East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.060317.020617a

Vol. 7, No. 3, pp. 603-614
August 2017

Primal-Dual Active Set Method for American Lookback Put Option Pricing

Haiming Song[1], Xiaoshen Wang[2], Kai Zhang[1,*] and Qi Zhang[3]

[1] *Department of Mathematics, Jilin University Changchun, Jilin, 130012, China.*
[2] *Department of Mathematics and Statistics, University of Arkansas at Little Rock, Little Rock, Arkansas, 72204, USA.*
[3] *School of Science, Shenyang University of Technology, Shenyang, Liaoning, 110870, China.*

Received 6 March 2017; Accepted (in revised version) 2 June 2017.

Abstract. The pricing model for American lookback options can be characterised as a two-dimensional free boundary problem. The main challenge in this problem is the free boundary, which is also the main concern for financial investors. We use a standard technique to reduce the pricing model to a one-dimensional linear complementarity problem on a bounded domain and obtain a corresponding variational inequality. The inequality is discretised by finite differences and finite elements in the temporal and spatial directions, respectively. By enforcing inequality constraints related to the options using Lagrange multipliers, the discretised variational inequality is reformulated as a set of semi-smooth equations, which are solved by a primal-dual active set method. One of the major advantages of our algorithm is that we can obtain the option values and the free boundary simultaneously, and numerical simulations show that our approach is as efficient as some other methods.

AMS subject classifications: 35A35, 90A09, 65K10, 65M12, 65M60

Key words: American lookback option, linear complementarity problem, variational inequality, finite element method, primal-dual active set method.

1. Introduction

The valuation of financial derivatives is important in financial markets such as the commodity market or the stock market. Options are special derivatives used to insure investments against a market downturn. According to the exercise time, options can be divided into two categories — viz. European style and American style. Due to its early exercise possibility, the American style is more popular with investors, corporations, mutual funds, and financial institutions. Here we focus on valuation of complex and flexible American lookback options, which are path-dependent with payoff functions depending on the maximum or minimum of the underlying asset price over the life of the options. Compared with

*Corresponding author. *Email address:* kzhang@jlu.edu.cn (K. Zhang)

 ©2017 Global-Science Press

https://doi.org/10.4208/eajam.060317.020617a Published online by Cambridge University Press

other options, the advantage of the lookback option model is twofold. On the one hand, it offers opportunities for investors whose concern is underlying asset price fluctuations during the life of the options but not the price on the maturity date. On the other hand, lookback options can be exercised at any time up to the maturity date as for other American options. However, in the absence of closed form solutions, the fast, efficient and reliable pricing methods for lookback options are in great demand.

Let S be the underlying asset price. In the classical framework of the Black-Scholes model [23, 26, 31], the payoff function of lookback options with a lifetime $[0, T]$ can be represented as

$$G(S,J,t) = \begin{cases} J-S & \text{for put option with } J = \max\limits_{0\le\varsigma\le t} S_\varsigma \\ S-J & \text{for call option with } J = \min\limits_{0\le\varsigma\le t} S_\varsigma \end{cases}, \quad 0 \le t \le T .$$

We consider lookback put option pricing problems — lookback call options can be treated similarly. Assume that the interest rate is a constant r, and let σ and q denote the volatility and dividend rate of S, respectively. Then the American lookback put option value $P(S,J,t)$ is the solution of the free boundary problem

$$\mathscr{L}P := P_t + \frac{\sigma^2}{2}S^2P_{SS} + (r-q)SP_S - rP = 0 , \quad B(t) < S \le J < +\infty , \quad 0 \le t < T . \quad (1.1)$$

This model has been solved by various numerical methods [7,20,22]. The binomial method, used to price classical American options [4, 6], was generalised and applied to American lookback options [1, 18, 19]. The finite difference and finite element methods have also been used in pricing American lookback options [5, 8].

In the free boundary problem (1.1), it is notable that one has to determine the optimal exercise boundary $B(t)$ in advance. To avoid the resulting computational difficulties, the problem (1.1) can be reformulated as a linear complementarity problem [17]:

$$\begin{cases} \mathscr{L}P \cdot (P-G) = 0 \\ \mathscr{L}P \le 0 , \ P \ge G \end{cases}, \quad 0 \le S \le J < +\infty , \quad 0 \le t < T . \quad (1.2)$$

This formulation does not contain any information about the free boundary. Numerical methods such as the Projected Successive Over Relaxation (PSOR) [12], projection and contraction [24], penalty [32], Lagrangian [15] and augmented Lagrangian [28] methods have been developed for the linear complementarity problem. These strategies have also been applied to the valuation of American lookback options [25,27,30]. A finite difference method to solve the linear complementarity problem (1.2) for American lookback options has been considered, where the option prices are obtained by the PSOR method [30]. The projection and contraction method has also been considered for the discretised variational inequality corresponding to American lookback options [25].

Here we also consider the linear complementarity problem to value American lookback options. According to Ref. [30], the linear complementarity problem (1.2) can be expressed as

$$\begin{cases} \mathscr{L}P \cdot (P-G) = 0 \\ \mathscr{L}P \le 0 , \ P \ge G \end{cases}, \quad JX \le S \le J < +\infty , \quad 0 \le t < T , \quad (1.3)$$

https://doi.org/10.4208/eajam.060317.020617a Published online by Cambridge University Press

where

$$X = \frac{-r+q+\frac{1}{2}\sigma^2 - \sqrt{(-r+q+\frac{1}{2}\sigma^2)^2+2r\sigma^2}}{-r+q-\frac{1}{2}\sigma^2 - \sqrt{(-r+q+\frac{1}{2}\sigma^2)^2+2r\sigma^2}}, \qquad B(T) = J\min\left(\frac{r}{q},\ 1\right).$$

Following Ref. [25], we obtain a variational inequality corresponding to the problem (1.3) and solve it by a finite element method. The resulting matrix is symmetric and positive definite, so the discretised variational inequality can be transformed into a convex minimisation problem and solved by the primal-dual active set method. It is well known that the primal-dual active set method is extremely efficient for quadratic programming problems with linear constraints [2,3,14], and it can be interpreted as a semi-smooth Newton method [11] — cf. also Refs. [9,13,21]. One of the main advantages of the primal-dual active set method is that the optimal exercise boundary can be obtained using a Lagrange multiplier. In Section 2, we apply the numeraire transformation to reduce the pricing model to a one-dimensional linear complementarity problem and obtain a corresponding variational inequality. Numerical approximations of the resulting variational inequality are also presented. In Section 3, the primal-dual active set method for solution of the discretised system is discussed. Section 4 presents the results of some numerical simulations, and our concluding remarks are provided in Section 5.

2. Discretised Model

In this section, numerical approximations of the two-dimensional parabolic linear complementarity problem (1.3) on an unbounded domain are considered. Our work can be split into two steps. First, we use the numeraire transformation and reduce the linear complementarity problem to a one-dimensional problem on a bounded domain. We then obtain a variational inequality corresponding to the American lookback options, and our finite element method. The resulting discretised system can be solved by the primal-dual active set method.

2.1. Variational inequality

Taking the underlying asset price S as the numeraire [17, 30] and applying the transformations

$$x = \ln\frac{J}{S}, \quad \tau = T-t, \quad P(S,J,t) = Su(x,\tau), \tag{2.1}$$

for the problem (1.3), we obtain the one-dimensional bounded linear complementarity problem

$$\begin{cases} (u_\tau - \gamma u_{xx} - \mu u_x + qu)\cdot(u-g) = 0 \\ u_\tau - \gamma u_{xx} - \mu u_x + qu \geq 0, \ \ u-g \geq 0 \end{cases}, \quad 0 \leq x < L, \ \ 0 < \tau \leq T, \tag{2.2}$$

https://doi.org/10.4208/eajam.060317.020617a Published online by Cambridge University Press

with the initial and boundary conditions

$$\begin{aligned} &u(x,0)=g(x)\,, && 0\le x\le L\,,\\ &u_x(0,\tau)=0\,, && u(L,\tau)=g(L)\,,\quad 0<\tau\le T\,, \end{aligned}$$

where

$$L=\ln\frac{1}{X}\,,\quad \gamma=\frac{\sigma^2}{2}\,,\quad \mu=q-r-\gamma\,,\quad g(x)=e^x-1\,.$$

In order to obtain a symmetric discretised matrix, we follow Ref. [30]. Using the transformation

$$\frac{1}{\kappa(x)}\big(\gamma\kappa(x)u_x\big)_x=\gamma u_{xx}+\mu u_x$$

with $\kappa(x):=e^{(\mu/\gamma)x}$, we eliminate the first order derivative term in the problem (2.2), so it takes the form

$$\begin{cases} \big(\kappa(x)u_\tau-\big(\gamma\kappa(x)u_x\big)_x+q\kappa(x)u\big)\cdot(u-g)=0\\ \kappa(x)u_\tau-\big(\gamma\kappa(x)u_x\big)_x+q\kappa(x)u\ge 0\,,\quad u-g\ge 0 \end{cases},\quad 0\le x<L\,,\ 0<\tau\le T\,. \tag{2.3}$$

Further, we want to rewrite this problem (2.3) as a variational inequality, and to do so we need an auxiliary result. Let $H_L^1(\Omega)$, $\Omega=[0,L]$ be the set

$$H_L^1(\Omega):=\big\{v\in H^1(\Omega);\ v\ge g(x),\ v(L)=g(L)\big\}\,.$$

Lemma 2.1. *If an element $u\in L^2(0,T;H^2(\Omega))$ meets the conditions $u_\tau\in L^2(0,T;L^2(\Omega))$ and $u(0,\tau)>g(0)$, $0<\tau\le T$, then u is a solution of the linear complementarity problem* (2.3) *if and only if it satisfies the variational inequality*

$$\big(\kappa(x)u_\tau,v-u\big)+\big(\gamma\kappa(x)u_x,v_x-u_x\big)+\big(q\kappa(x)u,v-u\big)\ge 0\,, \tag{2.4}$$

for all $v\in H_L^1(\Omega)$ and $0<\tau\le T$.

Remark 2.1. In the American lookback option pricing problem, the condition $u(0,\tau)>g(0)$ holds true naturally [25], so the linear complementarity problem (2.3) and the variational inequality (2.4) have the same solution.

Thus the linear complementarity problem for American lookback options is converted to the variational inequality (2.4) on a bounded rectangular domain. The next task is to derive approximations of this inequality (2.4) by a discretisation method.

2.2. Finite element approximations

We now present a completely discrete form of the variational inequality (2.4) involving finite elements and the backward Euler method in the spatial and temporal directions, respectively. The stability and convergence of this method for the inequality (2.4) have been established in Refs. [25, 29].

https://doi.org/10.4208/eajam.060317.020617a Published online by Cambridge University Press

Let I_τ and I_h be the partitions

$$I_\tau : 0 = \tau_0 < \tau_1 < \cdots < \tau_M = T\,,\ \ \Delta\tau = \frac{T}{M}\,,\ \ \tau_j = j\Delta\tau\,,\ j = 0, \cdots, M\,,$$
$$I_h : 0 = x_0 < x_1 < \cdots < x_N = L\,,\ \ h = \frac{L}{N}\,,\ \ x_i = ih,\ i = 0, \cdots, N\,,$$

where M and N are positive integers; and let $V_h(\Omega)$ denote the subset of piecewise linear polynomials from $H^1(\Omega)$. The piecewise linear finite element space is then defined by

$$S_L^1(\Omega) := \left\{ v_h \in V_h(\Omega);\ v(x_n) \geq g(x_n),\ \forall\ x_n\,,\ v_h(L) = g(L) \right\},$$

and the semi-discrete form of the problem (2.4) can be formulated as follows:

Find $u_h(\tau) \in S_L^1(\Omega)$ such that $u_h(x,0) = g_I(x)$, and for any $v_h \in S_L^1(\Omega)$ and any $\tau \in (0,T]$ the inequality

$$\left(\kappa(x)u_{h\tau}, v_h - u_h\right) + \left(\gamma\kappa(x)u_{hx}, v_{hx} - u_{hx}\right) + \left(q\kappa(x)u_h, v_h - u_h\right) \geq 0 \tag{2.5}$$

holds, where $g_I(x)$ denotes the piecewise linear interpolation of the function $g(x) \in S_L^1(\Omega)$.

Lemma 2.2 ([25,30]). *The solution of the variational inequality* (2.5) *is stable and satisfies the estimate*

$$C\|u_h(\tau)\|^2 \leq 4\|u_h(0)\|^2 + 6\|g_I\|^2 + \gamma\tau\|g_{Ix}\|^2 + q\tau\|g_I\|^2\,, \quad 0 < \tau \leq T\,,$$

where C is a constant independent of h.

As mentioned above, we discretise inequality (2.5) in the temporal direction by the backward Euler method. For any fixed point $\tau = \tau_m$, the fully discrete approximation of the variational inequality (2.4) is thus

$$\left(\kappa(x)\left((1 + q\triangle\tau)u_h^m - u_h^{m-1}\right), v_h - u_h^m\right) + \triangle\tau\left(\gamma\kappa(x)u_{hx}^m, v_{hx} - u_{hx}^m\right) \geq 0\,, \tag{2.6}$$

for all $v_h \in S_L^1(\Omega)$.

Let $\{\varphi_i(x)\}_{i=0,\cdots,N}$ be a basis in $S_L^1(\Omega)$. Then for any $\tau = \tau_m$, the finite element approximation of the solution u_h^m can be written in the form

$$u_h^m = \sum_{j=0}^{N-1} u_j^m \varphi_j(x) + g(L)\varphi_N(x), \quad m = 0, 1, \cdots, M\,.$$

Applying the finite element method to the variational inequality (2.6) for all $V \geq G$, we obtain

$$\left(\left(\gamma\triangle\tau A + (1 + q\triangle\tau)B\right)U^m, V - U^m\right) \geq (BU^{m-1} - F, V - U^m), \quad m = 1, \cdots, M\,, \tag{2.7}$$

https://doi.org/10.4208/eajam.060317.020617a Published online by Cambridge University Press

where

$$V=(v_0,v_1,\cdots,v_{N-1}), \qquad U^m=(u_0^m,u_1^m,\cdots,u_{N-1}^m),$$
$$A=\left(A_{i,j}\right)_{N\times N}, \qquad B=\left(B_{i,j}\right)_{N\times N}, \qquad i,j=0,\cdots,N-1,$$
$$G=\left(g(x_0),\cdots g(x_{N-1})\right)^T, \qquad F=\left(0,\cdots,0,g(L)\triangle\tau(qB_{N-1,N}+\gamma A_{N-1,N})\right)^T,$$
$$A_{i,j}=\left(\kappa(x)\varphi_j',\varphi_i'\right), \qquad B_{i,j}=\left(\kappa(x)\varphi_j,\varphi_i\right), \qquad i,j=0,\cdots,N-1.$$

Using the notation $D=\gamma\triangle\tau A+(1+q\triangle\tau)B$ and $W^m=BU^{m-1}-F$, we rewrite the inequality (2.7) as

$$(DU^m,V-U^m)\geq(W^m,V-U^m) \text{ for all } V\geq G, \tag{2.8}$$

a fully discretised version of the variational inequality (2.4). We next proceed to consider a fast solution method for inequality (2.8).

3. Primal-Dual Active Set Method

The matrix D above is a symmetric and positive definite [25], so the inequality (2.8) is the first order optimality condition for the minimisation problem

$$\min_{U\geq G}\mathscr{J}(U)=\frac{1}{2}U^TDU-U^TW^m. \tag{3.1}$$

Let

$$\mathscr{L}(U)=\frac{1}{2}U^TDU-U^TW^m-\lambda^T(U-G) \tag{3.2}$$

be the Lagrangian function of the problem (3.1) with the Lagrange multiplier $\lambda\in\mathbb{R}_+^N$. Then the Karish-Kuhn-Tucker (KKT) condition for (3.2) is

$$DU-W^m-\lambda=0, \quad \lambda\geq 0, \quad U-G\geq 0, \quad \lambda^T(U-G)=0. \tag{3.3}$$

The problem (3.3) can be approximated as follows [11]:

$$DU_\rho-W^m-\lambda_\rho=0, \quad \lambda_\rho=\max\left\{0,\lambda_\rho+\rho(G-U_\rho)\right\}\ \forall\,\rho>0. \tag{3.4}$$

To find a solution of this problem (3.4), the primal-dual active set method previously used in the American option problem [28] and the American basket option pricing problem [9] may be employed [11]. This approach is equivalent to a semi-smooth Newton method [11], and turns out to be very efficient.

For any primal-dual pair (U,λ), let us define the active set AS and inactive set IS by

$$AS:=\left\{i\in\mathscr{N}:\lambda_i+\rho(G_i-U_i)>0\right\}, \qquad IS:=\left\{i\in\mathscr{N}:\lambda_i+\rho(G_i-U_i)\leq 0\right\},$$

then the primal-dual active set method can be implemented as listed in Algorithm 3.1. The existence and uniqueness of the solution of the problem (3.4) has been established in Ref. [16]. On the other hand, if $\rho>0$ then any solution of (3.4) is also a solution of (3.3), and stated in the theorem below Algorithm 3.1.

https://doi.org/10.4208/eajam.060317.020617a Published online by Cambridge University Press

Algorithm 3.1 Primal-Dual Active Set Algorithm.

- 1. Let iteration step $k=0$, and initialise $\rho>0$, $\varepsilon>0$, $U^{(0)}\geq G$, $\lambda^{(0)}\geq 0$;
- 2. Set the inactive and active sets by
  $IS^{(k+1)}=\{i\in\mathcal{N},\ \lambda_i^{(k)}+\rho(G_i-U_i^{(k)})\leq 0\}$,
  $AS^{(k+1)}=\{i\in\mathcal{N},\ \lambda_i^{(k)}+\rho(G_i-U_i^{(k)})>0\}$,
  $\mathcal{N}=\{0,1,\cdots,N-1\}$.
- 3. Solve the following problem for $U^{(k+1)}$ and $\lambda^{(k+1)}$:
  $DU^{(k+1)}-\lambda^{(k+1)}=W^m$,
  $U_i^{(k+1)}=G_i,\quad i\in AS^{(k+1)}$,
  $\lambda_i^{(k+1)}=0,\quad i\in IS^{(k+1)}$.
- 4. If $\|U^{(k+1)}-U^{(k)}\|_\infty<\varepsilon$, $U^m=U^{(k+1)}$, stop;
  else, let $k=k+1$, and go to Step 2.

Theorem 3.1. *If $\rho>0$, then the solution (U_ρ,λ_ρ) of the problem* (3.4) *satisfies the KKT condition* (3.3).

Proof. We note that the first equation in the problems (3.3) and (3.4) are identical, so to show that (U_ρ,λ_ρ) meets the KKT condition we only have to verify that it satisfies the supplementary condition of the problem (3.3). From the condition of (3.4), we have that $\lambda_\rho=0$ or $\lambda_\rho=\lambda_\rho+\rho(G-U_\rho)\geq 0$. If $\lambda_\rho=0$, then $G-U_\rho\leq 0$ because $\rho>0$. On the other hand, if $\lambda_\rho=\lambda_\rho+\rho(G-U_\rho)\geq 0$ then $G-U_\rho=0$ such that

$$\lambda_\rho\geq 0\,,\quad U_\rho-G\geq 0\,,\quad \lambda_\rho^T(U_\rho-G)=0\,,$$

so from the problem (3.4) the result follows. □

Finally, let $X^*=(U^*,\lambda^*)$ be the unique exact solution of the problem (3.4), and let $X^0=(U^0,\lambda^0)$ denote the initial value of the iteration. The convergence of the primal-dual active set method is characterised by the following lemma.

Lemma 3.1 ([11]). *If $\|X^0-X^*\|$ is sufficiently small, then the primal-dual active set method converges superlinearly to $X^*=(U^*,\lambda^*)$.*

Algorithm 3.1 can be used for simultaneously finding the option prices and the optimal exercise boundary. Thus during computation of the discrete solution U^m corresponding to $u(x,\tau_m)$, we also obtain an index i_m representing the critical point of the inactive and active sets for the time level τ_m. If these indexed points are denoted by $\{x_{i_m}^m\}_{m=1}^M$ and their

https://doi.org/10.4208/eajam.060317.020617a Published online by Cambridge University Press

continuous counterpart by $b(\tau)$, then the option values and the optimal exercise boundary $B(t)$ follow from inverting the transformations (2.1) — i.e. from

$$P(S,J,t) = Je^{-x}u(x,\tau), \quad B(t) = Je^{-b(\tau)}, \quad S = Je^{-x}, \quad t = T-\tau\,. \tag{3.5}$$

In brief, in addition to option prices as in Ref. [25], we also establish the optimal exercise boundary that is one of the main concerns for investors [7,30].

4. Numerical Simulations

We now present some numerical experiments to verify the performance of our proposed method. Numerical simulations using the primal-dual active set and the projection and contraction methods show that the former method is also very efficient in American lookback option pricing. To give investors a better understanding of the American lookback option pricing problem, we then provide several visual illustrations of options and the optimal exercise boundary.

We simulated one-year American lookback options setting the interest rate r at 3%, the dividend rate q of the underlying asset at 1%, 3% or 9%, and the volatility σ of the underlying asset prices at 0.2, 0.3 or 0.4. Thus we considered nine combinations of the parameters $\{r,q,\sigma\}$, in examining the performance of our approach.

4.1. The validity of the primal-dual active set method

As there is no known closed-form solution for American lookback options, the binomial method is generally regarded as the benchmark [17]. We used the binomial method with 10000 nodes in the temporal direction. For a fixed $m \in \{1,\cdots,M\}$, the initial values $U^{(0)}$ and $\lambda^{(0)}$ in Algorithm 3.1 were set to $\max\{U^{m-1},G\}$ and $\mathbf{0} \in \mathbb{R}^N_+$. Moreover, in order to guarantee the accuracy of the approximate solutions obtained by our method, we chose $\rho = 10^6$ and $\varepsilon = 10^{-6}$.

We use BM to denote the binomial method, PDAS for the primal-dual active set method, and PCFEM for the projection contraction method combined with a finite element method. In Fig. 1, the solutions $u(x,\tau)$ of the problem (3.4) with $q = 3\%$ at $\tau = T$ is shown for $\sigma = 0.2$, $\sigma = 0.3$ and $\sigma = 0.4$. The red o-line represents the solution computed by Algorithm 3.1 with $M = 300$ and $N = 500$, the solid blue line the approximate solution found by the binomial method, and the solid green line is related to the payoff function $g(x)$. Fig. 2 shows the corresponding results for the free boundary $b(\tau)$. The approximations for $u(x,T)$ and $b(\tau)$ for the nine different parameter combinations from the various methods are shown in Table 1, and the satisfactory convergence of Algorithm 3.1 is evident in each case.

4.2. Efficiency of the PDAS

The projection and contraction method is commonly used for monotone variational inequalities [10]. For American lookback options in particular, the combination of this

https://doi.org/10.4208/eajam.060317.020617a Published online by Cambridge University Press

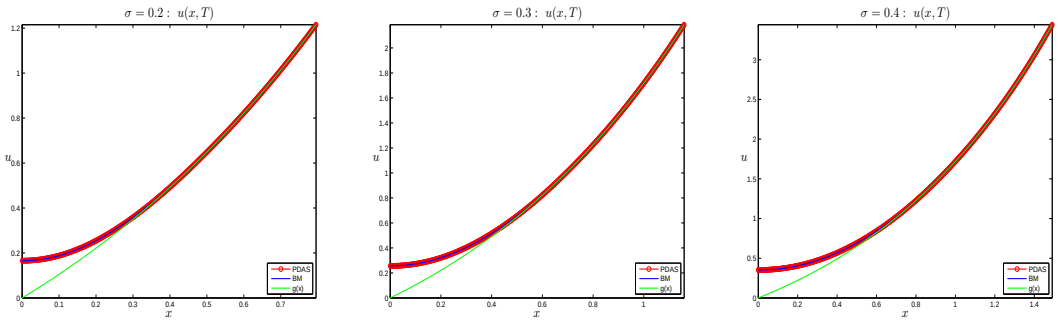


Figure 1: The solutions of (3.4) at $\tau = T$ for $\sigma = 0.2$, $\sigma = 0.3$ and $\sigma = 0.4$ computed by the PDAS with $M = 300$ and $N = 500$ and by the BM with 10000 nodes.

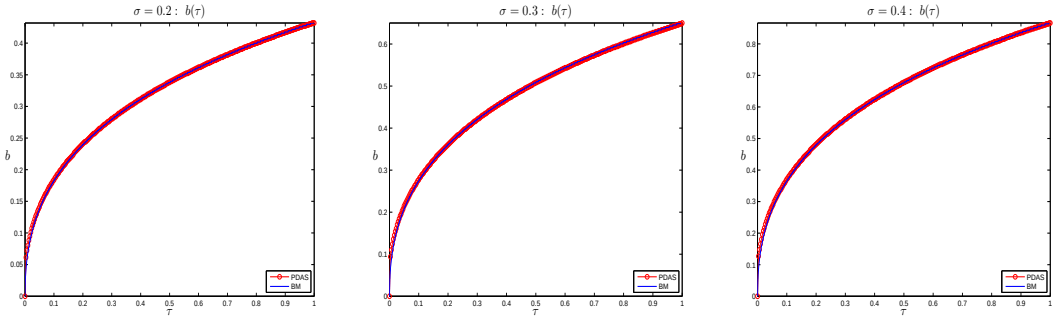


Figure 2: The optimal exercise boundary $b(\tau)$ of (3.4) for $\sigma = 0.2$, $\sigma = 0.3$ and $\sigma = 0.4$ computed by the PDAS with $M = 300$ and $N = 500$ and by the BM with 10000 nodes.

Table 1: L^2-difference of the approximations obtained by the PDAS.

σ	0.2			0.3			0.4		
q	1%	3%	9%	1%	3%	9%	1%	3%	9%
$u(x,T)$ Error/10^{-5}	0.550	0.154	0.237	1.089	0.800	0.490	1.370	0.681	2.266
$b(\tau)$ Error/10^{-4}	0.920	1.299	1.196	1.447	2.171	0.855	2.001	3.047	3.575

method with a finite element method was proposed in Ref. [25]. In order to demonstrate the efficiency of the primal-dual active set method, we provide the approximation error for the computation of $u(x,T)$ by the PDAS and the PCFEM in Table 2. We also show the computational cost for the PDAS, PCFEM and BM. Note that for the PDAS we chose $N = \left[L/(\sigma\sqrt{\Delta\tau})\right]$ such that N is determined by M since $\Delta\tau = T/M$. This choice is similar to Refs. [17] and [25], where the other two methods are considered.

Table 2 shows the corresponding results for $M = 100$ for the nine parameter combinations, showing the computational speed of the PDAS is comparable to PCFEM whereas its accuracy is slightly better. The primal-dual active set method can therefore be used as an alternative method for American lookback options, but the primal-dual active set method provides the option values and the free boundary simultaneously.

https://doi.org/10.4208/eajam.060317.020617a Published online by Cambridge University Press

Table 2: Computational cost of PDAS, PCFEM and BM.

σ	0.2			0.3			0.4		
q	1%	3%	9%	1%	3%	9%	1%	3%	9%
PCFEM Error/10^{-4}	1.424	0.981	0.717	0.976	0.907	0.523	1.263	0.398	1.625
PDAS Error/10^{-4}	0.359	0.074	0.213	0.712	0.549	0.282	0.995	0.254	1.235
PCFFM Time/s	0.078	0.076	0.516	0.087	0.083	0.093	0.089	0.088	0.081
PDAS Time/s	0.102	0.127	0.199	0.109	0.123	0.161	0.113	0.119	0.147
BM Time/s	7.707	10.398	17.477	8.669	9.978	13.733	8.837	9.665	12.066

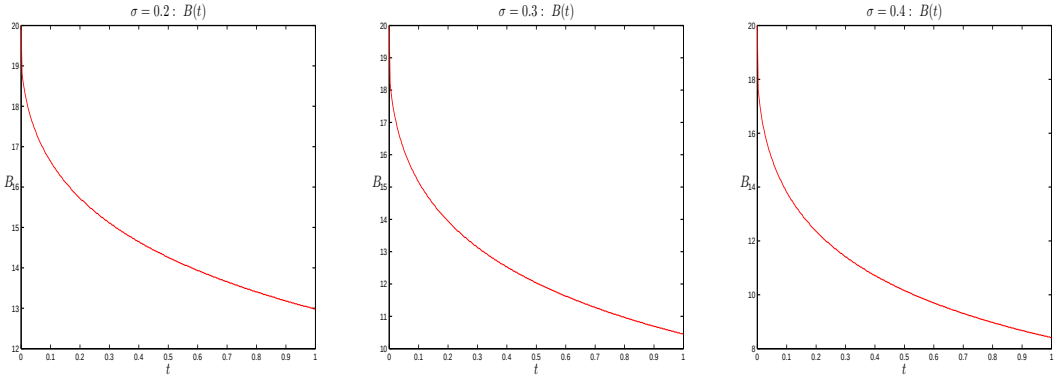


Figure 3: The optimal exercise boundary $B(t)$ with $J = 20$ for $\sigma = 0.2$, $\sigma = 0.3$ and $\sigma = 0.4$.

Remark 4.1. The last row in Table 2, showing the computational cost for the benchmark solution, indicates that the PDAS and PCFEM are superior to the binomial method in this respect.

Finally, we considered the optimal exercise boundary $B(t)$ and option values $P(S,J,t)$ with $J = 20$. For simplicity, we only present the cases $\sigma = 0.2, 0.3$ and 0.4 with $q = 3\%$ (the other cases were similar). Fig. 3 illustrates a new result in comparison with Ref. [25], while the results in Fig. 4 are similar. This again confirms that our method is efficient for the pricing of American lookback options.

5. Conclusions

We have proposed an effective numerical method for the valuation of American lookback options. The method simultaneously obtains the option values and the optimal exercise boundary with no additional computational cost. Using the numeraire transformation, we reformulated the American lookback option pricing model as a one-dimensional variational inequality problem on a bounded domain, which was solved by finite element method via the primal-dual active set algorithm. Numerical simulations demonstrated the efficiency of our method.

https://doi.org/10.4208/eajam.060317.020617a Published online by Cambridge University Press

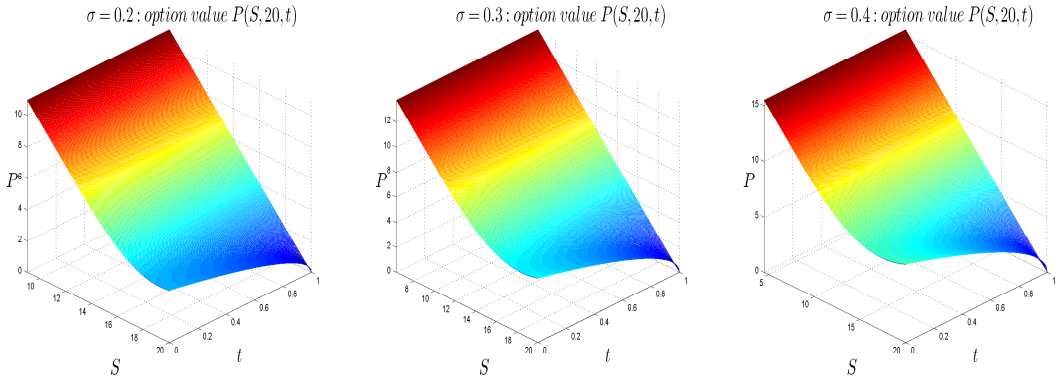


Figure 4: The valuation $P(S,20,t)$ computed by the PDAS with $M = 300$ and $N = 500$ for $\sigma = 0.2$, $\sigma = 0.3$ and $\sigma = 0.4$.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Grant No. 11471141, U1530116, 91630201) and the basic research of the science and technology development program of Jilin province (No. 20150101058JC). The authors also wish to thank the High Performance Computing Center of Jilin University, Computing Center of Jilin Province, and Key Laboratory of Symbolic Computation and Knowledge Engineering of the Ministry of Education for essential computing support.

References

[1] S. Babbs, *Binomial valuation of lookback options*, J. Econ. Dynam. Control **24**, 1499-1525 (2000).

[2] M. Bergounioux, K. Ito and K. Kunisch, *Primal-dual strategy for constrained optimal control problem*, SIAM J. Control Optim. **37**, 1176-1194 (1999).

[3] M. Bergounioux, M. Haddou, M. Hintermüller and K. Kunisch, *A comparison of a Moreau-Yosida-based active set strategy and interior point methods for constrained optimal control problem*, SIAM J. Optimiz. **11**, 495-521 (2000).

[4] P.P. Boyle, Y. Lai and K.S. Tan, *Pricing options using lattice rules*, N. Am. Actuar. J. **9**, 50-76 (2005).

[5] P. Buchen and O. Konstandatos, *A new method of pricing lookback options*, Math. Finance **15**, 245-259 (2005).

[6] J.C. Cox, S.A. Ross and M. Rubinstein, *Option pricing: A simplified approach*, J. Fin. Econ. **7**, 229-263 (1979).

[7] M. Dai and Y.K. Kwok, *American options with lookback payoff*, SIAM J. App. Math. **66**, 206-227 (2005).

[8] D.J. Duffy, *Finite Difference Methods in Financial Engineering: A Partial Differential Equation Approach*, Wiley (2006).

[9] C. Hager, S. Hüeber and B. Wohlmuth, *Numerical techniques for the valuation of basket options and their Greeks*, J. Comput. Finance **13**, 3-33 (2010).

https://doi.org/10.4208/eajam.060317.020617a Published online by Cambridge University Press

[10] B. He, *A class of projection and contraction methods for monotone variational inequalities*, Appl. Math. Opt. **35**, 69-76 (1997).
[11] M. Hintermüller, K. Ito and K. Kunisch, *The primal-dual active set strategy as a semi-smooth Newton method*, SIAM J. Optimiz. **13**, 865-888 (2003).
[12] *J. Huang and J.S. Pang*, *Option pricing and linear complementarity*, J. Comput. Finance **2**, 31-60 (1998).
[13] S. Hüeber and B.I. Wohlmuth, *A primal-dual active set strategy for non-linear multibody contact problems*, Comput. Method. Appl. Math. **194**, 3147-3166 (2005).
[14] K. Ito and K. Kunisch, *Optimal control of elliptic variational inequalities*, Appl. Math. Opt. **41**, 343-364 (2000).
[15] K. Ito and J. Toivanen, *Lagrange multiplier approach with optimisedfinite difference stencils for pricing American options under stochastic volatility*, SIAM J. Sci. Comput. **31**, 2646-2664 (2009).
[16] K. Ito and K. Kunisch, *Semi-smooth newton methods for variational inequality of the first kind*, Math. Model. Numer. Anal. **41**, 591-616 (2003).
[17] L.S. Jiang, *Mathematical Modeling and Methods of Option Pricing*, World Scientific Press, Singapore (2005).
[18] L.S. Jiang and M. Dai, *Convergence of binomial tree methods for European/American path-dependent options*, SIAM J. Numer. Anal. **42**, 1094-1109 (2004).
[19] K.I. Kim, H.S. Park and X.S. Qian, *A mathemathical modeling for the lookback option with jump-diffusion using binomial tree method*, J. Comput. Appl. Math. **235**, 5140-5154 (2011).
[20] T. Kimura, *American fractional lookback options: valuation and premium decomposition*, SIAM J. Appl. Math. **71**, 517-539 (2011).
[21] K. Kunisch and A. Rösch, *Primal-dual active set strategy for general class of contrained control problems*, SIAM J. Optimiz. **13**, 321-334 (2002).
[22] T.L. Lai and T.W. Lim, *Exercise regions and efficient valuation of American lookback options*, Math. Finance **14**, 249-269 (2004).
[23] D.P. Nicholls and A. Sward, *A discontinuous Galerkin method for pricing American options under the constant elasticity of variance model*, Commun. Comput. Phys. **17**, 761-778 (2015).
[24] H. M. Song and R. Zhang, *Projection and contraction method for the valuation of American options*, E. Asian J. Appl. Math. **5**, 48-60 (2015).
[25] H.M. Song, Q. Zhang and R. Zhang, *A fast numerical method for the valuation of American lookback put options*, Commun. Nonlinear Sci. **27**, 302-313 (2015).
[26] H.M. Song, K. Zhang and Y.T. Li, *Finite Element and Discontinuous Galerkin Methods with Perfect Matched Layers for American Options*, Numer. Math. Theory Methods Appl. (to appear).
[27] Y. Sun, Y. Shi and X. Gu, *An integro-differential parabolic variational inequality arising from the valuation of double barrier American option*, J. Sys. Sci. Complex. **27**, 276-288 (2014).
[28] K. Zhang, X.Q. Yang and K.L. Teo, *Augmented Lagrangian method applied to American option pricing*, Automatica **42**, 1407-1416 (2006).
[29] T. Zhang, *Numerical Methods for American Option Pricing*, Acta Math. Appl. Sin. **25**, 113-122 (2002).
[30] T. Zhang, S. Zhang, D. Zhu, *Finite difference approximation for pricing the American lookback option*, J. Comput. Math. **27**, 484-494 (2009).
[31] R. Zhang, H. Song. and N. Luan, *A weak Galerkin finite element method for the valuation of American options*, Front. Math. China **9**, 455-476 (2014).
[32] R. Zhang, Q. Zhang and H.M. Song, *An efficient finite element method for pricing American multi-asset put options*, Commun. Nonlinear Sci. **29**, 25-36 (2015).

https://doi.org/10.4208/eajam.060317.020617a Published online by Cambridge University Press