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On points with arbitrarily assigned mutual distances. By H. S. M. COXETER, Ph.D., Trinity College, and J. A. TODD, Ph.D., Trinity College.

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Let* S^sT^t denote a space of $s+t$ dimensions with the property that the square of the distance between any two points

$$(x_1, x_2, \ldots, x_{s+t}), \qquad (x_1', x_2', \ldots, x'_{s+t})$$

is of the form

$$\sum_{j=1}^{s+t} \epsilon_j (x_j - x_j')^2, \tag{1}$$

where s of the ϵ's are equal to $+1$ and t of them to -1.

It is sometimes convenient to consider a more general space $S^sT^tU^u$, of $s+t+u$ dimensions, in which the distance function is still given by (1), u of the coordinates being absent from the expression. Such a space evidently has the property that any set of points in it determines a corresponding set of points in S^sT^t, at the same mutual distances. It is easily seen that the most general n-dimensional subspace of S^sT^t (or of $S^sT^tU^u$) is of the form $S^pT^qU^r$ with $p \leqslant s$, $q \leqslant t$, $p+q+r=n$.

THEOREM 1. *Given* $\frac{1}{2}m(m+1)$ *real numbers* c_{ij} $(0 \leqslant i < j \leqslant m)$, *there exist* $m+1$ *points* $A^0, A^1, \ldots, A^m$, *in some space* S^sT^t $(s+t \leqslant m)$, *such that the square of the distance* A^iA^j *is equal to* c_{ij}.

When $m=1$ the theorem is trivial: if $c_{01}=0$, we take A^0 and A^1 to coincide with the origin; if $c_{01} \neq 0$, we take A^0 to be the origin, let A^1 have coordinate $|c_{01}|^{\frac{1}{2}}$, and put $\epsilon_1 = c_{01}/|c_{01}|$.

Using induction, we assume the theorem with m replaced by $m-1$. We now have points $A^0, A^1, \ldots, A^{m-1}$ at mutual distances $\sqrt{c_{ij}}$ $(i<j<m)$ in S^pT^q $(p+q<m)$. Let the coordinates of A^i be $(x_1^i, x_2^i, \ldots, x_{p+q}^i)$. Without loss of generality, we may take $x_j^0 = 0$ $(j=1, 2, \ldots, p+q)$.

Let us consider the following m points in space of $p+q+2m-1$ dimensions:

$$\begin{aligned}
B^0 \quad & (0, \quad 0, \ldots, \quad 0, 0, 0, 0, 0, 0, \ldots, 0, 0, 0),\\
B^1 \quad & (x_1^1, \quad x_2^1, \ldots, x_{p+q}^1, 1, 1, 0, 0, 0, \ldots, 0, 0, 0),\\
B^2 \quad & (x_1^2, \quad x_2^2, \ldots, x_{p+q}^2, 0, 0, 1, 1, 0, \ldots, 0, 0, 0),\\
& \ldots\ldots\ldots\ldots\\
B^{m-1} \quad & (x_1^{m-1}, x_2^{m-1}, \ldots, x_{p+q}^{m-1}, 0, 0, 0, 0, 0, \ldots, 1, 1, 0),
\end{aligned}$$

* Cf. Coxeter, *Proc. Camb. Phil. Soc.* 29 (1933), 3, 4.

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where the first $p+q$ ϵ's are the same as in the space of the A's, the next $2(m-1)$ ϵ's are alternately ± 1, and $\epsilon_{p+q+2m-1}$ is at present undetermined. Evidently these m points* have the same mutual distances as $A^0, A^1, \ldots, A^{m-1}$. We shall now show how to take a point B^m in this space so that

$$(B^iB^m)^2 = c_{im} \quad (i = 0, 1, \ldots, m-1).$$

Consider in fact the point

$$(0, 0, \ldots, 0, y_1, 0, y_2, 0, \ldots, y_{m-1}, 0, z).$$

Its distances from $B^0, B^1, \ldots, B^{m-1}$ will be of the required lengths if

$$\sum_{j=1}^{m-1} y_j^2 + \epsilon_{p+q+2m-1} z^2 = c_{0m},$$

$$c_{0i} + \sum_{j=1}^{m-1} y_j^2 - 2y_i + \epsilon_{p+q+2m-1} z^2 = c_{im} \quad (i = 1, 2, \ldots, m-1).$$

These equations are satisfied by

$$2y_i = c_{0i} + c_{0m} - c_{im},$$

$$\epsilon_{p+q+2m-1} z^2 = c_{0m} - \sum_{j=1}^{m-1} y_j^2.$$

We have thus obtained $m+1$ points $B^0, B^1, \ldots, B^m$, whose mutual distances are those required for $A^0, A^1, \ldots, A^m$. The B's are embedded in a space whose symbol is $S^{p+m}T^{q+m-1}$ or $S^{p+m-1}T^{q+m}$ according as $\epsilon_{p+q+2m-1}$ is positive or negative. Being only $m+1$ in number, they lie in a subspace of type $S^sT^tU^u$, with $s+t+u \leqslant m$. If $u > 0$, we can find a set of points at the same mutual distances in S^sT^t. This proves the theorem.

COROLLARY†. *Given $\frac{1}{2}m(m+1)$ real numbers c_{ij} $(0 \leqslant i < j \leqslant m)$, there exists a simplex $A^0A^1 \ldots A^m$, in some space $S^sT^tU^u$ $(s+t+u=m)$, such that the square of the edge A^iA^j is equal to c_{ij}.*

THEOREM, 2. *Given $\frac{1}{2}m(m+1)$ real numbers c_{ij} $(0 \leqslant i < j \leqslant m)$ satisfying*

$$\begin{vmatrix} 0 & 1 & 1 & 1 & \ldots & 1 \\ 1 & 0 & c_{01} & c_{02} & \ldots & c_{0m} \\ 1 & c_{01} & 0 & c_{12} & \ldots & c_{1m} \\ & & \ldots\ldots\ldots\ldots & & & \\ 1 & c_{0m} & c_{1m} & c_{2m} & \ldots & 0 \end{vmatrix} \neq 0, \qquad (2)$$

there exists a simplex $A^0A^1 \ldots A^m$, in some space S^sT^t $(s+t=m)$, such that the square of the edge A^iA^j is equal to c_{ij}.

* It is worth noticing that the B's, unlike the A's, are necessarily *independent*.

† This is the accurate statement of the remark made in parentheses in the first paragraph of the paper previously cited.

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By Theorem 1, it is possible to find points $A^0, A^1, \ldots, A^m$ at the proper mutual distances; but the space $S^s T^t$ might have $s+t<m$. Let A^i have coordinates $(x_1^i, x_2^i, \ldots, x_m^i)$, where, if $s+t<m$, we may take $x_{s+t+u}^i = 0$ for $u > 0$. An argument used by Sommerville* shows that the above determinant is a constant multiple of Δ^2, where

$$\Delta = \begin{vmatrix} x_1^0 & x_2^0 & \ldots & x_m^0 & 1 \\ x_1^1 & x_2^1 & \ldots & x_m^1 & 1 \\ \ldots\ldots & \ldots\ldots & \ldots & \ldots\ldots & \ldots \\ x_1^m & x_2^m & \ldots & x_m^m & 1 \end{vmatrix}$$

The inequality (2) then implies that

$$\Delta \neq 0.$$

Hence the $m+1$ A's are independent, and form the vertices of a simplex.†

* D. M. Y. Sommerville, *Geometry of n dimensions* (London, 1929), 124–125.

† Cf. A. Wald, *Ergebnisse eines mathematisches Kolloquiums*, 5 (1932/3), 36. His results are equivalent to ours if it is assumed that we can always find, in some *complex metric space*, $n+1$ points with arbitrarily assigned mutual distances.

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