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Triangle-Free Graphs with Large Degree

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A graph G is called an H-*type* graph for some graph H if there is a mapping from $V(G)$ to $V(H)$ preserving edges. In this paper, we shall prove that: (1) every triangle-free graph G of order n with $\chi(G) \leqslant 3$ and $\delta(G) > n/3$ is of F_d-type for some $d \geqslant 1$, where F_d is a certain d-regular triangle-free Hamiltonian Cayley graph of order $3d-1$, (2) every triangle-free graph G of order n with $\chi(G) \geqslant 4$ and $\delta(G) > n/3$ contains the Mycielski graph (see Figure 2) as a subgraph.

1. Introduction

Given a graph H, a graph G is called H-free if H is not a subgraph of G. The problem of classifying K_3-free (*i.e.* triangle-free) graphs with large degree has been studied by many researchers. Among them, Mantel [8] proved a result which implies that every graph G of order n with $\delta(G) > n/2$ contains a K_3. Indeed, this is just a special case of Zarankiewicz's result [13]. In 1974, Andrásfai, Erdős and Sós [1] showed that every K_3-free graph G of order n with $\delta(G) > 2n/5$ is bipartite.

In order to state further results, we have to give some definitions. A graph G is called a *complete* H-*type graph* for some graph H if G can be obtained from H first by replacing each vertex v in H by a nonempty independent set U_v, and then by joining every vertex in U_x to every vertex in U_y whenever $xy \in E(H)$, where $x, y \in V(H)$. A graph is said to be of H-*type* if it is a subgraph of a complete H-type graph. Trivially, G is of H-type if, and only if, G is homomorphic with H, *i.e.* there is a mapping from $V(G)$ to $V(H)$ preserving edges. Thus, a graph is bipartite if and only if it is of K_2-type.

For an integer $d \geqslant 1$, the graph F_d of order $3d-1$ was introduced in [5] as follows:

$$V(F_d) = \{a_1, a_2, a_3, \ldots, a_{3d-1}\} = \{a_i \ : \ i \in \mathbb{Z}_{3d-1}\}$$

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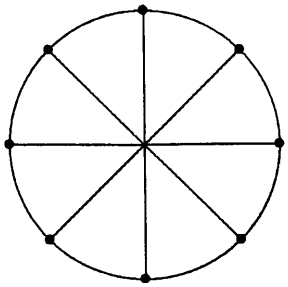


Figure 1 Möbius ladder F_3.

and

$$E(F_d) = \left\{ a_i a_{i+1}, a_i a_{i+4}, a_i a_{i+7}, \ldots, a_i a_{i+3\lceil \frac{d}{2} \rceil - 2} \ : \ i \in \mathbb{Z}_{3d-1} \right\},$$

where the indices are taken modulo $3d-1$. It is easy to check that $F_1 = K_2$, $F_2 = C_5$ and F_3 is a Möbius ladder (shown in Figure 1). Trivially, F_1 is 2-chromatic and F_i is 3-chromatic for $i \geqslant 2$. Clearly, F_i-type graphs are of F_j-type for $1 \leqslant i < j$.

In 1973, Woodall [12] conjectured that if the binding number of G is at least $3/2$ then G contains a triangle, which was proved by Shi [11] in 1985. Graphs F_d are among the graphs constructed by Woodall to explain that the number $3/2$ in his conjecture is the least possible. Pach [10] in 1981 rediscovered graphs F_d and proved that every triangle-free graph is of complete F_d-type for some d if and only if every independent set of its vertices has a common neighbour.

In 1982, Häggkvist [4] proved that every K_3-free graph G of order n with $\delta(G) > 3n/8$ is of F_2-type, *i.e.* C_5-type. Recently, Jin [5] showed that every K_3-free graph G of order n with $\delta(G) > 10n/29$ is of F_9-type. Note that the results of Andrásfai, Erdős and Sós [1], Häggkvist [4] and Jin [5] are sharp.

Clearly, the result in [5] implies that if G is a K_3-free graph of order n with $\delta(G) > 10n/29$, then $\chi(G) \leqslant 3$. In this paper, our first aim is to study the structure of K_3-free graphs G of order n with $\chi(G) \leqslant 3$ and $\delta(G) > n/3$. We shall show that every such graph is of F_i-type for some i. This result considerably extends those of Andrásfai, Erdős and Sós [1], Häggkvist [4] and Jin [5] mentioned above. Note that the bound $\delta(G) > n/3$ is best possible since $\delta(F_i) > |F_i|/3$ and $\lim_{i\to\infty} \delta(F_i)/|F_i| = 1/3$.

What can be said about the structure of K_3-free graphs G of order n with $\chi(G) \geqslant 4$ and $\delta(G) > n/3$? Our second aim here is to show that every such graph contains the Mycielski graph, shown in Figure 2, as a subgraph. For convenience, let M be the Mycielski graph [9].

In order to prove the results above, we have to do some preparation in the next section. A graph G is said to be *maximal K_3-free and M-free* if G does not contain either K_3 or M and $G + e$ contains K_3 or M for every $e \in E(\bar{G})$. In Theorem 2.2 we shall show that if G is maximal K_3-free and M-free then G does not contain a certain structure. In Section 3, instead of proving our results above we shall prove Theorem 3.1, which is somewhat stronger than our two results, but easier to prove. The techniques employed in our proofs of Theorems 2.2 and 3.1 are similar to those in [4], [5] and [6].

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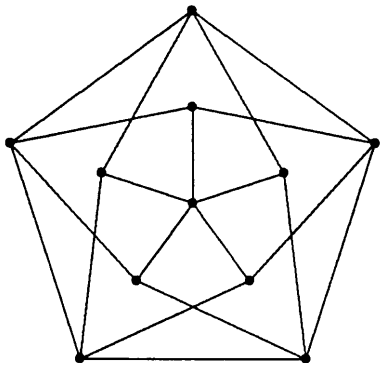


Figure 2 The Mycielski graph M.

2. Some basic results

In this section, we shall count all indices modulo 5. A graph G is said to be a *maximal K_3-free graph* if G is K_3-free and $G + e$ contains a K_3 for every edge $e \in E(\overline{G})$. The following trivial lemma is useful in proving Theorems 2.2 and 3.1.

Lemma 2.1. *If G is a maximal K_3-free graph, then every two nonadjacent vertices have a common neighbour.* □

We shall denote by $N(v)$ the set of neighbours of a vertex v of the graph G. If we wish to emphasize that the neighbourhood is taken in a graph G_0, then we write $N_{G_0}(v)$ instead of $N(v)$. For a vertex x and a vertex set (or subgraph) S of G, we define $d_S(x) = |N(x) \cap S|$. Note that if $S = G$, then $d_S(x)$ is just the degree of x in G. Also, we say that x is *adjacent* to S if and only if $d_S(x) > 0$.

We are now ready to prove the main theorem in this section which is essential in the subsequent discussions.

Theorem 2.2. *Let G be a maximal K_3-free and M-free graph of order n with $\delta(G) > \frac{n}{3}$. Then G has no vertex x such that $G - N(x)$ contains a 5-cycle.*

Proof. Suppose to the contrary that the graph G has a vertex x such that $G - N(x)$ contains a 5-cycle $a_1a_2a_3a_4a_5$.

For convenience, let $G' = G - N(x)$, and for $1 \leqslant i \leqslant 5$, let

$$\begin{aligned} X_i &= N(a_{i-1}) \cap N(a_{i+1}) \cap N(x), \\ P_i &= N_{G'}(a_{i-1}) \cap N_{G'}(a_{i+1}), \\ Y_i &= N(a_i) \cap N(x) - X_{i-1} - X_{i+1}, \\ Q_i &= N_{G'}(a_i) - P_{i-1} - P_{i+1}, \\ X &= V(G') - \bigcup_{i=1}^{5} (P_i \cup Q_i), \end{aligned}$$

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and

$$\bar{x}_i = |X_i|, \quad \bar{p}_i = |P_i|, \quad \bar{y}_i = |Y_i|, \quad \bar{q}_i = |Q_i|.$$

Also, let $X_i = \{x_i, \ldots\}$, $Q_i = \{q_i, \ldots\}$ and $Y_i = \{y_i, \ldots\}$, where the vertices x_i, q_i and y_i exist only if the corresponding sets are nonempty.

Since x is an isolated vertex of G', $x \in X$. Also,

$$n = |X| + d(x) + \sum_{i=1}^{5} (\bar{p}_i + \bar{q}_i) \tag{2.1}$$

and

$$d(x) \geqslant \sum_{i=1}^{5} (\bar{x}_i + \bar{y}_i). \tag{2.2}$$

Note that, if $X_i \neq \emptyset$, then, as G is K_3-free,

$$N(x_i) \subseteq P_{i-1} \cup P_{i+1} \cup Q_i \cup Q_{i-2} \cup Q_{i+2} \cup X. \tag{2.3}$$

By the definitions of Y_i and Q_i, we have

$$N(a_i) = P_{i-1} \cup P_{i+1} \cup X_{i-1} \cup X_{i+1} \cup Y_i \cup Q_i.$$

Since the sets on the right-hand side are disjoint,

$$d(a_i) = \bar{p}_{i-1} + \bar{p}_{i+1} + \bar{x}_{i-1} + \bar{x}_{i+1} + \bar{y}_i + \bar{q}_i. \tag{2.4}$$

With relations (2.1)–(2.4), we shall show now that they lead to a contradiction. Although it is long and tedious, the rest of our proof is essentially easy by checking several cases.

Denote by r the number of indices i such that $\bar{x}_i > 0$. We have $0 \leqslant r \leqslant 5$. We may assume that the vertex x and the 5-cycle C in $G - N(x)$ have been chosen so that r is maximum. If $r = 5$, then the graph G contains M as a subgraph. So $r \leqslant 4$.

Suppose that $r = 4$. Without loss of generality, we may assume that $\bar{x}_1 = 0$ and $\bar{x}_i \geqslant 1$ for $2 \leqslant i \leqslant 5$.

Assume that $d_{Q_1}(x_3) + d_{Q_1}(x_4) \leqslant \bar{q}_1$. Then, by (2.3),

$$\bar{q}_1 + \bar{p}_2 + \bar{p}_3 + \bar{p}_4 + \bar{p}_5 + \bar{q}_2 + \bar{q}_3 + \bar{q}_4 + \bar{q}_5 + 2|X| \geqslant d(x_3) + d(x_4) > \frac{2n}{3}.$$

So, by (2.1),

$$|X| > d(x) - \frac{n}{3} + \bar{p}_1 > \bar{p}_1. \tag{2.5}$$

On the other hand, by (2.4) and the fact that $\bar{x}_1 = 0$, we have

$$\bar{p}_1 + \sum_{i=1}^{5} \bar{p}_i + \sum_{i=2}^{5} \bar{x}_i + \bar{q}_1 + \bar{q}_2 + \bar{q}_5 + \bar{y}_1 + \bar{y}_2 + \bar{y}_5 \geqslant d(a_1) + d(a_2) + d(a_5) > n.$$

Hence,

$$\bar{p}_1 + \sum_{i=1}^{5} \bar{p}_i + \sum_{i=2}^{5} \bar{x}_i + \sum_{i=1}^{5} \bar{q}_i + \sum_{i=1}^{5} \bar{y}_i - \bar{q}_3 - \bar{q}_4 - \bar{y}_3 - \bar{y}_4 > n.$$

Thus, by (2.1), (2.2) and the fact that $\bar{x}_1 = 0$, we get

$$\bar{p}_1 > |X| + \bar{q}_3 + \bar{q}_4 + \bar{y}_3 + \bar{y}_4 > |X|.$$

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This contradicts (2.5). Hence $d_{Q_1}(x_3) + d_{Q_1}(x_4) > \bar{q}_1$. Therefore, there is a vertex $q_1 \in Q_1$ such that $q_1x_3, q_1x_4 \in E(G)$. However, if we choose the vertex a_1 instead of x and the cycle $xx_3a_4a_3x_4x$, we have $r = 5$. This contradicts our assumption that $r \leqslant 4$. Hence $r < 4$.

At this point, we would like to show that our graph G is in fact maximal K_3-free. Indeed, suppose that this is not so. Then, by the maximality of G, it contains a subgraph M' isomorphic to $M - e$, where $e \in E(M)$. Therefore, there exists a vertex $x \in V(G)$ such that $G - x$ contains a 5-cycle $a_1a_2a_3a_4a_5a_1$ in M' with the property that at most one of the five sets $N(a_i) \cap N(a_{i+2}) \cap N(x)$, $1 \leqslant i \leqslant 5$, is empty. This contradicts the fact that $r < 4$. Hence G is maximal K_3-free.

Now, the rest of the proof is divided into four cases according to the value of r.

Case 1: $r = 3$.
Without loss of generality, we may assume that either $\bar{x}_1 = \bar{x}_5 = 0$ and $\bar{x}_i \geqslant 1$ for $i = 2, 3, 4$; or $\bar{x}_2 = \bar{x}_5 = 0$ and $\bar{x}_i \geqslant 1$ for $i = 1, 3, 4$.

(**a**) Suppose first that $\bar{x}_1 = \bar{x}_5 = 0$ and $\bar{x}_i \geqslant 1$ for $i = 2, 3, 4$. It is easy to see that no vertex in Q_1 has a neighbour in $X_3 \cup X_4$. Otherwise, we can choose a_1 instead of x, and find a 5-cycle $xx_3a_4a_3x_4x$ such that $r \geqslant 4$. Therefore, for any vertices $x_3 \in X_3$ and $x_4 \in X_4$, we have $d_{Q_1}(x_3) + d_{Q_1}(x_4) \leqslant \bar{q}_1$. Hence, by (2.3),

$$\bar{q}_1 + \bar{p}_2 + \bar{p}_3 + \bar{p}_4 + \bar{p}_5 + \bar{q}_2 + \bar{q}_3 + \bar{q}_4 + \bar{q}_5 + 2|X| \geqslant d(x_3) + d(x_4) > \frac{2n}{3}.$$

So

$$|X| > d(x) - \frac{n}{3} + \bar{p}_1 > \bar{p}_1. \tag{2.6}$$

On the other hand, by (2.4) and the fact that $\bar{x}_1 = \bar{x}_5 = 0$, we have

$$\bar{p}_1 + \sum_{i=1}^{5} \bar{p}_i + \sum_{i=2}^{4} \bar{x}_i + \bar{q}_1 + \bar{q}_2 + \bar{q}_5 + \bar{y}_1 + \bar{y}_2 + \bar{y}_5 \geqslant d(a_1) + d(a_2) + d(a_5) > n.$$

Thus

$$\bar{p}_1 > |X| + \bar{q}_3 + \bar{q}_4 + \bar{y}_3 + \bar{y}_4 \geqslant |X|,$$

which contradicts (2.6).

(**b**) Suppose now that $\bar{x}_2 = \bar{x}_5 = 0$ and $\bar{x}_i \geqslant 1$ for $i = 1, 3, 4$. By Lemma 2.1, we have $\bar{y}_1 \geqslant 1$. Also, every vertex $y_1 \in Y_1$ has no neighbour in $P_3 \cup P_4$; for otherwise, let $y_1 \in Y_1$ with $a'_3 \in P_3 \cap N(y_1)$, say. Then, we can choose the 5-cycle $a_1a_2a'_3a_4a_5a_1$ instead of $a_1a_2a_3a_4a_5a_1$ and obtain $r = 4$, a contradiction. Clearly, the vertex y_1 has no neighbour in $Q_1 \cup P_2 \cup P_5$. Therefore, $N(y_1) \subseteq P_1 \cup \bigcup_{i=2}^{5} Q_i \cup X$ and so

$$\bar{p}_1 + \sum_{i=2}^{5} \bar{q}_i + |X| \geqslant d(y_1) > \frac{n}{3};$$

that is,

$$\bar{p}_1 > \frac{n}{3} - |X| - \sum_{i=2}^{5} \bar{q}_i. \tag{2.7}$$

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Let $x_3 \in X_3$ and $x_4 \in X_4$. Then, by (2.3),

$$\sum_{i=2}^{5} \bar{p}_i + \bar{q}_1 + \sum_{i=1}^{5} \bar{q}_i + 2|X| \geqslant d(x_3) + d(x_4) > \frac{2n}{3}.$$

Thus

$$\begin{aligned}
\sum_{i=1}^{5} \bar{q}_i + 2|X| &> \frac{2n}{3} + \bar{p}_1 - \sum_{i=1}^{5} \bar{p}_i - \bar{q}_1 \\
&> \frac{2n}{3} + \frac{n}{3} - |X| - \sum_{i=2}^{5} \bar{q}_i - \sum_{i=1}^{5} \bar{p}_i - \bar{q}_1, \quad \text{by (2.6)}, \\
&= n - |X| - \sum_{i=1}^{5} \bar{q}_i - \sum_{i=1}^{5} \bar{p}_i \\
&\geqslant n - (n - d(x)) \\
&> \frac{n}{3}. \qquad (2.8)
\end{aligned}$$

Since G is K_3-free, each vertex of G is adjacent to at most two vertices of the cycle C. By the definitions of Q_i and Y_i, each vertex in Q_i and Y_i is adjacent to only one vertex of the cycle for $1 \leqslant i \leqslant 5$, and no vertex in X is adjacent to any vertex of the cycle C. Therefore,

$$\sum_{i=1}^{5} (\bar{q}_i + \bar{y}_i) + 2\sum_{i=1}^{5} (\bar{x}_i + \bar{p}_i) \geqslant \sum_{v \in V(G)} d_C(v) = \sum_{i=1}^{5} d(a_i) > \frac{5n}{3}. \qquad (2.9)$$

Hence, we have

$$\begin{aligned}
\frac{5n}{3} &= 2n - \frac{n}{3} \\
&> 2n - \sum_{i=1}^{5} \bar{q}_i - 2|X| \\
&\geqslant \sum_{i=1}^{5} (\bar{q}_i + \bar{y}_i) + 2\sum_{i=1}^{5} (\bar{x}_i + \bar{p}_i) \\
&> \frac{5n}{3},
\end{aligned}$$

a contradiction, where the first inequality is a consequence of (2.8) and the last one follows from (2.9).

Case 2: $r = 2$.

Without loss of generality, we may assume that either $\bar{x}_1 = \bar{x}_2 = \bar{x}_5 = 0$ and $\bar{x}_i \geqslant 1$ for $i = 3, 4$; or $\bar{x}_1 = \bar{x}_3 = \bar{x}_5 = 0$ and $\bar{x}_i \geqslant 1$ for $i = 2, 4$.

Assume first that $\bar{x}_1 = \bar{x}_2 = \bar{x}_5 = 0$ and $\bar{x}_i \geqslant 1$ for $i = 3, 4$. If $d_{Q_1}(x_3) + d_{Q_1}(x_4) > \bar{q}_1$, then there is a vertex q_1 in Q_1 such that q_1 is adjacent to x_3 and x_4. Considering the vertex a_1 instead of x and the cycle $xx_3a_4a_3x_4x$ instead of the cycle C, we would have $r \geqslant 3$. This contradicts the assumption that $r = 2$. Hence $d_{Q_1}(x_3) + d_{Q_1}(x_4) \leqslant \bar{q}_1$.

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By (2.3), we have

$$\bar{q}_1 + \bar{p}_2 + \bar{p}_3 + \bar{p}_4 + \bar{p}_5 + \bar{q}_2 + \bar{q}_3 + \bar{q}_4 + \bar{q}_5 + 2|X| \geqslant d(x_3) + d(x_4) > \frac{2n}{3},$$

and so, by (2.1),

$$|X| > d(x) - \frac{n}{3} + \bar{p}_1 > \bar{p}_1. \tag{2.10}$$

On the other hand, by (2.4) and the fact that $\bar{x}_1 = \bar{x}_2 = \bar{x}_5 = 0$, we have

$$\bar{p}_1 + \sum_{i=1}^{5} \bar{p}_i + \sum_{i=3}^{4} \bar{x}_i + \bar{q}_1 + \bar{q}_2 + \bar{q}_5 + \bar{y}_1 + \bar{y}_2 + \bar{y}_5 \geqslant d(a_1) + d(a_2) + d(a_5) > n.$$

Thus, by (2.1), (2.2) and the fact that $\bar{x}_1 = \bar{x}_2 = \bar{x}_5 = 0$, we get

$$\bar{p}_1 > |X| + \bar{q}_3 + \bar{q}_4 + \bar{y}_3 + \bar{y}_4 \geqslant |X|,$$

contradicting (2.10).

Suppose now that $\bar{x}_1 = \bar{x}_3 = \bar{x}_5 = 0$ and $\bar{x}_i \geqslant 1$ for $i = 2, 4$. By Lemma 2.1, $\bar{y}_i \geqslant 1$ for $i = 2, 4$. Let $y_2 \in Y_2$. If we consider a_5 instead of x and the cycle $xy_2a_2a_3x_2x$ instead of the cycle C, we find that either $r \geqslant 3$ or we can have $\bar{x}_1 = \bar{x}_2 = \bar{x}_5 = 0$ and $\bar{x}_i \geqslant 1$ for $i = 3, 4$. We have shown that all these are impossible.

Case 3: $r = 1$.

We may assume that $\bar{x}_1 \geqslant 1$. Then $\bar{x}_i = 0$ for $2 \leqslant i \leqslant 5$. Let $x_1 \in X_1$. By Lemma 2.1, $\bar{y}_i \geqslant 1$ for $i = 1, 3, 4$. Since $r = 1$, the vertex y_1 is not adjacent to any vertex in $P_3 \cup P_4$; for otherwise, let $a_3' \in P_3 \cap N(a_1)$, say. Then, by considering the 5-cycle $a_1a_2a_3'a_4a_5a_1$ instead of $a_1a_2a_3a_4a_5a_1$, we would have $r = 2$, a contradiction. Clearly, y_1 is not adjacent to any vertex in $P_2 \cup P_5$, since G is K_3-free. Similarly, y_3 is not adjacent to any vertex in $P_1 \cup P_2 \cup P_4$, and y_4 is not adjacent to any vertex in $P_1 \cup P_3 \cup P_5$. Finally, since $r = 1$, the vertices y_3 and x_1 have no common neighbour in P_5, and the vertices y_4 and x_1 have no common neighbour in P_2. Therefore, by considering the possible neighbours of x_1, y_1, y_3 and y_4, we have

$$\sum_{i=1}^{5} \bar{p}_i + 3\sum_{i=1}^{5} \bar{q}_i + 4|X| \geqslant d(x_1) + d(y_1) + d(y_3) + d(y_4) > \frac{4n}{3}.$$

By (2.1), the inequality above implies that

$$2\sum_{i=1}^{5} \bar{q}_i + 3|X| > \frac{n}{3} + d(x) > \frac{2n}{3},$$

and so

$$\sum_{i=1}^{5} \bar{q}_i + 2|X| \geqslant \frac{n}{3}.$$

This means that the number of the vertices adjacent to (at least) two vertices of C is at most $2n/3$. However, this is impossible, since the number of the vertices adjacent to two

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vertices of C is at least

$$\sum_{i=1}^{5} d(a_i) - n > \frac{5n}{3} - n = \frac{2n}{3}.$$

Case 4: $r = 0$.
This means that no vertex in $N(x)$ is adjacent to two vertices of C. Therefore the number of the vertices adjacent to two vertices of C is at most $n-d(x)-1$ and at least $\sum_{i=1}^{5} d(a_i)-n$, that is,

$$n - d(x) \geqslant \sum_{i=1}^{5} d(a_i) - n > \frac{5n}{3} - n = \frac{2n}{3}.$$

It follows that

$$d(x) < \frac{n}{3},$$

a contradiction.

This completes the proof of our theorem. □

3. The main theorem

In this section, we shall prove the main result of the paper.

Theorem 3.1. *Let G be a maximal M-free and K_3-free graph of order n with $\delta(G) > n/3$. Then, for every $d \geqslant 1$, either G is of F_d-type or G contains F_{d+1} as a subgraph.*

To prove Theorem 3.1, we shall apply induction on d. But, the proofs of the theorem for $d = 2$ and for $d \geqslant 3$ are different. We shall state and prove Theorem 3.1 for $d = 1, 2$ separately, namely, Theorem 3.3 for $d = 1$; Theorem 3.4 for $d = 2$. In fact, Theorems 3.3 and 3.4 serve as the first step in the inductive proof of Theorem 3.1. After going through the proof of Theorem 3.4, readers may find that of Theorem 3.1 easier to understand since the methods employed are similar.

Before we state and prove our Theorem 3.3, we need the following lemma, which is taken from [1] and [4].

Lemma 3.2. *Every nonbipartite graph G of order n with $\delta(G) > \lfloor 2n/(2k+1) \rfloor$ contains an odd cycle of length at most $2k-1$, where $k \geqslant 2$.* □

As an immediate consequence of Lemma 3.2, we have the following result.

Theorem 3.3. *Let G be a K_3-free and M-free graph of order n with $\delta(G) > n/3$. Then, either G is bipartite or G contains a 5-cycle.*

Proof. Assume that G is not bipartite. Since $\delta(G) > n/3 \geqslant 2n/7$ and G is K_3-free, Lemma 3.2 implies that G contains a 5-cycle. □

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Theorem 3.4. *Let G be a maximal M-free and K_3-free graph of order n with $\delta(G) > n/3$. Then, either G is of F_2-type or G contains the Möbius ladder F_3 shown in Figure 1.*

Proof. Assume that G is not of F_2-type so, in particular, G is not bipartite. Thus, Theorem 3.3 implies that G contains a 5-cycle $C = a_1a_2a_3a_4a_5a_1$. Let T be the set of vertices in G adjacent to exactly two vertices of C. For $1 \leqslant i \leqslant 5$, define T_i by putting

$$T_i = N(a_{i-1}) \cap N(a_{i+1}),$$

with indices counted modulo 5. Clearly, $\{T_1, T_2, \ldots, T_5\}$ is a partition of T and the induced subgraph $G[T]$ is of F_2-type, since G is K_3-free. If $T = V(G)$, then G is of F_2-type, a contradiction. Thus $R = V(G) - T$ is not empty.

Call a vertex $y \in R$ a *T_i-vertex* if $N(y) \subseteq R \cup T_i$. By Theorem 2.2, every T_i-vertex y is adjacent to every vertex in T_i. Therefore the neighbours of a T_i-vertex are not adjacent to any vertex in T_i, since G is K_3-free. Let x be a vertex in R. By Theorem 2.2, the vertex x is adjacent to exactly one vertex of C, say to a_1. So x is adjacent to every vertex in T_1 and it is not adjacent to any vertex in $T_2 \cup T_5$. We now consider the following four cases. Our aim is either to find a F_3 or to obtain a contradiction in each case.

Case 1: x is a T_1-vertex, but there is a neighbour $x_1 \in R$ of x such that x_1 is not a T_i-vertex for any i.
Without loss of generality, we may assume that x_1 is adjacent to T_2 and T_4. By Lemma 2.1, x and a_3 have a common neighbour x_2. Clearly, x_2 is in R and $x_2 \neq x_1$. Let $y_2 \in T_2 \cap N(x_1)$ and $y_4 \in T_4 \cap N(x_1)$. Since a_5 is adjacent to a_1 and y_4, and G is K_3-free, $(N(a_1) \cup N(y_4)) \cap N(a_5) = \emptyset$. So $\{x, y_2, a_3, x_1\} \cap N(a_5) = \emptyset$. Since $y_2x_1xx_2a_3y_2$ is a 5-cycle and $\delta(G) > n/3$, Theorem 2.2 implies that a_5 is adjacent to x_2. Hence G contains F_3 with vertex set

$$\{a_5, a_1, y_2, a_3, x_2, x, x_1, y_4\}$$

and edge set

$$\{a_5a_1, a_1y_2, y_2a_3, a_3x_2, x_2x, xx_1, x_1y_4, y_4a_5, a_5x_2, a_1x, y_2x_1, a_3y_4\}.$$

Case 2: x is a T_1-vertex; every neighbour of a T_i-vertex in R is a T_j-vertex for some j.
First, we claim that if x_1 is a neighbour of x, then x and x_1 are not adjacent to consecutive T_is. Indeed, if x and x_1 are adjacent to consecutive T_is, then x_1 is adjacent to either T_2 or T_5, since x is a T_1-vertex. Without loss of generality, we may assume that x_1 is adjacent to T_2. Since x is a T_1-vertex, Lemma 2.1 implies that a_3 and x have a common neighbour $x_2 \in R$. Then x_1 and x_2 are T_i-vertices. Theorem 2.2 implies that a_5 is adjacent to at least one vertex of the 5-cycle $xx_1a_2a_3x_2x$. Since G is K_3-free, a_5 is not adjacent to any of x, a_2, a_3. Therefore, a_5 is adjacent to at least one of x_1 and x_2. This contradicts the fact that x_1 and x_2 are T_i-vertices, establishing our claim.

Thus, no T_j-vertex and its neighbour are adjacent to consecutive T_is. Lemma 2.1 implies that x and a_3 have a common neighbour $x_3 \in R$, and x and a_4 have a common neighbour $x_4 \in R$. Then x_3 and x_4 are T_i-vertices. Therefore, Lemma 2.1 implies that x_3 and a_5 have a common neighbour $x_5 \in R$, and x_4 and a_2 have a common neighbour $x_2 \in R$. Clearly x, x_2, x_3, x_4, x_5 are T_i-vertices. We notice that $xa_1a_2x_2x_4x$ is a 5-cycle. By Theorem 2.2, the

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vertex x_5 must be adjacent to one vertex of this cycle. Since G is K_3-free and no T_i-vertex and its neighbour are adjacent to consecutive T_is, the vertex x_5 is adjacent to x_2. It can now be checked that the subgraph induced by the set $P = \{a_1, a_2, a_3, a_4, a_5, x, x_2, x_3, x_4, x_5\}$ is the Petersen graph.

Note that every vertex in $T \cup \{x, x_2, x_3, x_4, x_5\}$ is adjacent to at most three vertices of P. Since G is K_3-free, every vertex y in $R - \{x, x_2, x_3, x_4, x_5\}$ is adjacent to at most two vertices in $\{a_1, a_2, a_3, a_4, a_5\}$ and two vertices in $\{x, x_2, x_3, x_4, x_5\}$. Furthermore, if y is adjacent to a vertex in $\{x, x_2, x_3, x_4, x_5\}$, then y is a T_j-vertex for some j. So y is adjacent to at most one vertex in $\{a_1, a_2, a_3, a_4, a_5\}$. Therefore, every vertex in G is adjacent to at most three vertices of P and we have

$$3n \geqslant \sum_{v \in P} d(v) > \frac{10n}{3} > 3n,$$

a contradiction.

Case 3: the vertex x is adjacent to at least three T_is.
Since x is adjacent to a_1, the vertex x is not adjacent to T_2 and T_5. Then x is adjacent to a vertex $y_3 \in T_3$ and a vertex $y_4 \in T_4$. Therefore G contains F_3 with vertex set

$$\{a_1, a_5, a_4, y_3, x, y_4, a_3, a_2\}$$

and edge set

$$\{a_1a_5, a_5a_4, a_4y_3, y_3x, xy_4, y_4a_3, a_3a_2, a_2a_1, a_1x, a_5y_4, a_4a_3, y_3a_2\}.$$

Case 4: every vertex in R is adjacent to exactly two T_is.
Since every vertex in R is adjacent to at least one vertex of C, no vertex in R is adjacent to consecutive T_is. Furthermore, if a vertex $x \in R$ is adjacent to a_i, then x is adjacent to all vertices in T_i, by Theorem 2.2. In this case, if

$$R_i = \{x \in R : \ N(x) \cap T_{i-1} \neq \emptyset \quad \text{and} \quad N(x) \cap T_{i+1} \neq \emptyset\}$$

for $1 \leqslant i \leqslant 5$, then $\{R_1, R_2, \ldots, R_5\}$ is a partition of R. Since G is K_3-free and every vertex in R_i is adjacent either to every vertex of T_{i-1} or to every vertex of T_{i+1}, the vertices of R_i are independent by the definition of R_i. Since G is not of F_2-type and $G[T]$ is of F_2-type, for some i, there is an edge xy between R_i and $R_{i+2} \cup T_{i+2}$.

Without loss of generality, we may assume that the edge xy is between R_1 and $R_3 \cup T_3$, with $x \in R_1$ and $y \in R_3 \cup T_3$. Let $x_2 \in N(x) \cap T_2$ and $y_2 \in N(y) \cap T_2$. Since G is K_3-free, we have $x_2 \neq y_2$. Then G contains F_3 with vertex set

$$\{x, a_5, a_4, a_3, x_2, a_1, y_2, y\}$$

and edge set

$$\{xa_5, a_5a_4, a_4a_3, a_3x_2, x_2a_1, a_1y_2, y_2y, yx, a_5a_1, a_4y, a_3y_2, x_2x\}.$$

This completes the proof of our theorem. □

Note that Theorem 3.4 extends the result of Häggkvist [4]. Based on Theorems 3.3 and 3.4, our main aim now is to prove Theorem 3.1. Although the proof of Theorem 3.1

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is quite similar to that of Theorem 3.4, it is much more complicated. Note that the vertex a_i of F_d is adjacent to the vertices a_{i+1+3j}, $0 \leqslant j \leqslant d-1$.

Proof of Theorem 3.1. We shall prove this theorem by induction on d. Theorems 3.3 and 3.4 imply that the result holds for $d = 1, 2$. Suppose that $d > 2$ and the result holds for values less than d. Then, either G is of F_{d-1}-type or G contains F_d as a subgraph.

Suppose that the graph G is not of F_d-type. Then G is not of F_{d-1}-type either. So it contains the graph F_d with vertex set

$$\{a_1, a_2, a_3, \ldots, a_{3d-1}\}$$

and edge set

$$\left\{a_i a_{i+1}, a_i a_{i+4}, a_i a_{i+7}, \ldots, a_i a_{i+3\lceil \frac{d}{2} \rceil - 2} \ : \ i \in \mathbb{Z}_{3d-1}\right\}.$$

For $1 \leqslant i \leqslant 3d-1$, let

$$Q_i = \bigcap_{j=1}^{d} N\left(a_{i+3j-2}\right).$$

Clearly, $a_i \in Q_i$ for $1 \leqslant i \leqslant 3d-1$. Let

$$Q = \bigcup_{i=1}^{3d-1} Q_i.$$

We claim that $G[Q]$ is an F_d-type graph. Indeed, it suffices to check that there is no edge between Q_1 and Q_i, where $3 \leqslant i \leqslant 3\lfloor d/2 \rfloor$ and $i \neq 5, 8, 11, \ldots, 3\lceil d/2 \rceil - 1$. Suppose then that $x_1 \in Q_1$ and $x_i \in Q_i$ are adjacent. Since x_1 is adjacent to a_2, a_5, $a_8, \ldots, 3\lceil d/2 \rceil - 1$, and x_i is adjacent to a_{i-1} and a_{i+1}, the vertex x_1 is adjacent to one of a_{i-1} and a_{i+1}, say a_{i-1}. Then $x_1 x_i a_{i-1}$ is a triangle of G, contradicting the fact that G is K_3-free. Thus $G(Q)$ is of F_d-type, as claimed.

Since G is not of F_d-type and $G[Q]$ is of F_d-type, $R = V(G) - Q$ is not empty. In the following three lemmas, we shall study the relation between the graph F_d and the vertices in R.

Lemma 3.5. *No vertex x of R is adjacent to both a_i and a_{i+2}.*

Proof. Suppose that a vertex $x \in R$ is adjacent to a_i and a_{i+2} for some i. Without loss of generality, we may assume that x is adjacent to the vertices a_1 and a_3.

For $1 \leqslant i \leqslant d-2$, the vertex x is not adjacent to a_{1+3i}, a_{2+3i}, a_{4+3i} and a_{5+3i} (since G is K_3-free, a_1 is adjacent to a_{2+3i} and a_{5+3i}, and a_3 is adjacent to a_{1+3i} and a_{4+3i}). Thus Theorem 2.2 implies that x is adjacent to the vertex a_{3+3i} of the 5-cycle $a_{1+3i}a_{2+3i}a_{3+3i}a_{4+3i}a_{5+3i}a_{1+3i}$. However, the vertex x is adjacent to each of the vertices

$$a_3,\ a_6,\ a_9,\ \ldots,\ a_{3d}(= a_1).$$

Thus, $x \in Q_2$, contradicting the fact that $x \in R$. □

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Lemma 3.6. *For every vertex x in R, there is an index $i \in \mathbb{Z}_{3d-1}$ such that x is adjacent to a_i and a_{i+5}, but not adjacent to any of a_{i+1}, a_{i+2}, a_{i+3} and a_{i+4}.*

Proof. Let x be a vertex in R, and let $a_{i_1}, \ldots, a_{i_k}$, where $i_1 < i_2 < \cdots < i_k$, be all the neighbours of x in F_d. Since F_d contains a 5-cycle, Theorem 2.2 implies that x is adjacent to at least one vertex of F_d, say a_1. Since $3d - 1 \geqslant 8$, Theorem 2.2 implies that x is adjacent to at least one vertex of the 5-cycle $a_2a_3a_4a_5a_6a_2$. So $k \geqslant 2$. Since a_i is adjacent to a_{i+4} and $a_ia_{i+1}a_{i+2}a_{i+3}a_{i+4}a_i$ is a 5-cycle, Lemma 3.6 and Theorem 2.2 imply that, for $1 \leqslant j \leqslant k-1$,

$$i_j - i_{j-1} = 3 \quad \text{or} \quad 5,$$

and

$$i_1 + 3d - 1 - i_k = 3 \quad \text{or} \quad 5.$$

Therefore, as $3 \nmid 3d-1$, we have either $i_j - i_{j-1} = 5$ for some $1 \leqslant j \leqslant k-1$, or $i_1 + 3d - 1 - i_k = 5$. □

Lemma 3.7. *For each vertex x of R, there is an index $i \in \mathbb{Z}_{3d-1}$ such that all the neighbours of x in F_d are the vertices a_{i+3j}, $0 \leqslant j \leqslant d-2$. Furthermore, $\bigcup_{j=0}^{d-2} Q_{i+3j} \subseteq N(x)$.*

Proof. Let x be a vertex in R. By Lemma 3.6, we may assume that x is adjacent to a_1 and a_6, and not to a_2, a_3, a_4 and a_5. For $1 \leqslant i \leqslant d-3$, x is not adjacent to a_{4+3i}, a_{5+3i}, a_{7+3i} and a_{8+3i} (since G is K_3-free, a_1 is adjacent to a_{5+3i} and a_{8+3i}, and a_6 is adjacent to the vertex a_{4+3i} and a_{7+3i}). Therefore x is adjacent to the vertex a_{6+3i} of the 5-cycle $a_{4+3i}a_{5+3i}a_{6+3i}a_{7+3i}a_{8+3i}a_{4+3i}$. Now, note that the graph G is K_3-free; a_1 is adjacent to each of a_8, a_{11}, $a_{14}, \ldots, a_{3d-1}$; a_6 is adjacent to each of a_7, a_{10}, $a_{13}, \ldots, a_{3d-2}$; and x is not adjacent to any of a_2, a_3, a_4 and a_5. Therefore the neighbours of x in F_d are precisely a_6, a_9, $a_{12}, \ldots, a_{3d} (= a_1)$. By a similar argument, we find that x is adjacent to every vertex of $\bigcup_{j=0}^{d-2} Q_{6+3j}$. □

Let us extend the notation of a T_i-vertex used in Theorem 3.4. Call a vertex $x \in R$ a *$(d-1)$-set vertex with classes Q_i, Q_{i+3}, $Q_{i+6}, \ldots, Q_{i+3(d-2)}$*, or simply a *$(d-1)$-set vertex*, if $N(x) \subseteq R \cup \bigcup_{j=0}^{d-2} Q_{i+3j}$.

We now return to the proof of Theorem 3.1. Let x be a vertex in R. By Lemma 2.1, we may assume that $\bigcup_{j=0}^{d-2} Q_{6+3j} \subseteq N(x)$. Consider the following three cases.

Case 1: the vertex x is a $(d-1)$-set vertex.

Since a_3 or a_4 is not adjacent to any vertex in $\bigcup_{j=0}^{d-2} Q_{6+3j}$, Lemma 2.1 implies that x and a_3 have a common neighbour x_3 in R, and that x and a_4 have a common neighbour x_4 in R. Since G is K_3-free, we have $x_3 \neq x_4$.

The vertex x is adjacent to a_6. So Lemma 3.7 implies that x_3 is adjacent to exactly a_8, a_{11}, $a_{14}, \ldots, a_{3d-1}, a_3$ in F_d. As x is adjacent to a_1, Lemma 3.7 implies that x_4 is adjacent to exactly a_4, a_7, $a_{10}, \ldots, a_{3d-2}$ in F_d. Since x_3 is not adjacent to a_4, a_5, a_6 and a_7, Theorem 2.2 implies that x_3 is adjacent to x_4 of the 5-cycle $a_4a_5a_6a_7x_4a_4$. Therefore $x_3x_4xx_3$ is a triangle of G, a contradiction.

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Case 2: the vertex x is adjacent to at least $d+1$ of the Q_is.
Since G is K_3-free and x is adjacent to the vertices a_6, a_9 $a_{12}, \ldots, a_{3d}(= a_1)$, the vertex x must be adjacent to one vertex y_3 in Q_3 and one vertex y_4 in Q_4. Clearly, $y_3 \neq a_3$ and $y_4 \neq a_4$. Then G contains the graph F_{d+1} with vertex set

$$\{a_1, a_2, a_3, y_4, x, y_3, a_4, a_5, \ldots, a_{3d-1}\}$$

and edge set

$$E(F_d) \cup X \cup Y_3 \cup Y_4 \cup \{y_4x, xy_3\},$$

where

$$\begin{aligned} X &= \{xa_{6+3i} \ : \ 0 \leqslant i \leqslant d-2\}, \\ Y_3 &= \{y_4a_{5+3i} \ : \ 0 \leqslant i \leqslant d-1\}, \end{aligned}$$

and

$$Y_4 = \{y_3a_{4+3i} \ : \ 0 \leqslant i \leqslant d-1\}.$$

Case 3: every vertex in R is adjacent to exactly d of the Q_is.
For $1 \leqslant i \leqslant 5$, let

$$W_i = \{x \in R : N(x) \cap Q_{i+1+3j} \neq \emptyset, \ 0 \leqslant j \leqslant d-1\}.$$

Then, by Lemma 3.7, $\{W_1, W_2, \ldots, W_5\}$ is a partition of R.

We claim that, for $i+3 \leqslant j \leqslant i+3\lfloor d/2 \rfloor - 1$ and $3 \nmid (j-i-1)$, the set $W_i \cup W_j \cup Q_j$ is independent. We may assume that $i = 1$, *i.e.* we shall show that, for $4 \leqslant j \leqslant 3\lfloor d/2 \rfloor$ and $3 \nmid j-2$, the set $W_1 \cup W_j \cup Q_j$ is independent.

Assume first that $3 \mid (j-1)$. Then every vertex in $W_j \cup Q_j$ is adjacent to a vertex in Q_{j+1} and a vertex in Q_{3d-1}, and every vertex in W_1 is adjacent to a vertex in Q_{j+1} and a vertex in Q_{3d-1}. Therefore, Lemma 3.7 and the definition of Q_i imply that every vertex in W_1 and every vertex in W_j have a common neighbour in $Q_{j+1} \cup Q_{3d-1}$. Since G is K_3-free, the set $W_1 \cup W_j \cup Q_j$ is independent.

Assume now that $3 \mid j$. Every vertex in $W_j \cup Q_j$ is adjacent to a vertex in Q_2 and a vertex in Q_{j-1}, and every vertex in W_1 is adjacent to a vertex in Q_2 and a vertex in Q_{j-1}. Therefore, Lemma 3.7 and the definition of Q_i imply that every vertex in W_1 and every vertex in W_j have a common neighbour in $Q_2 \cup Q_{j-1}$. Since G is K_3-free, the set $W_1 \cup W_j \cup Q_j$ is independent, completing the proof of our claim.

Now, we return to the proof for Case 3. Since G is not of F_d-type, the claim above implies that, for some $i \in \mathbb{Z}_{3d-1}$, there is an edge between W_i and $W_{i+2} \cup Q_{i+2}$. We may assume that a vertex $x \in W_1$ is adjacent to a vertex $y \in W_3 \cup Q_3$. Let $x_2 \in N(x) \cap Q_2$ and $y_2 \in N(y) \cap Q_2$. Then $y_2 \neq x_2$ since G is K_3-free and, by Lemma 3.7, the vertex x has to be adjacent to a_{2+3i}, $1 \leqslant i \leqslant 3(d-1)$, and y has to be adjacent to a_{1+3i}, $1 \leqslant i \leqslant 3(d-1)$. Then the graph G contains F_{d+1} with vertex set

$$\{a_1, y_2, y, x, x_2, a_3, a_4, \ldots, a_{3d-1}\}$$

and edge set

$$E(F_d) \cup Y_2 \cup Y \cup X \cup X_2 \cup \{xy, \ xx_2, \ yy_2\},$$

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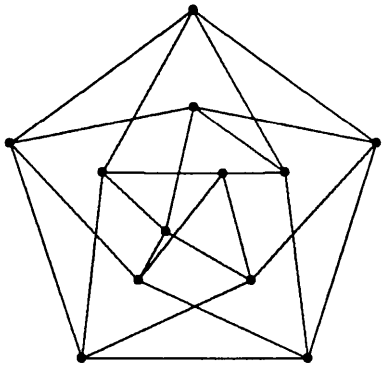


Figure 3 An extremal graph M'.

where

$$\begin{aligned} Y_2 &= \{y_2a_{3i} \ : \ 1 \leqslant i \leqslant d-1\}, \\ Y &= \{ya_{1+3i} \ : \ 1 \leqslant i \leqslant d-1\}, \\ X &= \{xa_{2+3i} \ : \ 1 \leqslant i \leqslant d-1\} \end{aligned}$$

and

$$X_2 = \{x_2a_{3i} \ : \ 1 \leqslant i \leqslant d-1\}.$$

This completes the inductive step and thus the proof of Theorem 3.1. □

Note that Theorem 3.1 is essentially best possible. Indeed, consider the graph M' as shown in Figure 3. It is easy to see that M' is maximal M-free and K_3-free and that $|V(M')| = n = 12$, $\delta(M') = 4 = \frac{n}{3}$. However, M' is not of F_d-type since $\chi(M') = 4$. Thus, M' is an extremal graph for the problem when $n = 12$. In fact, by considering graphs of M'-type, we may obtain extremal graphs for all n which are not congruent to 3, 6 or 9 modulo 12.

The following two results, as stated in the abstract, follow from Theorem 3.1.

Theorem 3.8. *Let G be a K_3-free graph of order n with $\chi(G) \leqslant 3$ and $\delta(G) > n/3$. Then G is of F_d-type for some $d \geqslant 1$.*

Proof. Let $d = \lfloor (n+1)/3 \rfloor$ and G_1 be a maximal K_3-free and M-free graph obtained from G by adding edges to G. Since $3(d+1) - 1 > n$, G_1 does not contain F_{d+1} as subgraph. Therefore, Theorem 3.1 implies G_1 is of F_d-type, so G is of F_d-type. □

Theorem 3.9. *Let G be a K_3-free graph of order n with $\chi(G) \geqslant 4$ and $\delta(G) > n/3$. Then G contains M as a subgraph.*

Proof. Suppose that, contrary to the assertion, G does not contain M as a subgraph. Let G_1 be a maximal K_3-free and M-free graph obtained from G by adding edges to G.

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Then, Theorem 3.1 implies that G_1 is of F_d-type for some $d \geqslant 1$. Therefore, we get a contradiction that $4 \leqslant \chi(G) \leqslant \chi(G_1) \leqslant \chi(F_d) \leqslant 3$. □

Finally, we have the following theorem.

Theorem 3.10. *Let $d \geqslant 1$ and let G be a K_3-free graph of order n with $\chi(G) \leqslant 3$ and $\delta(G) > \left\lfloor \frac{(d+1)n}{3d+2} \right\rfloor$. Then G is of F_d-type. Moreover, some F_{d+1}-type graphs are not of F_d-type with minimum degree $\left\lfloor \frac{(d+1)n}{3d+2} \right\rfloor$.*

Proof. Suppose that $\delta(G) > \frac{(d+1)n}{3d+2}$ and G is not of F_d-type. Without loss of generality, we may assume that G is maximal K_3-free. Since $d \geqslant 1$, we have $\frac{(d+1)n}{3d+2} > n/3$. Then Theorem 3.1 implies that G contains the graph F_{d+1}. Since the independence number of F_{d+1} is $d+1$ and G is K_3-free, every vertex of G is adjacent to at most $d+1$ vertices of F_{d+1}. Thus

$$(d+1)n \geqslant \sum_{x \in V(G)} d_{F_{d+1}}(x) = \sum_{x \in F_{d+1}} d(x) > (3d+2)((d+1)n/(3d-1)) = (d+1)n,$$

a contradiction.

Clearly, the complete F_{d+1}-type graph $F_{d+1}(n/(3d+2))$ is a $(d+1)n/(3d+2)$-regular K_3-free graph. Since $F_{d+1}(n/(3d+2))$ contains F_{d+1} as a subgraph for $n \geqslant 3d+2$, the graph $F_{d+1}(n/(3d+2))$ is not of F_d-type. Therefore, if $(3d+2) \mid n$, then $F_{d+1}(n/(3d+2))$ is a K_3-free, non-F_d-type graph of order n with minimum degree $(d+1)n/(3d+2)$. In fact, for a given d with $1 \leqslant d \leqslant 9$ and $n \geqslant 3d+1$, we can find a complete F_{d+1}-type graph of order n which is a K_3-free, non-F_d-type graph with minimum degree $\left\lfloor \frac{(d+1)n}{3d+2} \right\rfloor$. □

For 3-chromatic graphs we have the sequence $\{F_i\}_{i=1}^{\infty}$. But we do not know whether there is similar sequence for 4-chromatic graphs. By applying a result of this paper, Jin [7] very recently showed that every K_3-free graph G of order n with $\delta(G) > 11n/32$ is either of F_i-type for some $i \leqslant 10$, or of M-type with M as a subgraph.

Is there any graph G with chromatic number at least 5 and minimum degree greater than $|G|/3$? Erdős and Simonovits [3] (also see [2, page 366]) conjectured that it is unlikely.

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