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Matrix transformations between some classes of sequences

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Let $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk} $(n, k = 1, 2, \ldots)$ and X, Y two subsets of the space s of complex sequences. We say that the matrix A defines a (matrix) transformation from X into Y, and we denote it by writing $A: X \to Y$, if for every sequence $x = (x_k) \in X$ the sequence $Ax = (A_n(x))$ is in Y, where $A_n(x) = \Sigma a_{nk} x_k$ and the sum without limits is always taken from $k = 1$ to $k = \infty$. The sequence Ax is called the transformation of x by the matrix A. By (X, Y) we denote the class of matrices A such that $A: X \to Y$.

The problem of characterizing the matrices $A \in (X, Y)$ for X, Y arbitrary subsets of s remains unsolved in general although solutions have been given for many particular cases.

The main purpose of the present note is to determine the matrices of the classes $(l(p), l_\infty)$, $(l(p), c)$, $(l_\infty(p), l_\infty)$ and $(l_\infty(p), c)$, where l_∞, c are respectively the spaces of bounded and convergent complex sequences and for $p = (p_k)$ with $p_k > 0$,

$$l(p) = \{x = (x_k): \Sigma |x_k|^{p_k} < \infty\},$$

$$l_\infty(p) = \{x = (x_n): \sup_k |x_k|^{p_k} < \infty\}.$$

We shall also characterize the class $(w(p), c)$ in the case $0 < p_k \leqslant 1$, where $w(p)$ denotes the set of all $x = (x_k)$ such that

$$\sum_{k=1}^{n} |x_k - l|^{p_k} = o(n)$$

for some l.

By Theorem 1 (1) the boundedness of (p_k) is sufficient for the spaces $l(p)$, $l_\infty(p)$ to be linear. One can easily see that the condition $p_k = O(1)$ is also necessary for the linearity of the above spaces. Furthermore by Theorem 1 (Corollary 1) (2), if $H = \sup_k p_k < \infty$ and $M = \max(1, H)$ the space $l(p)$ is paranormed by

$$g(x) = (\Sigma |x_k|^{p_k})^{1/M}.$$

It is easy to see that $l(p)$ is also complete in the topology induced on it by the above paranorm. Finally, for $p_k = O(1)$ the set $w(p)$ is a linear metric space with metric function

$$g(x) = \sup_r \{2^{-r} \Sigma_r |x_k|^{p_k}\}^{1/M},$$

where Σ_r is the sum over the range $2^r \leqslant k < 2^{r+1}$, and it has been recently proved in (4) that $w(p)$ is also complete.

If E is a subset of s, then we shall write $E^\dagger$ for the generalized Köthe–Toeplitz dual of E, i.e.

$$E^\dagger = \{a: \Sigma a_k x_k \quad \text{converges for every } x \in E\}.$$

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If $0 < p_k \leqslant 1$, then $l^\dagger(p) = l_\infty(p)$ (see Theorem 7 (6)). If $p_k > 1$, then $l^\dagger(p) = M(p)$ (see Theorem 1 (3)), where

$$M(p) = \{a: \Sigma |a_k|^{q_k} N^{-q_k} < \infty \text{ for some integer } N > 1\}, \tag{1}$$

with $p_k^{-1} + q_k^{-1} = 1$.

If X is a topological linear space we shall denote by X^* the continuous dual of X, i.e. the set of all continuous linear functionals on X.

The following inequality (see Theorem 1 (3)) will be used. For any $B > 0$ and any two complex numbers a, x,

$$|ax| \leqslant B(|a|^q B^{-q} + |x|^p), \tag{2}$$

where $p > 1$ and $p^{-1} + q^{-1} = 1$.

To contract notation throughout the proofs we shall write

$$C(n, N, m, s) = \sum_{k=m}^{s} |a_{nk}|^{q_k} N^{-q_k}$$

for all integers $n, m \geqslant 1$, $1 \leqslant s \leqslant \infty$ and $N > 1$. We shall simply write $C(n, N)$ instead of $C(n, N, 1, \infty)$ and put $C(N) = \sup_n C(n, N)$ for every integer $N > 1$. We observe immediately that for each n, $C(n, N)$ decreases as N increases.

THEOREM 1. (i) *Let $1 < p_k \leqslant H < \infty$ for every k. Then $A \in (l(p), l_\infty)$ if and only if there exists an integer $B > 1$ such that $C(B) < \infty$.*

(ii) *Let $0 < p_k \leqslant 1$ for every k. Then $A \in (l(p), l_\infty)$ if and only if $\sup_{n,k} |a_{nk}|^{p_k} < \infty$.*

Proof. (i) For the sufficiency, by using the inequality (2) we see that for $x \in l(p)$ and for every n,

$$|A_n(x)| \leqslant B(C(n, B) + g^H(x)) \leqslant B(C(B) + g^H(x)).$$

For the necessity suppose that $A \in (l(p), l_\infty)$ but that $C(N) = \infty$ for every integer $N > 1$. Then $\Sigma a_{nk} x_k$ converges for every n and every $x \in l(p)$, whence $(a_{nk})_{k=1,2,\ldots} \in l^\dagger(p)$ for every n. By Theorem 2 (3) it follows that each A_n defined by $A_n(x) = \Sigma a_{nk} x_k$ is an element of $l^*(p)$. Since $l(p)$ is complete and since $\sup_n |A_n(x)| < \infty$ on $l(p)$, there exists by the uniform boundedness principle a number G independent of n and x and a number $\delta < 1$ such that

$$|A_n(x)| \leqslant G, \tag{3}$$

for every $x \in S[\theta, \delta]$ and every n, where by $S[\theta, \delta]$ we denote the closed sphere in $l(p)$ with centre the origin $\theta = (0, 0, 0, \ldots)$ and radius δ.

Now choose an integer $Q > 1$ such that

$$Q\delta^H > G.$$

By our supposition we have $C(Q) = \infty$ and so two cases are possible: either $C(n, Q) < \infty$ for every $n \geqslant 1$, or there exists $n \geqslant 1$ such that $C(n, Q) = \infty$. In the first case there exists $n \geqslant 1$ such that $C(n, Q) > 2$ and there exists $k_0 > 1$ such that

$$C(n, Q, k_0 + 1, \infty) < 1,$$

whence $C(n, Q, 1, k_0) > 1$. In the second case we may choose $k_0 > 1$ such that $C(n, Q, 1, k_0) > 1$, so that in either case there exist $n \geqslant 1$ and $k_0 > 1$ such that

$$S \equiv C(n, Q, 1, k_0) > 1. \tag{4}$$

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We now define using (4) a sequence $x = (x_k)$ as follows:

$$x_k = \delta^{H/p_k}(\operatorname{sgn} a_{nk})\,|a_{nk}|^{q_k-1} S^{-1}Q^{-q_k/p_k} \quad \text{for} \quad 1 \leqslant k \leqslant k_0$$
$$= 0 \quad \text{for} \quad k > k_0.$$

Then it is easy to see that $g(x) \leqslant \delta$ and that

$$|A_n(x)| = S^{-1}Q \Sigma\, |a_{nk}|^{q_k} Q^{-q_k}\, \delta^{H/p_k}$$
$$\geqslant Q\delta^H > G$$

which contradicts (3).

This completes the proof of Theorem 1 (i).

(ii) The sufficiency may be proved by the same kind of argument that has been used in (6) to prove that $\Sigma a_k x_k$ is in $l^*(p)$ whenever $a \in l_\infty(p)$ and $x \in l(p)$.

The necessity follows by a simple application of the uniform boundedness principle.

This completes the proof of the theorem.

We remark that the exact conditions for $A \in (l(p), l_\infty)$ in the general case $0 < p_k \leqslant \sup_k p_k < \infty$ may be obtained by combining the cases (i) and (ii) of the theorem.

COROLLARY. *Let $p = (p_k)$ be as in Theorem* 1. *Then $A \in (l(p), c)$ if and only if, together with the conditions of the theorem, we have*

$$a_{nk} \to \alpha_k \quad (n \to \infty, \, k \text{ fixed}). \tag{5}$$

Proof. (i) The necessity of (5) is trivial so that all we need to prove is the sufficiency.

For every integer $m \geqslant 1$ and every n we have

$$C(n, B, 1, m) \leqslant C(B) < \infty.$$

Hence
$$\lim_{m\to\infty} \lim_{n\to\infty} C(n, B, 1, m) \leqslant C(B),$$

i.e.
$$\Sigma\, |\alpha_k|^{q_k} B^{-q_k} \leqslant C(B).$$

Therefore $(\alpha_k) \in l^\dagger(p)$ and since also $(a_{nk})_{k=1,2,\ldots} \in l^\dagger(p)$, the series $\Sigma \alpha_k x_k$, $\Sigma a_{nk} x_k$ converge for every n and every $x \in l(p)$.

For each $x \in l(p)$ we can choose an integer $m \geqslant 1$ such that

$$\sum_{k=m+1}^{\infty} |x_k|^{p_k} < 1.$$

Then by the proof of Theorem 2 (3) and the inequality (2) we have

$$\sum_{k=m+1}^{\infty} |a_{nk} - \alpha_k|\, |x_k| \leqslant 2B(2C+1) \left(\sum_{k=m+1}^{\infty} |x_k|^{p_k} \right)^{1/H}.$$

It follows immediately that $\lim_{n\to\infty} \Sigma a_{nk} x_k = \Sigma \alpha_k x_k$.

(ii) The proof of this is similar to that in (i).

The general case $A \in (l(p), c)$, when $0 < p_k \leqslant H < \infty$, comes out again as a combination of the two cases.

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It is easy to see that, for $1 < p \leqslant p_k \leqslant H < \infty$, $A \in (l(p), l_\infty)$ if and only if

$$\sup_n \Sigma |a_{nk}|^{q_k} < \infty.$$

Also, if $p_k = p > 1$ for every k and $p^{-1} + q^{-1} = 1$ then the above condition becomes

$$\sup_n \Sigma |a_{nk}|^q < \infty,$$

which is the well-known condition for $A \in (l_p, l_\infty)$ given by Hahn.

Finally, we observe that $(l(p), l_\infty) \subseteq (l_1, l_\infty)$ if and only if $\inf p_k > 0$. The sufficiency is trivial, and if the inclusion holds but $\inf p_k = 0$ then there exists a sequence (k_i) such that $p_{k_i} < i^{-1}$ $(i = 1, 2, \ldots)$. Now define $a_{nk} = 0$ if $k \neq k_i$ and $a_{nk} = i$ for $k = k_i$. Then it is clear that $A \in (l(p), l_\infty) - (l_1, l_\infty)$, whence the necessity.

When $0 < \inf p_k \leqslant p_k \leqslant 1$ we see that the classes $(l(p), l_\infty)$ and (l_1, l_∞) are identical.

We now characterize the classes $(l_\infty(p), l_\infty)$ and $(l_\infty(p), c)$.

THEOREM 2. *Let $p_k > 0$ for every k. Then $l_\infty^\dagger(p) = M_\infty(p)$, where*

$$M_\infty(p) = \bigcap_{N=2}^{\infty} \{a: \Sigma |a_k| N^{1/p_k} < \infty\}.$$

Proof. Let $a \in M_\infty(p)$ and $x \in l_\infty(p)$. We choose an integer $N > \max(1, \sup_k |x_k|^{p_k})$. Then $|\Sigma a_k x_k| \leqslant \Sigma |a_k| N^{1/p_k} < \infty$ and therefore $M_\infty(p) \subset l_\infty^\dagger(p)$.

On the other hand if $a \in l_\infty^\dagger(p)$ but $a \notin M_\infty(p)$ then there exists an integer $N > 1$ such that $\Sigma |a_k| N^{1/p_k} = \infty$. Putting $x_k = N^{1/p_k} \operatorname{sgn} a_k$ we have $x \in l_\infty(p)$ but $\Sigma a_k x_k$ divergent.

Hence $l_\infty^\dagger(p) = M_\infty(p)$.

THEOREM 3. *Let $p_k > 0$ for every k. Then $A \in (l_\infty(p), l_\infty)$ if and only if*

$$D(N) = \sup_n \Sigma |a_{nk}| N^{1/p_k} < \infty,$$

for every integer $N > 1$.

Proof. For the sufficiency take an integer $N > \max(1, \sup_k |x_k|^{p_k})$. Then, for every n,

$$|A_n(x)| \leqslant \Sigma |a_{nk}| N^{1/p_k} \leqslant D(N)$$

and therefore $A \in (l_\infty(p), l_\infty)$.

For the necessity, suppose that $A \in (l_\infty(p), l_\infty)$ but that there exists an integer $N > 1$ such that $D(N) = \infty$. Then the matrix $(a_{nk} N^{1/p_k}) \notin (l_\infty, l_\infty)$ and so there exists an $x \in l_\infty$ with $\sup_k |x_k| = 1$, such that

$$\Sigma a_{nk} (N^{1/p_k} x_k) \neq O(1).$$

Hence although $y = (N^{1/p_k} x_k) \in l_\infty(p)$ the sequence $(A_n(y)) \notin l_\infty$, and this contradicts the fact that $A \in (l_\infty(p), l_\infty)$.

COROLLARY. *Let $p_k > 0$ for every k. Then $A \in (l_\infty(p), c)$ if and only if*

(i) *$\Sigma |a_{nk}| N^{1/p_k}$ converges uniformly in n, for all integers $N > 1$.*

(ii) *$a_{nk} \to \alpha_k$ $(n \to \infty, k$ fixed$)$.*

Proof. The sufficiency is trivial since by (i) $\Sigma a_{nk} x_k$ converges uniformly in n for each $x \in l_\infty(p)$, and therefore

$$\lim_{n \to \infty} \Sigma a_{nk} x_k = \Sigma \alpha_k x_k.$$

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The necessity of (ii) is also trivial.

Now if (i) fails then $(a_{nk}N^{1/p_k}) \notin (l_\infty, c)$ for some integer $N > 1$, whence as in Theorem 3 there exists $x \in l_\infty(p)$ such that $(\Sigma a_{nk}x_k) \notin c$.

We conclude this note by characterizing the class $(w(p), c)$ where $0 < p_k \leqslant 1$ for every k. The special case $p_k = p > 1$ for all k is important in connexion with Kuttner's theorem, which asserts that for every Toeplitz matrix T there is always a sequence in $w(p)$ which is not summable T (see (5)).

First we determine $w^\dagger(p)$.

THEOREM 4. *Let* $0 < p_k \leqslant 1$. *Then* $w^\dagger(p) = \mathscr{M}$, *where*

$$\mathscr{M} = \left\{a: \sum_{r=0}^{\infty} \max_r \{(2^r N^{-1})^{1/p_k} |a_k|\} < \infty \text{ for some integer } N > 1\right\}$$

and $\max_r$ *is the maximum taken over* $2^r \leqslant k < 2^{r+1}$.

Proof. Let $a \in \mathscr{M}$. Then there exists an integer $N > 1$ such that

$$\sum_{r=0}^{\infty} \max_r \{(2^r N^{-1})^{1/p_k} |a_k|\} < \infty.$$

Take $x_k \to l(w(p))$. Then there exists an integer $R > 0$ such that

$$2^{-r}\Sigma_r |x_k - l|^{p_k} < 1/2N.$$

and

$$2^{-r} \max(1, |l|) < 1/2N,$$

for every $r > R$. Hence $2^{-r}|x_k|^{p_k} \leqslant N^{-1}$, and since $p_k \leqslant 1$, we have for every $r > R$,

$$\Sigma_r |a_k x_k| \leqslant \max_r \{(2^r N^{-1})^{1/p_k} |a_k|\} N g(x),$$

where

$$g(x) = \sup_r \{2^{-r}\Sigma_r |x_k|^{p_k}\}.$$

Hence $\Sigma |a_k x_k| < \infty$.

To prove that $w^\dagger(p) \subseteq \mathscr{M}$ we observe that if $a \notin \mathscr{M}$ then

$$\sum_{r=0}^{\infty} \max_r \{(2^r N^{-1})^{1/p_k} |a_k|\} = \infty$$

for every integer $N > 1$ and therefore there exists a sequence $N_r \to \infty$ such that

$$\sum_{r=0}^{\infty} \max_r \{(2^r N_r^{-1})^{1/p_k} |a_k|\} = \infty.$$

One can now show that $a \notin w^\dagger(p)$ by an argument similar to that of Theorem 6 (1).

By using Theorem 4 it is now a straightforward matter to prove the following result.

THEOREM 5. *Let* $0 < p_k \leqslant 1$. *Then* $A \in (w(p), c)$ *if and only if*

(i) *There exists an integer* $B > 1$ *such that* $C = \sup_n \sum_{r=0}^{\infty} \max_r \{(2^r B^{-1})^{1/p_k} |a_{nk}|\} < \infty$.

(ii) $\lim_{n\to\infty} a_{nk} = \alpha_k$ *exists for every fixed* k.

(iii) $\lim_{n\to\infty} \Sigma a_{nk} = \alpha$ *exists*.

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We remark here that in the case when $0 < \inf p_k \leqslant p_k \leqslant 1$ for every k, the condition (i) of the theorem is equivalent to

$$\sup_n \sum_{r=0}^{\infty} \max_r \{2^{r/p_k} |a_{nk}|\} < \infty.$$

Also, if $0 < \inf p_k \leqslant \sup p_k < 1$, then (iii) of Theorem 5 is implied by (i) and (ii). In connexion with the last statement the referee has kindly pointed out that the condition $\inf p_k > 0$ is superfluous. For if we choose B such that (i) holds and a constant λ such that $0 < \lambda < 1 - H$ ($H = \sup p_k < 1$) then for all r large enough, $B < 2^{r\lambda}$ and therefore, with $2^r \leqslant k < 2^{r+1}$,

$$|a_{nk}| < A_{nr} 2^{(\lambda-1)r/H},$$

where $A_{nr} = \max_r \{(2^r B^{-1})^{1/p_k} |a_{nk}|\}$. Now since

$$\Sigma_r |a_{nk}| < [2^{r-(1-\lambda)r/H}] A_{nr}$$

and the expression in square brackets can be as small as we please for all sufficiently large r, it follows that (iii) is implied by (i) and (ii). In particular, when $0 < p_k = p < 1$ for every k, Theorem 5 reduces to Theorem $M(a)$ (5). We conclude by mentioning that, as in (5), we can deduce that if $\sup p_k < 1$ and if A is any coregular matrix there is a sequence summable $w(p)$ but not summable A.

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