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Local Multilevel Method on Adaptively Refined Meshes for Elliptic Problems with Smooth Complex Coefficients

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Abstract. In this paper, a local multilevel algorithm is investigated for solving linear systems arising from adaptive finite element approximations of second order elliptic problems with smooth complex coefficients. It is shown that the abstract theory for local multilevel algorithm can also be applied to elliptic problems whose dominant coefficient is complex valued. Assuming that the coarsest mesh size is sufficiently small, we prove that this algorithm with Gauss-Seidel smoother is convergent and optimal on the adaptively refined meshes generated by the newest vertex bisection algorithm. Numerical experiments are reported to confirm the theoretical analysis.

AMS subject classifications: 65F10; 65N30

Key words: Local multilevel algorithm, Adaptive finite element method, Complex coefficients, Gauss-Seidel smoother.

1. Introduction

Multilevel or multigrid methods are very efficient for solving the systems arising from the approximation of elliptic boundary value problems. The convergence properties of multigrid methods on quasi-uniform meshes have been studied by [2, 5, 7, 9, 11, 16, 18, 30, 31] and the references therein. Multigrid methods on locally refined meshes was first introduced by Brandt [10]. Subsequently, the techniques on locally refined meshes have been widely investigated, such as the multilevel adaptive technique (MLAT) in [1, 20, 27] and the fast adaptive composite grid (FAC) methods in [21–24]. Recently, Wu and Chen [33] proposed a multigrid V-cycle algorithm with Gauss-Seidel smoother on adaptive refined meshes generated by the newest vertex bisection algorithm, they proved that this algorithm is convergent and optimal when the smoother performing only on the new nodes and their "immediate" neighbors (i.e., the old nodes

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whose support or the nodal basis function has changed). Therefore, this multigrid algorithm is more robustly applied to practical problems. Based on the well known Schwarz theory [29], Xu and Chen [13, 32] proved the convergence of some local multilevel algorithms with Gauss-Seidel and Jacobi smoothers applied to second order elliptic boundary value problems.

The purpose of this paper is to study local multilevel algorithm applied to second order complex coefficient elliptic problems. Complex coefficient problems arising from time harmonic scattering and radiation have been studied in [4,8,14,19]. Gopalakrishnan and Pasciak [17] proved that multigrid algorithm is convergent on quasi-uniform meshes for second order elliptic problems with smooth complex coefficients. In this paper, adopting the techniques developed in [13], we analyze the convergence of local multilevel algorithm with Gauss-Seidel smoother applied to complex coefficient elliptic problems. The key is how to extend the assumption conditions presented in [13, 30] to complex coefficient case and verify them. Based on the perturbation arguments proposed in [17], it is shown that this algorithm converges if the mesh size of the coarsest grid is sufficiently small.

We shall use the standard notation for the Sobolev spaces $W_p^m(\Omega)$ with norm $\|\cdot\|_{m,p,\Omega}$ and the seminorm $|\cdot|_{m,p,\Omega}$ [15]. For $p = 2$, we denote $H^m(\Omega) = W_2^m(\Omega)$, $H_0^1(\Omega) = \{v \in H^1(\Omega) : v\ |_{\partial\Omega} = 0\}$, $\|\cdot\|_m = \|\cdot\|_{m,2}$ and $\|\cdot\| = \|\cdot\|_{0,2}$. Throughout this paper, the notation $x \lesssim y$ (or $x \gtrsim y$) represent the inequality $x \le Cy$ (or $x \ge Cy$), where the positive constant C is independent of all the variables in the inequality. The notation $x \approx y$ is equivalent to the statement that $x \lesssim y$ and $x \gtrsim y$.

The rest of the paper is organized as follows. In Section 2, we describe the multilevel method on adaptively refined meshes applied to a model of elliptic problem with complex coefficient. In Section 3, we present a convergence estimate for the local multilevel algorithm with Gauss-Seidel smoother. Finally, in Section 4, some numerical examples are given to show the efficiency of the local multilevel algorithm.

2. Multilevel algorithm on adaptively refined meshes

To illustrate the main idea, we consider the following second order elliptic problem [17]

$$\begin{cases} -\nabla\cdot(\alpha(x)\nabla u) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is a polygonal domain in R^2 and $\alpha(x) : \Omega \to C$ is a complex valued nonvanishing function in $C^2(\overline{\Omega})$, $f \in L^2(\Omega)$. Assuming that there exists a positive constant α_0 such that $|\alpha(x)| \ge \alpha_0$ for all $x \in \Omega$.

For brevity, we omit the variable x in the following exposition. The variational form of (2.1) is to find $u \in H_0^1(\Omega)$ such that

$$A(u, v) = (f, v) \quad \forall\, v \in H_0^1(\Omega), \tag{2.2}$$

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where the natural bilinear form is

$$A(u,v)=\int_\Omega \alpha\nabla u\cdot\nabla\overline{v}dx \quad \forall\ u,\ v\in H_0^1(\Omega)$$

and $(f,v)=\int_\Omega f\overline{v}dx$. Here, $\overline{v}$ denotes the complex conjugate of v, therefore, $(\cdot,\cdot)$ is the Hermitian inner product on $L^2(\Omega)$ and the corresponding norm is denoted by $\|\cdot\|$. For all $\phi\in H_0^1(\Omega)$, if $v\in H_0^1(\Omega)$ satisfies $A(v,\phi)=0$ then $v=0$, it follows that (2.1) has a unique solution [17].

Let $\mathcal{T}_k(k=0,1,2,\ldots,J)$ be a sequence of finite element triangulations of the domain Ω, where the initial mesh $\mathcal{T}_0$ is chosen quasi-uniform, and $\mathcal{T}_k(k\geq 1)$ is refined from $\mathcal{T}_{k-1}$ by the newest vertex bisection algorithm. It is shown in [3] that there exists a constant $\varphi>0$ such that

$$\varphi_K\geq\varphi \quad \forall\ K\in\mathcal{T}_k,\ (k=1,2,\ldots,J), \tag{2.3}$$

where φ_K is the minimum angle of the element K. Set $V_k=\left\{v_k\in H_0^1(\Omega)\middle|\ v_k|_K\in P_1(K),\ K\in\mathcal{T}_k\right\}$ is the continuous piecewise linear finite element space on $\mathcal{T}_k$. Denote the collection of the edges on $\mathcal{T}_k$ and the collection of the interior nodes of $\mathcal{T}_k$ by $\mathcal{E}_k$ and $\mathcal{N}_k$, respectively. For any $z\in\mathcal{N}_k$, let h_k^z be the shortest edge of $\mathcal{E}_k(\Omega_k^z)$, where $\mathcal{E}_k(\Omega_k^z)$ denotes the set of edges of $\mathcal{T}_k$ over Ω_k^z and the domain Ω_k^z is the union of elements containing $z\in\mathcal{N}_k$.

To maintain the optimal complexity of the multigrid algorithm on adaptively refined meshes, it is required to perform the smoothing only on local nodes [26, 33]. Let $\widehat{\mathcal{N}}_k$ be the set of new nodes and the old nodes whose support of basis function has been changed, i.e.,

$$\widehat{\mathcal{N}}_k=\left\{z\in\mathcal{N}_k: z\in\mathcal{N}_k\backslash\mathcal{N}_{k-1}\text{ or }z\in\mathcal{N}_{k-1}\text{ and }\phi_k^z\neq\phi_{k-1}^z\right\},$$

where ϕ_k^z is the basis function of node z. For convenience, we denote $\widehat{\mathcal{N}}_k=\{x_k^i,i=1,\ldots,\widehat{n}_k\}$, where $\widehat{n}_k$ is the cardinality of $\widehat{\mathcal{N}}_k$. Additionally, denote by $\phi_k^i=\phi_k^{x_k^i}$ the basis function corresponding to x_k^i, and denote by $h_{k,i}=h_k^{x_k^i}$ and Ω_k^i the support of ϕ_k^i.

Lemma 2.1. ([33]) *Let $\mathcal{T}_k(k=1,2,\ldots,J)$ be a sequence of nested conforming finite element triangulations of the domain Ω such that $\mathcal{T}_k$ is a refinement of $\mathcal{T}_{k-1}$ obtained by the newest vertex bisection algorithm. Then we have, for any $x_j^i\in\widehat{\mathcal{N}}_j$,*

$$\sum_{k=j+1}^{J}\sum_{x_k^l\in\widehat{\mathcal{N}}_k,x_k^l\in\mathcal{E}_j^i}\left(\frac{h_{k,l}}{h_{j,i}}\right)^{\frac{3}{2}}\lesssim 1,\qquad \sum_{k=j+1}^{J}\sum_{x_k^l\in\widehat{\mathcal{N}}_k,x_k^l\in\Omega_j^i}\left(\frac{h_{k,l}}{h_{j,i}}\right)^{3}\lesssim 1, \tag{2.4}$$

where $\mathcal{E}_j^i=\mathcal{E}_j(\Omega_j^i)$, and for any $x_k^l\in\widehat{\mathcal{N}}_k$,

$$\sum_{j=1}^{k-1}\sum_{x_j^i\in\widehat{\mathcal{N}}_j,x_k^l\in\mathcal{E}_j^i}\left(\frac{h_{k,l}}{h_{j,i}}\right)^{\frac{1}{2}}\lesssim 1,\qquad \sum_{j=1}^{k-1}\sum_{x_j^i\in\widehat{\mathcal{N}}_j,x_k^l\in\Omega_j^i}\left(\frac{h_{k,l}}{h_{j,i}}\right)\lesssim 1. \tag{2.5}$$

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Define the discretization operator $A_k : V_k \to V_k$ by

$$(A_k u, v) = A(u, v) \quad \forall\, u, v \in V_k$$

and the projection operators $P_k : H_0^1(\Omega) \mapsto V_k$ and $Q_k : L^2(\Omega) \mapsto V_k$ by

$$A(P_k u, v) = A(u, v) \text{ and } (Q_k u, v) = (u, v) \quad \forall\, u, v \in V_k.$$

The k^{th} level discrete problem for (2.1) is to find $u_k \in V_k$ such that

$$A(u_k, v) = (f, v) \quad \forall\, v \in V_k. \tag{2.6}$$

From the above analysis, (2.6) can be rewritten as

$$A_k u_k = f_k, \tag{2.7}$$

where $f_k = Q_k f$.

The V -cycle multigrid algorithm is an iterative solver for equations of the form (2.7): given $g \in V_k$ and initial guess $u^0 \in V_k$, defined a sequence approximating u by

$$u^{m+1} = MG_J(u^m, g), \tag{2.8}$$

where $MG_J(\cdot, \cdot) : V_J \times V_J \mapsto V_J$ is a linear map defined as follows.

Algorithm 2.1.

Set $MG_0(v, g) = A_0^{-1} g$. Let $k \geq 1$ and $v \in V_k$. Assuming that $MG_{k-1}(\cdot, \cdot)$ has been obtained, then $MG_k(v, g)$ is defined

(1) $v' = v + R_k(g - A_k v)$;

(2) $v'' = v' + MG_{k-1}(0, Q_{k-1}(g - A_k v'))$;

(3) $MG_k(v, g) = v'' + R_k^t(g - A_k v'')$,

where $R_k : V_k \mapsto V_k$ is a linear smoothing operator.

Let $T_k = R_k A_k P_k, k = 1, \ldots, J$ and $T_0 = P_0$. The error reduction operator deduced from Algorithm 2.1 is [6]

$$\mathcal{E} = EE^*, \tag{2.9}$$

where $E = (I - T_0)(I - T_1) \cdots (I - T_J)$, I is the identity operator in V_J.

When Algorithm 2.1 is applied to the problem

$$\widehat{A}(u, v) \equiv \int_\Omega \nabla u \cdot \nabla \overline{v} dx = (f, v) \quad \forall\, v \in H_0^1(\Omega), \tag{2.10}$$

which is a special case of the variable coefficient problems. Analogously, we can define the corresponding operators by replacing $A(u, v)$ with $\widehat{A}(u, v)$, to distinguish it from the general case, we add superscript "$\wedge$ "on the previously defined notation. Hence,

$$\widehat{E} = (I - \widehat{T}_0)(I - \widehat{T}_1) \cdots (I - \widehat{T}_J),$$

where $\widehat{T}_k = \widehat{R}_k \widehat{A}_k \widehat{P}_k, k = 1, \ldots, J$ and $\widehat{T}_0 = \widehat{P}_0$.

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3. Convergence for elliptic problems with complex coefficients

In this section, firstly, we extend the assumption conditions proposed in [13, 30] to complex coefficient cases. Secondly, based on the perturbation arguments presented in [17], we give convergence analysis of local multilevel algorithm with Gauss-Seidel smoother.

Chen and Xu presented the abstract theory for local multilevel algorithm applied to nonsymmetric and indefinite elliptic problems [13]. It is shown that this theory can be extended and applied to complex coefficient problems. We first present the following assumptions.

(A1) There exists a constant K_0 independent of k such that

$$\widehat{A}(v,v) \le K_0 \mathrm{Re}(\widehat{A}(Tv,v)) \ \ \forall v \in V_J,$$

where $T = \Sigma_{k=0}^{J} T_k$.

(A2) There exists a constant $0 < \omega < 2$ independent of k such that

$$\sum_{k=0}^{J} \widehat{A}(T_k v, T_k v) \le \omega \sum_{k=0}^{J} \mathrm{Re}(\widehat{A}(T_k v, v)) \ \ \forall v \in V_J.$$

(A3) For any $u, v \in V_J$, there exists a constant $K_1 > 0$ independent of k such that

$$\sum_{k=0}^{J}\sum_{j=0}^{k-1} \mathrm{Re}(\widehat{A}(T_k u, T_j v)) \le K_1 \Big(\sum_{k=0}^{J} \mathrm{Re}(\widehat{A}(T_k u, u)) \Big)^{\frac{1}{2}} \Big(\sum_{j=0}^{J} \mathrm{Re}(\widehat{A}(T_j v, v)) \Big)^{\frac{1}{2}}.$$

The following inequality is also required for the local multilevel algorithm with Gauss-Seidel smoother.

$$\begin{aligned} &\sum_{k=0}^{J} Re(\widehat{A}(T_k u, v)) \\ &\le K_2 \Big(\sum_{k=0}^{J} \mathrm{Re}(\widehat{A}(T_k u, u)) \Big)^{\frac{1}{2}} \Big(\sum_{k=0}^{J} \mathrm{Re}(\widehat{A}(T_k v, v)) \Big)^{\frac{1}{2}} \quad \forall u, v \in V_J, \end{aligned} \tag{3.1}$$

where $K_2 > 0$ is independent of k. The above assumptions are reduced to the assumptions presented in [13, 30] when applied to real coefficient problems.

Define $\| \cdot \|_{\widehat{A}}^2 = \widehat{A}(\cdot, \cdot)$, we can obtain the same convergence results as that proposed in [13, 30].

Theorem 3.1. *For sufficiently small h_0, if assumptions* (A1)-(A3) *and* (3.1) *are satisfied, then*

$$\widehat{A}(Eu, Eu) \le \delta \widehat{A}(u,u) \ \ \forall\ u \in V_J,$$

where $\delta = 1 - \frac{2-\omega}{K_0(K_1+K_2)^2}$. Hence, the error reduced operator derived from Algorithm 2.1 *can be bounded by*

$$\|\mathcal{E}\|_{\widehat{A}} = \|EE^*\|_{\widehat{A}} \le \delta.$$

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In the following, we obtain the convergence of the local multilevel Algorithm 2.1 with Gauss-Seidel smoother by verifying assumptions (A1)-(A3) and (3.1).

Decompose the space V_k into subspaces $V_k^i = span\{\phi_k^i\}$ such that

$$V_k = \sum_{i=1}^{\widehat{n}_k} V_k^i.$$

For any $v \in V_J$, considering the decomposition

$$\begin{aligned} v &= \sum_{k=0}^{J} v_k, \\ v_0 &= \Pi_0 v, \quad v_k = (\Pi_k - \Pi_{k-1})v, \quad k = 1, \ldots, J, \end{aligned}$$

where $\Pi_k : V_J \mapsto V_k$ is the Scott-Zhang interpolation operator [28]. Hence, it can be obtain that [33]

$$v_k = (\Pi_k - \Pi_{k-1})v = \sum_{i=1}^{\widehat{n}_k} v_k(x_k^i)\phi_k^i,$$

where $x_k^i \in \widehat{N}_k$ $(k = 1, \ldots, J)$. Define the operator $A_{k,i} : V_k^i \mapsto V_k^i$ by

$$(A_{k,i}u, \phi_k^i) = A(u, \phi_k^i) \ \ \forall\, u \in V_k^i$$

and the projection operator P_k^i and $Q_k^i : V_k \mapsto V_k^i$ by

$$A(P_k^i u, \phi_k^i) = A(u, \phi_k^i) \ \text{ and } \ (Q_k^i u, \phi_k^i) = (u, \phi_k^i) \quad \forall\, u \in V_k.$$

Analogously, we can also define $\widehat{A}_{k,i}, \widehat{P}_k^i, \widehat{Q}_k^i$ by replacing $A(u, v)$ by $\widehat{A}(u, v)$. Let

$$D_\kappa(u, v) = \int_\Omega (\frac{\kappa - \alpha(x)}{\kappa})\nabla u \cdot \nabla \overline{v} dx \quad \forall\, u, v \in H_0^1(\Omega), \tag{3.2}$$

where κ is a non-vanishing complex constant. Obviously, we have $A(u, v) = \kappa\widehat{A}(u, v) - \kappa D_\kappa(u, v)$. By the definitions of P_k^i and $\widehat{P}_k^i$, it can be obtained from [17] that

$$\widehat{A}((P_k^i - \widehat{P}_k^i)u, v) = D_\kappa((P_k^i - I)u, \widehat{P}_k^i v) \tag{3.3}$$

$$\widehat{A}((P_0 - \widehat{P}_0)u, v) = D_\kappa((P_0 - I)u, \widehat{P}_0 v). \tag{3.4}$$

If the parameter κ in (3.2) is chosen as $\kappa = \alpha_i := \alpha(x_k^i)$, since $\alpha(x)$ is smooth and the support of function in V_k^i extend only a distance $O(h_{k,i})$, which yields [17]

$$|\kappa - \alpha(x)| \le C h_{k,i} \|\nabla \alpha\|_{L^\infty(\Omega_k^i)}, \tag{3.5}$$

It follows from [17] that (3.3)-(3.5) imply

$$|D_{\alpha_i}((P_k^i - I)u, \widehat{P}_k^i v)| \le C h_{k,i} \|u\|_{\widehat{A}, \Omega_k^i} \|v\|_{\widehat{A}, \Omega_k^i}, \tag{3.6}$$

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$$|D_{\alpha_i}((P_0 - I)u, \widehat{P}_0 v)| \leq C\epsilon \|u\|_{\widehat{A}} \|v\|_{\widehat{A}}, \tag{3.7}$$

where $\epsilon = h_0^s (s > \frac{1}{2})$.

Now we introduce the Gauss-Seidel smoother defined on the adaptively refined meshes, which is

$$R_k = [I - (I - P_k^{\widehat{n}_k})(I - P_k^{\widehat{n}_k - 1}) \cdots (I - P_k^1)] A_k^{-1}. \tag{3.8}$$

Let

$$E_k \equiv E_k^{\widehat{n}_k} = (I - P_k^{\widehat{n}_k})(I - P_k^{\widehat{n}_{k-1}}) \cdots (I - P_k^1). \tag{3.9}$$

Obviously, we have $R_k = (I - E_k) A_k^{-1}$ and

$$T_0 = P_0, \;\; T_k = R_k A_k P_k = (I - E_k) P_k = I - E_k. \tag{3.10}$$

Set $E_k^0 = I$ and

$$E_k^i = (I - P_k^i)(I - P_k^{i-1}) \cdots (I - P_k^1) \text{ for } i = 1, \cdots, \hat{n}_k. (i = 1, \ldots, \widehat{n}_k).$$

Clearly, $E_k^i = (I - P_k^i) E_k^{i-1}$. Therefore, we obtain

$$\begin{aligned}
&\widehat{A}(E_k^{i-1} v, E_k^{i-1} v) \\
=&\widehat{A}(E_k^i v + P_k^i E_k^{i-1} v, E_k^i v + P_k^i E_k^{i-1} v) \\
=&\widehat{A}(E_k^i v, E_k^i v) + 2\text{Re}(\widehat{A}(E_k^i v, P_k^i E_k^{i-1} v)) + \widehat{A}(P_k^i E_k^{i-1} v, P_k^i E_k^{i-1} v) \\
=&\widehat{A}(E_k^i v, E_k^i v) + \widehat{A}(P_k^i E_k^{i-1} v, P_k^i E_k^{i-1} v) + 2\text{Re}(\widehat{A}((\widehat{P}_k^i - P_k^i) E_k^{i-1} v, P_k^i E_k^{i-1} v)).
\end{aligned}$$

Summing over all the local nodes $\widehat{N}_k$ on level k and since $E_k^0 = I$, it then follows from (3.3), (3.6) and (3.10) that

$$\begin{aligned}
&\sum_{i=1}^{\widehat{n}_k} \widehat{A}(P_k^i E_k^{i-1} v, P_k^i E_k^{i-1} v) \\
&= \widehat{A}(v, v) - \widehat{A}(E_k v, E_k v) + 2 \sum_{i=1}^{\widehat{n}_k} \text{Re}\Big(D_{\alpha_i}((P_k^i - I) E_k^{i-1} v, \widehat{P}_k^i P_k^i E_k^{i-1} v) \Big) \\
&\leq \widehat{A}(v, v) - \widehat{A}(E_k v, E_k v) + C \sum_{i=1}^{\widehat{n}_k} h_{k,i} \|E_k^{i-1} v\|^2_{\widehat{A}, \Omega_k^i} \\
&= 2\text{Re}\widehat{A}(T_k v, v) - \widehat{A}(T_k v, T_k v) + C \sum_{i=1}^{\widehat{n}_k} h_{k,i} \|E_k^{i-1} v\|^2_{\widehat{A}, \Omega_k^i}.
\end{aligned} \tag{3.11}$$

Lemma 3.1. ([13]) *For sufficiently small h_0, there holds*

$$\sum_{k=1}^{J} \sum_{i=1}^{\widehat{n}_k} h_{k,i} \left\| E_k^{i-1} v \right\|^2_{\widehat{A}, \Omega_k^i} \leq C h_0 \widehat{A}(v, v).$$

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Theorem 3.2. *Assume that h_0 is sufficiently small, if $R_k, T_k (k = 1, \ldots, J)$ are defined by* (3.8) *and* (3.10)*, respectively, then the assumptions* (A1)-(A3) *and* (3.1) *are satisfied.*

Proof. For convenience, the proof is mainly divided into there parts.

(1). Verification of assumption (A1). When Algorithm 2.1 is applied to the problem (2.10), there exists a constant $\widehat{K}_0 > 0$ such that [32]

$$\widehat{A}(v, v) \leq \widehat{K}_0 \mathrm{Re}(\widehat{A}(\widehat{T}v, v)) \ \ \forall v \in V_J, \tag{3.12}$$

where

$$\widehat{T} = \sum_{k=0}^{J} \widehat{T}_k, \quad \widehat{T}_0 = \widehat{P}_0, \quad \widehat{T}_k = \widehat{R}_k \widehat{A}_k \widehat{P}_k, \tag{3.13}$$

$$\widehat{\mathcal{R}}_k = [I - (I - \widehat{P}_k^{\widehat{n}_k})(I - \widehat{P}_k^{\widehat{n}_k - 1}) \cdots (I - \widehat{P}_k^1)] \widehat{A}_k^{-1}. \tag{3.14}$$

For any $v \in V_J$, due to (3.12), we have

$$\begin{aligned} \widehat{A}(v, v) &\leq \widehat{K}_0 \mathrm{Re}(\widehat{A}(\widehat{T}v, v)) = \widehat{K}_0 \mathrm{Re}(\widehat{A}(Tv, v)) + \widehat{K}_0 \mathrm{Re}(\widehat{A}((\widehat{T} - T)v, v)) \\ &= \widehat{K}_0 \mathrm{Re}(\widehat{A}(Tv, v)) + \widehat{K}_0 \mathrm{Re}(\widehat{A}((\widehat{P}_0 - P_0)v, v)) \\ &\quad + \widehat{K}_0 \sum_{k=1}^{J} \mathrm{Re}(\widehat{A}((\widehat{T}_k - T_k)v, v)). \end{aligned} \tag{3.15}$$

Let $\widehat{E}_k \equiv \widehat{E}_k^{\widehat{n}_k} = (I - \widehat{P}_k^{\widehat{n}_k})(I - \widehat{P}_k^{\widehat{n}_{k-1}}) \cdots (I - \widehat{P}_k^1)$. It follows from (3.13) and (3.14) that

$$\widehat{R}_k = (I - \widehat{E}_k)\widehat{A}_k^{-1}, \quad \widehat{T}_k = \widehat{R}_k \widehat{A}_k \widehat{P}_k = (I - \widehat{E}_k)\widehat{P}_k = I - \widehat{E}_k. \tag{3.16}$$

For $i = 1, \ldots, \widehat{n}_k$, it follows form (3.9) that

$$\begin{aligned} E_k^i - \widehat{E}_k^i &= (I - P_k^i)E_k^{i-1} - (I - \widehat{P}_k^i)\widehat{E}_k^{i-1} \\ &= (I - \widehat{P}_k^i)(E_k^{i-1} - \widehat{E}_k^{i-1}) + (\widehat{P}_k^i - P_k^i)E_k^{i-1}. \end{aligned}$$

Since $\|I - \widehat{P}_k^i\|_{\widehat{A}}$ is bounded by one, it follows that

$$\begin{aligned} &|\widehat{A}((E_k^i - \widehat{E}_k^i)v, v)| \\ \leq &|\widehat{A}((I - \widehat{P}_k^i)(E_k^{i-1} - \widehat{E}_k^{i-1})v, v)| + |\widehat{A}((\widehat{P}_k^i - P_k^i)E_k^{i-1}v, v)| \\ \leq &|\widehat{A}((E_k^{i-1} - \widehat{E}_k^{i-1})v, v)| + |\widehat{A}((\widehat{P}_k^i - P_k^i)E_k^{i-1}v, v)|. \end{aligned}$$

Summing over all the local nodes $\widehat{n}_k$ on level k yields

$$|\widehat{A}((E_k - \widehat{E}_k)v, v)| \leq \sum_{i=1}^{\widehat{n}_k} |\widehat{A}((\widehat{P}_k^i - P_k^i)E_k^{i-1}v, v)|. \tag{3.17}$$

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Combining (3.10), (3.16) with (3.17) implies

$$\begin{aligned}&\sum_{k=1}^{J}\Big|(\widehat{A}((\widehat{T}_k-T_k)v,v))\Big|\\ =&\sum_{k=1}^{J}\Big|(\widehat{A}((\widehat{E}_k-E_k)v,v))\Big|\\ \le&\sum_{k=1}^{J}\sum_{i=1}^{\widehat{n}_k}\Big|\widehat{A}((\widehat{P}_k^i-P_k^i)E_k^{i-1}v,v)\Big|.\end{aligned}\tag{3.18}$$

Since

$$I-E_k^{i-1}=\sum_{m=1}^{i-1}P_k^mE_k^{m-1},$$

we obtain

$$\begin{aligned}&\Big|\widehat{A}((\widehat{P}_k^i-P_k^i)E_k^{i-1}v,v)\Big|\\ \le&\Big|\widehat{A}((\widehat{P}_k^i-P_k^i)E_k^{i-1}v,E_k^{i-1}v)\Big|+\Big|\widehat{A}((\widehat{P}_k^i-P_k^i)E_k^{i-1}v,\sum_{m=1}^{i-1}P_k^mE_k^{m-1}v)\Big|.\end{aligned}\tag{3.19}$$

By Cauchy-Schwarz inequality and the mean value inequality, we have

$$\begin{aligned}&\sum_{i=1}^{\widehat{n}_k}\Big|\widehat{A}((\widehat{P}_k^i-P_k^i)E_k^{i-1}v,\sum_{m=1}^{i-1}P_k^mE_k^{m-1}v)\Big|\\ \le&\sum_{i,m=1}^{\widehat{n}_k}\Big|\widehat{A}((\widehat{P}_k^i-P_k^i)E_k^{i-1}v,P_k^mE_k^{m-1}v)\Big|\\ \le&C\Big(\sum_{i=1}^{\widehat{n}_k}\Big|\widehat{A}((\widehat{P}_k^i-P_k^i)E_k^{i-1}v,(\widehat{P}_k^i-P_k^i)E_k^{i-1}v)\Big|\Big)^{\frac12}\Big(\sum_{m=1}^{\widehat{n}_k}\Big\|P_k^mE_k^{m-1}v\Big\|_{\widehat{A}}^2\Big)^{\frac12}\\ \le&C\sum_{i=1}^{\widehat{n}_k}\Big|\widehat{A}((\widehat{P}_k^i-P_k^i)E_k^{i-1}v,(\widehat{P}_k^i-P_k^i)E_k^{i-1}v)\Big|+C\sum_{m=1}^{\widehat{n}_k}\Big\|P_k^mE_k^{m-1}v\Big\|_{\widehat{A}}^2,\end{aligned}\tag{3.20}$$

where the second inequality is derived form the following techniques (Lemma 5.1 of [6])

$$\begin{aligned}&\Big(\sum_{i,m=1}^{\widehat{n}_k}\Big|\widehat{A}(u_k^i,v_k^m)\Big|\Big)^2\\ \le&C\sum_{i=1}^{\widehat{n}_k}\widehat{A}(u_k^i,u_k^i)\sum_{m=1}^{\widehat{n}_k}\widehat{A}(v_k^m,v_k^m)\quad\forall u_k^i\in V_k^i,v_k^m\in V_k^m.\end{aligned}\tag{3.21}$$

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It follows from (3.3), (3.6), (3.11), (3.19) and (3.20) that

$$\begin{aligned}
&\sum_{i=1}^{\widehat{n}_k} |\widehat{A}((\widehat{P}_k^i - P_k^i)E_k^{i-1}v, v)| \\
&\le \sum_{i=1}^{\widehat{n}_k} |\widehat{A}((\widehat{P}_k^i - P_k^i)E_k^{i-1}v, E_k^{i-1}v)| \\
&\quad + C\sum_{i=1}^{\widehat{n}_k} |\widehat{A}((\widehat{P}_k^i - P_k^i)E_k^{i-1}v, (\widehat{P}_k^i - P_k^i)E_k^{i-1}v)| + C\sum_{m=1}^{\widehat{n}_k} \|P_k^m E_k^{m-1}v\|_{\widehat{A}}^2 \\
&\le C\sum_{i=1}^{\widehat{n}_k} \big(h_{k,i} + h_{k,i}^2\big)\|E_k^{i-1}v\|_{\widehat{A},\Omega_k^i}^2 + C\sum_{m=1}^{\widehat{n}_k} \|P_k^m E_k^{m-1}v\|_{\widehat{A}}^2 \\
&\le C(2\mathrm{Re}\widehat{A}(T_k v, v) - \widehat{A}(T_k v, T_k v)) + C\sum_{i=1}^{\widehat{n}_k} h_{k,i}\|E_k^{i-1}v\|_{\widehat{A},\Omega_k^i}^2 \\
&\le 2C\mathrm{Re}\widehat{A}(T_k v, v) + C\sum_{i=1}^{\widehat{n}_k} h_{k,i}\|E_k^{i-1}v\|_{\widehat{A},\Omega_k^i}^2. \qquad (3.22)
\end{aligned}$$

Thus, combining (3.4), (3.7), (3.15), (3.18), (3.22) and Lemma 3.1 yields

$$\begin{aligned}
&\widehat{A}(v, v) \\
&\le \widehat{K}_0\mathrm{Re}(\widehat{A}(Tv, v)) + \widehat{K}_0|D_{\alpha_i}((P_0 - I)v, \widehat{P}_k^i v)| + \widehat{K}_0\sum_{k=1}^{J} |(\widehat{A}((\widehat{T}_k - T_k)v, v))| \\
&\le (\widehat{K}_0 + 2C)\mathrm{Re}(\widehat{A}(Tv, v)) + C\epsilon\widehat{K}_0\widehat{A}(v, v) + C\sum_{k=1}^{J}\sum_{i=1}^{\widehat{n}_k} h_{k,i}\|E_k^{i-1}v\|_{\widehat{A},\Omega_k^i}^2 \\
&\le (\widehat{K}_0 + 2C)\mathrm{Re}(\widehat{A}(Tv, v)) + C(\epsilon\widehat{K}_0 + h_0)\widehat{A}(v, v). \qquad (3.23)
\end{aligned}$$

Therefore, when h_0 is sufficiently small, we have

$$\widehat{A}(v, v) \le K_0\mathrm{Re}(\widehat{A}(Tv, v)),$$

where

$$K_0 = \frac{(\widehat{K}_0 + 2C)}{1 - C(\epsilon\widehat{K}_0 + h_0)}.$$

The proof of assumption (A1) is complete.

(2). Verification of assumption (A2). Using the technique in (3.21), we have

$$\sum_{i,m=1}^{\widehat{n}_k} \widehat{A}(P_k^i E_k^{i-1}v, P_k^m E_k^{m-1}v) \le C\sum_{i=1}^{\widehat{n}_k} \widehat{A}(P_k^i E_k^{i-1}v, P_k^i E_k^{i-1}v). \qquad (3.24)$$

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It follows from (3.10), (3.11) and (3.24) that

$$\begin{aligned}
&\widehat{A}(T_k v, T_k v)\\
=&\widehat{A}((I-E_k)v,(I-E_k)v) = \widehat{A}(\sum_{i=1}^{\widehat{n}_k} P_k^i E_k^{i-1} v, \sum_{i=1}^{\widehat{n}_k} P_k^i E_k^{i-1} v)\\
\le& \sum_{i,m=1}^{\widehat{n}_k} \widehat{A}(P_k^i E_k^{i-1} v, P_k^m E_k^{m-1} v) \le C \sum_{i=1}^{\widehat{n}_k} \widehat{A}(P_k^i E_k^{i-1} v, P_k^i E_k^{i-1} v)\\
\le& 2C\mathrm{Re}\widehat{A}(T_k v, v) - C\widehat{A}(T_k v, T_k v) + C\sum_{i=1}^{\widehat{n}_k} h_{k,i} \|E_k^{i-1} v\|^2_{\widehat{A},\Omega_k^i}.
\end{aligned}$$

Combining the above inequality, Lemma 3.1 with assumption (A1) implies

$$\begin{aligned}
&\sum_{k=0}^{J} \widehat{A}(T_k v, T_k v)\\
\le& \frac{2C}{1+C} \sum_{k=0}^{J} \mathrm{Re}(\widehat{A}(T_k v, v)) + \frac{C}{1+C} h_0 \widehat{A}(v,v)\\
\le& \frac{2C}{1+C} \sum_{k=0}^{J} \mathrm{Re}(\widehat{A}(T_k v, v)) + \frac{C}{1+C} h_0 K_0 \sum_{k=0}^{J} \mathrm{Re}(\widehat{A}(T_k v, v)).
\end{aligned} \tag{3.25}$$

Hence, the proof of assumption (A2) is completed when h_0 is sufficiently small and

$$\omega = \frac{2C}{1+C} + \frac{C}{1+C} h_0 K_0.$$

(3). Verification of assumption (A3). In this part, we adopt the techniques developed in [13, 33]. Let $\xi = \sum_{k=j+1}^{J} T_k u$, since $T_j = I - E_j = \sum_{i=1}^{\widehat{n}_j} P_j^i E_j^{i-1}$, we obtain

$$\begin{aligned}
&\sum_{k=1}^{J}\sum_{j=1}^{k-1} \mathrm{Re}(\widehat{A}(T_k u, T_j v)) = \sum_{j=1}^{J}\sum_{k=j+1}^{J} \mathrm{Re}(\widehat{A}(T_k u, T_j v))\\
=&\sum_{j=1}^{J} \mathrm{Re}(\widehat{A}(\xi, (I-E_j)v)) = \sum_{j=1}^{J}\sum_{i=1}^{\widehat{n}_j} \mathrm{Re}(\widehat{A}(\widehat{P}_j^i \xi, P_j^i E_j^{i-1} v)).
\end{aligned}$$

By Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\sum_{k=1}^{J}\sum_{j=1}^{k-1} \mathrm{Re}(\widehat{A}(T_k u, T_j v))\\
\le& \Big(\sum_{j=1}^{J}\sum_{i=1}^{\widehat{n}_j} \widehat{A}(\widehat{P}_j^i \xi, \widehat{P}_j^i \xi)\Big)^{\frac{1}{2}} \Big(\sum_{j=1}^{J}\sum_{i=1}^{\widehat{n}_j} \widehat{A}(P_j^i E_j^{i-1} v, P_j^i E_j^{i-1} v)\Big)^{\frac{1}{2}}.
\end{aligned} \tag{3.26}$$

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It follows from (3.11) and Lemma 3.1 that

$$\begin{aligned}
&\sum_{j=1}^{J}\sum_{i=1}^{\widehat{n}_j}\widehat{A}(P_j^iE_j^{i-1}v,P_j^iE_j^{i-1}v)\\
&\le 2\sum_{j=1}^{J}\mathrm{Re}\widehat{A}(T_jv,v)+C\sum_{j=1}^{J}\sum_{i=1}^{\widehat{n}_j}h_{j,i}\|E_j^{i-1}v\|^2_{\widehat{A},\Omega_k^i}\\
&\le 2\sum_{j=1}^{J}\mathrm{Re}\widehat{A}(T_jv,v)+Ch_0\widehat{A}(v,v).
\end{aligned}\tag{3.27}$$

Due to

$$\widehat{P}_j^i\xi=\frac{\widehat{A}(\xi,\phi_j^i)}{\widehat{A}(\phi_j^i,\phi_j^i)}\phi_j^i\approx\widehat{A}(\xi,\phi_j^i)\phi_j^i,\quad \widehat{P}_k^l\phi_j^i=\frac{\widehat{A}(\phi_j^i,\phi_k^l)}{\widehat{A}(\phi_k^l,\phi_k^l)}\phi_k^l\approx\widehat{A}(\phi_j^i,\phi_k^l)\phi_k^l,$$

by Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\widehat{A}(\widehat{P}_j^i\xi,\widehat{P}_j^i\xi)\approx|\widehat{A}(\xi,\phi_j^i)|^2&\lesssim\Big(\sum_{k=j+1}^{J}\sum_{l=1}^{\widehat{n}_k}\Big|\widehat{A}(P_k^lE_k^{l-1}u,\widehat{P}_k^l\phi_j^i)\Big|\Big)^2\\
&\lesssim\Big(\sum_{k=j+1}^{J}\sum_{l=1}^{\widehat{n}_k}(\widehat{A}(P_k^lE_k^{l-1}u,P_k^lE_k^{l-1}u))^{\frac12}(\widehat{A}(\widehat{P}_k^l\phi_j^i,\widehat{P}_k^l\phi_j^i)^{\frac12}\Big)^2\\
&\approx\Big(\sum_{k=j+1}^{J}\sum_{l=1}^{\widehat{n}_k}\Big\|P_k^lE_k^{l-1}u\Big\|_{\widehat{A}}\Big|\widehat{A}(\phi_j^i,\phi_k^l)\Big|\Big)^2\\
&\approx\Big(\sum_{k=j+1}^{J}\sum_{x_k^l\in\widehat{N}_k,x_k^l\in\Omega_j^i}\Big\|P_k^lE_k^{l-1}u\Big\|_{\widehat{A}}\Big|\widehat{A}(\phi_j^i,\phi_k^l)\Big|\Big)^2.
\end{aligned}\tag{3.28}$$

Since $\phi_j^i\in\Omega_j^i$ is conforming and piecewise linear on $\mathcal{T}_j|_{\Omega_j^i}$, it follows from Green's formula that

$$\widehat{A}(\phi_j^i,\phi_k^l)=\sum_{K\subset\Omega_j^i,K\in\mathcal{T}_k}\int_K\nabla\phi_j^i\nabla\phi_k^ldx=\sum_{K\subset\Omega_j^i,K\in\mathcal{T}_k}\int_{\partial K}(\nabla\phi_j^i\cdot\nu)\phi_k^lds,$$

where ν is the outward normal to Ω_j^i. If $x_k^l\notin\mathcal{E}_j^i$ and $K\subset\Omega_j^i$, obviously

$$\phi_k^l|_{\partial K}=0,\quad|\nabla\phi_j^i|\lesssim(h_j^i)^{-1}.$$

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Hence,

$$\begin{aligned}
&\sum_{x_k^l\in\widehat{N}_k,x_k^l\in\Omega_j^i}\|P_k^lE_k^{l-1}u\|_{\widehat{A}}|\widehat{A}(\phi_j^i,\phi_k^l)|\\
&=\sum_{x_k^l\in\widehat{N}_k,x_k^l\in\Omega_j^i}\Big|\sum_{K\subset\Omega_j^i,K\in\mathcal{T}_k}\int_{\partial K}(\nabla\phi_j^i\cdot\nu)\phi_k^l ds\Big|\|P_k^lE_k^{l-1}u\|_{\widehat{A}}\\
&=\sum_{x_k^l\in\widehat{N}_k,x_k^l\in\mathcal{E}_j^i}\Big|\sum_{K\subset\Omega_j^i,K\in\mathcal{T}_k}\int_{\partial K}(\nabla\phi_j^i\cdot\nu)\phi_k^l ds\Big|\|P_k^lE_k^{l-1}u\|_{\widehat{A}}\\
&\lesssim\sum_{x_k^l\in\widehat{N}_k,x_k^l\in\mathcal{E}_j^i}\frac{h_{k,l}}{h_{j,i}}\|P_k^lE_k^{l-1}u\|_{\widehat{A}}.
\end{aligned}\tag{3.29}$$

It follows from (2.4), (3.28), (3.29) and Cauchy-Schwarz inequality that

$$\begin{aligned}
\widehat{A}(\widehat{P}_j^i\xi,\widehat{P}_j^i\xi)&\lesssim\Big(\sum_{k=j+1}^{J}\sum_{x_k^l\in\widehat{N}_k,x_k^l\in\mathcal{E}_j^i}\Big(\frac{h_{k,l}}{h_{j,i}}\Big)^{\frac{3}{4}}\Big(\frac{h_{k,l}}{h_{j,i}}\Big)^{\frac{1}{4}}\|P_k^lE_k^{l-1}u\|_{\widehat{A}}\Big)^2\\
&\lesssim\sum_{k=j+1}^{J}\sum_{x_k^l\in\widehat{N}_k,x_k^l\in\mathcal{E}_j^i}\Big(\frac{h_{k,l}}{h_{j,i}}\Big)^{\frac{1}{2}}\|P_k^lE_k^{l-1}u\|_{\widehat{A}}^2\\
&=\sum_{k=j+1}^{J}\sum_{x_k^l\in\widehat{N}_k}\Big(\frac{h_{k,l}}{h_{j,i}}\Big)^{\frac{1}{2}}\|P_k^lE_k^{l-1}u\|_{\widehat{A}}^2\delta(x_j^i,x_k^l),
\end{aligned}$$

where $\delta(x_j^i,x_k^l)=1$ if $x_k^l\in\mathcal{E}_j^i$ and $\delta(x_j^i,x_k^l)=0$ otherwise. Combining the above inequality with (2.5) implies

$$\begin{aligned}
&\sum_{j=1}^{J}\sum_{i=1}^{\widehat{n}_j}\widehat{A}(\widehat{P}_j^i\xi,\widehat{P}_j^i\xi)\lesssim\sum_{j=1}^{J}\sum_{i=1}^{\widehat{n}_j}\sum_{k=j+1}^{J}\sum_{x_k^l\in\widehat{N}_k}\Big(\frac{h_{k,l}}{h_{j,i}}\Big)^{\frac{1}{2}}\|P_k^lE_k^{l-1}u\|_{\widehat{A}}^2\delta(x_j^i,x_k^l)\\
&=\sum_{j=1}^{J}\sum_{x_j^i\in\widehat{N}_j}\sum_{k=j+1}^{J}\sum_{x_k^l\in\widehat{N}_k}\Big(\frac{h_{k,l}}{h_{j,i}}\Big)^{\frac{1}{2}}\|P_k^lE_k^{l-1}u\|_{\widehat{A}}^2\delta(x_j^i,x_k^l)\\
&=\sum_{k=2}^{J}\sum_{x_k^l\in\widehat{N}_k}\Big(\sum_{j=1}^{k-1}\sum_{x_j^i\in\widehat{N}_j}\Big(\frac{h_{k,l}}{h_{j,i}}\Big)^{\frac{1}{2}}\delta(x_j^i,x_k^l)\Big)\|P_k^lE_k^{l-1}u\|_{\widehat{A}}^2\lesssim\sum_{k=2}^{J}\sum_{l=1}^{\widehat{n}_k}\|P_k^lE_k^{l-1}u\|_{\widehat{A}}^2.
\end{aligned}\tag{3.30}$$

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It follows from (3.26), (3.27), (3.30) and assumption (A1) that

$$\begin{aligned}
&\sum_{k=1}^{J}\sum_{j=1}^{k-1}\mathrm{Re}(\widehat{A}(T_k u, T_j v))\\
\lesssim&\Big(2\sum_{k=2}^{J}\mathrm{Re}\widehat{A}(T_k u,u)+Ch_0\widehat{A}(u,u)\Big)^{\frac{1}{2}}\Big(2\sum_{j=1}^{J}\mathrm{Re}\widehat{A}(T_j v,v)+Ch_0\widehat{A}(v,v)\Big)^{\frac{1}{2}}\\
\lesssim&\Big(\sum_{k=0}^{J}\mathrm{Re}\widehat{A}(T_k u,u)\Big)^{\frac{1}{2}}\Big(\sum_{j=0}^{J}\mathrm{Re}\widehat{A}(T_j v,v)\Big)^{\frac{1}{2}}.
\end{aligned}$$

Analogously, we can prove that

$$\sum_{k=1}^{J}\mathrm{Re}(\widehat{A}(T_k u,T_0 v))\lesssim\Big(\sum_{k=0}^{J}\mathrm{Re}(\widehat{A}(T_k u,u))\Big)^{\frac{1}{2}}\Big(\sum_{k=0}^{J}\mathrm{Re}(\widehat{A}(T_k v,v))\Big)^{\frac{1}{2}}.$$

The proof of assumption (A3) is completed. Similar to the derivation of assumption (A3), estimation (3.1) can also be obtained.

4. Numerical examples

In this section, we present several numerical examples to illustrate the optimality of Algorithm 2.1. The implementation is based on FFW toolbox [12].

In the experiment, the domain Ω is refined into families $\mathcal{T}_k(k=0,1,2,\ldots,J)$ of triangulations at each level generated by the newest vertex bisection algorithm. The local error estimator is defined as in [25]. At the k^{th} level, the discrete equation is $\mathbf{A}_k\mathbf{u}_k=\mathbf{F}_k$. Set $\mathbf{u}_k^0=\mathbf{u}_{k-1}$, where $\mathbf{u}_{k-1}$ is the multigrid solution of the previous level. Let $\mathbf{r}_k^m=\mathbf{F}_k-\mathbf{A}_k\mathbf{u}_k^m(m=0,1,\ldots)$ be the residual at the m^{th} iteration, then the stopping rule of the multigrid algorithm is

$$\frac{|e_m|}{|e_0|}\leq 10^{-14}, \tag{4.1}$$

where $e_m=(\mathbf{r}_k^m)^t\mathbf{A}_k\mathbf{r}_k^m$ and $e_0=(\mathbf{r}_k^0)^t\mathbf{A}_k\mathbf{r}_k^0$. Moreover, if the number of the iteration exceeds 1000, the multigrid iteration is also terminated. The number of iteration steps required to achieve the condition (4.1) is denoted by "IT". And the average reduction factor is denoted by ρ, which is computed by $\rho=(\sqrt{|e_n|}/\sqrt{|e_0|})^{1/IT}$.

Example 4.1. Consider the complex coefficients equation

$$-\nabla\cdot((1+iK\sin(\pi(2y-1)/2))\nabla u)=f \tag{4.2}$$

on the L-shaped domain $\Omega=[-1,1]\times[-1,1]\backslash(0,1]\times[-1,0)$. where K is a real constant, the Dirichlet boundary condition and f are chosen such that the exact solution in polar conditions is given by $u(r,\theta)=(1+i)r^{\frac{2}{3}}\sin(\frac{2}{3}\theta)$.

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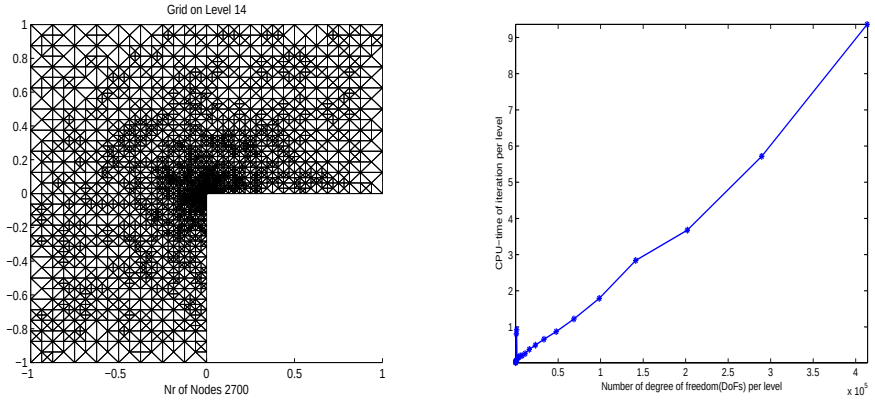


Figure 1: The adaptively refined mesh (left) at the eighth level and CPU time (right) of Algorthm 2.1 for solving (4.2)(K=0).

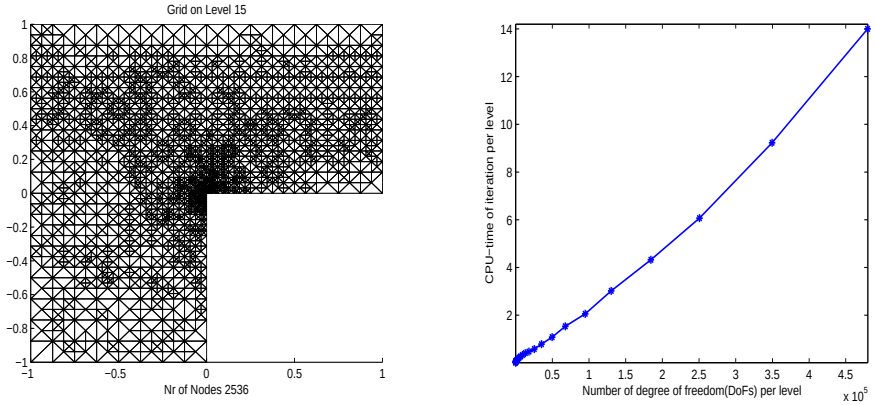


Figure 2: The adaptively refined mesh (left) at the eighth level and CPU time (right) of Algorthm 2.1 for solving (4.2)(K=1).

Table 1: Numerical results of Algorithm 2.1 for solving (4.2)($K = 0$).

level	DOFs	IT	ρ	$\|u - u_h\|_A$	$\|u - u_h\|$
11	818	12	0.2467	4.5609e-2	6.6804e-4
15	3736	12	0.2554	2.1506e-2	1.4526e-4
19	16147	12	0.2523	1.0341e-2	3.4124e-5
23	68649	12	0.2275	5.0177e-3	8.0038e-6
25	141273	12	0.2423	3.5023e-3	3.9127e-6
26	201989	11	0.2295	2.9281e-3	2.7722e-6
27	289500	11	0.2266	2.4471e-3	1.9485e-6
28	413424	11	0.2254	2.0472e-3	1.3035e-6

In the computation, the initial mesh size is chosen as $h_0 = \frac{1}{4}$, and the constant K in (4.2) is selected as $K = 0, 1, 10, 20$. The numerical results are given in Tables 1-4 and Figs. 1-4, respectively, where "DOFs" denotes the number of degrees of freedom, $\|u - u_h\|_A$ and $\|u - u_h\|$ are the error between the exact solution u and the discrete

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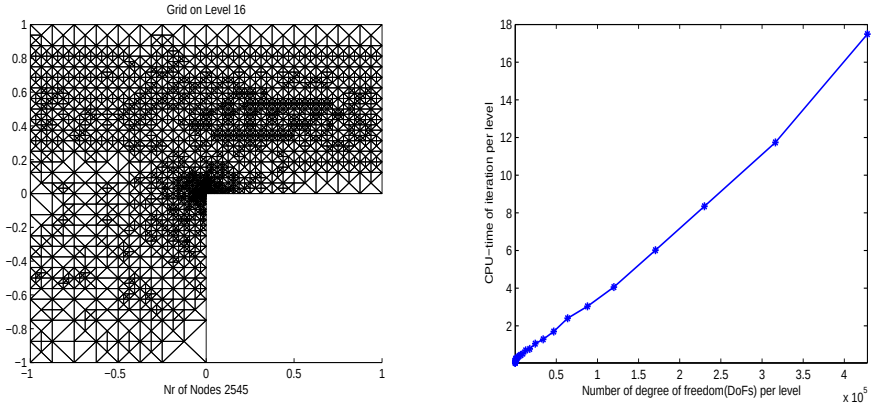


Figure 3: The adaptively refined mesh (left) at the eighth level and CPU time (right) of Algorithm 2.1 for solving (4.2)(K=10).

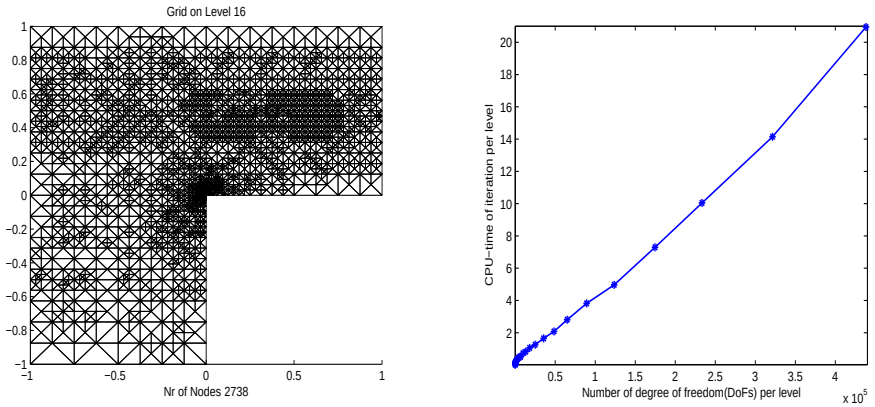


Figure 4: The adaptively refined mesh (left) at the eighth level and CPU time (right) of Algorithm 2.1 for solving (4.2)(K=20).

Table 2: Numerical results of Algorithm 2.1 for solving (4.2)($K = 1$).

level	DOFs	IT	ρ	$\|u - u_h\|_A$	$\|u - u_h\|$
14	1692	12	0.2481	4.5950e-2	7.2260e-4
18	6677	12	0.2559	2.3130e-2	1.9702e-4
22	25449	12	0.2538	1.1896e-2	5.0503e-5
26	94824	12	0.2498	6.1582e-3	1.3991e-5
28	184315	12	0.2459	4.4200e-3	7.7197e-6
29	250585	12	0.2442	3.8429e-3	5.7504e-6
30	349587	12	0.2410	3.1974e-3	3.7515e-6
31	479741	11	0.2291	2.7463e-3	2.7945e-6

solution u_h in the energy norm and L^2-norm, respectively. It clearly shows the optimal convergence of Algorithm 2.1, i.e., $\|u - u_h\| \approx DOFs^{-1}$ and $\|u - u_h\|_A \approx DOFs^{-\frac{1}{2}}$. It can be also observed that the number of the iteration on each level is bounded independently of the mesh levels, which confirms our theoretical analysis. We find that the

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Table 3: Numerical results of Algorithm 2.1 for solving (4.2)($K = 10$).

level	DOFs	IT	ρ	$\|u - u_h\|_A$	$\|u - u_h\|$
15	1695	23	0.4859	3.4357e-1	1.8684e-3
19	6552	21	0.4628	1.7341e-1	4.6106e-4
23	24408	21	0.4524	8.9994e-2	1.9315e-4
27	87860	19	0.4260	4.6354e-2	3.8600e-5
29	170455	19	0.4194	3.3258e-2	2.1409e-5
30	230037	19	0.4163	2.8558e-2	1.7710e-5
31	316315	18	0.4057	2.4204e-2	1.5211e-5
32	428048	18	0.4027	2.0832e-2	1.2236e-5

Table 4: Numerical results of Algorithm 2.1 for solving (4.2)($K = 20$).

level	DOFs	IT	ρ	$\|u - u_h\|_A$	$\|u - u_h\|$
15	1795	27	0.5468	6.2081e-1	1.9464e-3
19	6826	25	0.5217	3.2223e-1	4.6726e-4
23	25002	25	0.5187	1.7140e-1	1.8644e-4
27	89372	24	0.5013	9.0623e-2	3.9234e-5
29	174676	23	0.4901	6.6091e-2	2.1674e-5
30	233465	23	0.4863	5.6773e-2	1.8102e-5
31	321568	22	0.4769	4.7621e-2	1.5318e-5
32	438390	22	0.4746	4.0705e-2	1.3050e-5

number of the iteration increases as the constant K is chosen larger, it shows that the complex coefficients problem need more work than real coefficients problem(the case $K = 0$).

Figs. 1-4 show the locally refined meshes on the eighth level (left) and the CPU time of each iteration of Algorithm 2.1 (right). We observe that the CPU time is almost linear in terms of the number of interior nodes used, which clearly indicates the optimality of Algorithm 2.1.

Example 4.2. Consider the complex coefficient equation

$$-\nabla \cdot ((r^2 + r^4 e^{4i\theta})\nabla u) = 1 + i \tag{4.3}$$

on Hexagon domain with homogeneous Dirichlet boundary condition.

To illustrate the convergence of Algorithm 2.1, we give an example which is not coercive (the coefficient may be equal to zero in absolute value). The numerical results are reported in Table 5 and Fig. 5. From Table 5, we observe that the iteration steps and the average reduction factor on each level are bounded independently of the mesh levels, as expected from the theory. The linear relation between the CPU time and the number of interior nodes in Fig. 5 (right) indicates the optimality of Algorithm 2.1.

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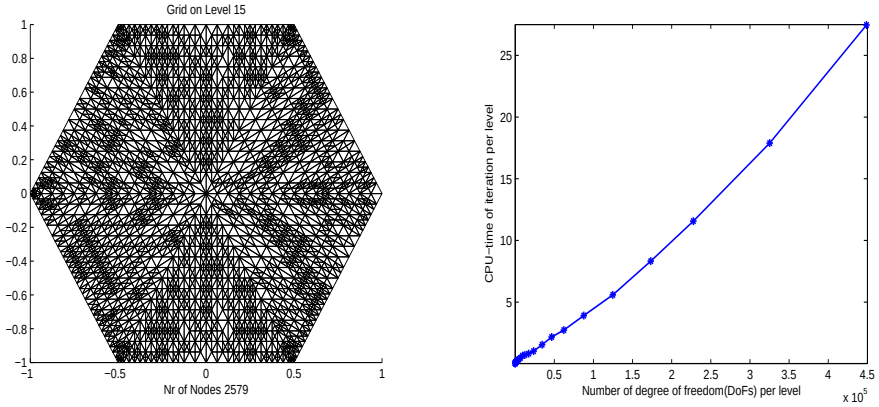


Figure 5: The adaptively refined mesh (left) at the eighth level and CPU time (right) of Algorthm 2.1 for solving (4.3).

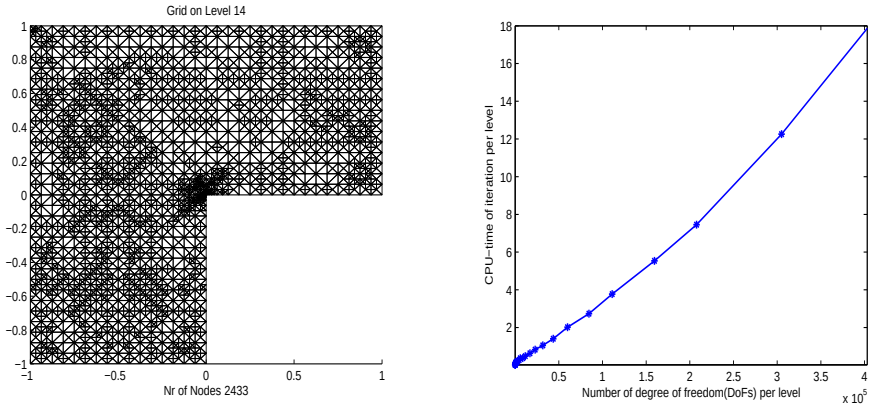


Figure 6: The adaptively refined mesh (left) at the eighth level and CPU time (right) of Algorthm 2.1 for solving (4.4).

Table 5: Iterations steps and average reduction factor for solving (4.3).

level	14	18	22	26	28	29	30	31
DOFs	1640	6416	23587	87644	173280	227673	325301	448935
IT	19	20	21	23	24	24	24	24
ρ	0.4085	0.4269	0.4616	0.4824	0.4951	0.4966	0.4994	0.5048

Example 4.3. Consider the complex coefficients equation

$$-\nabla\cdot((1-i\sin(\pi(2y-1)/2))\nabla u)=1+i \tag{4.4}$$

on the L-shaped domain $\Omega=[-1,1]\times[-1,1]\backslash(0,1]\times[-1,0)$ with homogeneous Dirichlet boundary condition.

Finally, we test this example without exact solution. The numerical result in Table 6 shows that the number of the iteration and the average reduction factor on each level is bounded independently of the mesh levels. Additionally, Fig. 6 shows that the CPU

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Table 6: Iterations steps and average reduction factor for solving (4.4).

level	13	17	21	25	27	28	29	30
DOFs	1528	6001	22926	84578	159464	207712	305040	403127
IT	15	17	18	19	19	19	20	20
ρ	0.3348	0.370	0.3931	0.4121	0.4208	0.4250	0.4236	0.4307

time is almost linear in terms of the number of interior nodes used. Therefore, it can be obtained that Algorithm 2.1 is optimal.

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