P})$ is called a *spectral projector*. One deduces from equality (1) that a bounded operator commuting with all operators U_t also commutes with all spectral projectors. In particular, the spectral projectors themselves commute with all the operators U_t. As a result, the range of any spectral projector is a closed $\mathbb{G}$-stable subspace of $L^2(X, m)$.

For each $\xi \in \widehat{\mathbb{G}}$ we simply denote by $\mathsf{E}(\xi)$ the spectral projector associated with the singleton $\{\xi\}$. As a consequence of (1), the range of $\mathsf{E}(\xi)$ consists exactly of the *eigenfunctions* of the system $(X, \mathbb{G}, m)$ with eigenvalue ξ, that is to say, functions $h \in L^2(X, m)$ such that $U_t(h) = \xi(t)h$ as L^2-functions for any $t \in \mathbb{G}$. A $\xi \in \widehat{\mathbb{G}}$ is called an *eigenvalue* of $(X, \mathbb{G}, m)$ if it admits a non-trivial eigenfunction in $L^2(X, m)$, that is, if the spectral projector $\mathsf{E}(\xi)$ is non-trivial. A measure dynamical system $(X, \mathbb{G}, m)$ is said to have a *pure point dynamical spectrum* if the eigenfunctions span $L^2(X, m)$, or equivalently, if the associated projection-valued measure E is purely atomic.

2.3. *The maximal Kronecker–Borel factor of an ergodic dynamical system.* When the measured dynamical system $(X, \mathbb{G}, m)$ is ergodic, each eigenvalue ξ has a unique eigenfunction up to a multiplicative constant, that is, the range of $\mathsf{E}(\xi)$ is one-dimensional in $L^2(X, m)$. Moreover, any such eigenfunction is almost everywhere constant in modulus and thus belongs to $L^\infty(X, m)$. Therefore, the product of an eigenfunction associated with ξ with an eigenfunction associated with ξ' is well-defined, and is an eigenfunction associated with $\xi.\xi'$. Hence, the set of eigenvalues of an ergodic dynamical system $(X, \mathbb{G}, m)$ always forms a (countable) subgroup $\mathcal{E}$ of $\widehat{\mathbb{G}}$.

Consider now any subgroup $\mathcal{E}$ of $\widehat{\mathbb{G}}$, endowed with the discrete topology, for which it is an LCA group. As a discrete Abelian group, it admits a Pontryagin dual $\mathrm{T}_{\mathcal{E}}$, which is a compact Abelian group, on which $\mathbb{G}$ acts *by rotation*: the natural continuous injection morphism of $\mathcal{E}$ in $\widehat{\mathbb{G}}$ leads by Pontryagin duality a continuous group morphism

$$\mathsf{i} : \mathbb{G} \longrightarrow \mathrm{T}_{\mathcal{E}}$$

with dense range, which in turn defines a $\mathbb{G}$-action (set on the right) by $z.t := z.\mathsf{i}(t)$. The resulting dynamical system $(\mathrm{T}_{\mathcal{E}}, \mathbb{G})$ is then a minimal *Kronecker system*, and the normalized Haar measure on $\mathrm{T}_{\mathcal{E}}$ is the unique invariant measure with respect to this action, providing so the ergodic dynamical system $(\mathrm{T}_{\mathcal{E}}, \mathbb{G}, m_{\mathrm{Haar}})$.

PROPOSITION 2.3. *Let $(X, \mathbb{G}, m)$ be an ergodic dynamical system and let $\mathcal{E} \subset \widehat{\mathbb{G}}$ be its eigenvalue group. Then $(\mathrm{T}_{\mathcal{E}}, \mathbb{G}, m_{\mathrm{Haar}})$ is a Borel factor of $(X, \mathbb{G}, m)$.*

A proof of this is given in §A.1. For $(X, \mathbb{G}, m)$, an ergodic dynamical system with eigenvalue group $\mathcal{E}$, the Borel factor $(\mathrm{T}_{\mathcal{E}}, \mathbb{G}, m_{\mathrm{Haar}})$ associated with $\mathcal{E}$, is called its *maximal Kronecker–Borel factor*. The term 'maximal' refers to the fact that any Kronecker–Borel factor of $(X, \mathbb{G}, m)$ factors through $(\mathrm{T}_{\mathcal{E}}, \mathbb{G}, m_{\mathrm{Haar}})$. The celebrated Halmos–von Neumann theorem asserts that an ergodic dynamical system has a pure point dynamical spectrum if and only if it is Borel conjugated with its maximal Kronecker–Borel factor.

Given an ergodic dynamical system $(X, \mathbb{G}, m)$ with eigenvalue group $\mathcal{E}$, any Borel factor map π on its maximal Kronecker–Borel factor $(\mathrm{T}_{\mathcal{E}}, \mathbb{G}, m_{\mathrm{Haar}})$ induces a

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$\mathbb{G}$-commuting isometric embedding of Hilbert spaces

$$L^2(\mathrm{T}_{\mathcal{E}}, m_{\mathrm{Haar}}) \ni h \longmapsto h \circ \pi \in L^2(X, m).$$

It maps a continuous character on $\mathrm{T}_{\mathcal{E}}$, which corresponds to some $\xi \in \mathcal{E}$ by Pontryagin duality, to a normalized eigenfunction of the system whose eigenvalue is ξ. As a byproduct, its range is precisely the closed $\mathbb{G}$-stable Hilbert subspace $\mathcal{H}_{\mathrm{p}}$ of $L^2(X, m)$ spanned by the eigenfunctions of the system. The orthogonal projection onto this subspace is the spectral projector $\mathsf{E}(\mathcal{E})$ associated to the countable subset $\mathcal{E} \subset \widehat{\mathbb{G}}$, and can be linked with the disintegration theorem applied to the factor map π over $\mathrm{T}_{\mathcal{E}}$ (regardless the choice of π, which is essentially unique).

PROPOSITION 2.4. *Let $(X, \mathbb{G}, m)$ be an ergodic dynamical system with eigenvalue group $\mathcal{E}$, and π a factor map over $\mathrm{T}_{\mathcal{E}}$. Then for any $f \in \mathcal{L}^2(X, m)$ the projection $\mathsf{E}(\mathcal{E})(f)$ in $L^2(X, m)$ is equal to the L^2-class of*

$$x \longmapsto \int_X f \, dm_{\pi(x)}.$$

This proposition provides a fundamental connection between disintegration over $\mathrm{T}_{\mathcal{E}}$ and the spectral projector $\mathsf{E}(\mathcal{E})$. A proof is given in §A.2.

3. *Translation-bounded measures*

3.1. *Basics on Fourier analysis.* Here we mainly follow [**13**, **54**]. The $\mathbb{C}$-vector space of compactly supported, complex-valued continuous functions on $\mathbb{G}$ will be denoted $\mathcal{C}_c(\mathbb{G})$, and is topologized as follows: for each compact subset $K \subset \mathbb{G}$ let $\mathcal{C}_K(\mathbb{G})$ be the subspace of functions supported on K equipped with the suppremum norm $\|.\|_\infty$, for which it is a Banach space. Then let $(K_n)_n$ be a nested sequence of compact regular sets in $\mathbb{G}$ whose interiors cover $\mathbb{G}$: it gives $\mathcal{C}_c(\mathbb{G}) = \bigcup_n \mathcal{C}_{K_n}(\mathbb{G})$ and defines an inductive limit topology $\mathcal{T}_{\mathrm{lim}}$ on $\mathcal{C}_c(\mathbb{G})$ which is the weakest topology making the natural inclusions

$$(\mathcal{C}_{K_n}(\mathbb{G}), \|.\|_\infty) \hookrightarrow (\mathcal{C}_c(\mathbb{G}), \mathcal{T}_{\mathrm{lim}})$$

continuous. This topology does not depend on the choice of nested sequence $(K_n)_n$, and the resulting space $(\mathcal{C}_c(\mathbb{G}), \mathcal{T}_{\mathrm{lim}})$ is a separable topological vector space. The space $\mathcal{C}_c(\mathbb{G})$ is closed under the operations $\phi_- := \phi(-.)$ and $\widetilde{\phi} := \overline{\phi(-.)}$, and under convolution product $\phi * \psi(t) := \int_{\mathbb{G}} \phi(s)\psi(t-s)\, ds$. One has the usual Fourier transform (see [**51**])

$$\mathcal{C}_c(\mathbb{G}) \ni \phi \longmapsto \widehat{\phi} \in \mathcal{C}_0(\widehat{\mathbb{G}}) \quad \text{with } \widehat{\phi}(\xi) := \int_{\mathbb{G}} \phi(t)\overline{\xi(t)}\, dt.$$

The space $\mathcal{C}_0(\widehat{\mathbb{G}})$ of continuous complex-valued functions on $\widehat{\mathbb{G}}$ vanishing at infinity is closed under pointwise product, and one has the relations

$$\widehat{(\phi_-)} = (\widehat{\phi})_-, \quad \widehat{\overline{\phi}} = \overline{\widehat{\phi}}, \quad \widehat{\phi * \psi} = \widehat{\phi}.\widehat{\psi}.$$

We now consider the vector space $\mathcal{M}(\mathbb{G})$ of (possibly unbounded) *complex Radon measures* on $\mathbb{G}$, that is, of σ-additive, regular complex-valued measures defined on the collection of relatively compact Borel subsets of $\mathbb{G}$ (see [**24**, Vol. II, Ch. 2, §§1 and 8]). By the Riesz–Markov representation theorem (see [**24**, Vol. II, Ch. 2, §8]), the

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space $\mathcal{M}(\mathbb{G})$ is under bijective linear correspondence with the dual space $\mathcal{C}_c(\mathbb{G})^*$ of $(\mathcal{C}_c(\mathbb{G}), \mathcal{T}_{\lim})$, the space of complex linear functionals on $\mathcal{C}_c(\mathbb{G})$ which are bounded on each subspace $(\mathcal{C}_{K_n}(\mathbb{G}), \|.\|_\infty)$ (with a constant depending on K_n). The functional in $\mathcal{C}_c(\mathbb{G})^*$ corresponding to the measure $\omega \in \mathcal{M}(\mathbb{G})$ is denoted $\int_{\mathbb{G}} \ldots d\omega$, and is given by usual Lebesgue integration when ω is a positive Radon measure.

The space $\mathcal{M}(\mathbb{G})$ carries a natural topology $\mathcal{T}_{\text{vague}}$, called the vague topology, which is the weak-$*$ topology on $\mathcal{C}_c(\mathbb{G})^*$ after identification, given by the weakest topology making the functions

$$\mathcal{N}_\phi(\omega) := \int_{\mathbb{G}} \phi_- \, d\omega = \int_{\mathbb{G}} \phi(-s) \, d\omega(s) \tag{2}$$

continuous on $\mathcal{M}(\mathbb{G})$ for each $\phi \in \mathcal{C}_c(\mathbb{G})$. Hence, the space defined $(\mathcal{M}(\mathbb{G}), \mathcal{T}_{\text{vague}})$ is a locally convex topological vector space. It carries a natural $\mathbb{G}$-action defined for $t \in \mathbb{G}$ by $\omega \longmapsto \omega * \delta_t$, where the convolution product of two convolable complex Radon measures is determined on $\phi \in \mathcal{C}_c(\mathbb{G})$ by

$$\int_{\mathbb{G}} \phi \, d\omega * \omega' := \int_{\mathbb{G}} \int_{\mathbb{G}} \phi(s+t) \, d\omega(s) \, d\omega'(t).$$

The functions given in equation (2) satisfy $\mathcal{N}_\phi(. * \delta_t) = \mathcal{N}_{\phi * \delta_t}$ for $t \in \mathbb{G}$ and $\phi \in \mathcal{C}_c(\mathbb{G})$. Finally, there is a natural involution on $\omega \in \mathcal{M}(\mathbb{G})$ set by $\widetilde{\omega}(\phi) = \overline{\omega(\widetilde{\phi})}$ for $\phi \in \mathcal{C}_c(\mathbb{G})$.

3.2. *Dynamical systems of translation-bounded measures.* Let (K, M) be a pair with $K \subset \mathbb{G}$ compact of non-empty interior and $M \geqslant 0$. A complex Radon measure $\omega \in \mathcal{M}(\mathbb{G})$ is called (K, M)-translation-bounded if

$$|\omega|(K + t) \leqslant M \quad \text{for each } t \in \mathbb{G}.$$

Here, $|\omega|$ is the absolute value, or total variation of ω, which is the least positive Radon measure on $\mathbb{G}$ such that $|\omega(B)| \leqslant |\omega|(B)$ for any relatively compact Borel set B. A complex Radon measure is then called a *translation-bounded measure* if it is (K, M)-translation-bounded for some (K, M). The collection of (K, M)-translation-bounded measures on $\mathbb{G}$ will be denoted $\mathcal{M}_{(K,M)}(\mathbb{G})$, and that of translation-bounded measures $\mathcal{M}^\infty(\mathbb{G})$. It is clear that $\mathcal{M}^\infty(\mathbb{G})$, when endowed with the vague topology, is a topological vector subspace of $\mathcal{M}(\mathbb{G})$ that remains stable under the natural $\mathbb{G}$-action.

PROPOSITION 3.1. [**13**] *Each set $\mathcal{M}_{(K,M)}(\mathbb{G})$ is a metrizable compact convex subset of $\mathcal{M}^\infty(\mathbb{G})$, on which the natural $\mathbb{G}$-action is well-defined and continuous.*

Compactness and metrizability are proved in [**13**, Theorem 2], and continuity of the $\mathbb{G}$-action in [**13**, Proposition 2]. The convexity statement is ensured by the inequality $|\alpha\omega + (1-\alpha)\omega'| \leqslant \alpha|\omega| + (1-\alpha)|\omega'|$ holding for any two measures ω, ω' and $\alpha \in [0, 1]$.

Definition 3.2. A measured dynamical system of translation-bounded measures is a measured dynamical system $(X, \mathbb{G}, m)$, with X a compact subset of some $\mathcal{M}_{(K,M)}(\mathbb{G})$ and the $\mathbb{G}$-action given by $X \times \mathbb{G} \ni (\omega, t) \longmapsto \omega * \delta_t \in X$.

A measured system $(X, \mathbb{G}, m)$ of translation-bounded measures will be called *trivial* whenever the support of m uniquely consists of the trivial measure on $\mathbb{G}$, and non-trivial otherwise.

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A way to generate dynamical systems of translation-bounded measures is to start with a particular measure ω in some $\mathcal{M}_{(K,M)}(\mathbb{G})$ and consider its hull, that is, the closure X_ω of its $\mathbb{G}$-orbit in $\mathcal{M}_{(K,M)}(\mathbb{G})$. Once a $\mathbb{G}$-invariant probability measure m on X_ω is chosen one ends up with a system of the desired form.

To a translation-bounded measure can be associated, as in [**13**], its *rubber local isomorphism class*, or RLI-class, which is the collection of all translation-bounded measures locally isomorphic with ω,

$$\mathrm{RLI}(\omega) := \{\omega' \in \mathcal{M}^\infty(\mathbb{G}) \mid X_\omega = X_{\omega'}\}. \tag{3}$$

It is not hard to show that the RLI-class of a translation-bounded measure ω is a Borel $\mathbb{G}$-stable subset of $\mathcal{M}^\infty(\mathbb{G})$ (it is even a G_δ), and moreover one clearly has $\mathrm{RLI}(\omega) \subseteq X_\omega$. Given a measured dynamical system of translation-bounded measures $(X, \mathbb{G}, m)$, it is clear that whenever ω belongs to X then the entire class $\mathrm{RLI}(\omega)$ also belongs to X, so that X always partitions into disjoint RLI-classes. Moreover, one easily sees that $(X, \mathbb{G})$ is minimal if and only if X contains a single RLI-class. If $(X, \mathbb{G}, m)$ is ergodic, one knows by Proposition 2.1 that almost all $\omega \in X$ have the support of m as their hull. These elements form, as it is easy to remark, a single RLI-class in X, yielding the following.

PROPOSITION 3.3. *An ergodic system of translation-bounded measures $(X, \mathbb{G}, m)$ admits a single* RLI*-class of full m-measure in X, formed of the $\omega \in X$ satisfying $X_\omega = \mathrm{Supp}(m)$.*

This will be of main importance in §5. Any dynamical system of translation-bounded measures $(X, \mathbb{G}, m)$ has a natural set of continuous functions on X given by the restriction of functions (2) on the compact subset X, which we denote $\mathcal{N}_\phi^X$, or simply $\mathcal{N}_\phi$ when the context is clear.

3.3. *Intensities.* Let $\mathcal{M}_{(K,M)}(\mathbb{G})$ be the compact and convex subset of (K, M)-translation-bounded measures in $\mathcal{M}^\infty(\mathbb{G})$. The following statement is a particular case of a general result on vector-valued integration theory, see [**52**, Theorem 3.27].

THEOREM 3.4. *Suppose that X is a compact subset of $\mathcal{M}_{(K,M)}(\mathbb{G})$. Then any bounded complex measure $\mu \in \mathcal{M}^b(X)$ defines a translation-bounded measure $I(\mu) \in \mathcal{M}^\infty(\mathbb{G})$ by*

$$\int_{\mathbb{G}} \phi \, dI(\mu) = \int_X \int_{\mathbb{G}} \phi \, d\omega \, d\mu(\omega).$$

If moreover μ is a probability measure then $I(\mu)$ belongs to $\mathcal{M}_{(K,M)}(\mathbb{G})$.

Given a compact subset X of some $\mathcal{M}_{(K,M)}(\mathbb{G})$ and μ a complex measure on X, the resulting translation-bounded measure $I(\mu)$ is sometimes called the integral, or barycenter, or also first moment of μ on X. However, we shall call it here *intensity*. This terminology comes from point-set theory, where one studies *Delone sets* of $\mathbb{G}$ (see, for instance, [**13**, **28**, **29**]): given a compact space $\mathbb{X}$ of Delone sets of $\mathbb{G}$ (with common radius of uniform discreteness), the intensity $I(\mu)$ of a bounded measure μ on $\mathbb{X}$ is the translation-bounded measure obtained, see [**29**, §4], on any Borel set $A \subseteq \mathbb{G}$ by

$$I(\mu)(A) = \int_{\mathbb{X}} \sharp(\Lambda \cap A) \, d\mu(\Lambda).$$

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A compact space $\mathbb{X}$ of Delone sets can be identified with a compact subset X of some $\mathcal{M}_{(K,M)}(\mathbb{G})$ by replacing a point set Λ by the uniform Dirac comb $\delta_\Lambda := \sum_{t\in\Lambda} \delta_t$ (see [**13**, §4 and Theorem 4]). Hence, a measure μ on $\mathbb{X}$ can alternatively be viewed as living on X, and its intensity as given just above is equal to the translation-bounded measure yield by Theorem 3.4.

Recall that if X is a compact $\mathbb{G}$-stable subset of $\mathcal{M}_{(K,M)}(\mathbb{G})$ then $\mathbb{G}$ acts on the space $\mathcal{M}^b(X)$ of bounded complex Radon measures of X via the formula $\mu.t := \mu(.*\delta_{-t})$, as in §2.1. The following is then straightforward to prove the following.

PROPOSITION 3.5. *Let X be a compact $\mathbb{G}$-stable subset of $\mathcal{M}_{(K,M)}(\mathbb{G})$, with $\mathcal{M}^b(X)$ its space of bounded complex measures. Then the map*

$$(\mathcal{M}^b(X),\, \mathcal{T}_{\text{vague}}) \ni \mu \longmapsto I(\mu) \in (\mathcal{M}^\infty(\mathbb{G}),\, \mathcal{T}_{\text{vague}})$$

is continuous and $\mathbb{G}$-commuting, that is, $I(\mu(.*\delta_{-t})) = I(\mu) * \delta_t$ for any $t \in \mathbb{G}$.

3.4. *Mathematical diffraction and connection with dynamical spectrum.* We provide here a very concise survey on diffraction, and refer the reader to [**9**, **13**, **17**, **31**, **54**] for more details, comments, and proofs. The diffraction of a translation-bounded measure ω is defined by first considering a van Hove sequence [**54**] of subsets $(\mathcal{A}_n)_{n\in\mathbb{N}}$, from which one considers the following vague limit in $\mathcal{M}^\infty(\mathbb{G})$ of truncated autocorrelations

$$\gamma_\omega := \lim_{n\to\infty} \frac{1}{|\mathcal{A}_n|}\omega|_{\mathcal{A}_n} * \widetilde{\omega|_{\mathcal{A}_n}}. \tag{4}$$

The above sequence may not converge in $\mathcal{M}^\infty(\mathbb{G})$ but it always will once one extracts an appropriate sub-sequence (one shows by mimicking the proof of [**31**, Proposition 2.2] and using [**54**, Lemma 1.1] that such a sequence belongs to a certain $\mathcal{M}_{(K,M)}(\mathbb{G})$, and the claim comes from compactness and metrizability of this latter set). Thus, by considering this sub-sequence in (4) one can get rid of this difficulty. Moreover, again up to extracting a sub-sequence, one can assume that the van Hove sequence is *tempered* (see [**40**, Proposition 1.4] for definition and proof). The limit γ_ω is the *autocorrelation measure* of $\omega \in \mathcal{M}^\infty(\mathbb{G})$ (with respect to the van Hove sequence $(\mathcal{A}_n)_{n\in\mathbb{N}}$), and is translation-bounded. Such a measure is moreover *positive-definite*, and hence Fourier transformable (see [**60**, Theorem 4.7]): there exists a unique Radon measure $\widehat{\gamma}_\omega$ on the Pontryagin dual $\widehat{\mathbb{G}}$, positive and translation-bounded, such that at any $\phi \in \mathcal{C}_c(\mathbb{G})$,

$$\int_{\widehat{\mathbb{G}}} |\widehat{\phi}_-|^2 \, d\widehat{\gamma}_\omega = \int_{\mathbb{G}} \phi * \widetilde{\phi}\, d\gamma_\omega. \tag{5}$$

We call $\widehat{\gamma}_\omega$ the *diffraction measure* of $\omega \in \mathcal{M}^\infty(\mathbb{G})$ (with respect to the sequence $(\mathcal{A}_n)_{n\in\mathbb{N}}$). Within a given dynamical system of translation-bounded measures $(X, \mathbb{G}, m)$ different measures will have, in principle, different diffraction measures. However, when m is ergodic there is a unique typical resulting diffraction, that is, a measure $\widehat{\gamma}$ on $\widehat{\mathbb{G}}$ such that $\widehat{\gamma} = \widehat{\gamma}_\omega$ for m-almost every $\omega \in X$ and along any tempered van Hove sequence (see [**13**, Theorem 5]). In fact, it is possible to get rid of the ergodicity assumption and provide a certain definition for the diffraction measure of a measured dynamical system of translation-bounded measures $(X, \mathbb{G}, m)$, in a way that provides the m-almost sure diffraction in the case of ergodicity.

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THEOREM 3.6. [**13**] *Let $(X, \mathbb{G}, m)$ be a measured dynamical system of translation-bounded measures. Then there exists a unique positive translation-bounded measure $\widehat{\gamma}$ on $\widehat{\mathbb{G}}$ satisfying the equality for each $\phi, \psi \in \mathcal{C}_c(\mathbb{G})$,*

$$\int_{\widehat{\mathbb{G}}} \widehat{\phi}\overline{\widehat{\psi}}\, d\widehat{\gamma} = \int_X \mathcal{N}_\phi \overline{\mathcal{N}_\psi}\, dm. \tag{6}$$

When $(X, \mathbb{G}, m)$ is ergodic the measure $\widehat{\gamma}$ is its m-almost sure diffraction measure.

The proof is given by combining [**13**, Theorem 5(b) and Lemma 7] with the formula defining the diffraction (5). The formula (6) defining the diffraction of a measured dynamical system of translation-bounded measures extends to the following formula, as shown in [**13**, Proposition 7], see also [**54**, equation 3.9], for any Borel subset $\mathcal{P} \subseteq \widehat{\mathbb{G}}$,

$$\int_{\mathcal{P}} \widehat{\phi}\overline{\widehat{\psi}}\, d\widehat{\gamma} = \int_X \mathsf{E}(\mathcal{P})(\mathcal{N}_\phi)\overline{\mathsf{E}(\mathcal{P})(\mathcal{N}_\psi)}\, dm \tag{7}$$

where $\mathsf{E}(\mathcal{P})$ stands for the spectral projector associated with $\mathcal{P}$ on $L^2(X, m)$. The theorem above provides an efficient tool to deal with the diffraction measure: as it was suggested in [**13**, **23**, **54**] and later on explicitly formulated in [**22**], see also [**15**, **38**], the equality (6) is nothing but an isometric embedding

$$\Theta : L^2(\widehat{\mathbb{G}}, \widehat{\gamma}) \lhook\joinrel\longrightarrow L^2(X, m) \tag{8}$$

where, on the dense subspace of $L^2(\widehat{\mathbb{G}}, \widehat{\gamma})$ formed of the Fourier transforms $\widehat{\phi}$ of compactly supported functions $\phi \in \mathcal{C}_c(\mathbb{G})$, the mapping Θ writes $\Theta(\widehat{\phi}) := \mathcal{N}_\phi$. This is the so-called *diffraction-to-dynamic map* of $(X, \mathbb{G}, m)$, called after [**38**]. Its range $\mathcal{H}_\Theta$ is the closed subspace of $L^2(X, m)$ spanned by functions of the form $\mathcal{N}_\phi$ with $\phi \in \mathcal{C}_c(\mathbb{G})$, which is in general not all $L^2(X, m)$. This subspace is $\mathbb{G}$-stable and thus its orthogonal projection, which we denote P_Θ, commutes with the $\mathbb{G}$-representation, and consequently also with any spectral projector. Now using the mapping Θ one can check that formula (7) admits the following equivalent form: denoting for $f \in L^\infty(\widehat{\mathbb{G}}, \widehat{\gamma})$ the associated multiplication operator on $L^2(\widehat{\mathbb{G}}, \widehat{\gamma})$ by M_f, one has for any Borel subset $\mathcal{P} \subseteq \widehat{\mathbb{G}}$,

$$\Theta \circ M_{\mathbb{1}_\mathcal{P}} \circ \Theta^{-1} = \mathsf{E}(\mathcal{P})P_\Theta. \tag{9}$$

An alternative way to see this is to note that $M_{\mathbb{1}_\mathcal{P}}$ and $\mathsf{E}(\mathcal{P})P_\Theta$ are both spectral projectors for appropriate $\mathbb{G}$-representations on $L^2(\widehat{\mathbb{G}}, \widehat{\gamma})$ and $\mathcal{H}_\Theta$ (see [**38**, §3] for more about these representations). These $\mathbb{G}$-representations are intertwined by the diffraction-to-dynamic map, hence naturally yielding the intertwining formula (9) for the associated spectral projectors. The fundamental connection between the dynamical and diffraction spectra of a measured dynamical system of translation-bounded measures can then be stated as follows, where point (i) below directly follows from formula (9), whereas point (ii) is shown in [**13**, Theorems 7 and 9].

THEOREM 3.7. [**13**] *Let $(X, \mathbb{G}, m)$ be a measured dynamical system of translation-bounded measures, with diffraction measure $\widehat{\gamma}$ and projection-valued measure E. Then, the following can be said.*

(i) *$\widehat{\gamma}$ is absolutely continuous with respect to E. In particular, the set $\mathcal{S}$ of Bragg peaks, the atoms of $\widehat{\gamma}$, belongs to the eigenvalue group $\mathcal{E}$, the atoms of E.*

(ii) *$\widehat{\gamma}$ is a pure point measure if and only if E is a pure point measure. In this case, the set of Bragg peaks $\mathcal{S}$ algebraically generates the eigenvalue group $\mathcal{E}$.*

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4. *The decomposition of translation-bounded measures*

4.1. *Existence of the decomposition.* In this section we give the proof of our main result.

THEOREM 4.1. *Let $(X, \mathbb{G}, m)$ be an ergodic system of translation-bounded measures with diffraction $\widehat{\gamma} = \widehat{\gamma}_{\mathrm{p}} + \widehat{\gamma}_{\mathrm{c}}$. There exist two ergodic systems of translation-bounded measures:*

- *$(X_{\mathrm{p}}, \mathbb{G}, m_{\mathrm{p}})$ having diffraction $\widehat{\gamma}_{\mathrm{p}}$;*
- *$(X_{\mathrm{c}}, \mathbb{G}, m_{\mathrm{c}})$ having diffraction $\widehat{\gamma}_{\mathrm{c}}$.*

Both of which are Borel factors of $(X, \mathbb{G}, m)$ under two Borel factor maps

$$X \ni \omega \longmapsto \omega_{\mathrm{p}} \in X_{\mathrm{p}}, \qquad X \ni \omega \longmapsto \omega_{\mathrm{c}} \in X_{\mathrm{c}},$$

such that the equality $\omega = \omega_{\mathrm{p}} + \omega_{\mathrm{c}}$ occurs for m-almost every $\omega \in X$.

Proof. Let us begin with the construction of the two desired dynamical systems. First, given $\mathcal{E}$ the eigenvalue group of $(X, \mathbb{G}, m)$ one has, by Proposition 2.3, a Borel factor map from X to $\mathrm{T}_{\mathcal{E}}$. Composing this factor map with the disintegration of the measure m over $\mathrm{T}_{\mathcal{E}}$, as given in Theorem 2.2, gives us a Borel $\mathbb{G}$-map

$$X \ni \omega \longmapsto \mu_\omega^{\mathrm{p}} := m_{\pi(\omega)} \in \mathcal{M}^b(X), \tag{10}$$

taking values in the subset of probability measures on X. Now the Borel factor map from X to $\mathrm{T}_{\mathcal{E}}$ dualizes in a $\mathbb{G}$-commuting isometric embedding of $L^2(\mathrm{T}_{\mathcal{E}})$ in $L^2(X, m)$, with the range the closed Hilbert subspace $\mathcal{H}_{\mathrm{p}}$ spanned by eigenfunctions of the system $(X, \mathbb{G}, m)$. The orthogonal projector onto $\mathcal{H}_{\mathrm{p}}$ is the spectral projector $P_{\mathrm{p}} := \mathsf{E}(\mathcal{E})$ associated with the Borel subset $\mathcal{E} \subset \widehat{\mathbb{G}}$, and for each $f \in \mathcal{C}(X)$, the L^2-class of $\omega \longmapsto \int_X f \, d\mu_\omega^{\mathrm{p}}$ is by Proposition 2.4 equal to $P_{\mathrm{p}}(f)$.

On the other hand, one has another Borel map, which is straightforwardly shown to be a $\mathbb{G}$-map,

$$X \ni \omega \longmapsto \mu_\omega^{\mathrm{c}} := \delta_\omega - m_{\pi(\omega)} \in \mathcal{M}^b(X) \tag{11}$$

with the property that for each $f \in \mathcal{C}(X)$ the L^2-class of $\omega \longmapsto \int_X f \, d\mu_\omega^{\mathrm{c}}$ is given by $f - \mathsf{E}(\mathcal{E})(f) = \mathsf{E}(\widehat{\mathbb{G}} \backslash \mathcal{E})(f) =: P_{\mathrm{c}}(f)$, the orthogonal projection of f onto the subspace $\mathcal{H}_{\mathrm{c}}$ orthogonal to $\mathcal{H}_{\mathrm{p}}$. It is obvious that both mappings (10) and (11) are valued in the compact $\mathbb{G}$-stable subset $\mathcal{M}_2^b(X)$ of $\mathcal{M}^b(X)$ of signed Radon measures of total variation less or equal to 2. Denote by $CV_2(X)$ the image under the intensity map of Proposition 3.5 of the compact $\mathbb{G}$-stable subset $\mathcal{M}_2^b(X)$: it is then a compact $\mathbb{G}$-stable subset of $\mathcal{M}^\infty(\mathbb{G})$. By composing the previous Borel $\mathbb{G}$-maps with the intensity map one gets two Borel $\mathbb{G}$-maps

$$\pi_{\mathrm{p}} : X \ni \omega \longrightarrow \pi_{\mathrm{p}}(\omega) := I(\mu_\omega^{\mathrm{p}}) \in CV_2(X),$$

$$\pi_{\mathrm{c}} : X \ni \omega \longrightarrow \pi_{\mathrm{c}}(\omega) := I(\mu_\omega^{\mathrm{c}}) \in CV_2(X).$$

Push-forwarding the ergodic measure m by these Borel $\mathbb{G}$-maps yield two ergodic probability measures $m_{\mathrm{p}} = \pi_{\mathrm{p}}(m)$ and $m_{\mathrm{c}} = \pi_{\mathrm{c}}(m)$ supported on $CV_2(X)$, whose support are two compact $\mathbb{G}$-stable subsets of $CV_2(X)$ which we denote X_{p} and X_{c}, respectively. The subset of $\omega \in X$ having images $\pi_{\mathrm{p}}(\omega)$ and $\pi_{\mathrm{c}}(\omega)$ in X_{p} and X_{c}, respectively, is a full Borel subset, so after possibly modifying the maps π_{p} and π_{c} on a set of measure 0 of X,

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we can assume that these are valued in X_{p} and X_{c}, respectively. We therefore have two ergodic systems of translation-bounded measures $(X_{\mathrm{p}}, \mathbb{G}, m_{\mathrm{p}})$ and $(X_{\mathrm{c}}, \mathbb{G}, m_{\mathrm{c}})$, as well as two Borel factor maps

$$X \xrightarrow{\pi_{\mathrm{p}}} X_{\mathrm{p}}, \qquad X \xrightarrow{\pi_{\mathrm{c}}} X_{\mathrm{c}}.$$

Our aim in the following is to show that these two Borel factors satisfy the desired properties. Let us first remark that the two summands $\widehat{\gamma}_{\mathrm{p}}$ and $\widehat{\gamma}_{\mathrm{c}}$, respectively, of the diffraction $\widehat{\gamma}$ are simply the restriction of $\widehat{\gamma}$ on the Borel subsets $\mathcal{E}$ and $\widehat{\mathbb{G}}\backslash\mathcal{E}$, respectively. As a result, with the notation $P_{\mathrm{p}} := \mathsf{E}(\mathcal{E})$ and $P_{\mathrm{c}} := \mathsf{E}(\widehat{\mathbb{G}}\backslash\mathcal{E})$ we obtain, by applying equation (7),

$$\int_{\widehat{\mathbb{G}}} \widehat{\phi}\overline{\widehat{\psi}}\, d\widehat{\gamma}_{\mathrm{p}} = \int_X P_{\mathrm{p}}(\mathcal{N}_\phi)\overline{P_{\mathrm{p}}(\mathcal{N}_\psi)}\, dm \qquad \int_{\widehat{\mathbb{G}}} \widehat{\phi}\overline{\widehat{\psi}}\, d\widehat{\gamma}_{\mathrm{c}} = \int_X P_{\mathrm{c}}(\mathcal{N}_\phi)\overline{P_{\mathrm{c}}(\mathcal{N}_\psi)}\, dm. \quad (12)$$

Let us then show that our two systems have diffraction $\widehat{\gamma}'_{\mathrm{p}}$ and $\widehat{\gamma}'_{\mathrm{c}}$ equal to $\widehat{\gamma}_{\mathrm{p}}$ and $\widehat{\gamma}_{\mathrm{c}}$, respectively. For each $\phi \in \mathcal{C}_c(\mathbb{G})$ and for either $\alpha = \mathrm{p}$ or c, consider the restriction $\mathcal{N}_\phi^\alpha$ of the function $\mathcal{N}_\phi$ to the space X_α (functions of the form $\mathcal{N}_\phi$ are introduced in equation (2)). We then observe that $\mathcal{N}_\phi^\alpha \circ \pi_\alpha$ has L^2-class equal to $P_\alpha(\mathcal{N}_\phi)$ in $L^2(X, m)$: indeed this follows from the series of almost everywhere equalities

$$\begin{aligned}\mathcal{N}_\phi^\alpha \circ \pi_\alpha(\omega) = \int_{\mathbb{G}} \phi_-\, d\pi_\alpha(\omega) &= \int_{\mathbb{G}} \phi_-\, dI(\mu_\omega^\alpha) \\ &= \int_X \left[\int_{\mathbb{G}} \phi_-\, d\omega'\right] d\mu_\omega^\alpha(\omega') = \int_X \mathcal{N}_\phi\, d\mu_\omega^\alpha\end{aligned}$$

whose L^2-class is, from what has been said earlier in this proof, equal to $P_\alpha(\mathcal{N}_\phi)$ in $L^2(X, m)$. One therefore has

$$\begin{aligned}\int \widehat{\phi}\overline{\widehat{\psi}}\, d\widehat{\gamma}'_\alpha = \int_{X_\alpha} \mathcal{N}_\phi^\alpha \overline{\mathcal{N}_\psi^\alpha}\, dm_\alpha &= \int_X \mathcal{N}_\phi^\alpha \circ \pi_\alpha \overline{\mathcal{N}_\psi^\alpha \circ \pi_\alpha}\, dm \\ &= \int_X P_\alpha(\mathcal{N}_\phi)\overline{P_\alpha(\mathcal{N}_\psi)}\, dm = \int \widehat{\phi}\overline{\widehat{\psi}}\, d\widehat{\gamma}_\alpha\end{aligned}$$

where, from left to right, the first equality follows from Theorem 3.6, the second follows from the equality $m_\alpha = \pi_\alpha(m)$, the third is from $\mathcal{N}_\phi^\alpha \circ \pi_\alpha = P_\alpha(\mathcal{N}_\phi)$ as shown earlier, and the fourth equality is from equation (12). This shows that integration with respect to $\widehat{\gamma}'_\alpha$ and $\widehat{\gamma}_\alpha$ agree on the collection of functions

$$\mathcal{F} := \{\widehat{\phi}\overline{\widehat{\psi}} : \phi, \psi \in \mathcal{C}_c(\mathbb{G})\}.$$

However, since $\mathcal{C}_c(\mathbb{G})$ is a convolution algebra that is stable under the operation $\phi \mapsto \widetilde{\phi}$, one easily checks that $\mathcal{F}$ is a subalgebra of $\mathcal{C}_0(\widehat{\mathbb{G}})$ that is invariant under complex conjugation. This algebra can be shown to separate elements in $\widehat{\mathbb{G}}$, and from the Stone–Weierstrass theorem $\mathcal{F}$ is consequently dense in $\mathcal{C}_0(\widehat{\mathbb{G}})$ in the uniform norm. In particular, any function of $\mathcal{C}_c(\widehat{\mathbb{G}})$ may be approximated uniformly by functions of $\mathcal{F}$, and consequently integration with respect to $\widehat{\gamma}'_\alpha$ and $\widehat{\gamma}_\alpha$ must agree on $\mathcal{C}_c(\widehat{\mathbb{G}})$. It follows from the Riesz representation theorem that $\widehat{\gamma}'_\alpha = \widehat{\gamma}_\alpha$ for $\alpha = \mathrm{p}$ or c, as desired.

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It then remains to show that $\omega = \pi_{\mathrm{p}}(\omega) + \pi_{\mathrm{c}}(\omega)$ for almost every $\omega \in X$, but this comes from the straightforward computation

$$\pi_{\mathrm{c}}(\omega) = I(\delta_\omega - m_{\pi(\omega)}) = I(\delta_\omega) - I(m_{\pi(\omega)}) = \omega - \pi_{\mathrm{p}}(\omega),$$

which concludes the proof. □

Let us make a few remarks on this result. Here we consider an ergodic system of translation-bounded measures $(X, \mathbb{G}, m)$ with eigenvalue group $\mathcal{E}$ and diffraction $\widehat{\gamma}$.

Remark 4.2. The first associated system $(X_{\mathrm{p}}, \mathbb{G}, m_{\mathrm{p}})$ has by construction pure point diffraction, and thus by Theorem 3.7 a pure point dynamical spectrum, with eigenvalue group $\mathcal{E}_{\mathrm{p}}$, a subgroup of $\mathcal{E}$. The inclusion of $\mathcal{E}_{\mathrm{p}}$ into $\mathcal{E}$ may however be strict: indeed the set of Bragg peaks of $(X_{\mathrm{p}}, \mathbb{G}, m_{\mathrm{p}})$ is precisely the set $\mathcal{S}$ of atoms of $\widehat{\gamma}$, and therefore the eigenvalue group $\mathcal{E}_{\mathrm{p}}$ is the subgroup of $\mathcal{E}$ algebraically generated by $\mathcal{S}$, which might not be the entire group $\mathcal{E}$. On the other hand, the second associated system $(X_{\mathrm{c}}, \mathbb{G}, m_{\mathrm{c}})$ has by construction no Bragg peak (not even at 0) but still can have a highly non-trivial eigenvalue group.

Remark 4.3. The autocorrelation γ of a generic element in X is positive-definite, and hence a weakly almost periodic measure on $\mathbb{G}$ [**21**]. As such, it admits a unique decomposition into the sum of a *strongly almost periodic* measure γ_{sap} and a *null-weakly almost periodic* one $\gamma_{0\mathrm{wap}}$, which are both Fourier transformable, and such that $\widehat{\gamma_{\mathrm{sap}}} = \widehat{\gamma}_{\mathrm{p}}$ and $\widehat{\gamma_{0\mathrm{wap}}} = \widehat{\gamma}_{\mathrm{c}}$, see [**21**] for definitions and proofs. In our situation, the two systems $(X_{\mathrm{p}}, \mathbb{G}, m_{\mathrm{p}})$ and $(X_{\mathrm{c}}, \mathbb{G}, m_{\mathrm{c}})$ have diffraction $\widehat{\gamma}_{\mathrm{p}}$ and $\widehat{\gamma}_{\mathrm{c}}$ respectively, so it straightly follows that the autocorrelation of a generic element of $(X_{\mathrm{p}}, \mathbb{G}, m_{\mathrm{p}})$ is exactly the strong almost periodic part γ_{sap} of γ while a generic element of $(X_{\mathrm{c}}, \mathbb{G}, m_{\mathrm{c}})$ has autocorrelation of the null weakly almost periodic part $\gamma_{0\mathrm{wap}}$ of γ.

Another observation, that is presented this time in details, is the following.

PROPOSITION 4.4. *If $(X, \mathbb{G}, m)$ is a non-trivial ergodic system whose translation-bounded measures are non-negative, then its factor $(X_{\mathrm{p}}, \mathbb{G}, m_{\mathrm{p}})$ is a non-trivial ergodic system whose translation-bounded measures are also non-negative.*

Proof. From the construction of the factor $(X_{\mathrm{p}}, \mathbb{G}, m_{\mathrm{p}})$ given in the proof of Theorem 4.1, a generic element of $(X_{\mathrm{p}}, \mathbb{G}, m_{\mathrm{p}})$ is given by the intensity $I(\mu_\omega^{\mathrm{p}})$ of a certain probability measure μ_ω^{p} on X, with the whole collection of these probabilities satisfying for a Borel bounded function f on X

$$\int_X \left[\int_X f \, d\mu_\omega^{\mathrm{p}} \right] dm(\omega) = \int_X f \, dm. \tag{13}$$

The space X_{p} therefore consists of non-negative translation-bounded measures. Let us now suppose for a contradiction that $(X_{\mathrm{p}}, \mathbb{G}, m_{\mathrm{p}})$ is trivial, that is, m_{p} is concentrated on the single trivial measure 0. Then a generic element $I(\mu_\omega^{\mathrm{p}})$ is trivial, and therefore

$$\int_X \mathcal{N}_\phi \, d\mu_\omega^{\mathrm{p}} = \int_{\mathbb{G}} \phi_- \, dI(\mu_\omega^{\mathrm{p}}) = 0$$

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for any $\phi \in \mathcal{C}_c(\mathbb{G})$. This shows that, for any compact subset $K \subset \mathbb{G}$, the collection of $\omega' \in X$ such that $\omega'(K) > 0$ has μ_ω^{p}-measure 0. Since μ_ω^{p} is carried by translation-bounded measures that are non-negative, it implies that μ_ω^{p} is supported over the trivial measure 0. We finally obtain from equation (13) that m itself is concentrated on the singleton $\{0\}$, yielding that $(X, \mathbb{G}, m)$ is trivial, a contradiction. □

In the case where translation-bounded measures of X are only signed, then the translation-bounded measures of both X_{p} and X_{c} are signed as well, but non-triviality of $(X_{\mathrm{p}}, \mathbb{G}, m_{\mathrm{p}})$ can no longer be guaranteed. This may be illustrated with the example of the Thue–Morse system.

Remark 4.5. The *Thue–Morse system* $(X_{\mathrm{TM}}, \mathbb{R}, m_{\mathrm{TM}})$ with labels in ± 1 is an example of an ergodic system of signed measures over $\mathbb{G} = \mathbb{R}$: here X_{TM} is the rubber local isomorphism class of a (bi-infinite) Thus–Morse sequence, which is a Dirac comb supported on the integers $\mathbb{Z}$, and with weights chosen here to belong to $\{\pm 1\}$. This system is ergodic, and consists of translation-bounded measures that are not non-negative. The factor map from this system onto its maximal Kronecker–Borel factor is m_{TM}-almost everywhere two-to-one (the system has *coincidence rank* equal to 2, see [**3**, §4] for more on this notion). The two-element fibers consist of a Dirac comb ω and its mirror comb $-\omega$, given by switching any weight ± 1 of ω by its opposed weight. The disintegration of m_{TM} over the maximal Kronecker–Borel factor yields a probability measure on each of these fibers, each time allocating half of the total mass to each of the two elements. As a result, the resulting intensity measures are m_{TM}-almost surely trivial on $\mathbb{G}$. This shows that the factor $(X_{\mathrm{p}}, \mathbb{G}, m_{\mathrm{p}})$ of the Thue–Morse system $(X_{\mathrm{TM}}, \mathbb{R}, m_{\mathrm{TM}})$ with weights in $\{\pm 1\}$ is actually trivial. An alternative way to get this is by invoking the fact that the so-weighted Thus–Morse system has (singular) continuous diffraction (see [**9**, §10] and references therein), that is, the summand $\widehat{\gamma}_{\mathrm{p}}$ of the diffraction of $(X_{\mathrm{TM}}, \mathbb{R}, m_{\mathrm{TM}})$ is trivial. But the diffraction of an ergodic system cannot be trivial unless the system itself is trivial, and thus the factor $(X_{\mathrm{p}}, \mathbb{G}, m_{\mathrm{p}})$ must be trivial. We are grateful to one of the referees for suggesting this example.

In Theorem 4.1 above, the uniqueness part of Theorem 1 in the introduction is absent. We shall prove this in §4.3 below. We first establish in §4.2 a general result, which will permit to prove point (ii) of Theorem 1 by only invoking standard arguments from operator theory.

4.2. *From Borel factors to operators.* Let $(X, \mathbb{G}, m)$ be a measured dynamical system of translation-bounded measures, with diffraction to dynamic map

$$\Theta : L^2(\widehat{\mathbb{G}}, \widehat{\gamma}) \longleftarrow \mathcal{H}_\Theta \subseteq L^2(X, m), \qquad \Theta(\widehat{\phi}) = \mathcal{N}_\phi^X$$

for any $\phi \in \mathcal{C}_c(\mathbb{G})$. Recall that P_Θ stands for the orthogonal projection onto $\mathcal{H}_\Theta$. Then assume we are given a dynamical system of translation-bounded measures $(X_\pi, \mathbb{G}, m_\pi)$, which is a Borel factor of $(X, \mathbb{G}, m)$ under a Borel factor map $\pi : X \longrightarrow X_\pi$. Suppose in addition that its diffraction measure $\widehat{\gamma}_\pi$ is absolutely continuous with respect to $\widehat{\gamma}$, with an essentially bounded Radon–Nikodym differential $f_\pi := d\widehat{\gamma}_\pi / d\widehat{\gamma} \in L^\infty(\widehat{\mathbb{G}}, \widehat{\gamma})$. Let us denote by $\mathcal{N}_\phi^\pi$ the restriction to X_π of the function $\mathcal{N}_\phi$ as set in equation (2).

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We shall here construct a bounded linear operator Q_π on $L^2(X, m)$, which characterizes the factor map π, and hence the factor system $(X_\pi, \mathbb{G}, m_\pi)$ itself, in a unique way.

PROPOSITION 4.6. *Let $\pi : X \longrightarrow X_\pi$ with $(X_\pi, \mathbb{G}, m_\pi)$ as above.*

(i) *There exists a unique $Q_\pi \in \mathcal{B}(L^2(X, m))$ such that $\mathcal{N}_\phi^\pi \circ \pi = Q_\pi(\mathcal{N}_\phi)$ in the L^2 sense for any $\phi \in \mathcal{C}_c(\mathbb{G})$ and subject to the restriction that $Q_\pi = Q_\pi P_\Theta$.*

(ii) *For π, π' as above then $Q_\pi = Q_{\pi'}$ if and only if $\pi = \pi'$ almost everywhere on X.*

(iii) *The following equality holds, with M_{f_π} the multiplication operator by f_π on $L^2(\widehat{\mathbb{G}}, \widehat{\gamma})$,*

$$\Theta \circ M_{f_\pi} \circ \Theta^{-1} = Q_\pi^* Q_\pi .$$

Proof. (i) First the factor map π induces a $\mathbb{G}$-commuting isometry

$$i_\pi : L^2(X_\pi, m_\pi) \ni f \longmapsto f \circ \pi \in L^2(X, m).$$

Denote the diffraction-to-dynamic map of $(X_\pi, \mathbb{G}, m_\pi)$ by

$$\Theta_\pi : L^2(\widehat{\mathbb{G}}, \widehat{\gamma}_\pi) \longleftrightarrow \mathcal{H}_{\Theta_\pi} \subseteq L^2(X_\pi, m_\pi).$$

Now $\widehat{\gamma}_\pi = f_\pi \widehat{\gamma}$ for some $f_\pi \in L^\infty(\widehat{\mathbb{G}}, \widehat{\gamma})$ and therefore $\|f\|_{L^2(\widehat{\mathbb{G}}, \widehat{\gamma}_\pi)} \leqslant \|f_\pi\|_{\infty,\mathrm{ess}}^{1/2} \|f\|_{L^2(\widehat{\mathbb{G}}, \widehat{\gamma})}$ for any $f \in L^2(\widehat{\mathbb{G}}, \widehat{\gamma})$. This shows that the identity map from $L^2(\widehat{\mathbb{G}}, \widehat{\gamma})$ to $L^2(\widehat{\mathbb{G}}, \widehat{\gamma}_\pi)$ is well-defined and a bounded linear map, and thus there exists a unique bounded operator $\widetilde{Q}_\pi$ making the diagram commutative

$$\begin{array}{ccc} L^2(\widehat{\mathbb{G}}, \widehat{\gamma}) & \xleftarrow{\ \Theta\ } & \mathcal{H}_\Theta \\ \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle \widetilde{Q}_\pi} \\ L^2(\widehat{\mathbb{G}}, \widehat{\gamma}_\pi) & \xleftarrow{\ \Theta_\pi\ } & \mathcal{H}_{\Theta_\pi} \end{array}$$

Now the operator $Q_\pi := i_\pi \circ \widetilde{Q}_\pi \circ P_\Theta$ is a well-defined linear operator of $L^2(X, m)$, and bounded with operator norm $\|Q_\pi\|_{\mathrm{op}} \leqslant \|f_\pi\|_{\infty,\mathrm{ess}}^{1/2}$, as it is not difficult to check. By construction one has $Q_\pi = Q_\pi P_\Theta$, and, for any $\phi \in \mathcal{C}_c(\mathbb{G})$, the almost everywhere equalities

$$Q_\pi(\mathcal{N}_\phi) = i_\pi \circ \widetilde{Q}_\pi \circ \Theta(\widehat{\phi}) = i_\pi(\Theta_\pi(\widehat{\phi})) = i_\pi((\mathcal{N}_\phi^\pi)) = \mathcal{N}_\phi^\pi \circ \pi. \tag{14}$$

Uniqueness of the operator Q_π having these properties is clear, as it is uniquely defined on the subspace $\mathcal{H}_\Theta$ and must vanish on its orthogonal space, giving (i).

(ii) The given equivalence statement easily follows, by observing that $\pi = \pi'$ almost everywhere on X if and only if, for any $\phi \in \mathcal{C}_c(\mathbb{G})$, one has for almost all $\omega \in X$,

$$\int \phi_-\, d\pi(\omega) = \int \phi_-\, d\pi'(\omega).$$

This may be written as $\mathcal{N}_\phi^{\pi'} \circ \pi' = \mathcal{N}_\phi^{\pi'} \circ \pi'$ for any $\phi \in \mathcal{C}_c(\mathbb{G})$, which, according to equalities (14), holds if and only if $Q_\pi(\mathcal{N}_\phi) = Q_{\pi'}(\mathcal{N}_\phi)$ for any $\phi \in \mathcal{C}_c(\mathbb{G})$. This latter condition holds if and only if Q_π and $Q_{\pi'}$ coincide on the closed subspace $\mathcal{H}_\Theta$, and since

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by construction one has always $Q_\pi = Q_\pi P_\Theta$, with P_Θ the orthogonal projection onto $\mathcal{H}_\Theta$, this is equivalent to have $Q_\pi = Q_{\pi'}$ on all $L^2(X, m)$, as desired.

(iii) For the last point it suffices to check the equality in scalar products against functions $\mathcal{N}_\phi, \mathcal{N}_\psi$ for $\phi, \psi \in \mathcal{C}_c(\mathbb{G})$: one has

$$\begin{aligned}\langle \Theta \circ M_{f_\pi} \circ \Theta^{-1}(\mathcal{N}_\phi), \mathcal{N}_\psi \rangle_m = \langle M_{f_\pi} \widehat{\phi}, \widehat{\psi} \rangle_{\widehat{\gamma}} &= \int f_\pi \widehat{\phi}\overline{\widehat{\psi}}\, d\widehat{\gamma} \\ &= \int \widehat{\phi}\overline{\widehat{\psi}}\, d\widehat{\gamma}_\pi = \langle \mathcal{N}_\phi^\pi, \mathcal{N}_\psi^\pi \rangle_{m_\pi},\end{aligned}$$

which is equal, using point (i), to

$$\langle \mathcal{N}_\phi^\pi \circ \pi, \mathcal{N}_\psi^\pi \circ \pi \rangle_m = \langle Q_\pi(\mathcal{N}_\phi),\, Q_\pi(\mathcal{N}_\psi) \rangle_m = \langle Q_\pi^* Q_\pi(\mathcal{N}_\phi), \mathcal{N}_\psi \rangle_m,$$

which settles the proof. □

4.3. *Uniqueness of decomposition of measures.* Using the result of the earlier paragraph we shall now prove uniqueness of the decomposition set in Theorem 4.1.

THEOREM 4.7. *Let $(X, \mathbb{G}, m)$ be an ergodic system of translation-bounded measures. Then the two ergodic systems provided in Theorem 4.1 exist in a unique way, as well as the two Borel factor maps up to almost everywhere equality.*

Proof. Recall the countable subset $\mathcal{S} \subset \widehat{\mathbb{G}}$ of atoms of $\widehat{\gamma}$, with complementary set $\widehat{\mathbb{G}}\backslash\mathcal{S}$. Suppose we are given two Borel factors $(X'_{\mathrm{p}}, \mathbb{G}, m'_{\mathrm{p}})$ and $(X'_{\mathrm{c}}, \mathbb{G}, m'_{\mathrm{c}})$ of $(X, \mathbb{G}, m)$, under two Borel factor maps π'_{p} and π'_{c}, with respective diffraction $\widehat{\gamma}_{\mathrm{p}} = \mathbb{I}_{\mathcal{S}}\widehat{\gamma}$ and $\widehat{\gamma}_{\mathrm{c}} = \mathbb{I}_{\widehat{\mathbb{G}}\backslash\mathcal{S}}\widehat{\gamma}$ and such that $\omega = \pi'_{\mathrm{p}}(\omega) + \pi'_{\mathrm{c}}(\omega)$ hold for almost every $\omega \in X$, as in Theorem 4.1. By Proposition 4.6(i) the factor maps π'_{p} and π'_{c} give rise to a pair of operators Q and Q' on $L^2(X, m)$. Then it will be sufficient to show that

$$Q = \mathsf{E}(\mathcal{S})P_\Theta, \quad Q' = \mathsf{E}(\widehat{\mathbb{G}}\backslash\mathcal{S})P_\Theta. \tag{15}$$

Indeed by the equivalence set in Proposition 4.6(ii), this will readily ensure uniqueness of the pair of Borel factors, and that of Borel factor maps up to almost everywhere equality. To show equalities (15), first observe that $Q + Q' = P_\Theta$: indeed this follows from the following equalities, holding for any $\phi \in \mathcal{C}_c(\mathbb{G})$ and almost every $\omega \in X$,

$$\begin{aligned}Q(\mathcal{N}_\phi)(\omega) + Q'(\mathcal{N}_\phi)(\omega) &= \mathcal{N}_\phi^{\pi'_{\mathrm{p}}}(\pi'_{\mathrm{p}}(\omega)) + \mathcal{N}_\phi^{\pi'_{\mathrm{c}}}(\pi'_{\mathrm{c}}(\omega)) \\ &= \int_{\mathbb{G}} \phi_-\, d(\pi'_{\mathrm{p}}(\omega) + \pi'_{\mathrm{c}}(\omega)) = \mathcal{N}_\phi(\omega),\end{aligned}$$

the last equality being given by the almost everywhere equality $\omega = \pi'_{\mathrm{p}}(\omega) + \pi'_{\mathrm{c}}(\omega)$. Second, observe that the assumption on diffraction implies, by Proposition 4.6(iii), that Q^*Q and Q'^*Q' are equal to the operators $\Theta \circ M_{\mathbb{I}_{\mathcal{S}}} \circ \Theta^{-1}$ and $\Theta \circ M_{\mathbb{I}_{\widehat{\mathbb{G}}\backslash\mathcal{S}}} \circ \Theta^{-1}$, respectively. From formula (9), one deduces that the absolute values of Q and Q' (see [**49**] for definition) satisfy $|Q| = |Q|^2 = Q^*Q = \mathsf{E}(\mathcal{S})P_\Theta$, whereas $|Q'| = |Q'|^2 = Q'^*Q' = \mathsf{E}(\widehat{\mathbb{G}}\backslash\mathcal{S})P_\Theta$. On the other hand, the bounded operators Q and Q' admit a polar decomposition (see, for instance, [**49**, Theorem VI.10]) of the form $Q = V|Q|$ and $Q' = V'|Q'|$, where V and V' are partial isometries with initial spaces of the ranges of $|Q|$

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and $|Q'|$ respectively, that is to say, the ranges of the projectors $\mathsf{E}(\mathcal{S})P_\Theta$ and $\mathsf{E}(\widehat{\mathbb{G}}\backslash\mathcal{S})P_\Theta$. Then, to show equalities (15) it only remains to show that these partial isometries are the identity on their respective initial space. Since $Q + Q' = P_\Theta$ one has for any h in the initial space of V that, minding that $\mathsf{E}(\mathcal{S})P_\Theta$ and $\mathsf{E}(\widehat{\mathbb{G}}\backslash\mathcal{S})P_\Theta$ have orthogonal ranges,

$$V(h) = V\mathsf{E}(\mathcal{S})P_\Theta(h) = V\mathsf{E}(\mathcal{S})P_\Theta(h) + V'\mathsf{E}(\widehat{\mathbb{G}}\backslash\mathcal{S})P_\Theta(h) = (Q + Q')(h) = h,$$

whereas for any h' in the initial space of V', one has that

$$V'(h') = V'\mathsf{E}(\widehat{\mathbb{G}}\backslash\mathcal{S})P_\Theta(h') = V\mathsf{E}(\mathcal{S})P_\Theta(h') + V'\mathsf{E}(\widehat{\mathbb{G}}\backslash\mathcal{S})P_\Theta(h') = (Q + Q')(h') = h',$$

as desired. □

Let us remark that, given a dynamical system of translation-bounded measures $(X, \mathbb{G}, m)$ with diffraction $\widehat{\gamma}$, if one drops the requirement of having a decomposition of (almost every) measures in X then one may exhibit several different Borel factors having diffraction $\widehat{\gamma_{\mathrm{p}}}$ and $\widehat{\gamma_{\mathrm{c}}}$, respectively. Indeed given a pair of such Borel factors, it suffices to take for each of them another system of translation-bounded measures which is Borel conjugated and *homometric*, that is, with the same diffraction, which even in the purely diffractive situation (in this case any two homometric systems are Borel conjugated) seems very likely to exist in general, see [**30**, **59**].

4.4. *Support of translation-bounded measures in* $(X_{\mathrm{p}}, \mathbb{G}, m_{\mathrm{p}})$. In this paragraph we shall exclusively deal with translation-bounded measures that are non-negative. Here, given a non-trivial ergodic system $(X, \mathbb{G}, m)$ of non-negative translation-bounded measures, we are interested by the support $S(\omega)$ of the translation-bounded measures ω of the system $(X_{\mathrm{p}}, \mathbb{G}, m_{\mathrm{p}})$. At first, one expects, as a consequence of ergodicity, that the support of measures in $(X_{\mathrm{p}}, \mathbb{G}, m_{\mathrm{p}})$ should be large. But large in which sense? For instance, do these measures have *relatively dense* support, or at least almost all of them? Recall that a subset $B \subset \mathbb{G}$ is relatively dense if there exists a compact set $K \subset \mathbb{G}$ with $B + K = \mathbb{G}$. This is not a trivial point and it seems judicious to deal with a weaker notion than relative density. Such a weaker notion is set in Proposition 4.8 just below. First select a van Hove sequence $(\mathcal{A}_k)_{k\in\mathbb{N}}$ of $\mathbb{G}$, which always exists in σ-compact LCA groups [**54**]. Then one defines the *lower density* of a subset $B \subseteq \mathbb{G}$ (along the van Hove sequence $(\mathcal{A}_k)_{k\in\mathbb{N}}$) is given by

$$\underline{\mathrm{dens}}(B) := \lim_{k\to\infty} \frac{|B \cap \mathcal{A}_k|}{|\mathcal{A}_k|}.$$

This definition of lower density may differ from the one used elsewhere in the literature, particularly for point sets, whose density always vanishes in the above sense. The lower density of a subset $B \subseteq \mathbb{G}$, as defined above, is a value between 0 and 1, and often depends on the choice of van Hove sequence made at the beginning.

PROPOSITION 4.8. *Let $(X, \mathbb{G}, m)$ be a non-trivial ergodic system of non-negative translation-bounded measures. Then almost all $\omega \in X$ have the following property: for any $\varepsilon > 0$ there is a compact set $K \subset \mathbb{G}$ such that* $\underline{\mathrm{dens}}(S(\omega) + K) > 1 - \varepsilon$.

Proof. Consider an increasing sequence of symmetric open sets $U_n \subset \mathbb{G}$ with compact closure and whose union is all $\mathbb{G}$, and set Q_n to be the subset of $\omega \in X$ such that

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$0 \in S(\omega) + U_n$. Then each Q_n is Borel in X. Indeed, given a translation-invariant compatible metric d on $\mathbb{G}$ consider the sequence of functions $\phi_n := d(., \mathbb{G}\backslash U_n) \in \mathcal{C}_c(\mathbb{G})$. Note that ϕ_n is non-negative, nowhere vanishing on U_n and entirely vanishing outside U_n, for every $n \in \mathbb{N}$. Then for each $\omega \in X$ and $s \in \mathbb{G}$ we have equivalence among $s \notin S(\omega) + U_n$, $\omega(U_n + s) = 0$ and $\int \phi_n(. - s)\, d\omega = 0$. Indeed, equivalence between the two last conditions is obvious, whereas if $\omega(U_n + s) > 0$ then $S(\omega)$ must intersect $U_n + s$, yielding $s \in S(\omega) - U_n = S(\omega) + U_n$. In the other direction, if $s \in \mathbb{G}$ satisfies $\omega(U_n + s) = 0$ then $S(\omega)$ cannot intersect the open set $U_n + s$ and thus s cannot belong to $S(\omega) + U_n$. As a result, one is allowed to write $Q_n = \mathcal{N}_{\phi_n}^{-1}(]0; +\infty[)$, which is open and thus Borel in X.

Now the pointwise ergodic theorem (see Theorem A.1 in the appendix, and [**48**] for further details), when applied to the Borel indicator function $\mathbb{I}_{Q_n}$, yields a Borel subset of full measure $X^{(n)} \subseteq X$ such that, for any $\omega \in X^{(n)}$, one has the convergence

$$\frac{1}{|\mathcal{A}_k|} \int_{\mathcal{A}_k} \mathbb{I}_{Q_n}(\omega.(-s))\, ds \xrightarrow{k \to \infty} \int_X \mathbb{I}_{Q_n}\, dm = m(Q_n).$$

However, $\mathbb{I}_{Q_n}(\omega.(-s)) \neq 0$ if and only if $0 \in S(\omega.(-s)) + U_n = S(\omega) - s + U_n$, which occurs if and only if $s \in S(\omega) + U_n$. Therefore, the left-hand side terms of the above limit are also given by

$$\frac{1}{|\mathcal{A}_k|} \int_{\mathcal{A}_k} \mathbb{I}_{Q_n}(\omega.(-s))\, ds = \frac{|(S(\omega) + U_n) \cap \mathcal{A}_k|}{|\mathcal{A}_k|}$$

so that we arrive by taking the limit to the equality, which holds for any $\omega \in X^{(n)}$,

$$\underline{\mathrm{dens}}(S(\omega) + U_n) = m(Q_n).$$

Let us show that $m(Q_n)$ converges to 1: first, as the sequence U_n is increasing in $\mathbb{G}$, the sequence Q_n is also increasing in X. Therefore, $m(Q_n)$ converges to $m(\bigcup_n Q_n)$. Now an element ω not belonging to $\bigcup_n Q_n$ must satisfy $0 \notin S(\omega) + U_n$ for each integer n, which is only possible for the everywhere trivial measure. Since $(X, \mathbb{G}, m)$ was assumed non-trivial it follows that $\bigcup_n Q_n$ has a complementary set of measure 0, so is of measure 1 in X, as desired.

As a result, for any element ω belonging in the Borel subset of full measure $X^{(\infty)}$ given by the countable intersection of the $X^{(n)}$ one has the convergence

$$\underline{\mathrm{dens}}(S(\omega) + U_n) = m(Q_n) \xrightarrow{n \to \infty} 1,$$

which gives the proof. □

This proposition, in particular, applies to the system $(X_{\mathrm{p}}, \mathbb{G}, m_{\mathrm{p}})$ associated to any non-trivial ergodic system $(X, \mathbb{G}, m)$ of non-negative translation-bounded measures. Now returning to relative density, to complete the above proposition we have the following.

PROPOSITION 4.9. *Let $(X, \mathbb{G}, m)$ be a non-trivial ergodic system of non-negative translation-bounded measures. Assume that almost any measure in X has relatively dense support. Then almost any measure in X_{p} also has relatively dense support.*

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Proof. Let $U \subset \mathbb{G}$ be a symmetric open subset with compact closure. Then the collection $X_U \subseteq X$ of translation-bounded measures having U-relatively dense support is a $\mathbb{G}$-stable Borel subset of X. Indeed, it is obviously $\mathbb{G}$-stable and moreover, from an argument of the proof of Proposition 4.8,

$$\omega \in X_U \Longleftrightarrow \omega(U + s) > 0 \quad \text{for all } s \in \mathbb{G}. \tag{16}$$

Given a translation-invariant compatible metric d on $\mathbb{G}$, letting $\phi := d(., \mathbb{G}\backslash U)$, one gets that the condition $\omega(U + s) > 0$ for any $s \in \mathbb{G}$ is equivalent to having $\int \phi(. + s)\, d\omega > 0$ for any $s \in \mathbb{G}$. Now consider a sequence $(K_k)_{k\in\mathbb{N}}$ of compact sets in $\mathbb{G}$ whose union covers $\mathbb{G}$. Since $\int \phi(. + s)\omega$ is continuous in the variable s, our condition is in turn equivalent to $\inf_{s\in K_k} \int \phi(. + s)\, d\omega \geqslant \delta_k$ for some $\delta_k > 0$, for each $k \in \mathbb{N}$. Again by continuity in the variable s its infimum over $s \in K_k$ can be set on a countable dense subset of K_k, independent of ω, showing that $F_k(\omega) := \inf_{s\in K_k} \int \phi(. + s)\, d\omega$ is the infimum of a countable collection of continuous functions on X and thus is Borel on X. One concludes that X_U is Borel by observing that

$$X_U = \bigcap_{k\in\mathbb{N}} F_k^{-1}(]0, +\infty[).$$

Then consider an increasing sequence of symmetric open subset $U_n \subset \mathbb{G}$ with compact closure and whose union covers $\mathbb{G}$. Since almost any measure in X has relatively dense support it follows that the increasing countable union of Borel sets X_{U_n} has measure 1. Therefore, by ergodicity there must exist some $U_{n_0} = V$ such that $m(X_V) = 1$. Now Theorem 2.2 ensures, when applied to the Borel indicator function $\mathbb{I}_{X_V}$, that the set X'_V of $\omega \in X$ such that X_V has $m_{\pi(\omega)}$-measure 1 is also of full measure in X. Let then X''_V be the Borel subset of full measure in X such that the equality $\omega_{\mathrm{p}} = I(m_{\pi(\omega)})$ holds. For any ω belonging to the intersection X'''_V of the sets X_V, X'_V, and X''_V one has

$$\omega_{\mathrm{p}}(V + t) = \int_X \omega'(V + t)\, dm_{\pi(\omega)}(\omega') = \int_{X_V} \omega'(V + t)\, dm_{\pi(\omega)}(\omega'),$$

which is the integral of a strictly positive function (by (16)) with respect to a probability measure, and thus must be strictly positive for any $t \in \mathbb{G}$. This shows that the image ω_{p} in X_{p} of an element $\omega \in X'''_V$ has V-relatively dense support in $\mathbb{G}$. But as X'''_V is of full measure in X, we conclude that the elements of X_{p} with V-relatively dense support form a set of full m_{p}-measures, as desired. □

5. *Decomposition of weighted Meyer sets*

In this section we focus on a particular type of translation-bounded measure. Very often, the translation-bounded measures considered in the literature are *Dirac combs*, with support being, from the most particular to the most general, a subset of a lattice [**4**, **6**], a point set with an extra geometric property such as the Meyer property, finite local complexity or simply uniform discreteness [**16**, **17**, **19**, **58**], or ultimately a possibly dense (yet countable) subset [**39**, **50**]. Here the translation-bounded measures we will consider are Dirac combs supported on a Meyer set, simply referred to as *weighted Meyer sets*. Our main result here is the claim that for an ergodic system of weighted Meyer sets $(X, \mathbb{G}, m)$, the two associated systems $(X_{\mathrm{p}}, \mathbb{G}, m_{\mathrm{p}})$ and $(X_{\mathrm{c}}, \mathbb{G}, m_{\mathrm{c}})$ also consist of weighted Meyer sets.

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5.1. *Weighted FLC sets.* A point set Λ of $\mathbb{G}$ is called uniformly discrete if there is an open set U such that any of its translates by an element of $\mathbb{G}$ contains at most one element of Λ. If one considers its Minkowski difference set

$$\Lambda - \Lambda := \{t - t' : t, t' \in \Lambda\}$$

then this means that 0 is isolated in $\Lambda - \Lambda$. A uniformly discrete set Λ is called of FLC if its Minkowski difference set $\Lambda - \Lambda$ is closed and all its points are isolated. In studying a particular FLC set Λ, one usually considers a whole ensemble of FLC sets X_Λ called its *hull* [**54**, §2], stable under the natural $\mathbb{G}$-action given for $\Lambda' \in X_\Lambda$ and $t \in \mathbb{G}$ by $\Lambda'.t := \Lambda' + t$. The hull X_Λ of a FLC set Λ is then a compact space with jointly continuous $\mathbb{G}$-action when equipped with a topology, the so-called *local topology*, for which a neighborhood basis at each $\Lambda' \in X_\Lambda$ is yielded by

$$\mathcal{O}_{U,K}(\Lambda') := \{\Lambda'' \in X_\Lambda : \exists\, s \in U \text{ such that } \Lambda' \cap K \equiv (\Lambda'' + s) \cap K\}$$

for $0 \in U \subset \mathbb{G}$ open and K compact (see [**54**] for details and a proof). An important remark that we will further need is that whenever a set Λ' belongs to the hull X_Λ of some FLC set Λ then its Minkowski difference set $\Lambda' - \Lambda'$ is included in $\Lambda - \Lambda$.

On the other hand, a FLC set Λ obviously defines a translation-bounded measure δ_Λ by setting a Dirac mass at any point of Λ. It is therefore natural to look at its associated dynamical system of translation-bounded measures $(X_{\delta_\Lambda}, \mathbb{G})$, and as discussed in [**13**, §4], this system is conjugated with $(X_\Lambda, \mathbb{G})$ under the natural map associating an FLC set with its corresponding Dirac comb. In light of this correspondence, a natural generalization of FLC sets is the following.

Definition 5.1. A weighted FLC set is a translation-bounded measure supported on an FLC set.

A weighted FLC set is thus always of the form

$$\omega = \sum_{t \in \Lambda} c_t \delta_t \quad \text{with} \quad \begin{cases} 0 \leqslant |c_t| \leqslant M, \\ \Lambda \text{ FLC set.} \end{cases}$$

Weighted lattices [**4**] are important examples of such measures.

PROPOSITION 5.2. *If a weighted FLC set ω is supported on some FLC set Λ then any $\omega' \in X_\omega$ is supported on some $\Lambda' \in X_\Lambda$.*

Proof. Let $\omega' \in X_\omega$ be chosen. It is a vague limit of a sequence of translates $\omega * \delta_{s_n}$ for $(s_n)_{n\in\mathbb{N}} \subset \mathbb{G}$, each respective translate having support in $\Lambda + s_n$. Since X_Λ is compact for the topology described earlier, the sequence $\Lambda + s_n$ must accumulate at some FLC set Λ', and after possibly extracting a subsequence one can suppose that $\Lambda + s_n$ also converges to Λ' in X_Λ. Select a decreasing sequence $(U_k)_k$ of neighborhoods of 0 in $\mathbb{G}$ whose intersection is the origin, and furthermore let $(K_k)_k$ be an increasing sequence of compact sets whose interiors cover $\mathbb{G}$. Then one is able to extract a subsequence $(s_{n_k})_k$ of $(s_n)_n$ such that for each $k \in \mathbb{N}$ one has $\Lambda' \cap K_k \equiv (\Lambda + s_{n_k} + \epsilon_k) \cap K_k$ for some $\epsilon_k \in U_k$. Let $s'_k := s_{n_k} + \epsilon_k$: then $\omega * \delta_{s'_k}$ converges vaguely to ω', each having support in $\Lambda + s'_k$ where $\Lambda' \cap K_k \equiv (\Lambda + s'_k) \cap K_k$ for all integers $k \in \mathbb{N}$. Therefore, on each compact K_{k_0},

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the restriction of $\omega * \delta_{s'_k}$ on K_{k_0} is eventually supported on the finite set $\Lambda' \cap K_{k_0}$, and it follows that the restriction of ω' on K_{k_0} is supported on $\Lambda' \cap K_{k_0}$. Therefore, ω' is a weighted Dirac comb supported on Λ', as desired. □

Remark 5.3. Note that if ω is a weighted FLC set with support the FLC set $\Lambda = S(\omega)$, then the mapping associating a $\omega' \in X_\omega$ to its support is in general neither continuous nor valued in X_Λ. In fact, such a mapping $X_\omega \longrightarrow X_\Lambda$ is well-defined and continuous if and only if any non-zero coefficient of the Dirac comb ω has an absolute value bounded from below by some positive constant.

Let ω be a weighted FLC set. It will later be convenient to have at our disposal a description of the topology of the hull X_ω as a 'local topology'. This local topology is constructed by considering the collection of subsets of $X_\omega \times X_\omega$,

$$\mathcal{O}_{U,K,\varepsilon} := \left\{(\omega', \omega'') \in X_\omega \times X_\omega : \inf_{s \in U} \left[\sup_{v \in K} |\omega'(v) - \omega''(v - s)|\right] < \varepsilon\right\}$$

for $0 \in U \subset \mathbb{G}$ open, K compact, and $\varepsilon > 0$. The collection of such sets constitutes a basis of a uniformity on X_ω (see [**33**, Ch. 6] for more on uniformities). Here, and all along this section, we use $\omega(v)$ for the ω-measure of the singleton $\{v\}$. This uniformity gives rise to a unique topology on X_ω, which we shall call the local topology and denote it by $\mathcal{T}_{\mathrm{loc}}$, such that a neighborhood basis at any $\omega' \in X_\omega$ is provided by

$$\mathcal{O}_{U,K,\varepsilon}(\omega') := \{\omega'' \in X_\omega : (\omega', \omega'') \in \mathcal{O}_{U,K,\varepsilon}\}$$

for $0 \in U \subset \mathbb{G}$ open, K compact, and $\varepsilon > 0$ [**33**, Ch. 6, Theorem 5]. Since $\mathbb{G}$ is σ-compact and second countable, one easily extracts a countable family of sets $\mathcal{O}_{U,K,\varepsilon}$ forming a basis of the same uniformity, and by [**33**, Ch. 6, Theorem 13], the local topology is consequently metrizable.

PROPOSITION 5.4. *Let ω be a weighted FLC set. Then the vague topology coincides with the local topology on X_ω. In particular, $(X_\omega, \mathcal{T}_{\mathrm{loc}})$ is compact.*

Proof. We only need to show the continuity of the identity map from $(X_\omega, \mathcal{T}_{\mathrm{vague}})$ to $(X_\omega, \mathcal{T}_{\mathrm{loc}})$, and the result will follow from the compactness of the former topology and Hausdorff property (which is straightforward to show since elements of X_ω are atomic measures) of the latter. Since both topologies are metrizable it suffices to show that if a sequence $(\omega_n)_n$ converges vaguely to ω_∞ then it converges to ω_∞ in the local topology. Thus, let $(\omega_n)_n$ be a sequence converging vaguely to a limit ω_∞: to show that this sequence also converges to ω_∞ in the local topology, it is sufficient to show that from any subsequence we can extract another subsequence converging to ω_∞ in the local topology.

Denote by Λ the support of the weighted FLC set ω. Let $(\omega_{n_k})_k$ be a subsequence, which still converges vaguely to ω_∞. Repeating the argument of the proof of Proposition 5.2, one can extract a subsequence $(\omega_{n_{k_l}})_l$ (still vaguely converging to ω_∞), as well as a sequence $(\Lambda_{k_l})_l$ in X_Λ with a limit $\Lambda_\infty \in X_\Lambda$, such that $\omega_{n_{k_l}}$ is supported on Λ_l, ω_∞ is supported on Λ_∞, and for a given decreasing sequence $(U_l)_l$ of neighborhoods of 0 in $\mathbb{G}$ whose intersection is the origin and a given increasing sequence $(K_l)_l$ of compact sets whose interiors cover $\mathbb{G}$ one has for each $l \in \mathbb{N}$ that

$$\Lambda_\infty \cap K_l \equiv (\Lambda_{k_l} + \epsilon_l) \cap K_l \quad \text{for some } \epsilon_l \in U_l.$$

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We now claim that the subsubsequence $(\omega_{n_{k_l}})_l$ converges to ω_∞ in the local topology. Indeed, for any $0 \in U \subset \mathbb{G}$ open, K compact, and $\varepsilon > 0$ there exists an integer L such that whenever $l \geqslant L$ then

$$\Lambda_\infty \cap K \equiv (\Lambda_{k_l} + \epsilon_l) \cap K \quad \text{for some } \epsilon_l \in U_l \subseteq U.$$

Therefore, for $l \geqslant L$, $\omega_{n_{k_l}}$ admits a $\epsilon_l \in U$ such that ω_∞ and $\omega_{n_{k_l}} * \delta_{\epsilon_l}$ have restrictions to the compact set K completely supported on the finite set $\Lambda_\infty \cap K$, that is to say, such that

$$|\omega_\infty(v) - \omega_{n_{k_l}}(v - \epsilon_l)| = |\omega_\infty(v) - \omega_{n_{k_l}} * \delta_{\epsilon_l}(v)| = 0 \quad \text{for all } v \in K \setminus \Lambda_\infty. \tag{17}$$

Hence, we will be done once we find an integer $L' \geqslant L$ such that for $l \geqslant L'$ one also has $|\omega_\infty(p) - \omega_{n_{k_l}}(p - \epsilon_l)| < \varepsilon$ at points $p \in K \cap \Lambda_\infty$. To do so consider an open set U' such that $((\Lambda_\infty \cap K) - (\Lambda_\infty \cap K)) \cap U' = \{0\}$, and consider an open set $0 \in U'' \subset \mathbb{G}$ whose closure is strictly included in U'. Then one can exhibit a non-negative, compactly supported continuous function ϕ identically equal to 1 on U'' and vanishing outside U'. Let $L' \geqslant L$ be such that

$$-\epsilon_l \in U''$$

and

$$\left| \int \phi * \delta_p \, d\omega_\infty - \int \phi * \delta_p \, d\omega_{n_{k_l}} \right| < \varepsilon \quad \text{for all } p \in \Lambda_\infty \cap K \text{ and } l \geqslant L',$$

which exists, since $(\epsilon_l)_l$ converges to 0 and $(\omega_{n_{k_l}})_l$ converges vaguely to ω_∞. From our particular choice of ϕ, for such an L' one has

$$|\omega_\infty(p) - \omega_{n_{k_l}}(p - \epsilon_l)| = \left| \int \phi * \delta_p \, d\omega_\infty - \int \phi * \delta_p \, d\omega_{n_{k_l}} \right| < \varepsilon \quad \text{for all } p \in \Lambda_\infty \cap K. \tag{18}$$

Hence, equations (17) and (18) show that the subsubsequence $(\omega_{n_{k_l}})_l$ converges to ω in the local topology, which finishes the proof. □

5.2. *Weighted Meyer sets and cut-and-project representations.* A strengthening of the FLC property for point sets requires two additional facts: first, the uniformly discrete set Λ should be relatively dense, meaning that there exists a compact set K such that $\Lambda + K$ covers the whole group $\mathbb{G}$ (we call such a set a *Delone set*). Second, the Minkowski difference set of Λ should be covered by finitely many translates of Λ, that is, there should exist a finite set F such that $\Lambda - \Lambda \subseteq \Lambda + F$. A uniformly discrete set fulfilling these two additional properties is nowadays called a *Meyer set*. Meyer sets have several different yet equivalent characterizations, see the original work of Meyer [**42**, **43**], as well as [**34**, **35**, **44**, **45**] for the case $\mathbb{G} = \mathbb{R}^d$, and the improvement [**58**] for the case of σ-compact LCA groups. If Λ is a Meyer set, then as an FLC set it has a hull X_Λ, and it is not hard to show that any $\Lambda' \in X_\Lambda$ is also a Meyer set.

We now review the widely studied *cut-and-project formalism* for point sets, see the different works [**2**, **9**, **10**, **14**, **17**, **18**, **26**, **31**, **32**, **36**, **39**, **42**, **45**–**47**, **53**, **54**, **58**]. Suppose we are given the triple $(\mathbb{G}, H, \mathcal{L})$ where H is an LCA group and $\mathcal{L}$ a lattice in $H \times \mathbb{G}$, that is, a discrete and co-compact subgroup of $H \times \mathbb{G}$. We furthermore require that the projection

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from $H \times \mathbb{G}$ onto $\mathbb{G}$ is one-to-one on the lattice $\mathcal{L}$, with image $L \subset \mathbb{G}$, and that the image of $\mathcal{L}$ by the projection from $H \times \mathbb{G}$ onto H forms a dense subset $L^\star \subset H$. Such a triple is called a *cut-and-project scheme* (CPS for short). The LCA group H of a CPS is commonly called the internal space (or internal group), and the subgroup $L \subset \mathbb{G}$, the structure group of the CPS. A CPS possesses a unique group morphism $L \longrightarrow L^\star$, the $\star$-*map* of the CPS, mapping any $t \in L$ onto the unique element $t^\star \in H$ such that $(t, t^\star) \in \mathcal{L}$.

Definition 5.5. Let Λ be a point set of $\mathbb{G}$. We call the cut-and-project representation of Λ any CPS $(\mathbb{G}, H, \mathcal{L})$ such that Λ belongs to the structure group L and whose image $\Lambda^\star$ under the $\star$-map has compact closure in H.

Not all point sets admit a cut-and-project representation, see Theorem 5.7 below. Whenever a point set Λ belongs to the structure group L for some CPS $(\mathbb{G}, H, \mathcal{L})$ then it can be lifted to a subset $\mathcal{L}$ in $H \times \mathbb{G}$, and the condition of compactness of the closure of $\Lambda^\star$ in H means that this lifting stand in a certain strip about $\mathbb{G}$ into $H \times \mathbb{G}$.

Given some point set Λ, we will call a cut-and-project representation $(\mathbb{G}, H, \mathcal{L})$ of Λ *irredundant* whenever the closure W_Λ of $\Lambda^\star$ in H is irredundant in H, that is, if there is no non-trivial element $v \in H$ satisfying $W_\Lambda + v = W_\Lambda$. It is always possible to turn a given cut-and-project representation into an irredundant one by simply modding out a certain compact subgroup of H, see [**14**, **36**] for details and a proof. Thus, a point set having a cut-and-project representation automatically also admits an irredundant cut-and-project representation.

Definition 5.6. A closed model set is a point set obtained from a CPS $(\mathbb{G}, H, \mathcal{L})$ and a compact topologically regular subset $W \subset H$ (a window) by

$$\mathfrak{P}(W) := \{t \in L : t^\star \in W\}.$$

We recall that a topologically regular subset $W \subset H$ is a compact subset which is the closure of its interior in H. It is known that a closed model set is always a Meyer set. Moreover, a closed model set Λ always comes with a cut-and-project representation, namely that of the CPS used to construct it, where $\Lambda^\star$ has compact closure W in H.

THEOREM 5.7. *Let Λ be a point set of $\mathbb{G}$. The following assertions are equivalent:*
(i) *Λ is a subset of a Meyer set;*
(ii) *Λ admits a cut-and-project representation;*
(iii) *Λ is a subset of a closed model set.*
If it holds true then the cut-and-project representation can be chosen irredundant.

The hard part of this theorem is (i) $\Rightarrow$ (ii), which is sufficient to prove when Λ is itself a Meyer set, and which was proved by Meyer by a combination of the results [**42**, Ch. II, §5, Proposition 4] and [**42**, Ch. II, §14, Theorem X]. This was later on followed by several works [**2**, **17**, **34**, **44**, **45**, **58**]. Assuming this, let us then provide a short proof of the remaining statements.

Proof. Assume (i): then Λ is contained in some Meyer set Λ', which by combining [**42**, Ch. II, §5, Proposition 4] and [**42**, Ch. II, §14, Theorem X] admits a cut-and-project representation $(\mathbb{G}, H, \mathcal{L})$. It follows that $\Lambda \subseteq \Lambda' \subseteq L$ and that the closure W_Λ of $\Lambda^\star$ is

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contained in the closure of $\Lambda'^{\star}$ which is compact in H, so that $(\mathbb{G}, H, \mathcal{L})$ is a cut-and-project representation of Λ, giving (ii). Then assuming (ii) one gives rise, by considering any compact topologically regular subset W of H containing $\Lambda^{\star}$, to a closed model set $\mathfrak{P}(W) := \{t \in L : t^{\star} \in W\}$ which contains Λ, yielding (iii). As any closed model set is a Meyer set this immediately gives (i). Finally, following [**14**, §9], given a cut-and-project representation $(\mathbb{G}, H, \mathcal{L})$ of Λ with closure W_{Λ} of $\Lambda^{\star}$ in H, one mods out the compact subgroup K_{Λ} of elements $v \in H$ such that $W_{\Lambda} + v = W_{\Lambda}$, which results in an irredundant cut-and-project representation $(\mathbb{G}, H', \mathcal{L}')$ of Λ with internal group $H' = H/K_{\Lambda}$ and lattice $\mathcal{L}' \subset H' \times \mathbb{G}$ the projection of $\mathcal{L}$ modulo $K_{\Lambda} \times \{0\}$. □

Definition 5.8. A weighted Meyer set is a translation-bounded measure supported on a Meyer set.

A weighted Meyer set is hence always of the form

$$\omega = \sum_{t \in \Lambda} c_t \delta_t \quad \text{with} \quad \begin{cases} 0 \leqslant |c_t| \leqslant M, \\ \Lambda \text{ Meyer set.} \end{cases}$$

Obviously, a weighted Meyer set is always a weighted FLC set, and its support admits *by definition* a Meyer super-set containing it. It may fail to have Meyer set support, since the latter may not be relatively dense. A weighted Meyer set could have also been called a *weighted model set*, but since this terminology makes explicit reference to a cut-and-project scheme representing it, we preferred the former appellation.

5.3. *The torus parametrization of a cut-and-project representation.* In [**12**], the authors illustrated in a particular case a relation, which is of main importance here and more generally in the whole point sets theory, between cut-and-project representations of a Meyer set Λ and its associated dynamical system $(X_{\Lambda}, \mathbb{G})$: they namely showed that existence of a cut-and-project representation of Λ always gives rise to a Kronecker factor of the system $(X_{\Lambda}, \mathbb{G})$ (and thus to a group of eigenvalues for $(X_{\Lambda}, \mathbb{G})$). Such a result has later on been proved in its greatest generality by Schlottmann in [**54**, §4].

To be more precise, it is shown in [**54**] that if a *repetitive* Meyer set Λ admits an irredundant cut-and-project representation $(\mathbb{G}, H, \mathcal{L})$, then there is a (continuous) factor map from X_{Λ} onto $[H \times \mathbb{G}]_{\mathcal{L}}$, the compact Abelian group obtained as the quotient of $H \times \mathbb{G}$ by its subgroup $\mathcal{L}$ and endowed with the $\mathbb{G}$-action 'by rotation', set for $s \in \mathbb{G}$ on an element $[v, t]$ by $[v, t].s := [v, t + s]$. Here repetitivity of a Meyer set Λ means minimality of its dynamical system $(X_{\Lambda}, \mathbb{G})$. The factor map from X_{Λ} onto the Kronecker factor issued from a cut-and-project representation of Λ is called a *torus parametrization*.

We propose here to set this result in a slightly more general situation, namely that of weighted Meyer sets, by carefully adapting the proof of [**54**] to our setting. The original proof has been set with the assumption of repetitivity on the Meyer sets, and we shall not assume this here. As a result, the torus parametrization yield by a cut-and-project representation of a weighted Meyer set ω will no longer be defined on the entire hull X_{ω} but rather only on the rubber local isomorphism class $\mathrm{RLI}(\omega)$ (one will notice that repetitivity of ω, that is to say, minimality of $(X_{\omega}, \mathbb{G})$, precisely means that $\mathrm{RLI}(\omega) = X_{\omega}$).

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This will, however, be sufficient to prove the main result of this section, namely Theorem 5.10 in the next paragraph (Theorem 2 in the introduction).

THEOREM 5.9. *Let ω be a weighted Meyer set with support $S(\omega)$ having an irredundant cut-and-project representation $(\mathbb{G}, H, \mathcal{L})$, with W the closure of $S(\omega)^\star$ in H.*

(i) *For each $\omega' \in X_\omega$ there exists $(v, t) \in H \times \mathbb{G}$ such that $S(\omega') \subseteq \mathfrak{P}(W + v) + t$.*

(ii) *When $\omega' \in \mathrm{RLI}(\omega)$ the element $(v, t) \in H \times \mathbb{G}$ is unique modulo $\mathcal{L}$.*

(iii) *The mapping $\mathrm{RLI}(\omega) \ni \omega' \longmapsto [v, t] \in [H \times \mathbb{G}]_{\mathcal{L}}$ is a continuous $\mathbb{G}$-map.*

Proof. (i) Let $\omega' \in X_\omega$, which we can assume not to be the trivial measure on $\mathbb{G}$, and supposed to be supported on the structure group L. Denote by $W(\omega')$, the closure of $S(\omega')^\star$ in H. Hence, with respect to this notation, W is nothing but $W(\omega)$. Then it is not hard to show, given $v \in H$, the equivalence of conditions

$$v \in \bigcap_{t \in S(\omega')} t^\star - W \Longleftrightarrow W(\omega') \subseteq W + v. \tag{19}$$

Let us show that any $\omega' \in X_\omega$ supported on L admits an element $v_{\omega'} \in H$ where these equivalent conditions hold: as $\omega' \in X_\omega$ one can find a sequence $(t_n)_n$ of elements in $\mathbb{G}$ such that $\omega * \delta_{t_n}$ converges to ω', which according to Proposition 5.4 means that for each compact K and $\varepsilon > 0$ there is a $N_{K,\varepsilon}$ such that, after possibly slightly moving each t_n, one has for each $n \geqslant N_{K,\varepsilon}$ that $|\omega'(s) - \omega * \delta_{t_n}(s)| < \varepsilon$ for any $s \in K$. Therefore, whenever $p \in S(\omega')$, we have $\omega * \delta_{t_n}(p)$ eventually non-zero, so that p eventually belongs to the support $S(\omega) + t_n$ of $\omega * \delta_{t_n}$. Since ω' is non-trivial by assumption, one can pick up some $p_0 \in S(\omega')$. Then the sequence t_n eventually belongs to $p_0 - S(\omega) \subset S(\omega') - S(\omega)$, which lies in the group L since both ω' and ω are supported on this group. Thus, $t_n^\star$ eventually makes sense, and since $p_0 \in S(\omega) + t_n$ then one eventually gets $p_0^\star \in W + t_n^\star$. Thus, eventually the sequence $t_n^\star$ lies in the compact set $p_0^\star - W$, and therefore accumulates at some element $v_{\omega'} \in H$. We can suppose, after possibly extracting a subsequence, that $t_n^\star$ converges to $v_{\omega'}$ in H. The latter must satisfy $p_0^\star \in W + v_{\omega'}$, and since this argument is independent upon the choice of $p_0 \in S(\omega')$, we deduce that $S(\omega')^\star \subset W + v_{\omega'}$, which yields $W(\omega') \subset W + v_{\omega'}$. This provides the existence of the desired $v_{\omega'} \in H$. Now given any $\omega' \in X_\omega$, one has for any chosen element $t \in S(\omega')$ that $\omega' * \delta_{-t}$ is supported on the structure group Γ, and as we just saw, there exists a $v \in H$ such that (19) holds, yielding a $(v, t) \in H \times \mathbb{G}$ with $S(\omega') \subseteq \mathfrak{P}(W + v) + t$, as desired.

(ii) We show that when $\omega' \in \mathrm{RLI}(\omega)$ with support in L, then the $v \in H$ satisfying (19) is unique, and that one in fact has $W(\omega') = W + v$. To that end, observe that from the above analysis, the set $W(\omega')$ is contained in some translate $W + v$ of the compact set W and thus is compact in H. Since $\omega \in \mathrm{RLI}(\omega')$, we can interchange the roles of ω' and ω in the previous argument, which shows that there equally exists some $v' \in H$ with $W \subseteq W(\omega') + v'$. It follows that $W \subseteq W(\omega') + v' \subseteq W + v + v'$ with W compact, and this forces $W = W + v + v'$: indeed one can easily check that the set

$$\{v + v' + v'' \mid W \subseteq W + v''\}$$

is a compact subsemigroup of H, which from a general result (see, for instance, [**27**, Theorem 1.1]) contains an idempotent element, that is, an element e such that $e + e = e$.

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This idempotent must be the identity of the group H, and as a result $-(v+v')$ must satisfy $W \subseteq W-(v+v')$, which yields $W = W+v+v'$. Now from the irredundancy assumption of W one gets $v+v'=0$, and this in turn shows that $W(\omega') = W+v$. Such an equality can of course hold for at most one element $v \in H$.

Now suppose that $\omega' \in \mathrm{RLI}(\omega)$ admits two pairs (v,t) and (v',t') in $H \times \mathbb{G}$ such that

$$S(\omega') \subseteq \mathfrak{P}(W+v)+t \quad \text{and} \quad S(\omega') \subseteq \mathfrak{P}(W+v')+t'.$$

Then, from what has just been said, one has

$$W(\omega'.(-t)) = W+v \quad \text{and} \quad W(\omega'.(-t')) = W+v'.$$

Given any $p \in S(\omega')$ one has

$$t-t' = (p-t')-(p-t) \in \mathfrak{P}(W+v')-\mathfrak{P}(W+v) \subseteq L-L = L.$$

Therefore, $(t-t')^\star$ makes sense and is an element of H such that

$$W+v+(t-t')^\star = W(\omega'.(-t))+(t-t')^\star = W(\omega'.(-t')) = W+v',$$

which shows by irredundancy that $v' = v+(t-t')^\star$. Thus, the element $(v,t) \in H \times \mathbb{G}$ for which inclusion of point (i) of the statement holds is unique modulo $\mathcal{L}$, as desired.

(iii) The mapping $\mathrm{RLI}(\omega) \ni \omega' \longmapsto [v,t] \in [H \times \mathbb{G}]_{\mathcal{L}}$ is from the conclusion of point (ii) well-defined. Moreover, when $\omega' \in \mathrm{RLI}(\omega)$ is such that $S(\omega') \subseteq \mathfrak{P}(W+v)+t$, then $S(\omega' * \delta_s) = S(\omega')+s \subseteq \mathfrak{P}(W+v)+t+s$ for any $s \in \mathbb{G}$, which shows that the map is a $\mathbb{G}$-map. It then remains to show continuity at any given $\omega' \in \mathrm{RLI}(\omega)$. Given such ω', let $(v,t) \in H \times \mathbb{G}$ be such that $S(\omega') \subseteq \mathfrak{P}(W+v)+t$: given a neighborhood U of 0 in H and a neighborhood U' of 0 in $\mathbb{G}$, since v satisfies

$$\{v\} = \bigcap_{p \in S(\omega')-t} p^\star - W \tag{20}$$

there exists, as one can check using a contradiction argument and σ-compactness of $\mathbb{G}$, a sufficiently large compact set K such that

$$\bigcap_{p \in (S(\omega')-t) \cap K} p^\star - W \subseteq v+U.$$

Now, since the set $S(\omega') \cap (K+t)$ is finite, one has a minimal value $M > 0$ for $|\omega(s)|$ on that set. Then we are done if we can show that, if $\omega'' \in \mathrm{RLI}(\omega)$ is such that

$$\inf_{\epsilon \in U'} \Big[\sup_{s \in K+t} |\omega'(s)-\omega''(s-\epsilon)| \Big] < \frac{M}{2}$$

(that is, if $\omega'' \in \mathrm{RLI}(\omega)$ belongs to a neighborhood of ω' uniquely defined by ω', U and U'), then it admits a representative $(v',t') \in H \times \mathbb{G}$ of its class such that

$$(v',t') \in (v,t)+U \times U'.$$

This is in turn true, since for such ω'' one has a $\epsilon \in U'$ such that, putting $t' := t-\epsilon$, $(S(\omega')-t) \cap K$ is included in $(S(\omega'')-t') \cap K$, and thus

$$v' \in \bigcap_{p \in (S(\omega'')-t') \cap K} p^\star - W \subseteq \bigcap_{p \in (S(\omega')-t) \cap K} p^\star - W \subseteq v+U,$$

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where v' is the unique element of H in the intersection $\bigcap_{p\in(S(\omega'')-t')} p^\star - W$. We therefore end up with a pair (v', t') such that, from the selection of v' satisfies

$$S(\omega'') \subseteq \mathfrak{P}(W + v') + t'$$

and such that $(v', t') \in (v, t) + U \times U'$. This shows continuity at any $\omega' \in \mathrm{RLI}(\omega)$, as desired. □

5.4. *The decomposition of weighted Meyer sets.* We shall show here our main result of this section, see Theorem 5.10 below. We will need to consider here the yield of the Borel $\mathbb{G}$-maps by Theorem 4.1. These maps are not uniquely defined, but, by the uniqueness statement of Theorem 4.7, any two choices agree m-almost everywhere on X. We assume here that such a choice is made.

THEOREM 5.10. *Let $(X, \mathbb{G}, m)$ be an ergodic system of weighted Meyer sets. Then the following property holds m-almost everywhere: whenever ω is supported on a closed model set, then so are its summands ω_{p} and ω_{c}.*

Proof. Let $(X, \mathbb{G}, m)$ be an ergodic system of weighted Meyer sets with eigenvalue group $\mathcal{E}$, and let $(X_{\mathrm{p}}, \mathbb{G}, m_{\mathrm{p}})$ and $(X_{\mathrm{c}}, \mathbb{G}, m_{\mathrm{c}})$ be as in Theorem 4.1. Recall from Theorem 4.7 that there exists a unique (up to almost everywhere equality) Borel factor map $X \longrightarrow X_{\mathrm{p}}$, and from the proof of Theorem 4.1 that it is almost everywhere given by the composition of the three Borel $\mathbb{G}$-maps

$$X \xrightarrow{\ \pi\ } \mathrm{T}_{\mathcal{E}} \longrightarrow \mathcal{M}^b(X) \xrightarrow{\ I\ } \mathcal{M}^\infty(\mathbb{G})$$

for some choice of Borel factor map π as in Proposition 2.3, the middle Borel $\mathbb{G}$-map being the disintegration of m over $\mathrm{T}_{\mathcal{E}}$ as in Theorem 2.2, and I the intensity map as in Proposition 3.5. The following proposition is at the core of our proof.

PROPOSITION 5.11. *Almost all $\omega \in X$ admit a Borel set $F_\omega \subset X$ of full $m_{\pi(\omega)}$-measure such that, whenever ω is supported on a closed model set Λ, then so is any $\omega' \in F_\omega$.*

Proof. First of all, an eigenvalue $\xi \in \mathcal{E}$ uniquely corresponds to a character θ_ξ on $\mathrm{T}_{\mathcal{E}}$, which determines a Borel eigenfunction $\Phi_\xi := \theta_\xi \circ \pi$ associated to the eigenvalue ξ on X. Now, as $(X, \mathbb{G}, m)$ is ergodic, the set

$$X^{(0)} := \{\omega \in X \mid X_\omega = \mathrm{Supp}(m)\}$$

is a $\mathbb{G}$-invariant Borel subset of full measure in X, and choosing any $\omega \in X^{(0)}$ yields $X^{(0)} = \mathrm{RLI}(\omega)$. We shall then consider the subset $\mathcal{E}_0 \subseteq \mathcal{E}$ of eigenvalues ξ admitting a continuous eigenfunction on $X^{(0)}$. To each $\xi \in \mathcal{E}_0$ a unique continuous eigenfunction can then be associated

$$\widetilde{\Phi}_\xi : X^{(0)} \longrightarrow \mathrm{T}_1$$

with $\widetilde{\Phi}_\xi \equiv \Phi_\xi$ almost everywhere on $X^{(0)}$. Clearly, for $\xi \in \mathcal{E}_0$ there can only be one such function. Since $\mathcal{E}$ is countable its subset $\mathcal{E}_0$ is countable as well, and the Borel set

$$X^{(1)} := \bigcap_{\xi\in\mathcal{E}_0} \{\omega \in X^{(0)} \mid \widetilde{\Phi}_\xi(\omega) = \Phi_\xi(\omega)\}$$

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is thus of full measure in X. On the other hand, by Theorem 2.2(D1), the Borel subset

$$X^{(2)} := \{\omega \in X \mid m_{\pi(\omega)} \text{ is supported on } \pi^{-1}(\pi(\omega))\}$$

is of full measure in X, and as a result the Borel set $X^{(3)} := X^{(1)} \cap X^{(2)}$ is of full measure in X. Moreover, applying Theorem 2.2 to the Borel almost everywhere 1 function $\mathbb{I}_{X^{(3)}}$, one deduces that the set

$$X^{\infty} := \{\omega \in X^{(3)} \mid m_{\pi(\omega)}(X^{(3)}) = 1\}$$

is a Borel set of full measure in X. From the very construction of this Borel set, for all $\omega \in X^{\infty}$ the Borel set

$$F_{\omega} := \pi^{-1}(\pi(\omega)) \cap X^{(3)}$$

has $m_{\pi(\omega)}$-measure equal to 1 and always contains the element ω.

We now claim that our statement holds for any $\omega \in X^{\infty}$ with corresponding set F_{ω}. Indeed, let $\omega \in X^{\infty}$ be given, and suppose it is supported on some closed model set Λ, that is, let $(\mathbb{G}, H, \mathcal{L})$ be a CPS and V a compact set in H such that the support $S(\omega)$ of ω is a subset of $\Lambda := \mathfrak{P}(V)$. Then $S(\omega)^{\star}$ makes sense and its closure W in H is contained in V, and thus is compact. Modding out the redundancy subgroup K_W of W in H leads to an irredundant cut-and-project representation $(\mathbb{G}, H', \mathcal{L}')$ of ω, with closure of $S(\omega)^{\star}$ in H' being a compact set W', and such that

$$\mathfrak{P}(W') = \mathfrak{P}(W) \subseteq \mathfrak{P}(V) = \Lambda.$$

Observe that ω is in fact supported on $\mathfrak{P}(W')$. It is then sufficient to show that

$$S(\omega') \subseteq \mathfrak{P}(W') \quad \text{for any } \omega' \in F_{\omega}. \tag{21}$$

Let us show this: from Theorem 5.9(iii) there is a continuous $\mathbb{G}$-map

$$X^{\infty} \subseteq X^{(0)} = \mathrm{RLI}(\omega) \ni \omega' \stackrel{p}{\longmapsto} [v, t] \in [H' \times \mathbb{G}]_{\mathcal{L}'}$$

where $p(\omega') = [v, t]$ is the unique $\mathcal{L}'$-class such that any representative $(v, t) \in H' \times \mathbb{G}$ satisfies

$$S(\omega') \subseteq \mathfrak{P}(W' + v) + t.$$

Since ω is supported on $\mathfrak{P}(W')$ one has $p(\omega) = [0, 0]$, and therefore equation (21) will hold once we have shown that $p(\omega') = p(\omega) = [0, 0]$ for all $\omega' \in F_{\omega}$. To do so, consider the eigenvalue group $\mathcal{E}'$ of the Kronecker action of $\mathbb{G}$ on $[H' \times \mathbb{G}]_{\mathcal{L}'}$, and given $\xi \in \mathcal{E}'$ consider some corresponding continuous eigenfunction $\chi_{\xi} : [H' \times \mathbb{G}]_{\mathcal{L}'} \longrightarrow \mathrm{T}_1$. It lifts to a continuous eigenfunction on $X^{(0)}$,

$$\chi_{\xi} \circ p : X^{(0)} \longrightarrow \mathrm{T}_1.$$

Hence $\xi \in \mathcal{E}_0$, and there exists $c \in \mathrm{T}_1$ (depending on ξ) such that $\chi_{\xi} \circ p \equiv c.\widetilde{\Phi}_{\xi}$ everywhere on $X^{(0)}$. Now, for $\omega' \in F_{\omega}$, one has $\pi(\omega') = \pi(\omega)$, and thus $\Phi_{\xi}(\omega') = \Phi_{\xi}(\omega)$, and since $\omega, \omega' \in X^{\infty} \subseteq X^{(1)}$ we consequently have $\widetilde{\Phi}_{\xi}(\omega') = \widetilde{\Phi}_{\xi}(\omega)$. This gives that $\chi_{\xi} \circ p(\omega') = \chi_{\xi} \circ p(\omega)$ for any $\xi \in \mathcal{E}'$, that is, $p(\omega)$ and $p(\omega')$ are identified under any eigenfunction of the Kronecker action of $\mathbb{G}$ on $[H' \times \mathbb{G}]_{\mathcal{L}'}$. One must then have $p(\omega') = p(\omega) = [0, 0]$, as desired. □

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From this we can easily settle the proof of Theorem 5.10: let $\omega \in X$ such that the conclusion of Proposition 5.11 holds true for some Borel set F_ω. Without loss of generality we can suppose that ω belongs to the Borel subset of full measure

$$\{\omega \in X \mid \omega_{\mathrm{p}} = I(m_{\pi(\omega)}) \text{ and } \omega_{\mathrm{c}} = \omega - \omega_{\mathrm{p}}\}.$$

We shall show that the supports of ω_{p} and ω_{c} satisfy

$$S(\omega_{\mathrm{p}}),\, S(\omega_{\mathrm{c}}) \subseteq \bigcup_{\omega' \in F_\omega} S(\omega'). \tag{22}$$

Indeed for any Borel set B of $\mathbb{G}$ one has $\omega_{\mathrm{p}}(B) = I(m_{\pi(\omega)})(B)$, and since F_ω is of full $m_{\pi(\omega)}$-measure one deduces by definition of I,

$$\omega_{\mathrm{p}}(B) = \int_X \omega'(B)\, dm_{\pi(\omega)}(\omega') = \int_{F_\omega} \omega'(B)\, dm_{\pi(\omega)}(\omega'),$$

which is null when B belongs to the complementary set of the right term of (22). This shows the desired inclusion for $S(\omega_{\mathrm{p}})$. Now since $\omega_{\mathrm{c}} = \omega - \omega_{\mathrm{p}}$, $S(\omega_{\mathrm{c}})$ is therefore supported on $S(\omega) \cup S(\omega_{\mathrm{p}})$, clearly included in the union of (22), giving the desired inclusion for ω_{c}. Therefore, whenever our ω is supported on some closed model set Λ, combining equation (22) and Proposition 5.11 one deduces that

$$S(\omega_{\mathrm{p}}),\, S(\omega_{\mathrm{c}}) \subseteq \left[\bigcup_{\omega' \in F_\omega} S(\omega') \right] \subseteq \Lambda,$$

as desired. □

Remark 5.12. From the above result we know that, given an ergodic system $(X, \mathbb{G}, m)$ of weighted Meyer sets with autocorrelation γ, then once a generic $\omega \in X$ is supported in a closed model set Λ then so are ω_{p} and ω_{c}. Therefore, using equation (4) of §3.4 for the autocorrelation, as well as the conclusion of Remark 4.3, one shows that both γ, its strong almost periodic part (the autocorrelation of a generic choice of $\omega_{\mathrm{p}} \in X_{\mathrm{p}}$) and its null weakly almost periodic part (the autocorrelation of a generic choice of $\omega_{\mathrm{c}} \in X_{\mathrm{c}}$) are supported on the Minkowski difference set $\Lambda - \Lambda$. This statement was obtained (among other things) in [**57**].

Remark 5.13. A lattice L of $\mathbb{G}$ is always a closed model set, for instance coming from the cut-and-project scheme having trivial internal group and L itself as a lattice in $\{0\} \times \mathbb{G}$. As a result, if X consists of weighted translates of some lattice L then so does X_{p} and X_{c}.

Remark 5.14. Suppose that $(X, \mathbb{G}, m)$ is an ergodic system of uniform Dirac combs of Meyer sets (and thus with constant weight equal to 1). Let Ξ_t stand for the subset of Dirac combs in X having t in its support, for any $t \in \mathbb{G}$. Then it is not difficult to show, for m-almost every $\omega \in X$, the formula

$$\omega_{\mathrm{p}} = \sum \omega_{\mathrm{p}}(t)\delta_t \quad \text{with } \omega_{\mathrm{p}}(t) = m_{\pi(\omega)}(\Xi_t).$$

Thus, from a probabilistic viewpoint, ω_{p} can be interpreted as the sum of Dirac combs of elements $t \in \mathbb{G}$ with each coefficient $\omega_{\mathrm{p}}(t)$ being the m-expectation of the point t to appear, conditioned by the event of belonging in the fiber of ω over $\mathrm{T}_{\mathcal{E}}$. In the general weighted case, a similar formula, yet less trivial, can be stated which involves the weight function $c : X \longrightarrow \mathbb{C}$, $c(\omega) := \omega(0)$, the measure of the singleton 0.

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Remark 5.15. Suppose that $(X, \mathbb{G}, m)$ is an ergodic system of weighted Meyer sets with non-negative weights. Then by Proposition 4.4 all weighted Meyer sets of $(X_{\mathrm{p}}, \mathbb{G}, m_{\mathrm{p}})$ also have non-negative weights. Suppose moreover that m-almost all measure in X have relatively dense support. Then it follows from Proposition 4.9 that m_{p}-almost all weighted Meyer set in X_{p} also have relatively dense support.

6. *Operator methods for the analysis of Borel factors*

In this section we will have a second look at the process associating an operator on $L^2(X, m)$ to certain Borel factors of a measured dynamical system of translation-bounded measures $(X, \mathbb{G}, m)$, as performed in §4.2. Our main result here is the identification of the class of operators involved in this process, see Theorem 6.6, as well as a relation between such operators and the diffraction of the corresponding Borel factors.

6.1. *Admissible operators.* Let $(X, \mathbb{G}, m)$ be a measured dynamical system of translation-bounded measures, with diffraction-to-dynamic map

$$\Theta : L^2(\widehat{\mathbb{G}}, \widehat{\gamma}) \longleftarrow \mathcal{H}_\Theta \subseteq L^2(X, m), \quad \Theta(\widehat{\phi}) = \mathcal{N}_\phi$$

for any $\phi \in \mathcal{C}_c(\mathbb{G})$. Recall that P_Θ stands for the $\mathbb{G}$-commuting orthogonal projection onto the Hilbert subspace $\mathcal{H}_\Theta$.

Definition 6.1. A bounded operator Q on the Hilbert space $L^2(X, m)$ is admissible if it is $\mathbb{G}$-commuting, if $Q = Q P_\Theta$ on $L^2(X, m)$ and if for each compact set $K \subset \mathbb{G}$ there is a constant $R_K \geqslant 0$ such that

$$\| Q(\mathcal{N}_\phi) \|_{\infty,\mathrm{ess}} \leqslant R_K \| \phi \|_\infty \quad \text{for all } \phi \in \mathcal{C}_K(\mathbb{G}).$$

It is obvious that the collection of admissible operators of the system $(X, \mathbb{G}, m)$ forms a linear subspace of the space of bounded operators on $L^2(X, m)$, and that P_Θ itself is admissible.

Remark 6.2. Any bounded operator on $L^2(X, m)$ admits a certain weak form of admissibility: if Q is any bounded operator, then, due to the continuity of the map $\phi \longrightarrow \mathcal{N}_\phi$, one always has, for any compact set $K \subset \mathbb{G}$, a certain constant $R'_K \geqslant 0$ such that $\| Q(\mathcal{N}_\phi) \|_2 \leqslant R'_K \| \phi \|_\infty$ for all $\phi \in \mathcal{C}_K(\mathbb{G})$.

Suppose we are given a Borel factor map $\pi : X \longrightarrow X_\pi$ towards a measured dynamical system of translation-bounded measures $(X_\pi, \mathbb{G}, m_\pi)$, with diffraction $\widehat{\gamma}_\pi \ll \widehat{\gamma}$, and essentially bounded Radon–Nikodym differential $f_\pi := d\widehat{\gamma}_\pi / d\widehat{\gamma} \in L^\infty(\widehat{\mathbb{G}}, \widehat{\gamma})$. Then Proposition 4.6 gives rise to a bounded operator Q_π on $L^2(X, m)$, and we have the following.

PROPOSITION 6.3. *The operator $Q_\pi \in \mathcal{B}(L^2(X, m))$ associated by Proposition 4.6 to any Borel factor map $\pi : X \longrightarrow X_\pi$ as above is admissible.*

Proof. One has $Q_\pi = Q_\pi P_\Theta$ by construction, whereas $\mathbb{G}$-commutativity, that is to say, $U_t Q_\pi = Q_\pi U_t$ for any $t \in \mathbb{G}$, is straightforward to check on functions of the form $\mathcal{N}_\phi$ for $\phi \in \mathcal{C}_c(\mathbb{G})$ and therefore holds on the closure $\mathcal{H}_\Theta$, and hence on the whole space $L^2(X, m)$.

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For each $\phi \in \mathcal{C}_K(\mathbb{G})$ supported on a given compact set $K \subset \mathbb{G}$ one has the following bound, holding for almost every $\omega \in X$,

$$|Q_\pi(\mathcal{N}_\phi)(\omega)| = |\mathcal{N}_\phi^\pi \circ \pi(\omega)| \leqslant \|\mathcal{N}_\phi^\pi\|_\infty \leqslant R_K \|\phi\|_\infty$$

where the constant R_K only depends on the translation-bound of measures in X_π and on the compact set K, settling the proof. □

6.2. *From admissible operators to Borel factors.* We shall now formulate the reciprocal statement of Proposition 4.6, that is, given a dynamical system $(X, \mathbb{G}, m)$ of translation-bounded measures on $\mathbb{G}$ and an admissible operator Q, we shall reconstruct the underlying Borel factor of translation-bounded measures $(X_Q, \mathbb{G}, m_Q)$. Anticipating things a little, for such a constructed dynamical system of translation-bounded measures we denote as usual

$$\mathcal{C}_c(\mathbb{G}) \ni \phi \longmapsto\!\!\!\!\!\!\rightarrow \mathcal{N}_\phi^Q \in \mathcal{C}(X_Q), \quad \mathcal{N}_\phi^Q(\omega_Q) = \int_{\mathbb{G}} \phi_-\, d\omega_Q.$$

The result concerning this is as follows.

PROPOSITION 6.4. *Let $Q \in \mathcal{B}(L^2(X, m))$ be admissible. Then there exists a dynamical system $(X_Q, \mathbb{G}, m_Q)$ of translation-bounded measures on $\mathbb{G}$ and a Borel factor map*

$$\pi_Q : X \ni \omega \longmapsto\!\!\!\!\!\!\rightarrow \omega_Q \in X_Q$$

such that $\mathcal{N}_\phi^Q \circ \pi_Q = Q(\mathcal{N}_\phi)$ in the L^2-sense for each $\phi \in \mathcal{C}_c(\mathbb{G})$.

Proof. We proceed by decomposing the proof into five steps.

Step 1: we fix here our notation. We choose U to be an open, symmetric and relatively compact subset of $\mathbb{G}$, and $M > 0$ a constant such that X consists of (U, M)-translation-bounded measures. We set $\mathbb{G}^0$ to be a dense countable subset of $\mathbb{G}$ containing 0, and let $(K_N)_N$ be a nested sequence of compact subsets which are the closure of their interior in $\mathbb{G}$, with the first term K_1 containing U in its interior, and whose union is $\mathbb{G}$. Next, as the group $\mathbb{G}$ is locally compact second countable, it is possible to pick a countable collection $(\phi_n^N)_{n\in\mathbb{N}} \subset \mathcal{C}_{K_N}(\mathbb{G})$, which is dense in $\mathcal{C}_{K_N}(\mathbb{G})$ for the uniform norm. In particular, the countable collection $(\phi_n^N)_{n,N\in\mathbb{N}}$ is dense in $\mathcal{C}_c(\mathbb{G})$ for the inductive limit topology. Thirdly, we denote by V the $\mathbb{G}^0$-stable $\mathbb{Q}[i]$-vector subspace of $\mathcal{C}_c(\mathbb{G})$ spanned by $(\phi_n^N * \delta_t)_{n,N\in\mathbb{N},t\in\mathbb{G}^0}$. It may be written as the countable union of countable sets

$$V = \bigcup_{l=1}^{\infty} \left\{ \sum_{j_1,j_2,j_3=1}^{l} \lambda_{j_1,j_2,j_3}\, \phi_{j_1}^{j_2} * \delta_{t_{j_3}} \mid \lambda_{j_1,j_2,j_3} \in \mathbb{Q}[i],\, t_{j_3} \in \mathbb{G}^0 \right\}$$

and is therefore itself countable. For each N, the $\mathbb{Q}[i]$-vector subspace $V_N := V \cap \mathcal{C}_{K_N}(\mathbb{G})$ is in particular countable, and dense in $\mathcal{C}_{K_N}(\mathbb{G})$ for the uniform norm. To finish these preparations, for each $\phi \in V$ we let $Q(\mathcal{N}_\phi)$ be a *chosen* Borel function of the L^2-class it represents. Since Q is assumed admissible, one has for each N a constant $R_N \geqslant 0$ such that

$$\|Q(\mathcal{N}_\phi)\|_{\infty,\mathrm{ess}} \leqslant R_N \|\phi\|_\infty \quad \text{for all } \phi \in V_N$$

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so we can assume without loss of generality that the chosen representative $Q(\mathcal{N}_\phi)$ is uniformly bounded on X by the value $R_N\|\phi\|_\infty$ whenever $\phi \in V_N$. In particular, $Q(\mathcal{N}_\phi)(\omega)$ makes sense for all $\phi \in V$ and all $\omega \in X$, and one has for each integer N

$$|Q(\mathcal{N}_\phi)(\omega)| \leqslant R_N\|\phi\|_\infty \quad \text{for all } \phi \in V_N,\ \omega \in X. \tag{23}$$

Step 2: we construct here our desired collection of measures ω_Q on $\mathbb{G}$, where $\omega \in X$. Given any two functions $\phi, \phi' \in V$ and $\lambda \in \mathbb{Q}[i]$, the Borel set of $\omega \in X$ where

$$Q(\mathcal{N}_{\lambda\phi+\phi'})(\omega) = \lambda Q(\mathcal{N}_\phi)(\omega) + Q(\mathcal{N}_{\phi'})(\omega) \tag{24}$$

is of full measure in X. As a byproduct, the countable intersection

$$X^{(0)} := \bigcap_{\phi,\phi' \in V,\ \lambda \in \mathbb{Q}[i]} \{\omega \in X \mid \text{equation (24) holds at } \omega\}$$

is a Borel subset of full measure in X. Since m is $\mathbb{G}$-invariant on X, the countable intersection $X^{(1)} := \bigcap_{t\in\mathbb{G}^0} X^{(0)} * \delta_t \subset X^{(0)}$ is Borel of full measure in X. Moreover, since Q commutes with the unitary operator U_t on $L^2(X, m)$, the set of $\omega \in X$ where

$$Q(\mathcal{N}_{\phi*\delta_t})(\omega) = QU_{-t}(\mathcal{N}_\phi)(\omega) \quad \text{is equal to } U_{-t}Q(\mathcal{N}_\phi)(\omega) = Q(\mathcal{N}_\phi)(\omega * \delta_t) \tag{25}$$

holds for all $\phi \in V$ and all $t \in \mathbb{G}^0$ is a Borel set of full measure in X, and thus intersects $X^{(1)}$ into a $\mathbb{G}^0$-stable Borel subset of full measure $X^{(2)}$ in X. Now, for any $\omega \in X^{(2)}$ the mapping $Q_\omega : \phi \longmapsto Q(\mathcal{N}_\phi)(\omega)$ is well-defined on the $\mathbb{Q}[i]$-vector space V, and is $\mathbb{Q}[i]$-linear. Thanks to equation (23) Q_ω is uniformly continuous on each $\mathbb{Q}[i]$-vector space $(V_N, \|.\|_\infty)$, so extends to a continuous $\mathbb{Q}[i]$-linear form on each $(\mathcal{C}_{K_N}(\mathbb{G}), \|.\|_\infty)$. It thus extends via the inductive limit topology to a continuous $\mathbb{Q}[i]$-linear form on $\mathcal{C}_c(\mathbb{G})$, which by continuity is in fact $\mathbb{C}$-linear on $\mathcal{C}_c(\mathbb{G})$. Thus, by the Riesz representation theorem, for each $\omega \in X^{(2)}$ there exists a complex Radon measure $\omega_Q \in \mathcal{M}(\mathbb{G})$ such that

$$Q(\mathcal{N}_\phi)(\omega) = \int_{\mathbb{G}} \phi_-\, d\omega_Q \quad \text{for all } \phi \in V,\ \omega \in X^{(2)}. \tag{26}$$

Setting $\pi_Q : X \longrightarrow \mathcal{M}(\mathbb{G})$ to be $\pi_Q(\omega) := \omega_Q$ as constructed when ω belongs to the $\mathbb{G}^0$-stable Borel subset of full measure $X^{(2)}$, and to be for instance the trivial measure when $\omega \in X\backslash X^{(2)}$, therefore yields a mapping.

Step 3: we show that π_Q is valued in the compact set $\mathcal{M}_{(U',R_1)}(\mathbb{G})$ of (U', R_1)-translation-bounded measures, where U' is any open subset of U with $\overline{U'} \subset U$ in $\mathbb{G}$, and R_1 is the constant involved in equation (23) at integer $N = 1$. Such a set U' always exists: indeed if one picks a point $s \in U$ then one gets a compact set $\{s\}$ disjoint from the closed set $\mathbb{G}\backslash U$. Then $\mathbb{G}$ being a topological group, it is a regular space and thus has an open subset U' containing s and disjoint from $\mathbb{G}\backslash U$, as desired. Let us now assume $\omega \in X^{(2)}$, and let $t \in \mathbb{G}^0$ be given: then any complex measure ω_Q satisfies (see [**13**, Proposition 1] for validity of the first equality)

$$\begin{aligned} |\omega_Q|(U - t) &= \sup\left\{\left|\int_{\mathbb{G}} \phi\, d\omega_Q\right| : \phi \in \mathcal{C}_{U-t}(\mathbb{G}),\ \|\phi\|_\infty \leqslant 1\right\} \\ &= \sup\left\{\left|\int_{\mathbb{G}} \phi(. + t)\, d\omega_Q\right| : \phi \in \mathcal{C}_U(\mathbb{G}),\ \|\phi\|_\infty \leqslant 1\right\}. \end{aligned}$$

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Since U was chosen symmetric we have $\phi \in \mathcal{C}_U(\mathbb{G})$ if and only if $\phi_- \in \mathcal{C}_U(\mathbb{G})$, and thus the above is equal to

$$\begin{aligned}
&\sup\left\{\left|\int_{\mathbb{G}} \phi_-(.+t)\, d\omega_Q\right| : \phi \in \mathcal{C}_U(\mathbb{G}),\ \|\phi\|_\infty \leqslant 1\right\}\\
&= \sup\left\{\left|\int_{\mathbb{G}} (\phi * \delta_t)_-\, d\omega_Q\right| : \phi \in \mathcal{C}_U(\mathbb{G}),\ \|\phi\|_\infty \leqslant 1\right\}\\
&\leqslant \sup\left\{\left|\int_{\mathbb{G}} (\phi * \delta_t)_-\, d\omega_Q\right| : \phi \in \mathcal{C}_{K_1}(\mathbb{G}),\ \|\phi\|_\infty \leqslant 1\right\}
\end{aligned}$$

the latter being true since K_1 was chosen to contain U in Step 1. This is in turn equal, with $V_1 := V \cap \mathcal{C}_{K_1}(\mathbb{G})$ as in Step 1 being dense in $\mathcal{C}_{K_0}(\mathbb{G})$, to

$$\sup\left\{\left|\int_{\mathbb{G}} (\phi * \delta_t)_-\, d\omega_Q\right| : \phi \in V_1,\ \|\phi\|_\infty \leqslant 1\right\}.$$

Now as $\phi \in V_1 \subset V$ and $t \in \mathbb{G}^0$ then $\phi * \delta_t$ is again in V, and by equations (25) and (26) this quantity is

$$\sup\{|Q(\mathcal{N}_\phi)(\omega * \delta_t)| : \phi \in V_1,\ \|\phi\|_\infty \leqslant 1\}.$$

This latter quantity is from equation (23) bounded by R_1. Therefore, $|\omega_Q|(U-t) \leqslant R_1$ holds for $t \in \mathbb{G}^0$. Now if an $s \in \mathbb{G}$ is given, then from the choice of U' one can in fact find a $t \in \mathbb{G}^0$ such that $U' - s \subset U - t$: indeed one easily proves (by contradiction) that there is an open neighborhood B of 0 in $\mathbb{G}$ such that $U' + B \subset U$, so by density of $\mathbb{G}^0$ one can pick up a $t \in \mathbb{G}^0$ with $t - s \in B$, yielding

$$U' - s = U' + (t-s) - t \subset U' + B - t \subset U - t,$$

as desired. This yields $|\omega_Q|(U'-s) \leqslant |\omega_Q|(U-t) \leqslant R_1$ for any $s \in \mathbb{G}$, whenever ω belongs to $X^{(2)}$, that is, any such ω_Q is (U', R_1)-translation-bounded. By construction ω_Q is trivial when $\omega \in X \backslash X^{(2)}$ so is also (U', R_1)-translation-bounded in that case. One has therefore, as desired,

$$\pi_Q : X \ni \omega \longmapsto \omega_Q \in \mathcal{M}_{(U', R_0)}(\mathbb{G}).$$

Step 4: we show here that this map is a Borel $\mathbb{G}$-map. To show that it is Borel, since $\mathcal{M}(\mathbb{G})$ carries the vague topology, it suffices to check that the map $\omega \longmapsto \int_{\mathbb{G}} \phi\, d(\omega_Q)$ is Borel for each ϕ taken in the dense subspace V of $\mathcal{C}_c(\mathbb{G})$. But this map is precisely the Borel map $Q(\mathcal{N}_\phi)$ on the Borel subset $X^{(2)}$ and the constant null map on $X \backslash X^{(2)}$, so the result follows. Now we infer that

$$\omega_Q * \delta_t = (\omega * \delta_t)_Q \quad \text{for all } \omega \in X^{(2)},\ t \in \mathbb{G}^0. \tag{27}$$

Here $(\omega * \delta_t)_Q$ makes sense since $X^{(2)}$ is $\mathbb{G}^0$-stable. That equation (27) holds comes from the fact that equation (25) holds for measures $\omega \in X^{(2)}$, by construction of $X^{(2)}$. This gives for each $\phi \in V$

$$\begin{aligned}
\int_{\mathbb{G}} \phi_-\, d(\omega_Q * \delta_t) &= \int_{\mathbb{G}} (\phi * \delta_t)_-\, d(\omega_Q) = Q(\mathcal{N}_{\phi * \delta_t})(\omega)\\
&= Q(\mathcal{N}_\phi)(\omega * \delta_t) = \int_{\mathbb{G}} \phi_-\, d(\omega * \delta_t)_Q
\end{aligned}$$

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so that, by density of V in $\mathcal{C}_c(\mathbb{G})$, the equality $\omega_Q * \delta_t = (\omega * \delta_t)_Q$ occurs on the whole space $\mathcal{C}_c(\mathbb{G})$, whenever $\omega \in X^{(2)}$ and $t \in \mathbb{G}^0$. Now to see that π_Q is a $\mathbb{G}$-map observe that, for each given $t_0 \in \mathbb{G}$, one can construct another mapping π'_Q as we just did, but with some $\mathbb{G}^{0'}$ containing t_0 playing the role of $\mathbb{G}^0$, some $\mathbb{G}^{0'}$-stable countable $\mathbb{Q}[i]$-vector subspace $V' \subset \mathcal{C}_c(\mathbb{G})$ containing our space V together with Borel representatives $Q(\mathcal{N}_\phi)'$ for each $\phi \in V'$ (which possibly differ from $Q(\mathcal{N}_\phi)$ when $\phi \in V \subseteq V'$). Then we also end up with a Borel map $\pi'_Q : X \ni \omega \longmapsto \omega'_Q \in \mathcal{M}_{(U',R_0)}(\mathbb{G})$ together with a Borel subset of full measure $X^{(2)'}$ in X such that, combining equations (26) and (27) in this case,

$$Q(\mathcal{N}_\phi)'(\omega) = \int_{\mathbb{G}} \phi_- \, d\omega'_Q$$

and

$$\omega'_Q * \delta_t = (\omega * \delta_t)'_Q \quad \text{for all } \phi \in V', \omega \in X^{(2)'}, t \in \mathbb{G}^{0'}.$$

Now the countable collection V being contained in V', and since the Borel maps $Q(\mathcal{N}_\phi)$ and $Q(\mathcal{N}_\phi)'$ are representatives of a same L^2-class so that they coincide almost everywhere on X, one has for each ω in the Borel set of full measure

$$X^{(2)} \cap X^{(2)'} \cap \bigcap_{\phi \in V} \{Q(\mathcal{N}_\phi) = Q(\mathcal{N}_\phi)'\}$$

equalities of ω_Q with ω'_Q on V, and thus on the whole space $\mathcal{C}_c(\mathbb{G})$. This shows that $\pi_Q = \pi'_Q$ almost everywhere on X, and since from its construction one has $\pi'_Q(\omega) * \delta_{t_0} = \pi'_Q(\omega * \delta_{t_0})$ for almost every ω, one deduces that $\pi_Q(\omega) * \delta_{t_0} = \pi_Q(\omega * \delta_{t_0})$ for almost every $\omega \in X$, for any $t_0 \in \mathbb{G}$, as desired.

Step 5: we now set properly the dynamical system $(X_Q, \mathbb{G}, m_Q)$ and the stated Borel $\mathbb{G}$-map π_Q, and prove validity of the formula of the statement. Let m_Q be the push-forward probability measure of m through π_Q, which is supported on the compact $\mathbb{G}$-stable set $\mathcal{M}_{(U',R_0)}(\mathbb{G})$ and ergodic. Its support X_Q is then a compact $\mathbb{G}$-stable set of (U', R_0)-translation-bounded measures, yielding our desired system $(X_Q, \mathbb{G}, m_Q)$. It now suffices to modify π_Q, if needed, on a set of null measures in order to have a Borel $\mathbb{G}$-map properly valued in X_Q, while having a Borel subset of full measure X^∞ where π_Q, as modified, still satisfies, from equation (26),

$$\pi_Q(\omega) = \omega_Q \quad \text{with } Q(\mathcal{N}_\phi)(\omega) = \int_{\mathbb{G}} \phi_- \, d\omega_Q \quad \text{for all } \phi \in V, \omega \in X^\infty.$$

This in particular ensures the equality $\mathcal{N}_\phi^Q \circ \pi_Q = Q(\mathcal{N}_\phi)$ in the L^2-sense for each $\phi \in V$. Since the linear maps

$$\mathcal{C}_c(\mathbb{G}) \ni \phi \longmapsto \mathcal{N}_\phi^Q \circ \pi_Q \in L^2(X, m), \qquad \mathcal{C}_c(\mathbb{G}) \ni \phi \longmapsto Q(\mathcal{N}_\phi) \in L^2(X, m)$$

are continuous and coincide on the dense collection V as we just saw, they must agree everywhere, concluding the proof. □

Given an admissible operator $Q \in \mathcal{B}(L^2(X, m))$, it may be shown that there is only one Borel map π_Q satisfying the statement of Proposition 6.4, up to almost everywhere equality. The Borel factor $(X_Q, \mathbb{G}, m_Q)$ of the statement is also uniquely defined (under the requirement that m_Q has full support).

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6.3. *Diffraction properties.* We check here the absolute continuity property for the diffraction $\widehat{\gamma}_Q$ arising from the system $(X_Q, \mathbb{G}, m_Q)$ previously constructed from an admissible operator Q:

PROPOSITION 6.5. *Any admissible operator $Q \in \mathcal{B}(L^2(X, m))$ has its associated system $(X_Q, \mathbb{G}, m_Q)$ having diffraction of the form $\widehat{\gamma}_Q = f_Q.\widehat{\gamma}$ for some $f_Q \in L^\infty(\widehat{\mathbb{G}}, \widehat{\gamma})$.*

Proof. One has the following set of equalities for any two $\phi, \psi \in \mathcal{C}_c(\mathbb{G})$, the first ensured by Theorem 3.6 and the third coming from the equalities $\mathcal{N}_\phi^Q \circ \pi_Q = Q(\mathcal{N}_\phi)$ and $\mathcal{N}_\psi^Q \circ \pi_Q = Q(\mathcal{N}_\psi)$,

$$\int \widehat{\phi}\overline{\widehat{\psi}}\, d\widehat{\gamma}_Q = \int_{X_Q} \mathcal{N}_\phi^Q \overline{\mathcal{N}_\psi^Q}\, dm_Q = \int_X \mathcal{N}_\phi^Q \circ \pi_Q\, \overline{\mathcal{N}_\psi^Q \circ \pi_Q}\, dm = \int_X Q(\mathcal{N}_\phi)\overline{Q(\mathcal{N}_\psi)}\, dm. \tag{28}$$

It directly follows from this equality that for each $\phi \in \mathcal{C}_c(\mathbb{G})$,

$$\|\widehat{\phi}\|_{L^2(\widehat{\mathbb{G}},\widehat{\gamma}_Q)} = \|Q(\mathcal{N}_\phi)\|_{L^2(X,m)} \leqslant \|Q\|_{\mathrm{op}} \|\mathcal{N}_\phi\|_{L^2(X,m)} = \|Q\|_{\mathrm{op}} \|\widehat{\phi}\|_{L^2(\widehat{\mathbb{G}},\widehat{\gamma})}$$

where $\|Q\|_{\mathrm{op}}$ is the operator norm of Q as a bounded operator on $L^2(X, m)$. Therefore, the identity map

$$L^2(\widehat{\mathbb{G}}, \widehat{\gamma}) \longrightarrow L^2(\widehat{\mathbb{G}}, \widehat{\gamma}_Q)$$

is well-defined and uniformly continuous on the subspace formed of functions $\widehat{\phi}$ with $\phi \in \mathcal{C}_c(\mathbb{G})$, which is dense in $L^2(\widehat{\mathbb{G}}, \widehat{\gamma})$ since $\widehat{\gamma}$ is translation-bounded (see [**38**, Proposition 3.8] and its proof). Thus, the identity map is well-defined and of operator norm less or equal to $\|Q\|_{\mathrm{op}}$ on all $L^2(\widehat{\mathbb{G}}, \widehat{\gamma})$. This means that $\|f\|_{L^2(\widehat{\mathbb{G}},\widehat{\gamma}_Q)} \leqslant \|Q\|_{\mathrm{op}} \|f\|_{L^2(\widehat{\mathbb{G}},\widehat{\gamma})}$ for any $f \in L^2(\widehat{\mathbb{G}}, \widehat{\gamma})$, and in particular one has for each $f \geqslant 0$ in $L^1(\widehat{\mathbb{G}}, \widehat{\gamma})$ the inequality

$$0 \leqslant \int f\, d\widehat{\gamma}_Q \leqslant \|Q\|_{\mathrm{op}}^2 \int f\, d\widehat{\gamma}. \tag{29}$$

Applying this to the indicator function of compact sets of the LCA group $\widehat{\mathbb{G}}$ shows that $\widehat{\gamma}_Q$ is absolutely continuous with respect to $\widehat{\gamma}$, and thus admits a Radon–Nikodym differential $f_Q := d\widehat{\gamma}_Q/d\widehat{\gamma}$ in $L^1_{\mathrm{loc}}(\widehat{\mathbb{G}}, \widehat{\gamma})$, which by an application of (29) is positive, essentially bounded by $\|Q\|_{\mathrm{op}}^2$, yielding the proof. □

6.4. *The correspondence between Borel factors and admissible operators.* Let as usual $(X, \mathbb{G}, m)$ be a measured dynamical system of translation-bounded measures, with diffraction-to-dynamic map

$$\Theta : L^2(\widehat{\mathbb{G}}, \widehat{\gamma}) \longleftarrow \mathcal{H}_\Theta \subseteq L^2(X, m)\ , \quad \Theta(\widehat{\phi}) = \mathcal{N}_\phi$$

for any $\phi \in \mathcal{C}_c(\mathbb{G})$. Then a combined formulation of Propositions 4.6, 6.4 and 6.5 is the following correspondence theorem.

THEOREM 6.6. *Let $(X, \mathbb{G}, m)$ be a dynamical system of translation-bounded measures with diffraction $\widehat{\gamma}$. There is a bijective correspondence between:*

(a) *Borel factor maps π over some dynamical system of translation-bounded measures having diffraction $\widehat{\gamma}_\pi \ll \widehat{\gamma}$, with Radon–Nikodym derivative $f_\pi := d\widehat{\gamma}_\pi/d\widehat{\gamma} \in L^\infty(\widehat{\mathbb{G}}, \widehat{\gamma})$; and*

(b) *admissible operators Q on $L^2(X, m)$.*

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A factor map π and its associated admissible operator Q determine each other according to the equality

$$Q(\mathcal{N}_\phi) = \mathcal{N}_\phi \circ \pi$$

for any $\phi \in \mathcal{C}_c(\mathbb{G})$. Moreover, the multiplication operator M_{f_π} by f_π on $L^2(\widehat{\mathbb{G}}, \widehat{\gamma})$ satisfies

$$\Theta \circ M_{f_\pi} \circ \Theta^{-1} = Q_\pi^* Q_\pi .$$

In particular, given a Borel subset $\mathcal{P} \subseteq \widehat{\mathbb{G}}$ we have $f_\pi = \mathbb{I}_\mathcal{P}$ if and only is $|Q_\pi| = \mathsf{E}(\mathcal{P}) P_\Theta$.

Here, $|Q|$ denotes the absolute value of the operator Q, that is, the unique positive operator whose square is Q^*Q, see, for instance, [**49**, §VI.4] for definition and existence.

Remark 6.7. If we are given a Borel factor map π with an associated dynamical system of translation-bounded measures having diffraction $f_\pi.\widehat{\gamma}$, $f_\pi \in L^\infty(\widehat{\mathbb{G}}, \widehat{\gamma})$, and we let Q_π be its associated admissible operator, then it is not difficult to show the equality of norms

$$\| Q_\pi \|_{\mathrm{op}} = \| f_\pi \|^{1/2}_{L^\infty(\widehat{\mathbb{G}}, \widehat{\gamma})}.$$

Indeed the operator norm is a C^*-norm so gives $\| Q_\pi \|^2_{\mathrm{op}} = \| Q_\pi^* Q_\pi \|_{\mathrm{op}}$, equal to the operator norm of the pullback of $Q_\pi^* Q_\pi$ under the isometry Θ, in turn equal to the operator norm of the multiplication operator by f_π on $L^2(\widehat{\mathbb{G}}, \widehat{\gamma})$ according to Theorem 6.6, which is nothing but $\| f_\pi \|_{L^\infty(\widehat{\mathbb{G}}, \widehat{\gamma})}$.

Remark 6.8. Given two admissible operators Q, Q' and a complex number λ, one can consider the mapping $\pi_Q + \lambda \pi_{Q'}$ from X to $\mathcal{M}^\infty(\mathbb{G})$, which is a Borel $\mathbb{G}$-map valued in a certain compact space of translation-bounded measures, and which satisfies

$$\pi_Q + \lambda \pi_{Q'} = \pi_{Q + \lambda Q'} \quad \text{(almost everywhere)}.$$

There is a similar yet less trivial result about diffraction measures, namely

$$\widehat{\gamma}_{\lambda Q} = |\lambda|^2 \widehat{\gamma}_Q, \quad \widehat{\gamma}_{Q+Q'} = \widehat{\gamma}_Q + \widehat{\gamma}_{Q'} + \nu$$

where ν is a measure of the form $g.\widehat{\gamma}$ such that the multiplication operator M_g is equal to the pullback of $Q^*Q' + Q'^*Q$ on $L^2(\widehat{\mathbb{G}}, \widehat{\gamma})$ under the isometry Θ. In particular, whenever two admissible projectors P and P' are orthogonal, meaning that $PP' = P'P = 0$ on $L^2(X, m)$, then the sum $P + P'$ is again an admissible projector and their associated Borel factor maps and diffraction of the Borel factors satisfy

$$\pi_{P+P'} = \pi_P + \pi_{P'} \text{ (almost everywhere)}, \quad \widehat{\gamma}_{P+P'} = \widehat{\gamma}_P + \widehat{\gamma}_{P'}.$$

This generalizes to any finite sums of pairwise orthogonal admissible projectors on $L^2(X, m)$.

6.5. *Examples of admissible operators: convolution operators.* We shall see here that, given a dynamical system of translation-bounded measures $(X, \mathbb{G}, m)$, one can define a whole class of admissible projectors, each associated with some bounded measure $\sigma \in \mathcal{M}^b(\mathbb{G})$. Given $\sigma \in \mathcal{M}^b(\mathbb{G})$, we define on $L^2(X, m)$ the convolution operator Q_σ

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by the operator-valued integral

$$Q_\sigma := \int_{\mathbb{G}} U_{-t} P_\Theta \, d\sigma(t).$$

Such an operator is well-defined, bounded on $L^2(X, m)$ and commutes with the unitary representation of $\mathbb{G}$. Recall that the convolution product between a bounded measure σ and a compactly supported continuous function ϕ is a continuous function belonging to $L^1(\mathbb{G})$. From [**1**, Theorem 1.2] one knows that the convolution $\omega * \sigma$ of any translation-bounded measure ω with any bounded measure σ is well-defined, and is again a translation-bounded measure. Also recall that any $\sigma \in \mathcal{M}^b(\mathbb{G})$ is Fourier transformable, with the Fourier transform given by a density $\widehat{\sigma}$ (its usual Fourier–Stieltjes transform) belonging to the space $\mathcal{C}_u^b(\widehat{\mathbb{G}})$ of bounded uniformly continuous functions on $\widehat{\mathbb{G}}$ (see, for instance, [**51**, §1.3.3]).

PROPOSITION 6.9. *The operator Q_σ is admissible, with the dynamical system of translation-bounded measures $(X_\sigma, \mathbb{G}, m_\sigma)$ and Borel $\mathbb{G}$-map given by*

$$X \ni \omega \longmapsto \pi_\sigma(\omega) = \omega * \sigma \in X_\sigma.$$

Moreover, $(X_\sigma, \mathbb{G}, m_\sigma)$ has diffraction $|\widehat{\sigma}|^2 \widehat{\gamma}$ with $|\widehat{\sigma}|^2 \in \mathcal{C}_u^b(\widehat{\mathbb{G}})$.

Proof. One has $Q_\sigma P_\Theta = Q_\sigma$ as on any $h \in L^2(X, m)$ we have

$$Q_\sigma P_\Theta(h) = \int_{\mathbb{G}} U_{-t} P_\Theta(P_\Theta(h)) \, d\sigma(t) = \int_{\mathbb{G}} U_{-t} P_\Theta(h) \, d\sigma(t) = Q_\sigma(h).$$

Let us show that $Q_\sigma(\mathcal{N}_\phi)(\omega) = \mathcal{N}_\phi(\omega * \sigma) = \mathcal{N}_{\sigma * \phi}(\omega)$ for any $\omega \in X$ and $\phi \in \mathcal{C}_c(\mathbb{G})$. Here the function $\mathcal{N}_{\sigma*\phi}$ is well-defined, since the middle term is always finite, but might not be continuous since it is not properly a function of the form given in equation (2). However, the function $\sigma * \phi$ is an absolutely integrable continuous function on $\mathbb{G}$, so by suitably approximating $\sigma * \phi$ by compactly supported continuous functions one sees that $\mathcal{N}_{\sigma*\phi}$ is a pointwise limit of functions of the form (2), and is therefore (at least) Borel on X. To show the equality above, observe that for any $\omega \in X$,

$$Q_\sigma(\mathcal{N}_\phi)(\omega) = \int_{\mathbb{G}} U_{-t}(\mathcal{N}_\phi)(\omega) \, d\sigma(t) = \int_{\mathbb{G}} \mathcal{N}_\phi(\omega * \delta_t) \, d\sigma(t)$$

in turn is equal to

$$\begin{aligned} \iint \phi(-s) \, d(\omega * \delta_t)(s) \, d\sigma(t) &= \iint \phi(-s - t) \, d\omega(s) \, d\sigma(t) \\ &= \iint \phi(-s - t) \, d\sigma(t) \, d\omega(s). \end{aligned}$$

One recognizes here the quantity $\mathcal{N}_\phi(\omega * \sigma)$, and it is a direct verification that this is precisely equal to $\mathcal{N}_{\sigma*\phi}(\omega)$, as desired.

This gives in particular for any compact set $K \subset \mathbb{G}$, K_σ and K'_σ appropriate constants only depending on σ, X and on K, the inequalities for any $\phi \in \mathcal{C}_K(\mathbb{G})$,

$$\| Q_\sigma(\mathcal{N}_\phi) \|_{\infty,\mathrm{ess}} = \| \mathcal{N}_{\sigma*\phi} \|_{\infty,\mathrm{ess}} \leqslant K_\sigma \| \sigma * \phi \|_\infty \leqslant K'_\sigma \| \phi \|_\infty.$$

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This shows admissibility of the operator Q_σ. We already have shown that the factor map π_σ of the statement satisfies

$$Q_\sigma(\mathcal{N}_\phi)(\omega) = \mathcal{N}_\phi(\omega * \sigma) = \mathcal{N}_\phi \circ \pi_\sigma(\omega)$$

so the factor and factor map associated to Q_σ are of the stated form. The diffraction $\widehat{\gamma_\sigma}$ of $(X_\sigma, \mathbb{G}, m_\sigma)$ satisfies for any pair $\phi, \psi \in \mathcal{C}_c(\mathbb{G})$

$$\begin{aligned}\int_{\widehat{\mathbb{G}}} \widehat{\phi}\overline{\widehat{\psi}}\, d\widehat{\gamma_\sigma} &= \int_X Q_\sigma(\mathcal{N}_\phi)\overline{Q_\sigma(\mathcal{N}_\psi)}\, dm = \int_X \mathcal{N}_{\sigma*\phi}\,\overline{\mathcal{N}_{\sigma*\psi}}\, dm \\ &= \int_{\widehat{\mathbb{G}}} \widehat{\sigma * \phi}\,\overline{\widehat{\sigma * \psi}}\, d\widehat{\gamma} = \int_{\widehat{\mathbb{G}}} \widehat{\phi}\overline{\widehat{\psi}}|\widehat{\sigma}|^2\, d\widehat{\gamma}\end{aligned}$$

showing that $\widehat{\gamma_\sigma} = |\widehat{\sigma}|^2\widehat{\gamma}$, as desired. □

A. *Appendix.*

A.1. *Existence of a maximal Kronecker Borel factor.* We wish here, as we could not find any reference for this statement which is certainly classical, to provide a proof of Proposition 2.3, that is, to show that for any ergodic dynamical system $(X, \mathbb{G}, m)$ with eigenvalue group $\mathcal{E} \subset \widehat{\mathbb{G}}$ the Kronecker system $(\mathrm{T}_\mathcal{E}, \mathbb{G}, m_{\mathrm{Haar}})$ is a Borel factor of $(X, \mathbb{G}, m)$.

Proof of Proposition 2.3. By ergodicity each eigenfunction $\Phi_\xi \in L^2(X, m)$, with $\xi \in \mathcal{E}$, is unique up to a multiplicative constant and can moreover be chosen of absolute value almost everywhere constant equal to 1 on X. Also, as discussed in §2.1, each Φ_ξ can also be chosen such that the equality $\Phi_\xi(x.t) = \xi(t)\Phi_\xi(x)$ occurs everywhere on a $\mathbb{G}$-stable Borel set $\tilde{X}$ and for any $t \in \mathbb{G}$. Now we have the celebrated pointwise ergodic theorem (see, for instance, [**48**, Theorem 2.14]).

THEOREM A.1. (Pointwise ergodic theorem) *Let $(X, \mathbb{G}, m)$ be an ergodic dynamical system. Then for any tempered van Hove sequence $(\mathcal{A}_n)_{n\in\mathbb{N}}$ in $\mathbb{G}$ and any Borel m-integrable function f on X, one has for m-almost every $x \in X$,*

$$\int_X f\, dm = \lim_{n\to\infty} \frac{1}{|\mathcal{A}_n|} \int_{\mathcal{A}_n} f(x.t)\, dt.$$

Using this theorem we can prove the following.

LEMMA A.2. *The eigenfunctions Φ_ξ can be chosen to satisfy*

$$\Phi_\xi.\Phi_{\xi'} = \Phi_{\xi.\xi'} \quad m\text{-almost everywhere for all } \xi, \xi' \in \mathcal{E}.$$

Proof. Consider a family $(\Phi_\xi)_{\xi\in\mathcal{E}}$ of eigenfunctions of modulus constant equal to 1 on X, and such that the equality $\Phi_\xi(x.t) = \xi(t)\Phi_\xi(x)$ occurs on a $\mathbb{G}$-stable Borel set $\tilde{X}$ and for any $t \in \mathbb{G}$. Since this collection of functions is countable there exists, by ergodicity of m, a Borel subset of full measure $X^0 \subseteq X$ such that $|\Phi_\xi| = 1$ and such that the conclusion of the pointwise ergodic theorem holds at any $x \in X^0$ and for any function in the collection

$$\{|\lambda_1\Phi_\xi.\Phi_{\xi'} - \lambda_2\Phi_{\xi.\xi'}| : \xi, \xi' \in \mathcal{E}, \lambda_1, \lambda_2 \in \mathbb{C}\}.$$

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Indeed one can find such a Borel subset where the conclusion of the pointwise ergodic theorem holds when λ_1, λ_2 are taken in a countable dense subset of $\mathbb{C}$, and one easily shows that on such a set the conclusion of the pointwise ergodic theorem also holds for any parameters $\lambda_1, \lambda_2 \in \mathbb{C}$. Now pick up such an element $x_0 \in X^0$ and set $\Phi'_\xi := \overline{\Phi_\xi(x_0)}\Phi_\xi$. Then $\Phi'_\xi.\Phi'_{\xi'} = \Phi'_{\xi.\xi'}$ along the orbit of x_0 as it is straightforward to observe, and it follows from the very choice of x_0 that

$$\int_X |\Phi'_\xi.\Phi'_{\xi'} - \Phi'_{\xi.\xi'}|\, dm = \lim_{n\to\infty} \frac{1}{|\mathcal{A}_n|} \int_{\mathcal{A}_n} |\Phi'_\xi.\Phi'_{\xi'} - \Phi'_{\xi.\xi'}|(x_0.t)\, dt = 0$$

which shows that $\Phi'_\xi.\Phi'_{\xi'} = \Phi'_{\xi.\xi'}$ almost everywhere on X. Thus, one has a new family $(\Phi'_\xi)_{\xi\in\mathcal{E}}$ of eigenfunctions, still of modulus constant equal to 1 on X and such that the equality $\Phi'_\xi(x.t) = \xi(t)\Phi_\xi(x)'$ occurs on a $\mathbb{G}$-stable Borel set $\tilde{X}$ and for any $t \in \mathbb{G}$, and obeying the stated equality. □

We complete our proof of Proposition 2.3: considering a family $(\Phi_\xi)_{\xi\in\mathcal{E}}$ of eigenfunctions as chosen and moreover satisfying the statement of Lemma A.2, one ends up with a $\mathbb{G}$-invariant Borel subset

$$X^0 := \bigcap_{\xi,\xi'\in\mathcal{E}} \{\Phi_\xi.\Phi_{\xi'} = \Phi_{\xi.\xi'}\}$$

of full m-measure in X. For each x in this Borel set, define $\pi(x)$ to be the unique character on $\mathcal{E}$ defined by $\xi \longmapsto \Phi_\xi(x)$, and to be the identity element elsewhere on X. This defines a map $\pi : X \longrightarrow \mathrm{T}_\mathcal{E}$ obeying the desired properties: indeed it is a Borel map which is $\mathbb{G}$-equivariant in the sense that for any $t \in \mathbb{G}$ one has $\pi(x.t) = \pi(x).\mathsf{i}(t)$ for m-almost every $x \in X$. From this latter property it comes that the probability measure m pushed forward on $\mathrm{T}_\mathcal{E}$ is translation invariant by the dense subgroup $\mathsf{i}(\mathbb{G})$ and thus is the Haar measure on the compact Abelian group $\mathrm{T}_\mathcal{E}$, completing the proof of the statement. □

A.2. *Disintegration over* $\mathrm{T}_\mathcal{E}$ *and the spectral projector* $\mathsf{E}(\mathcal{E})$. Given an ergodic dynamical system $(X, \mathbb{G}, m)$ with eigenvalue group $\mathcal{E}$, we show here the connection existing between disintegration of the probability measure m over $\mathrm{T}_\mathcal{E}$ and the spectral projector $\mathsf{E}(\mathcal{E})$ acting on $L^2(X, m)$, as given in Proposition 2.4:

Proof of Proposition 2.4. Given an ergodic dynamical system $(X, \mathbb{G}, m)$ with eigenvalue group $\mathcal{E}$, consider a Borel factor map $\pi : X \longrightarrow \mathrm{T}_\mathcal{E}$ as provided by Proposition 2.3, as well as the associated disintegration mapping $\mathrm{T}_\mathcal{E} \ni z \longmapsto m_z \in \mathrm{Prob}(X)$ as in Theorem 2.2. For a square-integrable Borel function f on X denote

$$F(z) := \int_X f\, dm_z.$$

One gets a function in $L^2(\mathrm{T}_\mathcal{E}, m_{\mathrm{Haar}})$, and what we need to prove here is that $\mathsf{E}(\mathcal{E})(f) = F \circ \pi$ in $L^2(X, m)$. From the general theory, the function $\mathsf{E}(\mathcal{E})(f)$ is characterized as the unique (equivalence class of) function of the form $h \circ \pi$, for some function $h \in L^2(\mathrm{T}_\mathcal{E}, m_{\mathrm{Haar}})$, such that

$$\int_X (f - \mathsf{E}(\mathcal{E})(f))(h' \circ \pi)\, dm = 0$$

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for each function $h' \in L^2(\mathrm{T}_{\mathcal{E}}, m_{\mathrm{Haar}})$. It is clear that, instead of testing the integral against any $h' \in L^2(\mathrm{T}_{\mathcal{E}}, m_{\mathrm{Haar}})$, one can restrict to indicator functions of the form $\mathbb{I}_{\pi^{-1}(B)}$ with B a Borel subset of $\mathrm{T}_{\mathcal{E}}$. Hence, it is sufficient to show that

$$\int_X (f - F \circ \pi)\mathbb{I}_{\pi^{-1}(B)}\, dm = 0$$

for any B a Borel subset of $\mathrm{T}_{\mathcal{E}}$, to get the conclusion. A quick computation gives

$$\begin{aligned}\int_X (f - F \circ \pi)\mathbb{I}_{\pi^{-1}(B)}\, dm &= \int_{\pi^{-1}(B)} f - F \circ \pi\, dm \\ &= \int_{\pi^{-1}(B)} f\, dm - \int_{\pi^{-1}(B)} F \circ \pi\, dm \\ &= \int_{\pi^{-1}(B)} f\, dm - \int_B F\, dm_{\mathrm{Haar}} \\ &= \int_{\pi^{-1}(B)} f\, dm - \int_B \left[\int_X f\, dm_z\right] dm_{\mathrm{Haar}}(z).\end{aligned}$$

Since, from Theorem 2.2(D1), the measure m_z is supported on $\pi^{-1}(z)$ for almost every $z \in \mathrm{T}_{\mathcal{E}}$ one can also write the last term as

$$\int_X f(x)\mathbb{I}_{\pi^{-1}(B)}(x)\, dm(x) - \int_{\mathrm{T}_{\mathcal{E}}} \left[\int_X f(x)\mathbb{I}_{\pi^{-1}(B)}(x)\, dm_z(x)\right] dm_{\mathrm{Haar}}(z),$$

which vanishes thanks to Theorem 2.2(D2), yielding the proof. □

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