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Hilbert polynomials of j-transforms

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Abstract

We study transformations of finite modules over Noetherian local rings that attach to a module M a graded module $\mathrm{H}_{(\mathbf{x})}(M)$ defined via partial systems of parameters $\mathbf{x}$ of M. Despite the generality of the process, which are called **j**-transforms, in numerous cases they have interesting cohomological properties. We focus on deriving the Hilbert functions of **j**-transforms and studying the significance of the vanishing of some of its coefficients.

1. *Introduction*

Let $(R, \mathfrak{m})$ be a Noetherian local ring and I an ideal of R. For a finite R-module M, our aim is to study the metrics and homological properties associated to the composition of the two functors

$$M \longmapsto \mathrm{gr}_I(M) = \bigoplus_{n\geqslant 0} I^n M/I^{n+1}M \longmapsto \mathrm{H}_I(M) = \mathrm{H}^0_{\mathfrak{m}}(\mathrm{gr}_I(M)).$$

Such transformation $\mathrm{H}_I(M)$ could be appropriately called the **j**-*transform* of M relative to I. The study of the **j**-transform was initiated by Achilles and Manaresi [**1**] who made use of the fact that $\mathrm{H}_I(M) = \bigoplus_{k\geqslant 0} \mathrm{H}_k$ has an associated numerical function $n \mapsto \psi_I^M(n) = \sum_{k\leqslant n} \lambda(\mathrm{H}_k)$, where $\lambda(-)$ denotes the length. This function is called the **j**-*function* of M relative to I and is a broad generalisation of the classical Hilbert function – the case where

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I is an $\mathfrak{m}$-primary ideal. Its Hilbert polynomial

$$\sum_{i=0}^{r}(-1)^i \mathrm{j}_i(I;M)\binom{n+r-i}{r-i}$$

will be referred to as the **j**-*polynomial* of M relative to I, where r stands for the analytic spread of I. In general it is very difficult to predict properties of the **j**-transform $\mathrm{H}_I(M)$, beginning with their Krull dimensions or the **j**-*coefficients* $\mathrm{j}_i(I;M)$. Nevertheless several authors have succeeded in applying the construction to extend the full array of classical integrality criteria for Rees algebras and modules [**1**, **3**, **4**, **16** and **21**].

Our aim here is to study a different facet of the **j**-polynomials. The specific aim is to derive explicit formulas for the **j**-coefficients $\mathrm{j}_i(I;M)$ in terms of properties of M known a priori and explore the significance of their vanishing. For that we limit ourselves to ideals generated by partial systems of parameters of M or even special classes of modules. Thus we let $\mathbf{x} = x_1, \ldots, x_r$ be a partial system of parameters of M, that is $\dim M = r + \dim M/(\mathbf{x})M$, and set $I = (\mathbf{x})$ and $\mathrm{j}_i(\mathbf{x};M) = \mathrm{j}_i(I;M)$.

A general issue is what the values of $\mathrm{j}_1(\mathbf{x};M)$ say about M itself. In [**5**], [**6**], [**7**] and [**22**], the authors and colleagues studied the values of a special class of these coefficients. For a Noetherian local ring R and a finite R-module M, we considered the Hilbert coefficients $\mathrm{e}_i(\mathbf{x};M)$ associated to filtrations defined by a system $\mathbf{x}$ of parameters of M, more precisely to the Hilbert functions

$$n \longmapsto \lambda(M/(\mathbf{x})^{n+1}M)$$

and made use of the values of $\mathrm{j}_1(\mathbf{x};M)$ as the means to detect various properties of M (e.g., Cohen–Macaulay, Buchsbaum, finite local cohomology, etc.). Here we seek to extend these probes to cases when $r < \dim M$. The significant distinction between $\mathrm{gr}_{(\mathbf{x})}(M)$ and $\mathrm{H}^0_{\mathfrak{m}}(\mathrm{gr}_{(\mathbf{x})}(M))$ is that when $r < \dim M$, the latter may not be homogeneous and therefore the vanishing of some of its Hilbert coefficients does not place them entirely in the context of [**5**], [**7**] and [**22**].

We illustrate these issues with a series of questions. Let R be a Noetherian local ring and M a finite R-module. Let I be an ideal generated by a partial system of parameters $\mathbf{x} = x_1, \ldots, x_r$ of M and let $\mathrm{G} = \mathrm{gr}_I(R)$ be the associated graded ring. Then the G-module $\mathrm{H}_I(M) = \mathrm{H}^0_{\mathfrak{m}}(\mathrm{gr}_I(M))$ has dimension at most r (Proposition 3·1). We list some questions similar to those raised in [**22**] for a full system of parameters:

(i) what are the possible values of $\dim_{\mathrm{G}} \mathrm{H}_I(M)$? Note that $\mathrm{H}_I(M) = (0)$ may happen or $\mathrm{H}_I(M) \neq (0)$ but of dimension zero;

(ii) what is the signature of $\mathrm{j}_1(\mathbf{x};M)$? If $r = \dim_{\mathrm{G}} \mathrm{H}_I(M)$, is $\mathrm{j}_1(\mathbf{x};M) \leqslant 0$? The answer is affirmative if $\mathrm{H}_I(M)$ is generated in degree 0, because $\mathrm{j}_1(\mathbf{x};M) = \mathrm{e}_1(\mathbf{f}, \mathrm{H}_I(M))$ and these coefficients are always non-positive according to [**15**, theorem 3·6] and [**6**, corollary 2·3], where $\mathbf{f} = f_1, \ldots, f_r$ denotes the initial forms of $x_i's$ relative to I;

(iii) if M is unmixed, $r = \dim_{\mathrm{G}} \mathrm{H}_I(M)$, and $\mathrm{j}_1(\mathbf{x};M) = 0$, then is $\mathbf{x}$ an M-regular sequence? The answer is obviously no. What additional restriction is required?

The questions (ii) and (iii) were dealt with in [**5**], [**6**], [**7**] and [**22**] for $r = \dim M$, but we do not know much in the other cases. We shall focus on a special kind of partial systems of parameters which will be shown to be ubiquitous. For a finite R-module M, we call a partial system $\mathbf{x} = x_1, \ldots, x_r$ of parameters of M *amenable* if the Koszul

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homology module $\mathrm{H}_1(\mathbf{x}; M)$ has finite length (and hence $\mathrm{H}_i(\mathbf{x}; M)$ have finite length for all $i \geqslant 1$).

Let us state one of our main questions.

CONJECTURE 1·1. Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finite unmixed R-module. Let $\mathbf{x} = x_1, \ldots, x_r$ be a partial system of parameters of M. Suppose that $\mathbf{x}$ is an amenable d-sequence relative to M, that $\dim \mathrm{H}^0_{\mathfrak{m}}(\mathrm{gr}_{(\mathbf{x})}(M)) = r$, and that $\mathrm{j}_1(\mathbf{x}; M) = 0$. Then $\mathbf{x}$ is a regular sequence on M.

The unmixedness hypothesis serves various purposes that begin from avoiding trivial counterexamples to the enabling of several constructions for cohomological calculations. There are two versions of the conjecture, the strong version as stated and a weak one. The latter assumes that $\mathrm{j}_1(\mathbf{x}; M) = 0$ for all partial systems of parameters of r elements.

In outline, our approach to describe some of the **j**-transform $\mathrm{H}^0_{\mathfrak{m}}(\mathrm{gr}_{(\mathbf{x})}(M))$ is the following. First we show how to find an amenable partial system of parameters of M that is a d-sequence relative to M (Proposition 2·5). Once provided with such a partial system $\mathbf{x}$ of parameters, $\mathrm{H}^0_{\mathfrak{m}}(\mathrm{gr}_{(\mathbf{x})}(M))$ occurs as the homology of the complex obtained by applying the section functor $\mathrm{H}^0_{\mathfrak{m}}(-)$ to the approximation complex $\mathcal{M}(\mathbf{x}; M)$ [**11**]:

$$0 \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(\mathrm{H}_r(\mathbf{x}; M)) \otimes S[-r] \longrightarrow \cdots \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(\mathrm{H}_1(\mathbf{x}; M)) \otimes S[-1] \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(\mathrm{H}_0(\mathbf{x}; M)) \otimes S \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(\mathrm{gr}_{\mathbf{x}}(M)) \longrightarrow 0,$$

where $S = R[\mathbf{T}_1, \ldots, \mathbf{T}_r]$ is the polynomial ring (see Theorem 3·2). When we add to this the assumption that $\mathrm{H}_1(\mathbf{x}; M)$ has finite length, we derive the Hilbert series $[\![\mathrm{H}]\!]$ of $\mathrm{H} = \mathrm{H}^0_{\mathfrak{m}}(\mathrm{gr}_{(\mathbf{x})}(M))$ in terms of the Koszul homology modules: If $h_i = \lambda(\mathrm{H}_i(\mathbf{x}; M))$, $i > 0$, and $h_0 = \lambda(\mathrm{H}^0_{\mathfrak{m}}(\mathrm{H}_0(\mathbf{x}; M))$, then

$$[\![\mathrm{H}]\!] = \frac{\sum_{i=0}^{r} (-1)^i h_i \mathbf{t}^i}{(1 - \mathbf{t})^r},$$

where $\mathbf{t}$ denotes the indeterminate. Several properties of H can be read right away from the complex, such as the vanishing of H and the Cohen–Macaulayness of H (Corollary 3·3). Both of which require that $\mathbf{x}$ be a regular sequence on M. Besides the expression of the Hilbert function of H in terms of the values of $\lambda(\mathrm{H}_i(\mathbf{x}; M))$, Corollary 3·4 interprets several properties of the relationship between $\mathbf{x}$ and M. For instance, if $\dim \mathrm{H} = r$, then $\mathrm{j}_1(\mathbf{x}; M) \leqslant 0$, the exact behavior of the case when $\mathbf{x}$ is a full system of parameters. However, it is not enough to determine the Krull dimension of H much less to resolve our conjecture on whether the vanishing of $\mathrm{j}_1(\mathbf{x}; M)$ means that $\mathbf{x}$ is a regular sequence on M. This would require that H be unmixed along with M. In the remaining of the section we establish the conjecture in several special cases that places restrictions on the projective dimension of M or the $\mathbf{x}$-depth of M.

In the following sections we develop formulas for the **j**-coefficients based on the finiteness of the local cohomology associated to certain partial systems of parameters. In Section 4, we derive (see Theorem 4·7) the Hilbert function of H as

$$\psi^M_{(\mathbf{x})}(n) = \sum_{i=0}^{r} k_i(\mathbf{x}; M) \binom{n + r - i}{r - i} \quad \text{for all } n \geqslant 0$$

with the $k_i(\mathbf{x}; M)$ being some positive linear combinations of the lengths of local cohomology modules. They are therefore very convenient to study relationships between depth conditions and the vanishing of the $\mathrm{j}_i(\mathbf{x}; M)$.

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The focus in Section 5 is on the boundedness of *all* **j**-coefficients $\mathrm{j}_i(\mathbf{x}; M)$ for *all* amenable partial systems of parameters which are d-sequences relative to the module M, a problem treated in [**5**] and [**6**] for full systems of parameters. The corresponding result (Theorem 5·1) in this more delicate setting requires modules admitting one strong d-sequence, a property that holds for all homomorphic images of Gorenstein rings. Finally, in Section 6 we give structure theorems for the **j**-transforms of Buchsbaum and sequentially Cohen-Macaulay modules. They come with a detailed description of their Hilbert functions.

2. *Amenable partial systems of parameters*

In this section we discuss a class of partial systems of parameters that provides a gateway to the calculation of numerous **j**-polynomials because of their flexibility in the calculation of local cohomology.

Definition 2·1. Let R be a Noetherian local ring and let M be a finite R-module of dimension $d > 0$. A partial system $\mathbf{x} = x_1, \ldots, x_r$ $(0 \leqslant r \leqslant d)$ of parameters of M is said to be *amenable* to M, if the Koszul homology modules $\mathrm{H}_i(\mathbf{x}; M)$ have finite length for all $i > 0$.

A simple example is that of a Noetherian local ring R of dimension 2, with x satisfying $0 : x = 0 : x^2$ and x not contained in any associated prime $\mathfrak{p}$ of R of codimension at most one. Similarly, if M is Cohen–Macaulay on the punctured support of M, then any partial system of parameters is amenable. Under mild additional conditions, these are the modules with *finite local cohomology* (FLC), that is the cohomology modules $\mathrm{H}^i_{\mathfrak{m}}(M)$, $i < \dim M$, have finite length.

Given the rigidity of Koszul complexes, the condition is equivalent to saying that $\mathrm{H}_1(\mathbf{x}; M)$ has finite length. These sequences are also called *filter regular* systems of parameters (see [**17**]), but the main source of amenable partial systems of parameters is found in a property of d-sequences. Let us recall briefly this notion which we pair to a related type of sequence. They are extensions of regular sequences, the notion of *d-sequence* invented by Huneke [**12**], and of a *proper sequence* [**11**]. They play natural roles in the theory of the *approximation complexes* (see [**11**]). We now collect appropriate properties of these sequences.

Definition 2·2. Let R be a commutative ring and M an R-module. A sequence $\mathbf{x} = x_1, \ldots, x_n$ of R-elements is called a *d-sequence* relative to M, if

$$(x_1, \ldots, x_{i-1})M :_M x_i x_j = (x_1, \ldots, x_{i-1})M :_M x_j \text{ for all } 1 \leqslant i \leqslant j \leqslant n.$$

A sequence $\mathbf{x} = x_1, \ldots, x_n$ of R-elements is called a *strong d-sequence* relative to M, if for every sequence $r_1, \ldots, r_n$ of positive integers the sequence $x_1^{r_1}, \ldots, x_n^{r_n}$ is a d-sequence relative to M. We say that $\mathbf{x} = x_1, \ldots, x_n$ is a *d^+-sequence* relative to M, if it is a strong d-sequence relative to M in any order.

Definition 2·3. Let R be a commutative ring and M an R-module. A sequence $\mathbf{x} = x_1, \ldots, x_n$ of R-elements is a *proper sequence* relative to M, if

$$x_{i+1}\mathrm{H}_j(x_1, \ldots, x_i; M) = 0 \text{ for all } 1 \leqslant i < n \text{ and } j > 0,$$

where $\mathrm{H}_j(x_1, \ldots, x_i; M)$ is the Koszul homology of M associated to $x_1, \ldots, x_i$.

PROPOSITION 2·4. *Let R be a Noetherian ring and $\mathbf{x} = x_1, \ldots, x_n$ a sequence in R.*

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(1) *If* $\mathbf{x}$ *is a d-sequence relative to* M*, then* $\mathbf{x}$ *is a proper sequence relative to* M.
(2) *If* $\mathbf{x}$ *is a proper sequence relative to* M*, then*

$$x_k \mathrm{H}_j(x_1, \ldots, x_i; M) = 0 \text{ for all } 1 \leqslant i < n,\ j > 0, \text{ and } k > i.$$

(3) *Suppose that* R *is a local ring and* M *a finite* R*-module of dimension* $n > 0$*. If* $\mathbf{x}$ *is a system of parameters of* M *that is also a proper sequence relative to* M*, then* $\mathrm{H}_j(x_1, \ldots, x_i; M)$ *has finite length for all* $1 \leqslant i \leqslant n$ *and* $j > 0$.

We make use of Proposition 2·4(3) to build amenable partial systems of parameters of M.

Ubiquity

The following assertion is based on [**11**, proposition 2·7]. That construction applied only to $M = R$ and had a missing sentence. We use the proof to build a system of parameters that is a d-sequence whose first r elements are given.

PROPOSITION 2·5. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring and* M *a finite* R*-module of dimension* $d > 0$*. Let* I *be an* $\mathfrak{m}$*-primary ideal of* R*. Then there exists a system of parameters of* M *which is contained in* I *and a* d*-sequence relative to* M.

Proof. For an R-module C we set $z(C) = \bigcup_{P \in \mathrm{Ass}_R C \setminus \{\mathfrak{m}\}} P$. Let $y_1 \in I \setminus z(M)$ and pick m large enough so that $(0) :_M y_1^m = (0) :_M y_1^{m+1}$. Let $x_1 = y_1^m$ and $J_1 = (0) :_R ((0) :_M x_1)$. For $1 \leqslant s \leqslant d-1$ we define the following recursively. Let

$$y_{s+1} \in (I \cap J_1 \cap \cdots \cap J_s) \setminus (z(M) \cup z(M/x_1M) \cup \cdots \cup z(M/(x_1, \ldots, x_s)M)),$$

where $J_i = (x_1, \ldots, x_{i-1})M :_R ((x_1, \ldots, x_{i-1})M :_M x_i)$ for each $1 \leqslant i \leqslant s$. Pick m large enough so that

$$(x_1, \ldots, x_i)M :_M y_{s+1}^m = (x_1, \ldots, x_i)M :_M y_{s+1}^{m+1} \quad \text{for all } 0 \leqslant i \leqslant s.$$

We set $x_{s+1} = y_{s+1}^m$. Then since the elements $x_1, \ldots, x_d$ as chosen generate an ideal of height d, it is enough to prove that every subsequence $x_1, \ldots, x_s$ is a d-sequence relative to M for all $1 \leqslant s \leqslant d$. We use induction on s. Let $s = 1$. Then by definition, $(0) :_M x_1 = (0) :_M x_1^2$, which means that x_1 is a d-sequence relative to M. Suppose that $x_1, \ldots, x_s$ is a d-sequence relative to M. In order to prove that $x_1, \ldots, x_s, x_{s+1}$ is a d-sequence relative to M, it is enough to show that for all $1 \leqslant i \leqslant s+1$

$$(x_1, \ldots, x_{i-1})M :_M x_i x_{s+1} \subseteq (x_1, \ldots, x_{i-1})M :_M x_{s+1}.$$

Let $u \in M$ such that $(x_i x_{s+1})u \in (x_1, \ldots, x_{i-1})M$. Then $x_{s+1}u \in ((x_1, \ldots, x_{i-1})M :_M x_i)$. Since $x_{s+1} \in J_i$ for all $1 \leqslant i \leqslant s$, we get

$$x_{s+1}(x_{s+1}u) \in (x_1, \ldots, x_{i-1})M.$$

Hence

$$u \in (x_1, \ldots, x_{i-1})M :_M x_{s+1}^2 = (x_1, \ldots, x_{i-1})M :_M x_{s+1}$$

as asserted.

COROLLARY 2·6. *Let* R *be a Noetherian local ring and* M *a finite* R*-module of dimension* $d > 0$*. Then for each* $0 \leqslant r \leqslant d$ *there exists an amenable partial system of parameters of* M *of length* r *that is a* d*-sequence relative to* M.

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Proof. By Proposition 2·5 there exists a system $x_1, \dots, x_d$ of parameters of M that is a d-sequence relative to M and by Proposition 2·4(3) the partial system $x_1, \dots, x_r$ is an amenable d-sequence relative to M.

COROLLARY 2·7. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finite R-module of dimension $d > 0$. Let $\mathbf{x} = x_1, \dots, x_r$ be an amenable partial system of parameters of M that is a d-sequence relative to M. Then the sequence $\mathbf{x}$ can be extended to a full system of parameters of M that is a d-sequence relative to M.*

Proof. Apply the construction of Proposition 2·5 to $I = \mathfrak{m}$, choosing $\mathbf{x}$ for the first r elements of the d-sequence.

Question 2·8. Let R be a Noetherian local ring and M a finite R-module. Given an amenable partial system $\mathbf{x} = x_1, \dots, x_r$ of parameters of M, is there an amenable partial system $y_1, \dots, y_r$ of parameters of M in the ideal $(\mathbf{x})$ that is a d-sequence relative to M?

Signature of $\mathbf{j}_1(\mathbf{x}; M)$

For an R-submodule N of M, we let $N :_M \langle \mathfrak{m} \rangle = \bigcup_{\ell>0}[N :_M \mathfrak{m}^\ell]$. Hence $\mathrm{H}^0_{\mathfrak{m}}(M/N) = [N :_M \langle \mathfrak{m} \rangle]/N$.

THEOREM 2·9. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finite R-module of dimension $d > 0$. Let $I = (\mathbf{x}) = (x_1, \dots, x_r)$ be an ideal of R generated by an amenable partial system of parameters of M that is a d-sequence relative to M, where $0 \leqslant r \leqslant d$. Let $\mathrm{G} = \mathrm{gr}_I(R)$. Then the following assertions hold true:*

(1) *$I^n M \bigcap [I^{n+1}M :_M \langle \mathfrak{m} \rangle] = I^n[IM :_M \langle \mathfrak{m} \rangle]$ for all $n \geqslant 0$;*

(2) *The* **j***-transform $\mathrm{H}_I(M)$ is generated by its homogeneous component of degree 0, i.e.,*

$$\mathrm{H}_I(M) = \mathrm{G}\cdot[\mathrm{H}_I(M)]_0.$$

Proof. (1) We have only to show $I^n M \bigcap [I^{n+1}M :_M \langle \mathfrak{m} \rangle] \subseteq I^n[IM :_M \langle \mathfrak{m} \rangle]$. By induction, we may assume that $n, r > 0$ and the assertion holds true for $n-1$ and $r-1$. We set $\overline{M} = M/x_1 M$ and consider the partial system $x_2, \dots, x_r$ of parameters of $\overline{M}$. Let $f \in I^{n+1}M :_M \langle \mathfrak{m} \rangle$ and let $\overline{f}$ denote the image of f in $\overline{M}$. Then $\overline{f} \in I^n \overline{M} \bigcap [I^{n+1}\overline{M} :_{\overline{M}} \langle \mathfrak{m} \rangle]$ and hence $f \in I^n[IM :_M \langle \mathfrak{m} \rangle] + x_1 M$ by the hypothesis on r. Therefore without loss of generality, we may assume that $f \in x_1 M \bigcap I^n M$. Hence, because $x_1 M \bigcap I^n M = x_1 I^{n-1} M$ (as $\mathbf{x}$ is a d-sequence relative to M; see [**13**, proposition 2·2]), we get $f = x_1 g$ for some $g \in I^{n-1}M$. Then for $\ell \gg 0$, $x_1(\mathfrak{m}^\ell g) \subseteq I^{n+1}M \bigcap x_1 M = x_1 I^n M$. Let $a \in \mathfrak{m}^\ell$ and write $x_1(ag) = x_1 h$ with $h \in I^n M$. Then $ag - h \in [(0) :_M x_1] = \mathrm{H}_r(\mathbf{x}; M)$. Since $\mathbf{x}$ is amenable, by using the hypothesis of induction, we get

$$g \in I^{n-1}M \bigcap [I^n M :_M \langle \mathfrak{m} \rangle] = I^{n-1}[IM :_M \langle \mathfrak{m} \rangle].$$

This means that $f = x_1 g \in I^{n-1}[IM :_M \langle \mathfrak{m} \rangle]$ as asserted.

(2) It follows from $\mathrm{H}^0_{\mathfrak{m}}(I^n M/I^{n+1}M) = \{I^n[IM :_M \mathfrak{m}]\}/I^{n+1}M$ for all $n \geqslant 0$ by (1).

COROLLARY 2·10. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let M be a finite R-module of dimension $d > 0$. Let $\mathbf{x} = x_1, \dots, x_r$ be an amenable partial system of parameters of M that is a d-sequence relative to M. Suppose that $\dim \mathrm{H}_{(\mathbf{x})}(M) = r$. Then $\mathrm{j}_1(\mathbf{x}; M) \leqslant 0$.*

Proof. Let $\mathrm{H} = \mathrm{H}_{(\mathbf{x})}(M)$ and let $\mathbf{f} = f_1, \dots, f_r$ denote the initial forms of $x_i's$ relative to I in $\mathrm{G} = \mathrm{gr}_I(R)$. Then $\mathrm{gr}_{(\mathbf{x}^*)}(\mathrm{H}) = \mathrm{H}$ by (2·9), so that $\mathrm{j}_1(\mathbf{x}; M) = \mathrm{e}_1(\mathbf{x}^*, \mathrm{H}) \leqslant 0$ by [**15**, theorem 3·6] and [**6**, corollary 2·3].

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To get $\mathrm{H}_I(M) = \mathrm{G}\cdot[\mathrm{H}_I(M)]_0$, one really needs the assumption that $\mathbf{x}$ is a d-sequence relative to M which is amenable. Let us explore an example. See Example 6·12 to agree with the independence of the d-sequence property and amenability.

Example 2·11. Let $(S, \mathfrak{n})$ be a regular local ring of dimension $q+3$ ($q \geqslant 0$). We write $\mathfrak{n} = (X, Y, Z, W_1, \ldots, W_q)$. Let $\mathfrak{a} = (X) \cap (Y^2, Z) \cap (X^2, Y, Z^2) = (X^2Z, XY^2, XYZ, XZ^2)$ and set $R = S/\mathfrak{a}$. Then $\dim R = q + 2$. Let x, y, z and w_i denote, respectively, the images of X, Y, Z, and W_i in R. We consider the partial system $z, w_1, \ldots, w_q$ of parameters of R. Let $I = (z, w_1, \ldots, w_q)$ and $\mathrm{G} = \mathrm{gr}_I(R)$. We set $\mathrm{H} = \mathrm{H}^0_{\mathfrak{m}}(\mathrm{G})$ and let $\mathfrak{m} = \mathfrak{n}/\mathfrak{a}$ denote the maximal ideal of R. We then have the following.

LEMMA 2·12. *The following assertions hold true:*

(1) *R/I is a Cohen-Macaulay ring and $0 \neq \overline{xz} \in \mathrm{H}^0_{\mathfrak{m}}(I/I^2)$, where $\overline{xz}$ denotes the image of xz in I/I^2;*

(2) *$w_1, \ldots, w_q$ is a super-regular sequence with respect to I, that is the initial forms of w_i's constitute a regular sequence in* G*;*

(3) *if $q = 0$, then $\mathrm{H} = \mathrm{H}_1$;*

(4) *$\mathrm{H} = \mathrm{G}\cdot\mathrm{H}_1$ and $\lambda(\mathrm{H}_1) = 1$.*

Proof. (1) The first assertion is clear, as $R/I = S/[(Z, W_1, \ldots, Wq) + (XY^2)]$. We set $K = (W_1, \ldots, W_q)$ and $J = (Z, W_1, \ldots, W_q)$. Then $I/I^2 \simeq J/[J^2 + (\mathfrak{a} \cap J)]$. Since $J^2+(\mathfrak{a}\cap J) = J^2+(X^2Z, XYZ, XZ^2)+(XYW_i \mid 1 \leqslant i \leqslant q)$, we have $XZ \notin J^2+(\mathfrak{a}\cap J)$, while $\mathfrak{n}\cdot XZ \subseteq J^2 + [\mathfrak{a} \cap J]$.

(2) We may assume that $q > 0$. As w_1 is R-regular and $R/(w_1) = S/[(W_1) + \mathfrak{a}]$, we have by induction on q that $w_1, \ldots, w_q$ is an R-regular sequence. We must show that $(w_1, \ldots, w_q) \cap I^n = (w_1, \ldots, w_q)I^{n-1}$ for all $n \geqslant 2$. Notice that

$$(K + \mathfrak{a}) \cap (J^n + \mathfrak{a}) = \mathfrak{a} + \big(K \cap [KJ^{n-1} + (Z^n) + \mathfrak{a}]\big) = (\mathfrak{a} + KJ^{n-1}) + K \cap [(Z^n) + \mathfrak{a}].$$

We then have $(K + \mathfrak{a}) \cap (J^n + \mathfrak{a}) = \mathfrak{a} + KJ^{n-1}$, since $K \cap [(Z^n) + \mathfrak{a}] = K\cdot[(Z^n) + \mathfrak{a}]$.

(3), (4) We have $\mathfrak{a} :_S Z = X(X, Y, Z)$ and $\mathfrak{a} :_S Z^n = (X)$ for all $n \geqslant 2$, whence $(0) :_R z = x(x, y, z)$ and $(0) :_R z^n = (x)$. Therefore $(z^n)/(z^{n+1}) \cong R/(x, z) = S/(X, Z)$, so that $\mathrm{H}^0_{\mathfrak{m}}((z^n)/(z^{n+1})) = (0)$ for $n \geqslant 2$, which shows that if $q = 0$, then $\mathrm{H} = \mathrm{H}_1$ with $\lambda(\mathrm{H}_1) = 1$ (remember that $(z)/(z^2) \simeq S/(X\mathfrak{n} + (Z))$, if $q = 0$). Suppose now that $q > 0$ and that the assertion holds true for $q - 1$. Let $\overline{R} = R/(w_1)$ and set $\mathrm{G}(\overline{R}) = \mathrm{gr}_{I\overline{R}}(\overline{R})$. We consider the exact sequence

$$0 \longrightarrow \mathrm{G}[-1] \xrightarrow{f} \mathrm{G} \longrightarrow \mathrm{G}(\overline{R}) \longrightarrow 0$$

of graded G-modules, where f denotes the initial form of w_1. Then applying $\mathrm{H}^0_{\mathfrak{m}}(-)$, we get the exact sequence

$$(\sharp_1) \quad 0 \longrightarrow \mathrm{H}[-1] \xrightarrow{f} \mathrm{H} \longrightarrow \mathrm{H}(\overline{R})$$

of graded G-modules, where $\mathrm{H}(\overline{R}) = \mathrm{H}^0_{\mathfrak{m}}(\mathrm{G}(\overline{R}))$. Then because $\mathrm{H}_0 = (0)$, we have an embedding $0 \to \mathrm{H}_1 \to [\mathrm{H}(\overline{R})]_1$, which must be surjective, as $\mathrm{H}_1 \neq (0)$ by assertion (1) and $\lambda([\mathrm{H}(\overline{R})]_1) = 1$ by the hypothesis of induction. Therefore, because $\mathrm{H}(\overline{R}) = \mathrm{G}(\overline{R})\cdot[\mathrm{H}(\overline{R})]_1$ by the hypothesis of induction, exact sequence $(\sharp_1)$ is actually a short exact sequence

$$(\sharp_2) \quad 0 \longrightarrow \mathrm{H}[-1] \xrightarrow{f} \mathrm{H} \longrightarrow \mathrm{H}(\overline{R}) \longrightarrow 0,$$

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so that $H = G{\cdot}H_1 + fH$, whence $H = G{\cdot}H_1$ by Nakayama's lemma. The equality $\lambda(H_1) = 1$ is now clear, since $H_1 \simeq [H(\overline{R})]_1$.

We set $\overline{G} = G/\mathfrak{m}G$. Notice that H is a graded $\overline{G}$-module, as $\mathfrak{m}H = (0)$ (Lemma 2·12(4)). Let $h = xz + I^2 \in H_1$ and let F denote the image of $z + I^2$ in $\overline{G}$. With this notation our conclusion is the following.

PROPOSITION 2·13. $[\![H]\!] = \dfrac{\mathbf{t}}{(1-\mathbf{t})^q}$ *and* $(\overline{G}/F\overline{G})[-1] \simeq H$ *as a graded* $\overline{G}$*-module.*

Proof. If $q = 0$, then $H = H_1$ and $\lambda(H_1) = 1$, so that the assertions are certainly true. Suppose that $q > 0$ and that our assertion holds true for $q - 1$. Then, by the exact sequence $(\sharp_2)$ in the proof of Lemma 2·12, we have

$$[\![H]\!] = \frac{[\![H(\overline{R})]\!]}{1-\mathbf{t}}.$$

Since, by the hypothesis of induction, $[\![H(\overline{R})]\!] = \dfrac{\mathbf{t}}{(1-\mathbf{t})^{q-1}}$, we obtain

$$[\![H]\!] = \frac{\mathbf{t}}{(1-\mathbf{t})^q}.$$

Let $\varphi : \overline{G}[-1] \to H$ be the $\overline{G}$-linear map defined by $\varphi(1) = h$. Then the homomorphism φ is surjective, since $H_1 = Rh$ (Lemma 2·12(1), (4)). Therefore, as $FH = (0)$ (remember that $xz^2 = 0$ in R), we get an epimorphism

$$(\overline{G}/F\overline{G})[-1] \longrightarrow H \longrightarrow 0$$

of graded $\overline{G}$-modules, which forces $(\overline{G}/F\overline{G})[-1] \simeq H$, because the Hilbert series of both sides are the same.

Here we notice that $(0) :_R z = x(x, y, z)$ and $(0) :_R z^2 = (x)$, whence $z, w_1, \ldots, w_q$ is not a d-sequence relative to R. We also see that $z, w_1, \ldots, w_q$ is not amenable, as $zR_{\mathfrak{p}} = (0)$ for the prime ideal $\mathfrak{p} = (y, z, w_1, \ldots, w_q)$ of R.

Amenable sequences and regular sequences

The following elementary observation shows the strength of a condition to decide which partial system of parameters is already a regular sequence.

PROPOSITION 2·14. *Let* $(R, \mathfrak{m})$ *be a Cohen–Macaulay local ring of dimension* n *and* M *a finite* R*-module of projective dimension* s*. Let* $\mathbf{x} = x_1, \ldots, x_r$, $1 \leqslant r \leqslant n - s$, *be a partial system of parameters for* R *and* M*. If* $\mathbf{x}$ *is amenable to* M*, then it is an* M*–regular sequence.*

Proof. Let K be the module of syzygies of M:

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0.$$

Since $H_i(\mathbf{x}; F) = (0)$, $i \geqslant 1$, we have $H_i(\mathbf{x}; K) = H_{i+1}(\mathbf{x}; M)$ for $i \geqslant 1$. This means that $\mathbf{x}$ is amenable for K, and by induction on the projective dimensions, $\mathbf{x}$ is a regular sequence on K. Since $r < \operatorname{depth} K$, $H^0_{\mathfrak{m}}(K/(\mathbf{x})K) = (0)$. From the the exact sequence

$$0 \longrightarrow H_1(\mathbf{x}; M) \longrightarrow K/(\mathbf{x})K \longrightarrow F_0/(\mathbf{x})F_0 \longrightarrow M/(\mathbf{x})M \longrightarrow 0$$

we get $H^0_{\mathfrak{m}}(H_1(\mathbf{x}; M)) = (0)$, which by assumption means $H_1(\mathbf{x}; M) = (0)$.

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Amenable sequences and finite local cohomology

Let R be a Noetherian local ring and M a finite R-module of dimension $d > 0$. The authors in [**10**] attached to a system $\mathbf{x} = x_1, \ldots, x_d$ of parameters of M (which is naturally amenable to M) the quantity

$$\chi_1(\mathbf{x}; M) = \sum_{i=1}^{d} (-1)^{i-1} \lambda(\mathrm{H}_i(\mathbf{x}; M)),$$

which is called the *partial* Euler characteristic. If $\mathbf{x} = x_1, \ldots, x_r$ $(0 < r \leqslant d)$ is the initial subsequence of a full system $x_1, \ldots, x_d$ of parameters that is a d-sequence relative to R, they lead to explicit formulas for *some* of the standard Hilbert coefficients $\mathrm{e}_i(\mathbf{x}; R)$ [**10**, theorem 3·7] and their global bounds. In similar situations, we will obtain bounds for all the $\mathrm{j}_i(\mathbf{x}; M)$ (see Section 5).

3. *Koszul homology*: *Explicit formulas for* **j***-coefficients*

In this section we derive a general formula for the **j**-coefficients of a general class of modules. Let us begin by introducing some notation. Throughout this section, let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finite R-module of dimension $d > 0$. Let $\mathbf{x} = x_1, \ldots, x_r$ be a partial system of parameters of M, where $0 \leqslant r \leqslant d$. We denote by $(\mathbf{x})$ the ideal generated by $\mathbf{x}$. Let $\mathrm{G} = \mathrm{gr}_{(\mathbf{x})}(R)$, $\mathrm{G}(M) = \mathrm{gr}_{(\mathbf{x})}(M)$, and $\mathrm{H} = \mathrm{H}^0_{\mathfrak{m}}(\mathrm{G}(M))$. Finally, let $\mathrm{H}_i(\mathbf{x}; M)$ denote the Koszul homology modules of M relative to $\mathbf{x}$. Let us begin by highlighting some of its general properties.

PROPOSITION 3·1. *We have the following:*

(1) $\mathfrak{m}\mathrm{G} \in \mathrm{Spec}\mathrm{G}$ *and* $\mathrm{Ass}_{\mathrm{G}}\mathrm{H} = \{P \in \mathrm{Ass}_{\mathrm{G}}\mathrm{G}(M) \mid \mathfrak{m}\mathrm{G} \subseteq P\}$. *In particular,* $\dim \mathrm{G}/P \leqslant r$ *for all* $P \in \mathrm{Ass}_{\mathrm{G}}\mathrm{H}$*;*

(2) *if* $\mathrm{H} \neq (0)$*, then* $\dim_{\mathrm{G}} \mathrm{H} \leqslant r$*, where the equality holds if and only if* $\mathfrak{m}\mathrm{G} \in \mathrm{Ass}_{\mathrm{G}}\mathrm{H}$*;*

(3) *if* $r < d$ *and* $\dim \mathrm{G}/P = d$ *for all* $P \in \mathrm{Ass}_{\mathrm{G}}\mathrm{G}(M)$*, then* $\mathrm{H} = (0)$.

Proof. (1) Since $\mathrm{G}/\mathfrak{m}\mathrm{G} \simeq (R/\mathfrak{m})[\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_r]$ is the polynomial ring, we have $\mathfrak{m}\mathrm{G} \in \mathrm{Spec}\mathrm{G}$. Let $P \in \mathrm{Ass}_{\mathrm{G}}\mathrm{H}$. Then $P \in \mathrm{Ass}_{\mathrm{G}}\mathrm{G}(M)$ and $P = (0) :_{\mathrm{G}} \varphi$ for some $\varphi \in \mathrm{H} \setminus (0)$, whence $\mathfrak{m}\mathrm{G} \subseteq P$, because $\mathfrak{m}^{\ell}\varphi = (0)$ for $\ell \gg 0$. Conversely, assume that $P \in \mathrm{Ass}_{\mathrm{G}}\mathrm{G}(M)$ with $\mathfrak{m}\mathrm{G} \subseteq P$. We write $P = (0) :_{\mathrm{G}} \varphi$ for some $\varphi \in \mathrm{G}(M) \setminus (0)$. Then $\mathfrak{m}\varphi = (0)$ since $\mathfrak{m}\mathrm{G} \subseteq P$, so that $P \in \mathrm{Ass}_{\mathrm{G}}\mathrm{H}$. Of course, $\dim \mathrm{G}/P \leqslant r$ if $P \in \mathrm{Ass}_{\mathrm{G}}\mathrm{H}$, because for all $P \in \mathrm{Ass}_{\mathrm{G}}\mathrm{H}$, G/P is a homomorphic image of $\mathrm{G}/\mathfrak{m}\mathrm{G}$ and $\dim \mathrm{G}/\mathfrak{m}\mathrm{G} = r$. Assertions (2) and (3) now follow from (1).

One of our main techniques to determine **j**-polynomials is the following, where unadorned tensor products are over R.

THEOREM 3·2. *Let* $\mathbf{x} = x_1, \ldots, x_r$ $(0 < r \leqslant d)$ *be a partial system of parameters of* M *that is a* d*-sequence relative to* M*. Let* $S = R[\mathbf{T}_1, \ldots, \mathbf{T}_r]$ *be the polynomial ring. Then there exists a complex of graded* S*-modules:*

$$0 \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(\mathrm{H}_r(\mathbf{x}; M)) \otimes S[-r] \longrightarrow \cdots \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(\mathrm{H}_1(\mathbf{x}; M)) \otimes S[-1] \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(\mathrm{H}_0(\mathbf{x}; M)) \otimes S \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(\mathrm{gr}_{\mathbf{x}}(M)) \longrightarrow 0.$$

Furthermore, if $\mathbf{x}$ *is amenable to* M*, then the following complex is acyclic*:

$$0 \longrightarrow \mathrm{H}_r(\mathbf{x}; M) \otimes S[-r] \longrightarrow \cdots \longrightarrow \mathrm{H}_1(\mathbf{x}; M) \otimes S[-1] \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(\mathrm{H}_0(\mathbf{x}; M)) \otimes S \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(\mathrm{gr}_{(\mathbf{x})}(M)) \longrightarrow 0.$$

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Proof. We refer to [**11**] for details about the approximation complexes $\mathcal{M}(\mathbf{x}; M)$ used here:

$$0 \longrightarrow \mathrm{H}_r(\mathbf{x}; M) \otimes S[-r] \longrightarrow \cdots \longrightarrow \mathrm{H}_1(\mathbf{x}; M) \otimes S[-1] \longrightarrow \mathrm{H}_0(\mathbf{x}; M) \otimes S \longrightarrow \mathrm{gr}_{(\mathbf{x})}(M) \longrightarrow 0.$$

This complex is an acyclic complex of graded S-modules, because $\mathbf{x}$ is a d-sequence relative to M [**11**, theorem 4·1]. Our complex arises from applying the functor $\mathrm{H}^0_{\mathfrak{m}}(-)$ to $\mathcal{M}(\mathbf{x}; M)$. Now assume that $\mathbf{x}$ is amenable to M, so that $\mathrm{H}_i(\mathbf{x}; M) = \mathrm{H}^0_{\mathfrak{m}}(\mathrm{H}_i(\mathbf{x}; M))$ for all $i \geqslant 1$. We note that the image L of $\mathrm{H}_1(\mathbf{x}; M) \otimes S[-1]$ in $\mathrm{H}_0(\mathbf{x}; M) \otimes S$ has finite length and therefore $\mathrm{H}^1_{\mathfrak{m}}(L) = 0$. Since for all $i \geqslant 1$, $\mathrm{H}_i(\mathbf{x}; M) \otimes_R S[-i]$ is supported in the set $\{\mathfrak{m}\}$, we obtain the exact complex $\mathrm{H}^0_{\mathfrak{m}}(\mathcal{M}(\mathbf{x}; M))$:

$$0 \longrightarrow \mathrm{H}_r(\mathbf{x}; M) \otimes S[-r] \longrightarrow \cdots \longrightarrow \mathrm{H}_1(\mathbf{x}; M) \otimes S[-1] \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(\mathrm{H}_0(\mathbf{x}; M)) \otimes S \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(\mathrm{gr}_{(\mathbf{x})}(M)) \longrightarrow 0.$$

as asserted.

Several properties of the graded $\mathrm{gr}_{(\mathbf{x})}(R)$-module $\mathrm{H} = \mathrm{H}^0_{\mathfrak{m}}(\mathrm{gr}_{(\mathbf{x})}(M))$ are read off the exactness of the complex. More explicitly, H is a module over $(R/J)[\mathbf{T}]$, J taken as an $\mathfrak{m}$-primary ideal annihilating the coefficient modules of the complex. Let us first explain the vanishing of H and the (maximal) Cohen–Macaulayness of H.

COROLLARY 3·3. *Let $I = (\mathbf{x})$ be the ideal of R, where $\mathbf{x} = x_1, \ldots, x_r$ is an amenable partial system of parameters of M that is a d-sequence relative to M. Let $\mathbf{T} = \mathbf{T}_1, \ldots, \mathbf{T}_r$ be variables, $S = R[\mathbf{T}]$ the polynomial ring and $T = (R/J)[\mathbf{T}]$, where J taken as an $\mathfrak{m}$-primary ideal annihilating the coefficient modules of the complex given in Theorem* 3·2. *Then we have the following*:

(1) $\mathrm{H}_I(M) = (0)$ *if and only if* $\operatorname{depth} M/IM > 0$;
(2) $\mathrm{H}_I(M)$ *is a Cohen-Macaulay T-module of dimension r if and and only if $\mathbf{x}$ is an M-regular sequence and* $\operatorname{depth} M/IM = 0$;
(3) *the $\mathbf{T}$-depth of $\mathrm{H}_I(M)$ is equal to the $\mathbf{x}$-depth of M.*

Proof. (1) Suppose that $\mathrm{H}_I(M) = (0)$. Then by Theorem 3·2 we have an exact complex

$$0 \longrightarrow \mathrm{H}_r(\mathbf{x}; M) \otimes S[-r] \longrightarrow \cdots \longrightarrow \mathrm{H}_1(\mathbf{x}; M) \otimes S[-1] \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(\mathrm{H}_0(\mathbf{x}; M)) \otimes S \longrightarrow 0$$

of graded S-modules. Reduction mod $(\mathbf{T})$ yields an acyclic complex of R-modules graded in different degrees, which therefore must be trivial. Conversely, suppose that $\operatorname{depth} M/(\mathbf{x})M > 0$. Then $\mathrm{H}^0_{\mathfrak{m}}(\mathrm{H}_0(\mathbf{x}; M)) = \mathrm{H}^0_{\mathfrak{m}}(M/(\mathbf{x})M) = (0)$, so that $\mathrm{H}_I(M)$ vanishes by Theorem 3·2.

(2) The same argument will show that $\mathrm{H}_I(M) \simeq \mathrm{H}^0_{\mathfrak{m}}(M/(\mathbf{x})M) \otimes S$.

(3) By [**11**, proposition 3·8], the $\mathbf{x}$-depth M is equal to the $\mathbf{T}$-depth of $\mathrm{gr}_{(\mathbf{x})}(M)$. This value is also obtainable from the $\mathbf{T}$-depth of the image of $\mathrm{H}_1(\mathbf{x}; M) \otimes S[-1]$ in $\mathrm{H}_0(\mathbf{x}; M) \otimes S$.

COROLLARY 3·4. *Let $\mathbf{x} = x_1, \ldots, x_r$ $(0 \leqslant r \leqslant d)$ be an amenable partial system of parameters of M that is a d-sequence relative to M. We set*

$$h_i = \begin{cases} \lambda(\mathrm{H}_i(\mathbf{x}; M)) & \text{if } i > 0, \\ \lambda(\mathrm{H}^0_{\mathfrak{m}}(\mathrm{H}_0(\mathbf{x}; M)) & \text{if } i = 0. \end{cases}$$

We then have the following:

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(1) *the Hilbert series* $[\![\mathrm{H}]\!]$ *of the* **j***-transform* $\mathrm{H} = \mathrm{H}^0_{\mathfrak{m}}(\mathrm{gr}_{(\mathbf{x})}(M))$ *is*

$$[\![\mathrm{H}]\!] = \frac{\sum_{i=0}^{r}(-1)^i h_i \mathbf{t}^i}{(1-\mathbf{t})^r}$$

and the **j***-polynomial of* M *with respect to* $\mathbf{x}$ *is*

$$\sum_{i=0}^{r}(-1)^i \mathrm{j}_i(\mathbf{x};M)\binom{n+r-i}{r-i}, \quad \text{where } \mathrm{j}_i(\mathbf{x};M) = \sum_{k=i}^{r}(-1)^k h_k \binom{k}{i}.$$

In particular

$$\mathrm{j}_0(\mathbf{x};M) = \sum_{k=0}^{r}(-1)^k h_k \text{ and } \mathrm{j}_1(\mathbf{x};M) = \sum_{k=1}^{r}(-1)^k \cdot k \cdot h_k;$$

(2) $\mathbf{x}$ *is an* M*-regular sequence if and only if* $\mathrm{j}_1(\mathbf{x};M) = \cdots = \mathrm{j}_r(\mathbf{x};M) = 0$.

Proof. (1) It is read directly from the acyclic complex given in Theorem 3·2.

(2) The vanishing of the $\mathrm{j}_i(\mathbf{x};M)$'s for $i \geqslant 1$ is equivalent to the vanishing of the h_i's for $i \geqslant 1$.

Question 3·5. Let $\mathrm{H} = \mathrm{H}^0_{\mathfrak{m}}(\mathrm{gr}_{(\mathbf{x})}(M))$, where $\mathbf{x}$ is an amenable partial system of parameters of M that is a d-sequence relative to M. As given in the proof of Corollary 2·10, we have $\mathrm{j}_1(\mathbf{x};M) = \mathrm{e}_1(\mathbf{x}^*, \mathrm{H}) \leqslant 0$, where $\mathbf{f} = f_1, \ldots, f_r$ denote the sequence of the initial forms x_i^*'s of x_i's. Hence, if H is unmixed of dimension r, then the vanishing of $\mathrm{j}_1(\mathbf{x};M)$ means that H is Cohen–Macaulay by [**6**, theorem 3·1]. It will follow that $\lambda(\mathrm{H}_i(\mathbf{x};M)) = 0$ for all $i > 0$ and thus $\mathbf{x}$ is M-regular by Theorem 3·2 and Corollary 3·3). How do we guarantee that H is unmixed? We need a condition on M such that $\mathrm{gr}_{(\mathbf{x})}(M)$ satisfies Serre's condition (S_1). Then H, being a submodule, inherits the condition.

Example 3·6. Let R be a Cohen-Macaulay local ring, F a finite free R-module, and M a submodule of F such that $C = F/M$ has finite length. Let $\mathbf{x} = x_1, \ldots, x_r$, $0 < r \leqslant \dim R$, be a partial system of parameters of R that is an amenable d-sequence relative to M such that $(\mathbf{x})C = (0)$. Then the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathbf{x})^{n+1}M & \longrightarrow & (\mathbf{x})^{n+1}F & \longrightarrow & C_{n+1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (\mathbf{x})^{n}M & \longrightarrow & (\mathbf{x})^{n}F & \longrightarrow & C_{n} & \longrightarrow & 0 \end{array}$$

with exact rows yields the exact sequence

$$0 \to C_{n+1} \longrightarrow (\mathbf{x})^n M/(\mathbf{x})^{n+1}M \longrightarrow (\mathbf{x})^n F/(\mathbf{x})^{n+1}F \longrightarrow C_n \to 0,$$

since the rightmost vertical mapping is trivial. Hence $[\mathrm{H}^0_{\mathfrak{m}}(\mathrm{gr}_{(\mathbf{x})}(M))]_n = C_{n+1}$ for all $n \geqslant 0$. Corollary 3·4 gives a formula for the **j**-polynomial in terms of the Koszul homology modules $\mathrm{H}_i(\mathbf{x};M)$. Let us try to extract it from the exact sequence

$$0 \longrightarrow M \longrightarrow F \longrightarrow C \longrightarrow 0.$$

We have the isomorphisms

$$\mathrm{H}_i(\mathbf{x};M) \simeq \mathrm{H}_{i+1}(\mathbf{x};C) = \wedge^{i+1}(R^{\oplus r}) \otimes C \text{ for } i \geqslant 1,$$

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so that

$$\lambda(\mathrm{H}_i(\mathbf{x};M)) = \lambda(C)\binom{r}{i+1} \text{ for } 1 \leqslant i \leqslant r-1 \quad \text{and} \quad \lambda(\mathrm{H}_r(\mathbf{x};M)) = 0.$$

Also from the exact sequence

$$0 \longrightarrow \mathrm{H}_1(\mathbf{x};C) = C^{\oplus r} \longrightarrow \mathrm{H}_0(\mathbf{x};M) \longrightarrow \mathrm{H}_0(\mathbf{x};F) \to \mathrm{H}_0(\mathbf{x};C) = C \longrightarrow 0,$$

we get

$$\mathrm{H}^0_{\mathfrak{m}}(\mathrm{H}_0(\mathbf{x};M)) = C^{\oplus r}.$$

By Corollary 3·4 the Hilbert series $[\![\mathrm{H}]\!]$ of $\mathrm{H}^0_{\mathfrak{m}}(\mathrm{gr}_{(\mathbf{x})}(M))$ is

$$[\![\mathrm{H}]\!] = \frac{\sum_{i=0}^{r}(-1)^i h_i \mathbf{t}^i}{(1-\mathbf{t})^r} = \lambda(C)\frac{\sum_{i=0}^{r-1}(-1)^i\binom{r}{i+1}\mathbf{t}^i}{(1-\mathbf{t})^r}.$$

We furthermore have

$$\mathrm{j}_0(\mathbf{x};M) = \lambda(C)\sum_{i=0}^{r-1}(-1)^i\binom{r}{i+1} = \lambda(C)$$

and

$$\mathrm{j}_1(\mathbf{x};M) = \lambda(C)\sum_{i=1}^{r-1}(-1)^i\binom{r}{i+1}i = -\lambda(C).$$

COROLLARY 3·7. *Let R be a Cohen–Macaulay local ring of dimension $d \geqslant 3$ and M a finite torsion-free R-module of projective dimension two. Let $\mathbf{x} = x_1, \ldots, x_r$ $(0 < r \leqslant d)$ be a partial system of parameters of R and M that is an amenable d-sequence relative to M. We then have the following*:

(1) *if $r \leqslant d-2$, then $\mathbf{x}$ is an M-regular sequence*;
(2) *if $r \geqslant d-1$, then $\mathrm{j}_1(\mathbf{x};M) \neq 0$.*

Proof. (1) This follows from Proposition 2·14.

(2) If $r = d$, then $\mathrm{j}_1(\mathbf{x};M) = \mathrm{e}_1(\mathbf{x};M)$ which cannot vanish, since M is unmixed but not Cohen-Macaulay [**6**, theorem 3·1]. Suppose that $r = d-1$. Let K be the module of syzygies of M:

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0.$$

Since $\mathrm{H}_i(\mathbf{x};F) = (0)$, $i \geqslant 1$, we have $\mathrm{H}_i(\mathbf{x};K) = \mathrm{H}_{i+1}(\mathbf{x};M)$ for $i \geqslant 1$. This means that $\mathbf{x}$ is amenable for K. By Proposition 2·14, $\mathbf{x}$ is a regular sequence on K. In particular, $\mathrm{H}_{i+1}(\mathbf{x};M) = 0$ for all $i \geqslant 1$. Since depth $M = d-2$, it follows that $\mathrm{H}_1(\mathbf{x};M)$ is the only non-vanishing Koszul homology module. By Corollary 3·4(1) we obtain $\mathrm{j}_1(\mathbf{x};M) = -\lambda(\mathrm{H}_1(\mathbf{x};M)) \neq 0$.

Let us give another application.

COROLLARY 3·8. *Let R be a Noetherian local ring and M a Buchsbaum R-module of dimension $d > 0$. Let $0 < r \leqslant d$ be a positive integer. Assume that* $\dim \mathrm{H}_{(\mathbf{x})}(M) = s$ *and* $\mathrm{j}_1(\mathbf{x};M) = 0$ *for every partial system $\mathbf{x} = x_1, \ldots, x_s$ of parameters of M of length s, $0 < s \leqslant r$. Then M satisfies the condition (S_r) of Serre.*

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Proof. We argue by induction on r. We may assume $r > 1$ by Corollary 3·4(1). By induction on the case $r-1$, we have $\mathrm{H}_i(\mathbf{x}; M) = 0$ for $i > 1$, so that $\mathrm{j}_i(\mathbf{x}; M) = 0$ for $i > 1$. Therefore, if $\mathrm{j}_1(\mathbf{x}; M) = 0$ as well, then by Corollary 3·4(2), $\mathbf{x}$ is an M-regular sequence.

4. *Local cohomology*: *Additional formulas for* **j***-coefficients*

Let $(R, \mathfrak{m})$ be a Noetherian local ring and let M be a finite R-module of dimension $d > 0$. Throughout this section, let $\mathbf{x} = x_1, \ldots, x_r$ $(0 \leqslant r \leqslant d)$ be an amenable partial system of parameters of M that is a d-sequence relative to M. We set $I = (\mathbf{x})$ and $\mathrm{G} = \mathrm{gr}_I(R)$. Notice that when $r > 0$, the sequence $\mathbf{x}' = x_2, \ldots, x_r$ is naturally an amenable partial system of parameters of M/x_1M that is a d-sequence relative to M/x_1M. Let $I_i = (x_1, \ldots, x_i)$ for each $0 \leqslant i \leqslant r$. We begin with the following.

LEMMA 4·1. *The following assertions hold true*:

(1) *suppose that* $r > 0$ *and let* $\overline{M} = M/x_1M$. *Then there is a long exact sequence*

$$\begin{aligned} 0 \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(M) \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(\overline{M}) \longrightarrow \mathrm{H}^1_{\mathfrak{m}}(M) \xrightarrow{x_1} \mathrm{H}^1_{\mathfrak{m}}(M) \longrightarrow \mathrm{H}^1_{\mathfrak{m}}(\overline{M}) \\ \longrightarrow \mathrm{H}^2_{\mathfrak{m}}(M) \xrightarrow{x_1} \mathrm{H}^2_{\mathfrak{m}}(M) \longrightarrow \mathrm{H}^2_{\mathfrak{m}}(\overline{M}) \longrightarrow \cdots \\ \longrightarrow \mathrm{H}^i_{\mathfrak{m}}(M) \xrightarrow{x_1} \mathrm{H}^i_{\mathfrak{m}}(M) \longrightarrow \mathrm{H}^i_{\mathfrak{m}}(\overline{M}) \longrightarrow \cdots \end{aligned}$$

of local cohomology modules;

(2) *let* $1 \leqslant p \leqslant r$ *be an integer. Then* $h^0(M/I_{p-1}M) = 0$ *if and only if* $x_1, \ldots, x_p$ *is an* M*-regular sequence.*

Proof. (1) This follows from the fact that $\mathrm{H}^0_{\mathfrak{m}}(M) = (0) :_M x_1 = (0) :_M I$.

(2) It suffices to check the *only if part*. Since $(0) :_M x_1 = \mathrm{H}^0_{\mathfrak{m}}(M)$, we may assume that $p > 1$ and that the assertion holds true for $p-1$. We consider the exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(M/I_{p-2}M) \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(M/I_{p-1}M) \longrightarrow \mathrm{H}^1_{\mathfrak{m}}(M/I_{p-2}M) \xrightarrow{x_{p-1}} \mathrm{H}^1_{\mathfrak{m}}(M/I_{p-2}M) \\ \longrightarrow \cdots \end{aligned}$$

given by assertion (1). Then since $h^0(M/I_{p-1}M) = 0$, we get $h^0(M/I_{p-2}M) = 0$, so that by the hypothesis of induction the sequence $x_1, \ldots, x_{p-1}$ is M-regular, while x_p is $M/I_{p-1}M$-regular, as $(0) :_{M/I_{p-1}M} x_p = \mathrm{H}^0_{\mathfrak{m}}(M/I_{p-1}M) = (0)$.

For each R-module C, we set $\mathrm{G}(C) = \mathrm{gr}_I(C)$ and let $\mathrm{H}(C) = \mathrm{H}^0_{\mathfrak{m}}(\mathrm{G}(C))$ denote the **j**-transform of C relative to I.

PROPOSITION 4·2. *The following conditions are equivalent*:

(i) $\mathrm{H}(M) = (0)$;

(ii) $\mathrm{H}^0_{\mathfrak{m}}(M/IM) = (0)$;

(iii) $\mathrm{depth}\, M \geqslant r + 1$.

When this is the case, $r < d$ *and* $\mathbf{x}$ *is an* M*-regular sequence.*

Proof. (i) $\Leftrightarrow$ (ii) See Theorem 2·9.

(i) $\Rightarrow$ (iii) We have $r < d$, since $H \neq \mathrm{G}(M)$. On the other hand, $x_1, x_2 \ldots, x_r$ is an M-regular sequence, because

$$[I_{i-1}M :_M x_i]/I_{i-1}M \simeq [I_{i-1}M :_M x_i + IM]/IM \subseteq \mathrm{H}^0_{\mathfrak{m}}(M/IM) = (0)$$

for all $1 \leqslant i \leqslant r$ (remember that $\mathbf{x}$ is a d-sequence relative to M which is amenable for M).

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Therefore $x_1, x_2, \ldots, x_r$ is an M-regular sequence, whence depth $M > r$, as $\mathrm{H}^0_{\mathfrak{m}}(M/IM) = (0)$.

(iii) $\Rightarrow$ (ii) This is clear.

Let $W = \mathrm{H}^0_{\mathfrak{m}}(M)$ and let $\psi : \mathrm{G}(M) \to \mathrm{G}(M/W)$ be the canonical epimorphism of graded G-modules. We set $W^* = \mathrm{Ker}\, \psi$. Then since $W \cap IM = (0)$, we have

$$W^* = \{w + IM \mid w \in W\} \subseteq M/IM = [\mathrm{G}(M)]_0,$$

so that $W^* = [W^*]_0 \simeq W$ as an R-module.

LEMMA 4·3. *The following assertions hold true*:

(1) *there is an exact sequence*

$$0 \longrightarrow W^* \longrightarrow \mathrm{H}(M) \xrightarrow{\varphi} \mathrm{H}(M/W) \longrightarrow 0,$$

of graded G*-modules, where φ denotes the homomorphism induced from the canonical epimorphism $\psi : \mathrm{G}(M) \to \mathrm{G}(M/W)$;*

(2) *suppose that $r > 0$ and let $\overline{M} = M/x_1M$. Then there is an exact sequence*

$$0 \longrightarrow W^*[-1] \longrightarrow \mathrm{H}(M)[-1] \xrightarrow{f_1} \mathrm{H}(M) \xrightarrow{\tau} \mathrm{H}(\overline{M}) \longrightarrow 0$$

of graded G*-modules, where τ denotes the homomorphism induced from the canonical epimorphism $\sigma : \mathrm{G}(M) \to \mathrm{G}(\overline{M})$ and f_1 denotes the initial form of x_1.*

Proof. (1) Apply the functors $\mathrm{H}^i_{\mathfrak{m}}(-)$ to the exact sequence

$$0 \longrightarrow W^* \longrightarrow \mathrm{G}(M) \xrightarrow{\psi} \mathrm{G}(M/W) \longrightarrow 0$$

of graded G-modules and we get the exact sequence as asserted, since $\mathrm{H}^0_{\mathfrak{m}}(W^*) = W^*$ and $\mathrm{H}^1_{\mathfrak{m}}(W^*) = (0)$.

(2) Remember that $\mathrm{G}(M)/f_1\mathrm{G}(M) \simeq \mathrm{G}(\overline{M})$, since $x_1M \cap I^{n+1}M = x_1I^nM$ for all $n \in \mathbb{Z}$ and that $(0) :_{\mathrm{G}(M)} f_1 = W^*$, since $[(0) :_M x_1] \cap I^nM = (0)$ for all $n > 0$. We then have the exact sequence

$$0 \longrightarrow W^*[-1] \xrightarrow{\iota} \mathrm{G}(M)[-1] \xrightarrow{f_1} \mathrm{G}(M) \xrightarrow{\sigma} \mathrm{G}(\overline{M}) \longrightarrow 0$$

of graded G-modules, and applying the functors $\mathrm{H}^i_{\mathfrak{m}}(-)$ to the sequence, we get the required exact sequence, because by Theorem 2·9, the homomorphism $\tau : \mathrm{H}(M) \to \mathrm{H}(\overline{M})$ is surjective.

For an R-module C and $i \in \mathbb{Z}$ we set $h^i(C) = \lambda(\mathrm{H}^i_{\mathfrak{m}}(C))$. Then by Lemma 4·3 an induction on r gives the following description of the Hilbert series $[\![\mathrm{H}]\!]$ of $\mathrm{H} = \mathrm{H}(M)$. We note a brief proof.

THEOREM 4·4. *Let $\mathbf{x} = x_1, \ldots, x_r$ $(0 \leqslant r \leqslant d)$ be an amenable partial system of parameters of M that is a d-sequence relative to M. We set $I = (\mathbf{x})$. Then the Hilbert series of the* **j***-transform* $\mathrm{H} = \mathrm{H}(M)$ *is*

$$[\![\mathrm{H}]\!] = \frac{h^0(M/IM) + \sum_{i=1}^{r}(-1)^i \left[\sum_{j=i}^{r} h^0(M/I_{r-j}M)\binom{j-1}{i-1}\right]\mathbf{t}^i}{(1-\mathbf{t})^r}.$$

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Proof. We may assume that $r > 0$ and the assertion holds true for $r-1$. Let $\overline{M} = M/x_1M$. We write

$$[\![\mathrm{H}]\!] = \frac{Q(\mathbf{t})}{(1-\mathbf{t})^r}$$

with $Q(\mathbf{t}) \in \mathbb{Z}[\mathbf{t}]$. Then by the exact sequence

$$0 \longrightarrow W^*[-1] \longrightarrow \mathrm{H}(M)[-1] \longrightarrow \mathrm{H}(M) \longrightarrow \mathrm{H}(\overline{M}) \longrightarrow 0$$

given in Lemma 4·3(2) and by the hypothesis of induction we get

$$\begin{aligned} 0 = h^0(M)\mathbf{t} - \frac{Q(\mathbf{t})\mathbf{t}}{(1-\mathbf{t})^r} + \frac{Q(\mathbf{t})}{(1-\mathbf{t})^r} \\ - \frac{h^0(\overline{M}/J\overline{M}) + \sum_{i=1}^{r-1}(-1)^i \left[\sum_{j=i}^{r-1} h^0(\overline{M}/J_{r-1-j}\overline{M})\binom{j-1}{i-1}\right]\mathbf{t}^i}{(1-\mathbf{t})^{r-1}}, \end{aligned}$$

where $J = (x_2, \ldots, x_r)$ and $J_{r-1-j} = (x_2, \ldots, x_{r-j})$ for $1 \leqslant j \leqslant r-1$. Since

$$M/IM \simeq \overline{M}/J\overline{M} \quad \text{and} \quad M/I_{r-j}M \simeq \overline{M}/J_{r-1-j}\overline{M} \ \text{ for } \ 1 \leqslant j \leqslant r-1,$$

we obtain

$$\begin{aligned} Q(\mathbf{t}) &= h^0(M/IM) + \sum_{i=1}^{r-1}(-1)^i \left[\sum_{j=i}^{r-1} h^0(M/I_{r-j}M)\binom{j-1}{i-1}\right]\mathbf{t}^i - h^0(M)\mathbf{t}(1-\mathbf{t})^{r-1} \\ &= h^0(M/IM) + \sum_{i=1}^{r-1}(-1)^i \left[\sum_{j=i}^{r-1} h^0(M/I_{r-j}M)\binom{j-1}{i-1}\right]\mathbf{t}^i + h^0(M)\sum_{i=1}^{r}(-1)^i\binom{r-1}{i-1}\mathbf{t}^i \\ &= h^0(M/IM) + \sum_{i=1}^{r-1}(-1)^i \left[\sum_{j=i}^{r} h^0(M/I_{r-j}M)\binom{j-1}{i-1}\right]\mathbf{t}^i + (-1)^r h^0(M)\mathbf{t}^r, \end{aligned}$$

as claimed.

Remark 4·5. Comparing Theorem 4·4 with the formula given by Corollary 3·4, we have

$$\lambda(\mathrm{H}_i(\mathbf{x}; M)) = \sum_{j=i}^{r} h^0(M/I_{r-j}M)\binom{j-1}{i-1}$$

for all $1 \leqslant i \leqslant r$, which one can prove also by induction on r.

We set

$$k_i(\mathbf{x}; M) = \begin{cases} h^0\left(M/[(I_{r-i-1}M :_M x_{r-i}) + x_{r-i}M]\right) & \text{if} \quad 0 \leqslant i \leqslant r-1, \\ h^0(M) & \text{if} \quad i = r. \end{cases}$$

We then have the following.

LEMMA 4·6. *$k_i(\mathbf{x}; M) = h^0(M/I_{r-i}M) - h^0(M/I_{r-i-1}M)$ for all $0 \leqslant i \leqslant r-1$. Hence $k_i(\mathbf{x}'; \overline{M}) = k_i(\mathbf{x}; M)$ for all $0 \leqslant i \leqslant r-2$, where $\mathbf{x}' = x_2, \ldots, x_r$ and $\overline{M} = M/x_1M$.*

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Proof. Set $L = M/I_{r-i-1}M$. Then we get the short exact sequence

$$0 \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(L) \longrightarrow L/x_{r-i}L \longrightarrow L/[\mathrm{H}^0_{\mathfrak{m}}(L) + x_{r-i}L] \longrightarrow 0.$$

Notice that $\mathrm{H}^0_{\mathfrak{m}}(L) = [I_{r-i-1}M :_M x_{r-i}]/I_{r-i-1}M$ and we have

$$k_i(\mathbf{x}; M) = h^0\,(M/[I_{r-i-1}M :_M x_{r-i} + x_{r-i}M]) = h^0(M/I_{r-i}M) - h^0(M/I_{r-i-1}M),$$

because $h^0\,(M/(I_{r-i-1}M :_M x_{r-i} + x_{r-i}M) = \lambda(L/[\mathrm{H}^0_{\mathfrak{m}}(L) + x_{r-i}L])$.

We now have the following formula of the **j**-function of M.

THEOREM 4·7. *Let* $\mathbf{x} = x_1, \ldots, x_r$ $(0 \leqslant r \leqslant d)$ *be an amenable partial system of parameters of* M *that is a* d*-sequence relative to* M*. Then the* **j***-function of* M *relative to* $I = (\mathbf{x})$ *is*

$$\psi^M_I(n) = \sum_{i=0}^{r} k_i(\mathbf{x}; M)\binom{n+r-i}{r-i}$$

for $n \geqslant 0$*. Therefore* $(-1)^i \mathrm{j}_i(\mathbf{x}; M) = k_i(\mathbf{x}; M)$ *for all* $0 \leqslant i \leqslant r$.

Proof. If $r = 0$, then the **j**-function of M is $\psi^M_{(0)}(n) = \sum_{\ell=0}^{n} h^0(I^\ell M/I^{\ell+1}M) = h^0(M)$ for $n \geqslant 0$. Assume that $r > 0$ and the assertion holds true for $r-1$. Let $\overline{M} = M/x_1M$ and $\mathbf{x}' = x_2, \ldots, x_r$. Then by Lemma 4·3, the hypothesis of induction, and Lemma 4·6 we get

$$\begin{aligned}
\psi^M_I(n) &= \sum_{\ell=0}^{n} h^0(I^\ell M/I^{\ell+1}M) \\
&= h^0(M/IM) + \sum_{\ell=1}^{n} h^0(I^\ell M/I^{\ell+1}M) \\
&= \psi^{\overline{M}}_{(\mathbf{x}')}(0) + \sum_{\ell=1}^{n} \left(\psi^{\overline{M}}_{(\mathbf{x}')}(\ell) - h^0(M)\right) \\
&= \sum_{\ell=0}^{n} \left(\psi^{\overline{M}}_{(\mathbf{x}')}(\ell)\right) - n \cdot h^0(M) \\
&= \sum_{\ell=0}^{n} \left(\sum_{i=0}^{r-2} k_i(\mathbf{x}'; \overline{M})\binom{\ell+r-1-i}{r-1-i} + k_{r-1}(\mathbf{x}'; \overline{M})\right) - n \cdot h^0(M) \\
&= \sum_{i=0}^{r-2} k_i(\mathbf{x}'; \overline{M}) \left(\sum_{\ell=0}^{n} \binom{\ell+r-i-1}{r-i-1}\right) + (n+1)h^0(\overline{M}) - n \cdot h^0(M) \\
&= \sum_{i=0}^{r-2} k_i(\mathbf{x}'; \overline{M})\binom{n+r-i}{r-i} + (n+1)(h^0(\overline{M}) - h^0(M)) + h^0(M) \\
&= \sum_{i=0}^{r-2} k_i(\mathbf{x}; M)\binom{n+r-i}{r-i} + k_{r-1}(\mathbf{x}; M)\binom{n+1}{1} + k_r(\mathbf{x}; M),
\end{aligned}$$

as wanted.

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COROLLARY 4·8. *Suppose that* $r > 0$. *Then*

$$-\mathrm{j}_1(\mathbf{x}; M) = \begin{cases} h^0(M) & \text{if} \quad r = 1, \\ k_1(\mathbf{x}; M) = h^0\left(M/[(I_{r-2}M :_M x_{r-1}) + x_{r-1}M]\right) & \text{if} \quad r > 1, \end{cases}$$

whence $\mathrm{j}_1(\mathbf{x}, M) \leqslant 0$.

COROLLARY 4·9. *Let* $r > 0$. *Then the following assertions hold true*:

(1) *suppose* $r = 1$. *Then* $\mathrm{j}_1(\mathbf{x}; M) = 0$ *if and only if* depth $M > 0$.

(2) *suppose* $r > 1$. *Then* $\mathrm{j}_1(\mathbf{x}; M) = 0$ *if and only if*

$$\mathrm{H}^0_{\mathfrak{m}}\left(M/[(I_{r-2}M :_M x_{r-1}) + x_{r-1}M]\right) = (0).$$

Therefore, if $\mathbf{x} = x_1, \ldots, x_r$ *is an* M*-regular sequence, then* $\mathrm{j}_1(\mathbf{x}; M) = 0$.

The following gives a partial answer to Conjecture 1·1.

THEOREM 4·10. *Let* $\mathbf{x} = x_1, \ldots, x_r$ *be an amenable partial system of parameters of* M *that is a* d*-sequence relative to* M. *Suppose that either* (a) $r \geqslant 2$ *and* depth $M \geqslant r - 1$ *or that* (b) $r = 3$ *and* M *is unmixed. Then* $\mathrm{j}_1(\mathbf{x}; M) = 0$ *if and only if* $\mathbf{x} = x_1, \ldots, x_r$ *is an* M*-regular sequence.*

Proof. Case (a). Suppose that $r \geqslant 2$, depth $M \geqslant r - 1$, and $\mathrm{j}_1(\mathbf{x}; M) = 0$. If $r = 2$, then depth $M \geqslant 1$, whence

$$h^0(M/x_1M) = k_1(\mathbf{x}, M) = -\mathrm{j}_1(\mathbf{x}, M) = 0$$

by Lemma 4·6, so that by Lemma 4·1(2), $\mathbf{x} = x_1, x_2$ is M-regular. Assume that $r \geqslant 3$ and the assertion holds true for $r - 1$. Then x_1 is M-regular, because $(0) :_M x_1 = \mathrm{H}^0_{\mathfrak{m}}(M) = (0)$, while Lemma 4·6 and Theorem 4·7 show

$$\mathrm{j}_1(\mathbf{x}'; \overline{M}) = -k_1(\mathbf{x}'; \overline{M}) = -k_1(\mathbf{x}; M) = \mathrm{j}_1(\mathbf{x}; M) = 0,$$

where $\mathbf{x}' = x_2, \ldots, x_r$ and $\overline{M} = M/x_1M$. Hence the hypothesis of induction shows $\mathbf{x}'$ is $\overline{M}$-regular, so that $\mathbf{x}$ is M-regular.

Case (b). Suppose that $r = 3$ and M is unmixed. By Case (a) it is enough to show that depth $M \geqslant 2$. We set $\overline{M} = M/x_1M$ and $L = \overline{M}/[(0) :_{\overline{M}} x_2]$. Then x_2 is L-regular and depth $L \geqslant 1$. Let $\mathbf{x}' = x_2, x_3$ and notice that

$$-\mathrm{j}_1(\mathbf{x}'; L) = h^0(L/x_2L) - h^0(L) = h^0(L/x_2L)$$

$$= h^0(\overline{M}/((0 :_{\overline{M}} x_2) + x_2\overline{M})) = -\mathrm{j}_1(\mathbf{x}', \overline{M}) = -\mathrm{j}_1(\mathbf{x}, M) = 0.$$

Then by Case (a), $\mathbf{x}' = x_2, x_3$ is L-regular, whence $\mathrm{H}^1_{\mathfrak{m}}(L) = (0)$, so that $\mathrm{H}^1_{\mathfrak{m}}(\overline{M}) = (0)$ (remember that $L = \overline{M}/\mathrm{H}^0_{\mathfrak{m}}(\overline{M})$). Since $\mathrm{H}^1_{\mathfrak{m}}(M)$ is finitely generated as M is unmixed, we get $\mathrm{H}^1_{\mathfrak{m}}(M) = (0)$ by Nakayama's lemma, thanks to the exact sequence

$$\mathrm{H}^1_{\mathfrak{m}}(M) \xrightarrow{x_1} \mathrm{H}^1_{\mathfrak{m}}(M) \longrightarrow \mathrm{H}^1_{\mathfrak{m}}(\overline{M}) = 0$$

given in Lemma 4·1(1). Hence depth $M \geqslant 2$.

The condition (a) that depth $M \geqslant r - 1$ in Theorem 4·10 is not too much. Let us explore one example.

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Example 4·11. Let S be a Buchsbaum local ring of dimension 3 and a, b, c a system of parameters of S. We assume $h^0(S) \neq 0$ but $h^1(S) = 0$. Let $R = S[[X, Y]]$ be the formal power series ring and set $x_1 = X, x_2 = Y, x_3 = a, x_4 = b, x_5 = c$. Then x_1, x_2, x_3, x_4, x_5 is a system of parameters of R, forming a d-sequence. Let $\mathbf{x} = x_1, x_2, x_3, x_4$. Since

$$R/[(x_1, x_2) : x_3 + x_3R] \simeq S/[(0) : a + aS],$$

we get

$$\begin{aligned} h^0(R/[(x_1, x_2) : x_3 + x_3R]) &= h^0(S/[(0) : a + aS]) \\ &= h^0(S/(0) : a) + h^1(S/(0) : a) \\ &= h^1(S) \\ &= 0, \end{aligned}$$

so that $j_1(\mathbf{x}; R) = 0$, while depth $R = 2$. Notice that R is a mixed local ring.

Let $I = (\mathbf{x})$ and set $\mathrm{H} = \mathrm{H}_{\mathrm{I}}(\mathrm{R})$. Then

$$\mathrm{H} \simeq \left[\overline{\mathrm{G}}/(f_3, f_4)\overline{\mathrm{G}}\right]^{\oplus h^0(S)} \oplus \overline{\mathrm{G}}^{\oplus h^2(S)}$$

as a graded $\overline{\mathrm{G}}$-module, where $\overline{\mathrm{G}} = \mathrm{G}/\mathfrak{m}\mathrm{G}$. Therefore $\mathrm{H} \neq (0)$ and

$$\dim \mathrm{H} = \begin{cases} 4 & \text{if} \quad h^2(S) > 0, \\ 2 & \text{if} \quad h^2(S) = 0. \end{cases}$$

Proof. We have $\mathfrak{m}\mathrm{H} = (0)$, since $\mathfrak{m}\mathrm{H}^0_{\mathfrak{m}}(R/I) = (0)$ (remember that $R/I \simeq S/(a, b)$ is a Buchsbaum local ring). Let $J = (0) :_S \mathfrak{n}$, where $\mathfrak{n}$ is the maximal ideal of S. Then $J + I = JR + I \subseteq I : \mathfrak{m}$. We also have $J \bigcap I \subseteq [(X, Y) :_R a] \bigcap (X, Y, a, b) = (X, Y)$, so that $J \bigcap I = (0)$, because $J \bigcap (X, Y) = (0)$. Therefore $[JR + I]/I \simeq J$ as an S-module, whence $\lambda_R([JR + I]/I) = \lambda_S(J) = h^0(S)$. Consequently, because $(f_3, f_4)\,[(JR + I)/I] = (0)$ in H, we get a homomorphism

$$\varphi : \left[\overline{\mathrm{G}}/(f_3, f_4)\overline{\mathrm{G}}\right]^{\oplus h^0(S)} \oplus \overline{\mathrm{G}}^{\oplus h^2(S)} \longrightarrow \mathrm{H}$$

of graded $\overline{\mathrm{G}}$-modules, which sends $(\mathbf{e}_i \bmod (f_3, f_4)\overline{\mathrm{G}}, 0)$ to v_j and $(0, \mathbf{e}_k)$ to w_k, where $\{v_j, w_k\}$ is a k-basis of $(I :_R \mathfrak{m})/I$ which extends a k-basis $\{v_j\}$ of $[JR + I]/I$. Then φ is surjective, since $\mathrm{H} = \overline{\mathrm{G}}\cdot\mathrm{H}_0$ and $\mathrm{H}_0 = [I :_R \mathfrak{m}]/I$. Therefore φ must be an isomorphism, as the Hilbert series of both sides are the same, which follows from Theorem 4·7 (remember that $h^0(R) = h^0(R/I_1) = 0$, $h^0(R/I_2) = h^0(R/I_3) = h^0(S)$, and $h^0(R/I) = h^0(S) + h^2(S)$).

Finite local cohomology

A finite R-module M is said to have *finite local cohomology* (FLC), if $\lambda(\mathrm{H}^i_{\mathfrak{m}}(M)) < \infty$ for all $i \neq \dim M$. This condition is equivalent to saying that M possesses a system of parameters which is a d^+-sequence relative to M (see [**18**, **19**]). We call such a system of parameters of M is *standard* [**20**]. Remember that every partial system of parameters of M is amenable, once M has FLC [**18**].

PROPOSITION 4·12. *Suppose that M has* FLC *and that $\mathbf{x} = x_1, \ldots, x_r$ is a part of a standard system of parameters of M. Then for each $0 \leqslant i \leqslant r - 1$ we have*

$$(-1)^i \mathrm{j}_i(\mathbf{x}; M) = k_i(\mathbf{x}; M) = \sum_{j=1}^{r-i} \binom{r - i - 1}{j - 1} h^j(M).$$

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Proof. By [**19**, lemma 4·14], $h^0(M/I_kM) = \sum_{j=0}^{k} \binom{k}{j} h^j(M)$ for $0 \leqslant k < d$. Hence by Lemma 4·6 and Theorem 4·7 we get

$$(-1)^i \mathrm{j}_i(\mathbf{x}; M) = k_i(\mathbf{x}; M) = h^0(M/I_{r-i}M) - h^0(M/I_{r-i-1}M)$$

$$= \sum_{j=0}^{r-i} \binom{r-i}{j} h^j(M) - \sum_{j=0}^{r-i-1} \binom{r-i-1}{j} h^j(M)$$

$$= \sum_{j=1}^{r-i-1} \left(\binom{r-i}{j} - \binom{r-i-1}{j} \right) h^j(M) + h^{r-i}(M)$$

$$= \sum_{j=1}^{r-i} \binom{r-i-1}{j-1} h^j(M).$$

THEOREM 4·13. *Suppose that M has* FLC *and* depth $M > 0$. *Let $\mathbf{x} = x_1, \ldots, x_r$ be a partial system of parameters of M that is a d-sequence relative to M. Then $\mathrm{j}_1(\mathbf{x}; M) = 0$ if and only if $\mathbf{x} = x_1, \ldots, x_r$ is an M-regular sequence.*

Proof. We have only to show that $\mathbf{x}$ is an M-regular sequence, if $j_1(\mathbf{x}, M) = 0$ (Corollary 4·9(3)). Notice that M is unmixed, since depth $M > 0$ [**18**]. Therefore by Theorem 4·10, we may assume that $r \geqslant 4$ and that our assertion holds true for $r - 1$. We set $\overline{M} = M/x_1M$, $L = \overline{M}/[(0) :_{\overline{M}} x_2]$, and $\mathbf{x}' = x_2, \ldots, x_r$. Then L has FLC, depth $L > 0$, and

$$\mathrm{j}_1(\mathbf{x}'; L) = \mathrm{j}_1(\mathbf{x}'; \overline{M}) = \mathrm{j}_1(\mathbf{x}; M) = 0.$$

Therefore by the hypothesis of induction on r, the sequence $\mathbf{x}'$ is L-regular so that depth $L \geqslant r - 1$. Thus $\mathrm{H}^i_{\mathfrak{m}}(\overline{M}) = (0)$ for all $1 \leqslant i \leqslant r - 2$, so that by the exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(M) \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(\overline{M}) \longrightarrow \mathrm{H}^1_{\mathfrak{m}}(M) \xrightarrow{x_1} \mathrm{H}^1_{\mathfrak{m}}(M) &\longrightarrow 0 \longrightarrow \mathrm{H}^2_{\mathfrak{m}}(M) \xrightarrow{x_1} \mathrm{H}^2_{\mathfrak{m}}(M) \\ &\longrightarrow 0 \longrightarrow \cdots \\ &\longrightarrow 0 \longrightarrow \mathrm{H}^{r-2}_{\mathfrak{m}}(M) \xrightarrow{x_1} \mathrm{H}^{r-2}_{\mathfrak{m}}(M) \\ &\longrightarrow 0 \longrightarrow \end{aligned}$$

given in Lemma 4·1(1), we get $\mathrm{H}^i_{\mathfrak{m}}(M) = (0)$ for all $1 \leqslant i \leqslant r - 2$ (use Nakayama's lemma). Thus depth $M \geqslant r - 1$.

Partial Euler characteristics

We are now in a position to compare $\mathrm{j}_1(\mathbf{x}; M)$ to partial Euler characteristics. Suppose that $r > 0$ and we set

$$\chi_1(x_1, \ldots, x_k; M) = \sum_{i=1}^{k} (-1)^{i+1} \lambda(\mathrm{H}_i(x_1, \ldots, x_k; M))$$

for each $1 \leqslant k \leqslant r$. We then have the following.

LEMMA 4·14. *Let $\mathbf{x} = x_1, \ldots, x_r$ $(0 < r \leqslant d)$ be an amenable partial system of parameters of M that is a d-sequence relative to M. Let $1 \leqslant k \leqslant r$ be an integer. Then*

$$\chi_1(x_1, \ldots, x_k; M) = h^0(M/I_{k-1}M).$$

Hence $\chi_1(x_1, \ldots, x_k; M) = 0$ if and only if $x_1, \ldots, x_k$ is an M-regular sequence.

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Proof. By Remark 4·5

$$\chi_1(x_1, \dots, x_k : M) = \sum_{1 \leqslant i \leqslant j \leqslant k} (-1)^{i+1} h^0(M/I_{k-j}M) \binom{j-1}{i-1}$$

$$= \sum_{j=1}^{k} h^0(M/I_{k-j}M) \left[\sum_{i=1}^{j} (-1)^{i+1} \binom{j-1}{i-1} \right]$$

$$= \sum_{j=1}^{k} h^0(M/I_{k-j}M) \left[\sum_{i=0}^{j-1} (-1)^{i} \binom{j-1}{i} \right]$$

$$= h^0(M/I_{k-1}M)$$

The second assertion follows from Lemma 4·1(2).

The following results extend [**10**, theorems 3·7 and 4·2] to partial amenable systems of parameters.

COROLLARY 4·15. *Let* $\mathbf{x} = x_1, \dots, x_r$ $(0 < r \leqslant d)$ *be an amenable partial system of parameters of* M *that is a* d*-sequence relative to* M*. Then, for all* $0 \leqslant i \leqslant r$,

$$(-1)^i \mathrm{j}_i(\mathbf{x}; M) = \chi_1(x_1, \dots, x_{r+1-i}; M) - \chi_1(x_1, \dots, x_{r-i}; M).$$

Proof. For $0 \leqslant i \leqslant r-1$

$$(-1)^i \mathrm{j}_i(\mathbf{x}; M) = k_i(\mathbf{x}; M) = h^0(M/I_{r-i}M) - h^0(M/I_{r-i-1}M) \qquad \text{(by (4·6))}$$

$$= \chi_1(x_1, \dots, x_{r+1-i}; M) - \chi_1(x_1, \dots, x_{r-i}; M) \qquad \text{(by (4·14))},$$

while for $i = r$

$$(-1)^r \mathrm{j}_r(\mathbf{x}; M) = k_r(\mathbf{x}; M) = h^0(M) = \chi_1(x_1; M),$$

since $\mathrm{H}^0_{\mathfrak{m}}(M) = (0) :_M x_1$.

We close this section with the following.

PROPOSITION 4·16. *Let* $\mathbf{x} = x_1, \dots, x_r$ $(2 \leqslant r \leqslant d)$ *be an amenable partial system of parameters of* M *that is a* d*-sequence relative to* M*. Then*

$$\chi_1(\mathbf{x}; M) \geqslant -\mathrm{j}_1(\mathbf{x}; M),$$

where the equality holds true if and only if $x_1, \dots, x_{r-1}$ *is an* M*-regular sequence.*

Proof. Notice that

$$\chi_1(x_1, \dots, x_r; M) = \chi_1(x_1, \dots, x_{r-1}; M) - \mathrm{j}_1(\mathbf{x}; M) \geqslant -\mathrm{j}_1(\mathbf{x}; M)$$

(Corollary 4·15) and we get $\chi_1(x_1, \dots, x_r; M) = -\mathrm{j}_1(\mathbf{x}; M)$ if and only if $\chi_1(x_1, \dots, x_{r-1}; M) = 0$. Hence the assertion follows from Lemma 4·14.

5. *Boundedness of* **j***-coefficients*

Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finite R-module of dimension $d = \dim M \geqslant 2$. Let $0 < r < d$ be an integer. In this section firstly we discuss the problem

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of when the set $\Lambda(M)$ of non-negative integers

$$k_i(\mathbf{x}; M) = (-1)^i \mathrm{j}_i(\mathbf{x}; M)$$

is finite, where $\mathbf{x} = x_1, \ldots, x_r$ $(0 \leqslant i \leqslant r-1)$ is an amenable partial system of parameters of M which is a d-sequence relative to M. The goal is to prove the following.

THEOREM 5·1. *Assume that there exists a system of parameters of M which is a strong d-sequence relative to M. Then the following conditions are equivalent*:

(i) *the set $\Lambda(M)$ is finite*;

(ii) $\mathrm{H}^i_{\mathfrak{m}}(M)$ *is a finite R-module for every $1 \leqslant i \leqslant r$.*

When this is the case, one has $\mathfrak{m}^\ell \mathrm{H}^i_{\mathfrak{m}}(M) = (0)$ for all $1 \leqslant i \leqslant r$, where $\ell = \max \Lambda(M)$.

For the implication (ii) $\Rightarrow$ (i) we do not need the assumption of the existence of a system of parameters of M which is a strong d-sequence relative to M.

Proof. (ii) $\Rightarrow$ (i) Let $\mathbf{x} = x_1, \ldots, x_r$ be an amenable partial system of parameters of M that is a d-sequence relative to M. We will show that for all $0 \leqslant i \leqslant r-1$

$$k_i(\mathbf{x}; M) \leqslant \sum_{j=1}^{r-i} h^j(M) \binom{r-i-1}{j-1}.$$

If $r = 1$, then by Lemma 4·6 and the exact sequence

$$(\sharp_0) \quad 0 \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(M) \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(M/x_1M) \longrightarrow \mathrm{H}^1_{\mathfrak{m}}(M) \xrightarrow{x_1} \mathrm{H}^1_{\mathfrak{m}}(M)$$

stated in Lemma 4·1(1), we have

$$k_0(\mathbf{x}; M) = h^0(M/x_1M) - h^0(M) = \lambda((0) :_{\mathrm{H}^1_{\mathfrak{m}}(M)} x_1) \leqslant h^1(M).$$

Suppose that $r > 1$ and that our assertion holds true for $r-1$. We consider $\overline{M} = M/x_1M$. Then, by the exact sequence

$$(\sharp_i) \quad \mathrm{H}^i_{\mathfrak{m}}(M) \xrightarrow{x_1} \mathrm{H}^i_{\mathfrak{m}}(M) \longrightarrow \mathrm{H}^i_{\mathfrak{m}}(\overline{M}) \longrightarrow \mathrm{H}^{i+1}_{\mathfrak{m}}(M) \xrightarrow{x_1} \mathrm{H}^{i+1}_{\mathfrak{m}}(M)$$

given in Lemma 4·1(1), $\mathrm{H}^i_{\mathfrak{m}}(\overline{M})$ is finitely generated and $h^i(\overline{M}) \leqslant h^i(M) + h^{i+1}(M)$ for all $1 \leqslant i \leqslant r-1$. Let $\mathbf{x}' = x_2, \ldots, x_r$. Then for all $0 \leqslant i \leqslant r-2$, by Lemma 4·6 and the hypothesis of induction we get

$$\begin{aligned} k_i(\mathbf{x}; M) = k_i(\mathbf{x}'; \overline{M}) &\leqslant \sum_{j=1}^{r-i-1} h^j(\overline{M}) \binom{r-i-2}{j-1} \\ &\leqslant \sum_{j=1}^{r-i-1} \left[h^j(M) + h^{j+1}(M)\right] \binom{r-i-2}{j-1} \\ &= \sum_{j=1}^{r-i} h^j(M) \binom{r-i-1}{j-1}, \end{aligned}$$

while we have by Lemma 4·6

$$k_{r-1}(\mathbf{x}; M) = h^0(M/x_1M) - h^0(M) \leqslant h^1(M),$$

thanks to exact sequence $(\sharp_0)$ above.

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(i) $\Rightarrow$ (ii) We choose a system $a_1, \dots, a_d$ of parameters of M which is a strong d-sequence relative to M. Let Λ_0 denote the set of $k_i(a_1^{n_1}, a_2^{n_2}, \dots, a_r^{n_r}; M)'s$, where $0 \leqslant i \leqslant r-1$ and $n_i's$ are positive integers. Then $\Lambda_0 \subseteq \Lambda$ and hence Λ_0 is finite. We will show by induction on r that $\mathfrak{m}^\ell \mathrm{H}^i_{\mathfrak{m}}(M) = (0)$ for all $1 \leqslant i \leqslant r$, where $\ell = \max \Lambda_0$.

Let $r = 1$. Consider exact sequence $(\sharp_0)$ above where $x_1 = a_1^{n_1}$ and $n_1 > 0$. Then we have

$$\lambda\left(\left[(0) :_{\mathrm{H}^1_{\mathfrak{m}}(M)} a_1^{n_1}\right]\right) = h^0(M/a_1^{n_1}M) - h^0(M) = k_0(a_1^{n_1}; M) \in \Lambda_0,$$

so that $\lambda\left(\left[(0) :_{\mathrm{H}^1_{\mathfrak{m}}(M)} a_1^{n_1}\right]\right) \leqslant \ell$, whence

$$\mathfrak{m}^\ell\left[(0) :_{\mathrm{H}^1_{\mathfrak{m}}(M)} a_1^{n_1}\right] = (0).$$

Therefore $\mathfrak{m}^\ell \mathrm{H}^1_{\mathfrak{m}}(M) = (0)$, since $n_1 > 0$ is arbitrary. Assume now that $r > 1$ and that our assertion holds true for $r-1$. Let $\overline{M} = M/a_1^{n_1}M$. Then the set of

$$k_i(a_2^{n_2}, \dots, a_r^{n_r}; \overline{M}) = k_i(a_1^{n_1}, \dots, a_r^{n_r}; M)$$

where $0 \leqslant i \leqslant r-2$ and $n_i's$ are positive integers is a subset of Λ_0, whence the hypothesis of induction shows $\mathfrak{m}^\ell \mathrm{H}^i_{\mathfrak{m}}(\overline{M}) = (0)$ for all $1 \leqslant i \leqslant r-1$. Since $\left[(0) :_{\mathrm{H}^{i+1}_{\mathfrak{m}}(M)} a_1^{n_1}\right]$ is a homomorphic image of $\mathrm{H}^i_{\mathfrak{m}}(\overline{M})$ as shown in exact sequence $(\sharp_i)$, we have

$$\mathfrak{m}^\ell\left[(0) :_{\mathrm{H}^{i+1}_{\mathfrak{m}}(M)} a_1^{n_1}\right] = (0)$$

for all $1 \leqslant i \leqslant r-1$, while by exact sequence $(\sharp_0)$ we have

$$\lambda\left(\left[(0) :_{\mathrm{H}^1_{\mathfrak{m}}(M)} a_1^{n_1}\right]\right) = h^0(M/a_1^{n_1}M) - h^0(M) = k_{r-1}(a_1^{n_1}, a_2^{n_2}, \dots, a_r^{n_r}; M) \in \Lambda_0.$$

Therefore

$$\mathfrak{m}^\ell\left[(0) :_{\mathrm{H}^i_{\mathfrak{m}}(M)} a_1^{n_1}\right] = (0) \quad \text{for all } \ 1 \leqslant i \leqslant r.$$

Thus $\mathfrak{m}^\ell \mathrm{H}^i_{\mathfrak{m}}(M) = (0)$ for all $1 \leqslant i \leqslant r$, since $n_1 > 0$ is arbitrary.

Let us describe in Proposition 5·3 below a broad class of modules for which the existence of strong d-sequences is guaranteed. It is based on a result of T. Kawasaki [**14**, theorem 4·2(1)], which deals with a case of rings. We extend this result to a case of modules and include a brief proof for the reader's convenience. We start with the following observation.

LEMMA 5·2. *Let $0 \to X \to Y \to Z \to 0$ be an exact sequence of R-modules and let $f_1, \dots, f_n$ be a sequence of elements in R. If $f_1, \dots, f_n$ is a Z-regular sequence which is a d-sequence relative to Z, then $f_1, \dots, f_n$ is a d-sequence relative to X.*

PROPOSITION 5·3. *Suppose that R is a homomorphic image of a Gorenstein local ring and let M be a finite R-module of dimension $d > 0$. Then there is a system of parameters of M which is a strong d-sequence relative to M.*

Proof. We may assume that R is a Gorenstein local ring with $\dim R = \dim M = d$ and consider the idealisation $A = R \ltimes M$ of M over R. Then, since A is a homomorphic image of a Gorenstein ring, by [**14**, theorem 4·2(1)] A contains a system $f_1, \dots, f_d$ of parameters which is a strong d-sequence relative to A. We write $f_i = (a_i, x_i)$ with $a_i \in \mathfrak{m}$ and $x_i \in M$. Let $p : A \to R, (a, x) \mapsto a$ and let ${}_pM$ denote M which is regarded as an A-module via the ring homomorphism $p : A \to R, (r, m) \mapsto r$. Then ${}_pM = (0) \times M \subseteq A$ and we get the exact sequence

$$0 \longrightarrow {}_pM \longrightarrow A \xrightarrow{p} R \longrightarrow 0$$

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of A-modules. Since R is a Gorenstein local ring possessing $a_1, \ldots, a_d$ as a system of parameters, by Lemma 5·2, $f_1, \ldots, f_d$ is a strong d-sequence relative to ${}_pM$. Hence $a_1, \ldots, a_d$ is a strong d-sequence relative to M.

We note a consequence of Theorem 5·1.

COROLLARY 5·4. *Let* $\mathbf{x} = x_1, \ldots, x_r$ *be an amenable partial system of parameters of* M *which is a* d*-sequence relative to* M. *Suppose that* $\lambda(\mathrm{H}^i_{\mathfrak{m}}(M)) < \infty$ *for all* $1 \leqslant i \leqslant r$. *Let* $0 \leqslant p < r$ *be an integer. Then the following conditions are equivalent*:

(i) $\mathrm{j}_p(\mathbf{x}; M) = 0$;

(ii) $\mathrm{H}^i_{\mathfrak{m}}(M) = (0)$ *for all* $1 \leqslant i \leqslant r - p$.

Therefore $\mathrm{j}_1(\mathbf{x}; M) = 0$ *if and only if* $\mathbf{x}$ *is* M*-regular, provided* depth $M > 0$.

Proof. Suppose that $r = 1$ and consider the exact sequence

$$0 \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(M) \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(M/x_1M) \longrightarrow \mathrm{H}^1_{\mathfrak{m}}(M) \xrightarrow{x_1} \mathrm{H}^1_{\mathfrak{m}}(M).$$

Then $\mathrm{j}_0(\mathbf{x}; M) = k_0(\mathbf{x}; M) = h^0(M/x_1M) - h^0(M) = \lambda((0) :_{\mathrm{H}^1_{\mathfrak{m}}(M)} x_1)$ by Lemma 4·6, whence $\mathrm{j}_0(\mathbf{x}; M) = 0$ if and only if $\mathrm{H}^1_{\mathfrak{m}}(M) = (0)$.

Suppose that $r > 1$ and that our assertion holds true for $r - 1$. Let $\overline{M} = M/x_1M$ and $\mathbf{x}' = x_2, \ldots, x_r$. Then, by the exact sequence

$$(\sharp_i) \quad \mathrm{H}^i_{\mathfrak{m}}(M) \xrightarrow{x_1} \mathrm{H}^i_{\mathfrak{m}}(M) \longrightarrow \mathrm{H}^i_{\mathfrak{m}}(\overline{M}) \longrightarrow \mathrm{H}^{i+1}_{\mathfrak{m}}(M) \xrightarrow{x_1} \mathrm{H}^{i+1}_{\mathfrak{m}}(M)$$

given by Lemma 4·1(1), $\mathrm{H}^i_{\mathfrak{m}}(\overline{M})$ is finitely generated for all $1 \leqslant i \leqslant r - 1$. Therefore for all $0 \leqslant p \leqslant r - 2$ we get by Lemma 4·6, Theorem 4·7, and the hypothesis of induction that

$$\mathrm{j}_p(\mathbf{x}'; \overline{M}) = \mathrm{j}_p(\mathbf{x}; M) = 0$$

if and only if

$$\mathrm{H}^i_{\mathfrak{m}}(\overline{M}) = (0) \quad \text{for all} \quad 1 \leqslant i \leqslant r - 1 - p.$$

By exact sequence $(\sharp_i)$ the latter condition is equivalent to saying that $\mathrm{H}^i_{\mathfrak{m}}(M) = (0)$ for all $1 \leqslant i \leqslant r - p$, because each $\mathrm{H}^i_{\mathfrak{m}}(M)$ $(1 \leqslant i \leqslant r)$ has finite length, while for $p = r - 1$, by Lemma 4·6 we get

$$(-1)^{r-1}\mathrm{j}_{r-1}(\mathbf{x}; M) = k_{r-1}(\mathbf{x}; M) = h^0(M/x_1M) - h^0(M) = \lambda((0) :_{\mathrm{H}^1_{\mathfrak{m}}(M)} x_1).$$

Thus equivalence of assertions (i) and (ii) follows. See Theorem 4·10 for the last assertion.

We now consider the problem of when the set $\Gamma(M)$ of

$$\chi_1(\mathbf{x}; M) = \sum_{i=1}^{r} (-1)^{i-1} \lambda(\mathrm{H}_i(\mathbf{x}; M)) = h^0(M/(x_1, \ldots, x_{r-1})M)$$

is finite, where $\mathbf{x} = x_1, \ldots, x_r$ is an amenable partial system of parameters of M which is a d-sequence relative to M.

THEOREM 5·5. *Assume that there exists a system of parameters of* M *which is a strong* d*-sequence relative to* M. *Then the following conditions are equivalent*:

(i) *the set* $\Gamma(M)$ *is finite*;

(ii) $\mathrm{H}^i_{\mathfrak{m}}(M)$ *is finitely generated for every* $0 \leqslant i \leqslant r - 1$.

For the implication (ii) $\Rightarrow$ (i) we do not need the assumption of the existence of a system of parameters of M which is a strong d-sequence relative to M.

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Proof. (ii) $\Rightarrow$ (i) Let $\mathbf{x} = x_1, \dots, x_r$ be an amenable partial system of parameters of M which is a d-sequence relative to M. We will show by induction on r that

$$\chi_1(\mathbf{x}; M) \leqslant \sum_{i=0}^{r-1} \binom{r-1}{i} h^i(M).$$

If $r = 1$, we have nothing to prove. Suppose that $r > 1$ and that our assertion holds true for $r-1$. Let $\overline{M} = M/x_1M$. Then by Lemma 4·1(1), $\mathrm{H}^i_{\mathfrak{m}}(\overline{M})$ $(0 \leqslant i \leqslant r-2)$ is finitely generated with $h^i(\overline{M}) \leqslant h^i(M) + h^{i+1}(M)$. Let $\mathbf{x}' = x_2, \dots, x_r$. Then

$$\chi_1(\mathbf{x}; M) = \chi_1(\mathbf{x}'; \overline{M}) \leqslant \sum_{i=0}^{r-2} \binom{r-2}{i} h^i(\overline{M}) \leqslant \sum_{i=0}^{r-2} \binom{r-2}{i} \left[h^i(M) + h^{i+1}(M)\right]$$

$$= \sum_{i=0}^{r-1} \binom{r-1}{i} h^i(M).$$

(i) $\Rightarrow$ (ii) Let $a_1, \dots, a_d$ be a system of parameters of M which is a strong d-sequence relative to M and let $\Gamma_0(M)$ be the set of all $\chi_1(a_1^{n_1}, \dots, a_r^{n_r}; M)$ with integers $n_i > 0$. Then $\Gamma_0(M) \subseteq \Gamma(M)$. We set $\ell = \max \Gamma_0(M)$ and show by induction on r that

$$\lambda(\mathrm{H}^i_{\mathfrak{m}}(M)) \leqslant \ell$$

for all $0 \leqslant i \leqslant r-1$. If $r = 1$, then $h^0(M) = \chi_1(a_1^{n_1}; M) \in \Gamma_0(M)$ and the assertion is clear. Suppose that $r > 1$ and that the assertion holds true for $r-1$. Choose an arbitrary integer $n_1 > 0$ and let $\overline{M} = M/a_1^{n_1}M$ and consider the exact sequence

$$(\sharp_i) \quad \cdots \longrightarrow \mathrm{H}^i_{\mathfrak{m}}(M) \xrightarrow{a_1^{n_1}} \mathrm{H}^i_{\mathfrak{m}}(M) \longrightarrow \mathrm{H}^i_{\mathfrak{m}}(\overline{M}) \longrightarrow \mathrm{H}^{i+1}_{\mathfrak{m}}(M) \xrightarrow{x_1} \mathrm{H}^{i+1}_{\mathfrak{m}}(M) \longrightarrow \cdots.$$

Let $\Gamma_0(\overline{M})$ be the set of $\chi_1(a_2^{n_2}, \dots, a_r^{n_r}; \overline{M})$ with n_i positive integers. Then since

$$\chi_1(a_2^{n_2}, \dots, a_r^{n_r}; \overline{M}) = \chi_1(a_1^{n_1}, \dots, a_r^{n_r}; M),$$

$\Gamma_0(\overline{M}) \subseteq \Gamma_0(M)$, whence the hypothesis of induction on r guarantees

$$\lambda(\mathrm{H}^i_{\mathfrak{m}}(\overline{M})) \leqslant \ell$$

for all $0 \leqslant i \leqslant r-2$. From the exact sequence $(\sharp_i)$, since $\left[(0) :_{\mathrm{H}^{i+1}_{\mathfrak{m}}(M)} a_1^{n_1}\right]$ is a homomorphic image of $\mathrm{H}^i_{\mathfrak{m}}(\overline{M})$, we get

$$\lambda\left(\left[(0) :_{\mathrm{H}^{i+1}_{\mathfrak{m}}(M)} a_1^{n_1}\right]\right) \leqslant \ell$$

for all $0 \leqslant i \leqslant r-2$. Since n_1 is arbitrary, this means that

$$\lambda\left(\mathrm{H}^i_{\mathfrak{m}}(M)\right) \leqslant \ell \quad \text{for all} \quad 1 \leqslant i \leqslant r-1.$$

Because $\mathrm{H}^0_{\mathfrak{m}}(M) \subseteq \mathrm{H}^0_{\mathfrak{m}}(\overline{M})$ by Lemma 4·1(1), we get $\lambda(\mathrm{H}^0_{\mathfrak{m}}(M)) \leqslant \ell$, which completes the proof.

The proof of the implication (ii) $\Rightarrow$ (i) of (5·5) shows

$$0 \leqslant \chi_1(\mathbf{x}; M) \leqslant \sum_{i=0}^{r-1} \binom{r-1}{i} h^i(M)$$

for every amenable partial system $\mathbf{x} = x_1, \dots, x_r$ of parameters of M which is a d-sequence relative to M. Let us add the following.

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PROPOSITION 5·6. *Suppose that* $\mathrm{H}^i_{\mathfrak{m}}(M)$ *is finitely generated for every* $0 \leqslant i \leqslant r-1$.

(1) *Suppose that there exists a system of parameters* $a_1, \dots, a_d$ *of* M *which is a strong* d*-sequence relative to* M. *Then*

$$\sup_{n_1,\dots,n_r>0} \chi_1(a_1^{n_1}, \dots, a_r^{n_r}; M) = \sum_{i=0}^{r-1} \binom{r-1}{i} h^i(M).$$

(2) *There always exists a system* $x_1, \dots, x_d$ *of parameters of* M *which is a* d*-sequence relative to* M *such that*

$$\chi_1(x_1, \dots, x_r; M) = \sum_{i=0}^{r-1} \binom{r-1}{i} h^i(M).$$

Therefore $\max \Gamma(M) = \sum_{i=0}^{r-1} \binom{r-1}{i} h^i(M)$.

Proof. (1) We may assume that $r > 1$ and that the assertion holds true for $r-1$. Let $n > 0$ be an integer such that $\mathfrak{m}^n \mathrm{H}^i_{\mathfrak{m}}(M) = (0)$ for all $0 \leqslant i \leqslant r-1$. Let $\overline{M} = M/a_1^n M$. By Lemma 4·1(1), we then have $h^i(\overline{M}) = h^i(M) + h^{i+1}(M)$ for $0 \leqslant i \leqslant r-2$. Now choose integers $n_2, \dots, n_r > 0$ so that

$$\chi_1(a_2^{n_2}, \dots, a_r^{n_r}; \overline{M}) = \sum_{i=0}^{r-2} \binom{r-2}{i} h^i(\overline{M}).$$

Let $\mathbf{a} = a_1^{n_1}, \dots, a_r^{n_r}$ and $\mathbf{a}' = a_2^{n_2}, \dots, a_r^{n_r}$. Then

$$\chi_1(\mathbf{a}; M) = \chi_1(\mathbf{a}'; \overline{M}) = \sum_{i=0}^{r-2} \binom{r-2}{i} h^i(\overline{M}) = \sum_{i=0}^{r-2} \binom{r-2}{i} \left[h^i(M) + h^{i+1}(M)\right]$$

$$= \sum_{i=0}^{r-1} \binom{r-1}{i} h^i(M).$$

(2) Let $\mathfrak{a}_i(M) = \operatorname{Ann}_R \mathrm{H}^i_{\mathfrak{m}}(M)$ for each $0 \leqslant i \leqslant r-1$ and set $\mathfrak{a}(M) = \prod_{i=0}^{r-1} \mathfrak{a}_i(M)^{\binom{r-1}{i}}$. Then since the ideal $\mathfrak{a}(M)$ contains a high power of $\mathfrak{m}$, by (2·5) there is a system $x_1, \dots, x_d$ of parameters of M which is contained in $\mathfrak{a}(M)$ and is a d-sequence relative to M. We will show by induction on r that

$$\chi_1(x_1, \dots, x_r; M) = \sum_{i=0}^{r-1} \binom{r-1}{i} h^i(M).$$

We may assume that $r > 1$ and that our assertion holds true for $r-1$. Let $\overline{M} = M/x_1 M$. We then have, by Lemma 4·1(1), for each $0 \leqslant i \leqslant r-2$ the short exact sequence

$$0 \longrightarrow \mathrm{H}^i_{\mathfrak{m}}(M) \longrightarrow \mathrm{H}^i_{\mathfrak{m}}(\overline{M}) \longrightarrow \mathrm{H}^{i+1}_{\mathfrak{m}}(M) \longrightarrow 0,$$

because $x_1 \mathrm{H}^j_{\mathfrak{m}}(M) = (0)$ for $j = i, i+1$. Hence $h^i(\overline{M}) = h^i(M) + h^{i+1}(M)$ and $\mathfrak{a}_i(M)\mathfrak{a}_{i+1}(M) \subseteq \mathfrak{a}_i(\overline{M})$. Consequently $\mathfrak{a}(M) \subseteq \mathfrak{a}(\overline{M})$. Let $\mathbf{x} = x_1, \dots, x_r$ and $\mathbf{x}' =$

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$x_2, \dots, x_r$. Then, since $x_2, \dots, x_d \in \mathfrak{a}(\overline{M})$, by the hypothesis of induction on r, we get

$$\chi_1(\mathbf{x}; M) = \chi_1(\mathbf{x}'; \overline{M}) = \sum_{i=0}^{r-2} \binom{r-2}{i} h^i(\overline{M}) = \sum_{i=0}^{r-2} \binom{r-2}{i} \left[h^i(M) + h^{i+1}(M)\right]$$

$$= \sum_{i=0}^{r-1} \binom{r-1}{i} h^i(M).$$

6. *The structure of some* **j***-transforms*

The general outline of **j**-transforms $\mathrm{H} = \mathrm{H}^0_{\mathfrak{m}}(\mathrm{gr}_I(M))$ is still unclear. In two cases however, Buchsbaum and sequentially Cohen-Macaulay modules, one has a satisfying vista.

Buchsbaum modules

Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and M a finite R-module of dimension $d \geqslant 2$. Let $x_1, \dots, x_d$ be a system of parameters of M. We fix an integer $0 < r < d$ and set $I = (x_1, \dots, x_r)$, $\mathrm{G} = \mathrm{gr}_I(R)$, $\mathrm{G}(M) = \mathrm{gr}_I(M)$, and

$$\overline{\mathrm{G}} = \mathrm{G}/\mathfrak{m}\mathrm{G} = k[\mathbf{T}_1, \mathbf{T}_2, \dots, \mathbf{T}_r],$$

where $\mathbf{T}_i$ denotes the image of $f_i = x_i + I^2$ in $\overline{\mathrm{G}}$. Let us consider the **j**-transform $\mathrm{H} = \mathrm{H}^0_{\mathfrak{m}}(\mathrm{G}(M))$. The goal is the following.

THEOREM 6·1. *Suppose that M is a Buchsbaum R-module. Then the following assertions hold true*:

(1) *suppose that* $\mathrm{H} \neq (0)$. *Then*

$$\dim \mathrm{H} = \begin{cases} 0 & \text{if} \quad h^1(M) = h^2(M) = \cdots = h^r(M) = 0, \\ r & \text{otherwise}; \end{cases}$$

(2) $\mathrm{H} \simeq \bigoplus_{i=0}^{r} [Z_i(i)]^{\oplus h^i(M)}$ *as a graded* G*-module.*

Here $Z_i = \mathrm{Syz}^{\overline{\mathrm{G}}}_i(k)$ denotes the i-th syzygy module of the residue class field $k = \overline{\mathrm{G}}/[\overline{\mathrm{G}}]_+$. The second exact sequence in Theorem 3·2 *gives a graded minimal free resolution of the* $\overline{\mathrm{G}}$*-module* H*, where we identify* $\overline{\mathrm{G}} = R/\mathfrak{m} \otimes_R S$ (cf. Corollary 3·3).

Let us begin with the following.

LEMMA 6·2. *Let $\mathbf{x} = x_1, \dots, x_r$ be an amenable partial system of parameters of M that is a d-sequence relative to M. Let $f_i = x_i + I^2$ in* G *for each $i = 1, \dots, r$.*

(1) *The sequence $f_1, \dots, f_r$ is a d-sequence relative to* H,

$$\mathrm{H}/(f_1, f_2, \dots, f_r)\mathrm{H} = [\mathrm{H}/(f_1, f_2, \dots, f_r)\mathrm{H}]_0,$$

and $[\mathrm{H}/(f_1, f_2, \dots, f_r)\mathrm{H}]_0 \simeq \mathrm{H}^0_{\mathfrak{m}}(M/IM)$ *as an R-module.*

(2) $(f_1, \dots, f_i)\mathrm{G}(M) \bigcap \mathrm{H} = (f_1, \dots, f_i)\mathrm{H}$ *for all* $1 \leqslant i \leqslant r$.

Proof. (1) Note that $(0) :_{\mathrm{H}} f_1 = [(0) :_{\mathrm{H}} f_1]_0$ and $\mathrm{H}(\overline{M}) = \mathrm{H}/f_1\mathrm{H}$ when $r > 0$. The assertion follows from Lemma 4·3 and induction on r.

(2) Let $h \in \mathrm{H}$ and assume that $h = f_1 g$ for some $g \in \mathrm{G}(M)$. We choose an integer $\ell \gg 0$ so that $\mathfrak{m}^\ell \mathrm{H} = (0)$. Then since $\mathfrak{m}^\ell h = (0)$, we get

$$\mathfrak{m}^\ell g \subseteq (0) :_{\mathrm{G}(M)} f_1 \subseteq H.$$

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Thus $\mathfrak{m}^{2\ell}g = (0)$ and hence $g \in \mathrm{H}$. Assume that $i > 1$ and that the assertion holds true for $i-1$. Let $\overline{M} = M/(x_1, \ldots, x_{i-1})M$. Let $h \in (f_1, \ldots, f_i)\mathrm{G}(M) \bigcap \mathrm{H}$. Then $\overline{h} \in f_i\mathrm{G}(\overline{M}) \bigcap \mathrm{H}(\overline{M})$, where $\overline{h}$ denotes the image of h in $\mathrm{G}(\overline{M})$ under the canonical map $\psi : \mathrm{G}(M) \to \mathrm{G}(\overline{M})$ and $\mathrm{H}(\overline{M})$ denotes the **j**-transform of $\overline{M}$ relative to I. Because $\mathrm{H}(\overline{M}) = \psi(\mathrm{H})$ by Lemma 4·3(2), from the case where $i = 1$ it follows that

$$h \in f_i\mathrm{H} + (f_1, \ldots, f_{i-1})\mathrm{G}(M).$$

Hence $h - f_ih' \in (f_1, \ldots, f_{i-1})\mathrm{G}(M) \bigcap \mathrm{H}$ for some $h' \in \mathrm{H}$, so that the hypothesis of induction on i implies $h - f_ih' \in (f_1, \ldots, f_{i-1})\mathrm{H}$. Thus $h \in (f_1, \ldots, f_i)\mathrm{H}$.

PROPOSITION 6·3. *$f_1, \ldots, f_r$ is a $\mathrm{G}(M)/\mathrm{H}$-regular sequence.*

Proof. Let $1 \leqslant i \leqslant r$ be an integer. Let $g \in \mathrm{G}(M)$ and assume that

$$f_ig \in (f_1, \ldots, f_{i-1})\mathrm{G}(M) + H.$$

Then

$$f_ig - \sum_{j=1}^{i-1} f_jg_j \in (f_1, \ldots, f_i)\mathrm{G}(M) \bigcap \mathrm{H}$$

with $g_j \in \mathrm{G}(M)$. Since $(f_1, \ldots, f_i)\mathrm{G}(M) \bigcap \mathrm{H} = (f_1, \ldots, f_i)\mathrm{H}$ by Lemma 6·2, we get

$$f_ig - \sum_{j=1}^{i-1} f_jg_j = \sum_{j=1}^{i} f_jh_j$$

with $h_j \in \mathrm{H}$. Hence $f_i(g - h_i) = \sum_{j=1}^{i-1} f_j(h_j + g_j)$, so that

$$g - h_i \in (f_1, \ldots, f_{i-1})\mathrm{G}(M) :_{\mathrm{G}(M)} f_i.$$

Let $\overline{M} = M/I_{i-1}M$ and identify

$$\mathrm{G}(M)/(f_1, \ldots, f_{i-1})\mathrm{G}(M) = \mathrm{G}(\overline{M}).$$

Then since $(0) :_{\mathrm{G}(\overline{M})} f_i \subseteq \mathrm{H}(\overline{M})$ and $\mathrm{H}(\overline{M})$ coincides with the image of H in $\mathrm{G}(\overline{M})$, we get

$$g - h_i \in (f_1, \ldots, f_{i-1})\mathrm{G}(M) + H.$$

Thus $g \in (f_1, \ldots, f_{i-1})\mathrm{G}(M) + \mathrm{H}$, whence $f_1, \ldots, f_r$ is a $\mathrm{G}(M)/\mathrm{H}$-regular sequence.

COROLLARY 6·4. *Suppose that $x_1, \ldots, x_r$ is a d^+-sequence relative to M. Then $f_1, \ldots, f_r$ is a d^+-sequence relative to* H.

Proof. It suffices to check that $f_1, \ldots, f_r$ is a strong d-sequence relative to H, which readily follows by definition from the facts that $f_1, \ldots, f_r$ is a strong d-sequence on $\mathrm{G}(M)$ [**20**, corollary 5·5] and that $f_1, \ldots, f_r$ is a $\mathrm{G}(M)/\mathrm{H}$-regular sequence (Proposition 6·3).

Let $\mathrm{H}^i_{\mathrm{G}_+}(*)$ denote the local cohomology functor relative to the graded ideal $\mathrm{G}_+ = (f_1, \ldots, f_r)\mathrm{G}$.

COROLLARY 6·5. *We have the following*:

(1) $\mathrm{H}^i_{\mathrm{G}_+}(\mathrm{H}) \simeq \mathrm{H}^i_{\mathrm{G}_+}(\mathrm{G}(M))$ *as a graded* G*-module for all* $i < r$;

(2) $[\mathrm{H}^r_{\mathrm{G}_+}(\mathrm{H})]_n = (0)$ *for all* $n > -r$.

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Proof. Apply $\mathrm{H}^i_{\mathrm{G}_+}(*)$ to the exact sequence

$$0 \longrightarrow \mathrm{H} \longrightarrow \mathrm{G}(M) \longrightarrow \mathrm{G}(M)/\mathrm{H} \longrightarrow 0$$

of graded G-modules and we get the assertion (1), because $f_1, \ldots, f_r$ is a G(H)/H-regular sequence. Assertion (2) follows from the fact that $\mu_{\mathrm{G}}(\mathrm{G}_+) \leqslant r$ and $\mathrm{H} = \mathrm{G}{\cdot}\mathrm{H}_0$.

When $x_1, \ldots, x_r$ is a d^+-sequence relative to M, we furthermore have the following.

PROPOSITION 6·6. *Let $I = (\mathbf{x}) = (x_1, \ldots, x_r)$ be an ideal generated by a d^+-sequence relative to M.*

(1) $\mathrm{H}^i_{\mathrm{G}_+}(\mathrm{H}) = [\mathrm{H}^i_{\mathrm{G}_+}(\mathrm{H})]_{-i}$ *for all* $i < r$.

(2) *Suppose that* $\dim \mathrm{H} = r$. *Then the* G*-module* H *has* FLC*, so that* H *is a Cohen-Macaulay* G*-module, if* $\mathrm{j}_1(\mathbf{x}; M) = 0$ *and* $\operatorname{depth} M > 0$.

Proof. By Corollary 6·5 and [**2**, (5·1) theorem] we get $\mathrm{H}^i_{\mathrm{G}_+}(\mathrm{H}) \simeq \mathrm{H}^i_{\mathrm{G}_+}(\mathrm{G}(M)) = [\mathrm{H}^i_{\mathrm{G}_+}(\mathrm{G}(M))]_{-i}$, whence the assertion follows. The second assertion is clear, because $\mathrm{e}_1(\mathbf{f}; \mathrm{H}) = \mathrm{j}_1(\mathbf{x}; M)$, where $\mathbf{f} = f_1, \ldots, f_r$ (notice that H is unmixed, if depth $M > 0$).

We are in a position to prove Theorem 6·1.

Proof of Theorem 6·1. We remember that $x_1, \ldots, x_d$ is a standard system of parameters of M, since M is a Buchsbaum R-module.

(1) By Theorem 4·7

$$\dim \mathrm{H} = r - \min\{0 \leqslant i \leqslant r \mid k_i(\mathbf{x}; M) \neq 0\}$$

and by Proposition 4·12

$$k_i(\mathbf{x}; M) = \sum_{j=1}^{r-i} \binom{r-i-1}{j-1} h^j(M) \quad \text{for all } \ 0 \leqslant i \leqslant r-1.$$

Therefore, if $h^1(M) = h^2(M) = \cdots = h^r(M) = 0$, then

$$k_i(\mathbf{x}; M) = 0 \quad \text{for all } \ 0 \leqslant i \leqslant r-1,$$

while $k_r(\mathbf{x}; M) = h^0(M) \neq 0$ by Theorem 4·7, because $\mathrm{H} \neq (0)$. Hence $\dim \mathrm{H} = 0$. If $h^i(M) \neq 0$ for some $1 \leqslant i \leqslant r$, then $k_0(\mathbf{x}; M) \neq 0$ by Proposition 4·12, whence $\dim \mathrm{H} = r$.

(2) We may assume that $\mathrm{H} \neq (0)$ and $\dim \mathrm{H} = r$. Let $S = \overline{\mathrm{G}}$ and $\mathfrak{M} = S_+$. Then H is a graded S-module, because $\mathfrak{m}\mathrm{H} = (0)$ (remember that $\mathfrak{m}\mathrm{H}^0_{\mathfrak{m}}(M/IM) = (0)$, since $r < d$). Hence H is a Buchsbaum S-module by Proposition 6·6 and [**8**, proposition 3·1]. Consequently, by [**9**, theorem (1·1)] applied to the $S_{\mathfrak{M}}$-module $\mathrm{H}_{\mathfrak{M}}$, we get an isomorphism

$$(D_0) \quad \mathrm{H}_{\mathfrak{M}} \simeq \bigoplus_{i=0}^{r-1} (E_i)_{\mathfrak{M}}^{\oplus h^i(\mathrm{H})} \oplus S_{\mathfrak{M}}^{\oplus q}$$

of $S_{\mathfrak{M}}$-modules, where $q \geqslant 0$ is an integer. Taking the graded module $\mathrm{gr}_{\mathfrak{n}}(*)$ (here $\mathfrak{n} = \mathfrak{M}S_{\mathfrak{M}}$) of both sides in (D_0), we get a required direct sum decomposition

$$(D) \quad \mathrm{H} \simeq \bigoplus_{i=0}^{r-1} E_i^{\oplus h^i(\mathrm{H})} \oplus S^{\oplus q}$$

of H as a graded S-module, because $\mathrm{gr}_{\mathfrak{n}}(\mathrm{H}_{\mathfrak{M}}) = \mathrm{H}$ and $\mathrm{gr}_{\mathfrak{n}}((E_i)_{\mathfrak{M}}) = E_i$ (remember that

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$\mathrm{H} = S{\cdot}\mathrm{H}_0$ and $E_i = S{\cdot}[E_i]_0$). On the other hand, since $f_1\mathrm{H}^i_{\mathfrak{M}}(\mathrm{H}) = (0)$ for all $i \neq r$ and $\mathrm{H}/f_1\mathrm{H} = \mathrm{H}(M/x_1M)$ by Lemma 4·3, we have the exact sequence

$$0 \longrightarrow \mathrm{H}^{i-1}_{\mathfrak{M}}(\mathrm{H}) \longrightarrow \mathrm{H}^i_{\mathfrak{M}}(\mathrm{H}(M/x_1M)) \longrightarrow \mathrm{H}^{i+1}_{\mathfrak{M}}(\mathrm{H})(-1) \longrightarrow 0$$

of graded S-modules. The induction on i now shows that $h^i(\mathrm{H}) = h^i(M)$ for all $i < r$, while we have by decomposition (D) of H that

$$\operatorname{rank}_S \mathrm{H} = \sum_{i=1}^{r-1} \binom{r-1}{i-1} h^i(M) + q,$$

since $\operatorname{rank}_S E_i = \binom{r-1}{i-1}$. Let $\mathbb{I}(\mathrm{H}) = \sum_{i=0}^{r-1} \binom{r-1}{i} h^i(\mathrm{H})$ be the Stückrad-Vogel invariant of H. Therefore, because H is a Buchsbaum S-module, we have

$$\begin{aligned}
\operatorname{rank}_S \mathrm{H} &= \mathrm{e}_0(\mathbf{T}_1, \ldots, \mathbf{T}_r; \mathrm{H}) \\
&= \lambda(\mathrm{H}/(\mathbf{T}_1, \ldots, \mathbf{T}_r)\mathrm{H}) - \mathbb{I}(\mathrm{H}) && [\textbf{19}, \text{proposition } 2{\cdot}6] \\
&= h^0(M/IM) - \sum_{i=0}^{r-1} \binom{r-1}{i} h^i(M) && \text{Lemma } 6{\cdot}2 \\
&= \sum_{i=0}^{r} \binom{r}{i} h^i(M) - \sum_{i=0}^{r-1} \binom{r-1}{i} h^i(M) && [\textbf{19}, \text{lemma } 4{\cdot}14] \\
&= \sum_{i=1}^{r-1} \binom{r-1}{i-1} h^i(M) + h^r(M).
\end{aligned}$$

Thus $q = h^r(M)$, which completes the proof of Theorem 6·1.

Remark 6·7. (1) Suppose that M has FLC and our system $x_1, \ldots, x_d$ of parameters of M is standard. If $\mathrm{H} \neq (0)$ and $\dim \mathrm{H} = r$, then H has FLC and $h^i(\mathrm{H}) = h^i(M)$ for all $i < r$.

(2) It is known that M is a quasi-Buchsbaum R-module, that is $\mathfrak{m}\mathrm{H}^i_{\mathfrak{m}}(M) = (0)$ for all $i \neq d$ if and only if for each $1 \leqslant i \leqslant d$ and for every system $a_1, \ldots, a_d$ of parameters of M such that $a_1, \ldots, a_d \in \mathfrak{m}^2$, one has the equality

$$(a_1, \ldots, a_{i-1})M :_M a_i = (a_1, \ldots, a_{i-1})M :_M \mathfrak{m}.$$

Hence if M is quasi-Buchsbaum and $x_1, \ldots, x_r \in \mathfrak{m}^2$, then $x_1, \ldots, x_r$ forms a d^+-sequence relative to M, $\mathfrak{m}\mathrm{H}^0_{\mathfrak{m}}(M/IM) = (0)$, and $h^0(M/IM) = \sum_{i=0}^{r} \binom{r}{i} h^i(M)$. Therefore the proof of Theorem 6·1 works also in this case.

We explore one example.

Example 6·8. Let $P = k[\![Y_1, Y_2, Y_3, Z_1, Z_2, Z_3]\!]$ be the formal power series ring over a field k and consider the local ring

$$R = P/\left[\mathfrak{p} \cap \mathfrak{q} \cap (Y_1^2, Y_2, Y_3, Z_1^2, Z_2, Z_3)\right],$$

where $\mathfrak{p} = (Y_1, Y_2, Y_3)$ and $\mathfrak{q} = (Z_1, Z_2, Z_3)$. Then R is a quasi-Buchsbaum but non-Buchsbaum local ring of dimension 3 such that

$$h^0(R) = h^1(R) = 1 \text{ and } h^i(R) = 0 \text{ if } i \neq 0, 1, 3$$

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[**19**, p. 87, example]. For each $1 \leqslant i \leqslant 3$ let y_i and z_i respectively denote the images of Y_i and Z_i in R. Let $x_i = y_i + z_i$ $(1 \leqslant i \leqslant 3)$. Then

$$\lambda(R/(x_1, x_2, x_3)) = 5, \quad \mathrm{e}_0((x_1, x_2, x_3), R) = 2, \quad \text{and} \quad \mathbb{I}(R) = 3,$$

so that the system x_1, x_2, x_3 of parameters of R is standard [**20**, theorem 2·1].

We set $I = (x_1, x_2)$ and consider the **j**-transform $\mathrm{H}(R)$ of R relative to I. Let $W = \mathrm{H}^0_{\mathfrak{m}}(R)$, $\mathfrak{M} = \mathfrak{m}\mathrm{G} + \mathrm{G}_+$, and $\overline{\mathrm{G}} = \mathrm{G}/\mathfrak{m}\mathrm{G}$. We then have the following. Remember that f_1, f_2 acts on $\mathrm{H}(R)$ as a d^+-sequence, since x_1, x_2 is a d^+-sequence relative to R.

(1) There exists an exact sequence

$$(\sharp) \quad 0 \longrightarrow \overline{\mathrm{G}}/\overline{\mathrm{G}}_+ \longrightarrow \mathrm{H}(R) \longrightarrow [\overline{\mathrm{G}}_+](1) \longrightarrow 0$$

of graded G-modules. Hence $\mathrm{H}(R)$ is a quasi-Buchsbaum G-module.

(2) $\mathfrak{m}\mathrm{H}(R) \neq (0)$ and $\mathfrak{M}\mathrm{H}^0_{\mathfrak{M}}(\mathrm{H}(R)/f_1\mathrm{H}(R)) \neq (0)$. Hence $\mathrm{H}(R)$ is not a Buchsbaum G-module and exact sequence $(\sharp)$ above cannot be split.

Proof. (1) Since $R/W = P/\mathfrak{p} \cap \mathfrak{q}$, R/W is a Buchsbaum local ring. Hence, by Theorem 6·1,

$$\mathrm{H}(R/W) \simeq [\overline{\mathrm{G}}_+](1)$$

as a graded G-module. The exact sequence now follows from the canonical exact sequence

$$0 \longrightarrow W^* \longrightarrow \mathrm{H}(R) \longrightarrow \mathrm{H}(R/W) \longrightarrow 0$$

of **j**-transforms, since $W^* \simeq \overline{\mathrm{G}}/\overline{\mathrm{G}}_+$. Therefore, applying the functor $\mathrm{H}^i_{\mathfrak{M}}(*)$, we get

$$\mathrm{H}^0_{\mathfrak{M}}(\mathrm{H}(R)) = W^* \text{ and } \mathrm{H}^1_{\mathfrak{M}}(\mathrm{H}(R)) = \mathrm{H}^1_{\mathfrak{M}}(\overline{\mathrm{G}}_+)(1) \simeq \left[\overline{\mathrm{G}}/\overline{\mathrm{G}}_+\right](1).$$

Thus $\mathrm{H}(R)$ is a quasi-Buchsbaum G-module.

(2) We have $\mathfrak{m}\mathrm{H}^0_{\mathfrak{m}}(R/x_1R) \neq (0)$ and $\mathfrak{m}\mathrm{H}^0_{\mathfrak{m}}(R/(x_1, x_2)) \neq (0)$, both of which we can check by a direct computation. Hence exact sequence $(\sharp)$ above is not split, as $\mathfrak{m}\mathrm{H}(R) \neq (0)$ (notice that $\mathfrak{m}[\mathrm{H}(R)]_0 = \mathfrak{m}\mathrm{H}^0_{\mathfrak{m}}(R/(x_1, x_2)) \neq (0)$), while

$$\mathfrak{M}\mathrm{H}^0_{\mathfrak{M}}(\mathrm{H}(R)/f_1\mathrm{H}(R)) = \mathfrak{M}\mathrm{H}^0_{\mathfrak{M}}(\mathrm{H}(R/x_1R)) \neq (0)$$

because $\mathfrak{m}\mathrm{H}^0_{\mathfrak{m}}(R/x_1R)) \neq (0)$. Thus $\mathrm{H}(R)$ is not a Buchsbaum G-module.

In Example 6·8, if we choose a partial system x_1, x_2 of parameters of R so that $x_1, x_2 \in \mathfrak{m}^2$, we always have

$$\mathrm{H}(R) \simeq \left[\overline{\mathrm{G}}/\overline{\mathrm{G}}_+\right] \oplus \left[\overline{\mathrm{G}}_+\right](1),$$

which shows the structure of **j**-transforms depends more or less on the choice of partial systems of parameters.

Sequentially Cohen–Macaulay modules

Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finite R-module of dimension $d = \dim SM > 0$. Let

$$\mathcal{S} = \{\dim N \mid (0) \neq N \subseteq M, \text{an } R\text{-submodule of } M\}.$$

We set $\ell = \sharp\mathcal{S}$ and write $\mathcal{S} = \{d_1 < d_2 < \cdots < d_\ell = d\}$. Let $d_0 = 0$. We then have the dimension filtration

$$\mathcal{D}_0 = (0) \subsetneq \mathcal{D}_1 \subsetneq D_2 \subsetneq \cdots \subsetneq \mathcal{D}_\ell = M$$

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of M, where each $\mathcal{D}_i$ $(1 \leqslant i \leqslant \ell)$ is the largest R-submodule of M with $\dim \mathcal{D}_i = d_i$. We put $\mathcal{C}_i = \mathcal{D}_i/\mathcal{D}_{i-1}$ $(1 \leqslant i \leqslant \ell)$ and assume that $\mathcal{C}_i$ is a Cohen–Macaulay R-module, necessarily of dimension d_i, for each $1 \leqslant i \leqslant \ell$. Hence M is a sequentially Cohen–Macaulay R-module.

We choose a system $x_1, \ldots, x_d$ of parameters of M so that

$$(x_j \mid d_i < j \leqslant d)M \cap \mathcal{D}_i = (0)$$

for all $1 \leqslant i \leqslant \ell$. Such a system of parameters exists and called a *good* system of parameters of M. Here we notice that the condition $(x_j \mid d_i < j \leqslant d)M \cap \mathcal{D}_i = (0)$ for all $1 \leqslant i \leqslant \ell$ is equivalent to saying that

$$(x_j \mid d_i < j \leqslant d)\mathcal{D}_i = (0)$$

for all $1 \leqslant i \leqslant \ell$, because M is a sequentially Cohen–Macaulay R-module.

Let $0 \leqslant r \leqslant d$ and $I = (x_1, \ldots, x_r)$. We are trying to find what the **j**-transform $\mathrm{H}(M)$ of M relative to I is. The goal is the following.

THEOREM 6·9. *Let* $q = \max\{0 \leqslant i \leqslant \ell \mid d_i \leqslant r\}$. *Then*

$$\mathrm{H}(M) \simeq \mathrm{gr}_I(\mathcal{D}_q)$$

as a graded G*-module, where* $\mathrm{G} = \mathrm{gr}_I(R)$. *If* $q > 0$, *that is if* $d_1 \leqslant r$, *then* $\mathrm{H}(M) \neq (0)$ *and is a sequentially Cohen–Macaulay* G*-module with dimension filtration* $\{\mathrm{gr}_I(\mathcal{D}_i)\}_{0 \leqslant i \leqslant q}$; *hence* $\dim \mathrm{H}(M) = d_q \leqslant r$ *and the Hilbert function of* $\mathrm{H}(M)$ *is given by*

$$\sum_{k=0}^{n} \lambda([\mathrm{H}(M)]_k) = \sum_{i=1}^{q} \lambda(\mathcal{C}_i/\mathfrak{q}\mathcal{C}_i)\binom{n+d_i}{d_i}$$

for all $n \geqslant 0$, *where* $\mathfrak{q} = (x_1, \ldots, x_d)$.

To prove this, we need the following.

LEMMA 6·10. $I^n M \cap \mathcal{D}_i = I^n \mathcal{D}_i$ *for* $0 \leqslant i \leqslant \ell$ *and* $n \in \mathbb{Z}$.

Proof. We may assume that $\ell > 1$ and that the assertion holds true for $\ell - 1$. Since $\mathcal{D}_{\ell-1}$ is a sequentially Cohen–Macaulay R-module with dimension filtration $\{\mathcal{D}_i\}_{0 \leqslant i \leqslant \ell-1}$ and the system $x_1, \ldots, x_{d_{\ell-1}}$ is a good system of parameters of $\mathcal{D}_{\ell-1}$, it suffices to show that

$$I^n M \cap \mathcal{D}_{\ell-1} = I^n \mathcal{D}_{\ell-1}$$

for all $n \geqslant 0$, which follows from the fact that $\mathcal{D}_\ell$ is a Cohen-Macaulay R-module of dimension $d = d_\ell$.

Since $I^n \mathcal{C}_i = [I^n \mathcal{D}_i + \mathcal{D}_{i-1}]/\mathcal{D}_{i-1}$, we have the exact sequence

$$0 \longrightarrow I^n \mathcal{D}_{i-1} \longrightarrow I^n \mathcal{D}_i \longrightarrow I^n \mathcal{C}_i \longrightarrow 0$$

of R-modules for each $n \in \mathbb{Z}$, which yields the exact sequence

$$0 \longrightarrow \mathrm{G}(\mathcal{D}_{i-1}) \longrightarrow \mathrm{G}(\mathcal{D}_i) \longrightarrow \mathrm{G}(\mathcal{C}_i) \longrightarrow 0$$

of graded G-modules, where $\mathrm{G}(\mathcal{D}_i) = \mathrm{gr}_I(\mathcal{D}_i)$ and $\mathrm{G}(\mathcal{C}_i) = \mathrm{gr}_I(\mathcal{C}_i)$. Remember that

$$\mathrm{G}(\mathcal{D}_i) = \mathrm{gr}_J(\mathcal{D}_i) \text{ and } \mathrm{G}(\mathcal{C}_i) = \mathrm{gr}_J(\mathcal{C}_i),$$

where $J = I$ if $r \leqslant d_i$ and $J = (x_1, \ldots, x_{d_i})$ if $r > d_i$. Hence we get the following.

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LEMMA 6·11. *Let* $1 \leqslant i \leqslant \ell$. *Then*

$$\mathrm{H}^0_{\mathfrak{m}}(\mathrm{G}(\mathcal{C}_i)) = \begin{cases} \mathrm{G}(\mathcal{C}_i) & \text{if } d_i \leqslant r, \\ (0) & \text{if } r < d_i. \end{cases}$$

We are now ready to prove Theorem 6·9.

Proof of Theorem 6·9. We apply the functor $\mathrm{H}^0_{\mathfrak{m}}(*)$ to the exact sequences

$$0 \longrightarrow \mathrm{G}(\mathcal{D}_{i-1}) \longrightarrow \mathrm{G}(\mathcal{D}_i) \longrightarrow \mathrm{G}(\mathcal{C}_i) \longrightarrow 0$$

$(1 \leqslant i \leqslant \ell)$ and get by (5·3) that

$$\mathrm{H}(M) \cong \mathrm{G}(\mathcal{D}_q),$$

since $\mathrm{H}^0_{\mathfrak{m}}(\mathrm{G}(\mathcal{C}_i)) = (0)$ if $q < i$. Suppose $q > 0$, that is $d_1 \leqslant r$. Then $\{\mathrm{G}(\mathcal{D}_i)\}_{0 \leqslant i \leqslant q}$ gives rise to the dimension filtration of $\mathrm{G}(\mathcal{D}_i)$, since the graded module $\mathrm{G}(\mathcal{C}_i)$ $(1 \leqslant i \leqslant q)$ is Cohen-Macaulay and $\dim \mathrm{G}(\mathcal{C}_i) = d_i$. Hence $\dim \mathrm{H}(M) = d_q \leqslant r$, whose Hilbert function is given by

$$\sum_{k=0}^{n} \lambda([\mathrm{H}(M)]_k) = \sum_{k=0}^{n} \left\{ \sum_{i=1}^{q} \lambda([\mathrm{G}(\mathcal{C}_i)]_k) \right\} = \sum_{i=1}^{q} \lambda(\mathcal{C}_i/\mathfrak{q}\mathcal{C}_i) \binom{n+d_i}{d_i}$$

for all $n \geqslant 0$.

We end this paper with the following.

Example 6·12. Let $(R, \mathfrak{n})$ be a regular local ring of dimension $d+1$ $(d \geqslant 2)$ and let $X, Y_1, Y_2, \ldots, Y_d$ be a regular system of parameters of R. We set $\mathfrak{p} = (Y_1, Y_2, \ldots, Y_d)$, $M = R/[(X) \cap \mathfrak{p}]$, and $V = R/\mathfrak{p}$. Then $\dim M = d$ and the dimension filtration of M is given by

$$\mathcal{D}_0 = (0) \subsetneq \mathcal{D}_1 = Rx \subsetneq \mathcal{D}_2 = M,$$

where x is the image of X in M and $V \simeq Rx$ as an R-module, so that M is a sequentially Cohen–Macaulay R-module. Let $\mathbf{x} = x_1, \ldots, x_r$ $(0 \leqslant r \leqslant d)$ be an arbitrary partial system of parameters of M and set $I = (\mathbf{x})$. We then have

$$\mathrm{H}(M) \simeq \begin{cases} (0) & \text{if} \quad I \subseteq \mathfrak{p}, \\ \mathrm{gr}_I(V) & \text{if} \quad I \nsubseteq \mathfrak{p} \text{ and } r < d, \\ \mathrm{gr}_I(M) & \text{if} \quad r = d. \end{cases}$$

Notice that when $0 < r$ and $I \subseteq \mathfrak{p}$, $\mathbf{x}$ forms a d-sequence relative to M, but $\mathbf{x}$ is not amenable for M, because $(0) :_M I = Rx$ and $\dim Rx = 1$.

Proof. Let $N = Rx$, $C = R/(X)$, and we consider the exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow C \longrightarrow 0$$

of R-modules. Suppose $I \subseteq \mathfrak{p}$. Then $r < d$ and $\mathbf{x}$ is a C-regular sequence. Therefore since $I^{n+1}M \cap N = I^{n+1}N = (0)$ for all $n \geqslant 0$, we get the exact sequence

$$0 \longrightarrow N \longrightarrow M/I^{n+1}M \longrightarrow C/I^{n+1}C \longrightarrow 0$$

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of R-modules, which shows $\mathrm{H}^0_{\mathfrak{m}}(M/I^{n+1}M) = (0)$, because $C/I^{n+1}C$ is a Cohen–Macaulay R-module of dimension $d - r > 0$. Thus $\mathrm{H}(M) = (0)$.

Suppose $I \nsubseteq \mathfrak{p}$. Then $IV \neq (0)$ and $0 < r$. We choose $1 \leqslant i \leqslant r$ so that $IV = x_i V$ (this choice is possible, because V is a discrete valuation ring). Let $1 \leqslant j \leqslant r$. Then $x_j \in x_i R + \mathfrak{p}$. We write $x_j = x_i c_j + y_j$ with $c_j \in R$ and $y_j \in \mathfrak{p}$. Hence $I = (x_i) + (y_j \mid 1 \leqslant j \leqslant r,\ j \neq i)$. Because $y_j \in \mathfrak{p}$ for all j and $x_i N \neq (0)$, the sequence $x_i, \{y_j\}_{1 \leqslant j \leqslant r, j \neq i}$ is extended to a good system of parameters of M. Therefore, if $r < d$, $\mathrm{H}(M) \cong \mathrm{gr}_I(V)$ by Theorem 6·9, while $\mathrm{H}(M) = \mathrm{gr}_I(M)$, if $r = d$.

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