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Minimal flows with a closed proximal cell

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Abstract. It is shown that regional proximality is an equivalence relation in a minimal flow which contains a closed proximal cell. Also, a certain group theoretic condition is shown to be equivalent to a 'local Bronstein' condition.

A flow (X, T) is a (right) topological action $(x, t) \mapsto xt$ of the discrete group T on the compact Hausdorff space X. The flow is *minimal* if every orbit is dense: $\overline{xT} = X$ for all $x \in X$.

If (X, T) and (Y, T) are flows, a *homomorphism* is a continuous equivariant map $\pi : X \to Y$, $\pi(xt) = \pi(x)t$ $(x \in X, t \in T)$. If π is surjective (which is always the case when (Y, T) is minimal) (Y, T) is said to be a factor of (X, T).

We recall the definitions of proximal and regionally proximal. If (X, T) is a flow, x and y in X are said to be *proximal* if for every neighborhood W of the diagonal Δ in $X \times X$ there is a $t \in T$ such that $(xt, yt) \in W$. The points x and y are *regionally proximal* if for every neighborhood U of x, V of y and W of Δ there are $x' \in U$, $y' \in V$ and $t \in T$ such that $(x't, y't) \in W$.

Let $P = P(X)$ and $Q = Q(X)$ denote, respectively, the proximal and regionally proximal relations. Clearly P and Q are reflexive, symmetric and T invariant, with $P \subset Q$. It is also easy to see that Q is closed. In general, neither P nor Q is an equivalence relation.

It is immediate that a flow is equicontinuous if and only if $Q = \Delta$. Moreover, when Q is an equivalence relation, it coincides with the equicontinuous structure relation. That is, the flow $(X/Q, T)$ is equicontinuous, and every equicontinuous factor of (X, T) is a factor of $(X/Q, T)$.

If (X, T) is a minimal flow, a number of sufficient conditions for $Q(X)$ to be an equivalence relation are known [**A, AEE**]. An example due to McMahon [**M**] shows that Q is not always an equivalence relation in minimal flows.

In this paper, we give another sufficient condition for regionally proximal to be an equivalence relation in a minimal flow, namely that some proximal cell be closed (Theorem 1). (If $x \in X$ the proximal cell of x is the set $P(x) = [y \in X \mid (x, y) \in P]$.)

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The proof of our theorem depends on the Galois theory of minimal flows, as developed by Ellis, which we briefly review. General references are [**A**] and [**E**].

Let βT denote the Stone–Čech compactification of T. T acts on βT, and the action of T on X extends to an action of βT on X. Moreover, the group structure on T gives rise to a semigroup structure on βT, for which the maps $p \mapsto qp$ are continuous. The minimal right ideals in βT coincide with the minimal sets in the flow $(\beta T, T)$.

Fix a minimal set M in βT (all are isomorphic). The flow (M, T) is the universal minimal flow—every minimal flow (X, T) is a factor. Let G be the group of flow automorphisms of (M, T). The group G is equipped with a T_1 (but not Hausdorff) topology. (Convergence in G will be described below.) If u is an idempotent in M, the group G may be identified with Mu. (Mu acts on M by left translations.)

If (X, T) is a minimal flow, and $\pi : M \to X$ is a homomorphism, then the Ellis group of (X, T) is the subgroup of G defined by $\mathcal{G}(X) = [\alpha \in G \mid \pi\alpha = \pi]$. (A different homomorphism from M to X gives rise to a conjugate subgroup.) The Ellis groups are closed subgroups of G and for every closed subgroup A of G there is a minimal flow with Ellis group A [**A**, Chapter 10].

The universal minimal flow is *regular*—all almost periodic points (x, y) of the product flow $(M \times M, T)$ are on graphs of automorphisms. That is, $y = \varphi(x)$ for some $\varphi \in G$. (Thus a regular minimal flow is characterized by having a maximal supply of automorphisms.)

The proof of Theorem 1 is preceded by several lemmas.

LEMMA 1. *Let (X, T) be a regular minimal flow, let $(x', y') \in P(X)$ and let $x \in X$. Then there is a minimal right ideal I in βT, idempotents v and w in I and an automorphism η of (X, T) such that $x' = \eta(xv)$ and $y' = \eta(xw)$.*

Proof. Let I be a minimal right ideal in βT such that $x'p = y'p$ for $p \in I$. Let $J(I)$ denote the idempotents in I. Let $v, w \in J(I)$ such that $x'v = x'$ and $y'w = y'$. Let $u \in J(I)$ such that $xu = x$. By regularity, there are automorphisms η and ψ of (X, T) such that $\eta(xv) = x'$ and $\psi(xw) = y'$. Now if $p \in I$, $x'p = \eta(xvp) = \eta(xp)$ and similarly $y'p = \psi(xp)$. Since $x'p = y'p$ we have $\eta(xp) = \psi(xp)$ so $\psi = \eta$. □

LEMMA 2. *Let $\pi : X \to Y$ be a homomorphism of minimal flows. Let $(y, y') \in P(Y)$ and let $(x, x') \in X \times X$ with $\pi(x, x') = (y, y')$. Then there is an $x'_0 \in X$ with $(x, x'_0) \in P(X)$ such that $\pi(x'_0) = y'$ and (x', x'_0) almost periodic.*

Proof. Let I be a minimal right ideal in βT such that $yp = y'p$ for all $p \in I$. Let $u, v \in J(I)$ such that $xu = x$ and $x'v = x'$. Note that $yu = y$ and $y'v = y'$, so $yv = y'$. Let $x'_0 = xv$. Clearly $(x, x'_0) \in P(X)$ and (x', x'_0) is almost periodic. Also, $\pi(x'_0) = \pi(xv) = yv = y'$. □

Now we define convergence in G. Let $\{\alpha_j\}$ be a net in G, and let $\alpha \in G$. Then $\alpha_j \to \alpha$ if there is a net $\{m_j\}$ in M and $m \in M$ such that (in M) $m_j \to m$ and $\alpha_j(m_j) \to \alpha(m)$.

Note that convergence in G is defined in terms of convergence in the topology of M. In order to avoid confusion we will use lower case Greek letters for elements of G (except for id, the identity of G).

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Let G' be the intersection of closed neighborhoods of id. Note that $\eta \in G'$ if and only if there is a net $\{\beta_n\}$ in G such that $\beta_n \to \eta$ and $\beta_n \to \mathrm{id}$. G' is a closed normal subgroup of G.

We note that if $\alpha \in G'$, then $\mathrm{gr}(\alpha)$, the graph of α, is contained in $Q(M)$. If $\alpha_j \to \alpha$, id, then $(m_j, \alpha_j(m_j)) \to (m, \alpha(m))$ and $(m'_j, \alpha_j(m'_j)) \to (m', m')$ for some nets $\{m_j\}$ and $\{m'_j\}$, and $m, m' \in M$. Now we may approximate $(m'_j, \alpha_j(m'_j))$ by $(m_j t_j, \alpha_j(m_j)t_j)$ for some net $\{t_j\}$ in T, so $(m, \alpha(m)) \in Q$. This also shows that the net which affects the regional proximality of $(m, \alpha(m))$ may be chosen to consist of almost periodic points. This property will be used in the proof of Theorem 2.

LEMMA 3.

(i) *Suppose* $(m_i, \alpha_i(m_i)) \to (m, n)$ *and* $(m', \zeta(m')) \in \overline{(m, n)T}$, *where* $\alpha_i, \zeta \in G$. *Then* $\alpha_i \to \zeta$.

(ii) *Let* $\alpha, \eta \in G$, *and suppose there are nets* $\{m_i\}$ *in* M *and* $\{\eta_i\}$ *in* G *such that* $(m_i, \eta_i(m_i)) \to (\alpha(m), \eta(n))$ *where* $(m, n) \in P(M)$. *Then* $\eta_i\alpha \to \eta$.

Proof. (i) If $s \in T$, $(m_i, \alpha_i(m_i))s \to (ms, ns)$, so there is a net $\{s_i\}$ in T such that $(m_i, \alpha_i(m_i))s_i \to (m', \zeta(m'))$.

(ii) It follows from (i) that if $(m_i, \sigma_i(m_i)) \to (m, \theta(n))$ where $(m, n) \in P$, then $\sigma_i \to \theta$. Now $(m_i, \eta_i(m_i)) \to (\alpha(m), \eta(n))$ so $(\alpha^{-1}(m_i), \eta_i\alpha(\alpha^{-1}(m_i))) \to (m, \eta(n))$. Therefore $\eta_i\alpha \to \eta$. □

LEMMA 4. *Let* A *be a closed subgroup of* G, *and let* $\eta \in G$. *Let* $\{\eta_i\}$ *be a net in* G *such that, for some* α, $a \in A$, $\eta_i\alpha \to \eta$ *and* $\eta_i a \to \mathrm{id}$. *Then* $\eta \in AG'$.

Proof. By continuity of right multiplication, $\eta_i a \to \eta\alpha^{-1}a$. Also $\eta_i a \to \mathrm{id}$, so $\eta\alpha^{-1}a \in G'$ and $\eta \in G'A = AG'$, by normality of G'. □

Let D and E be the groups, respectively, of the universal distal and universal equicontinuous minimal flows. Since an equicontinuous flow is distal, $D \subset E$, and since $\mathrm{gr}(\alpha) \subset Q(M)$ for $\alpha \in G'$ we have $G' \subset E$. In [**AEE**, Lemma 1.14] it is proved that $E = DG'$. A generalization will be proved below (Theorem 2).

The main result in [**AEE**] is a sufficient condition for regional proximality to be an equivalence relation in the minimal flow (X, T), namely that $E \subset \mathcal{G}(X)G'$. This is the condition which we will verify in Theorem 1.

LEMMA 5. *Let* A *be a closed subgroup of* G. *Let* $D_1 = [\beta \in G | \mathrm{gr}(\beta) \subset \overline{P(M)}]$. *Then the condition* $E \subset AG'$ *is equivalent to* $D_1 \subset AG'$.

Proof. It is easy to see that D is the closed subgroup of G generated by D_1. (For a proof, see [**AG**, Corollary 4.3].) Suppose $D_1 \subset AG'$. Since AG' is a group, $D \subset AG'$, and since $E = DG'$, $E \subset AG'G' = AG'$. The converse follows from the inclusions $D_1 \subset D \subset E$. □

THEOREM 1. *Let* (X, T) *be minimal, and suppose that for some* $x \in X$ *the proximal cell* $P(x)$ *is closed. Then* $Q(X)$ *is an equivalence relation.*

Proof. Let $A = \mathcal{G}(X)$. We show that $E \subset AG'$. By Lemma 5 it is sufficient to show that $D_1 \subset AG'$.

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Let $\eta \in D_1$. Let $\pi : M \to X$ be a homomorphism, and let $m \in M$ such that $\pi(m) = x$. Then $(m, \eta(m)) \in \overline{P(M)}$. Let $(m_i, n_i) \in P(M)$ such that $(m_i, n_i) \to (m, \eta(m))$.

By Lemma 1 there are minimal idempotents v_i and w_i and $\eta_i \in G$ such that $m_i = \eta_i(mv_i)$ and $n_i = \eta_i(mw_i)$. Let (subnets of) $mv_i \to m'$ and $mw_i \to n'$. Then $m' \in \overline{P(m)}$ and $n' \in \overline{P(m)}$. Let $x' = \pi(m')$ and $y' = \pi(n')$, so $x' \in \overline{P(x)} = P(x)$ and, similarly, $y' \in P(x)$. Now $\pi(m, m') = (x, x') \in P(X)$, so by Lemma 2 there is an $m'_0 \in M$ with $(m, m'_0) \in P(M)$, (m', m'_0) almost periodic, and $\pi(m, m'_0) = (x, x')$. Then $m' = \alpha(m'_0)$ for some $\alpha \in A$. Now $(mv_i, \eta_i(mv_i)) = (mv_i, m_i) \to (m', m) = (\alpha(m'_0), m)$, so by Lemma 3 $\eta_i\alpha \to \mathrm{id}$, and a similar argument (using $\eta_i(mw_i) = n_i \to \eta(m)$) shows that for some $a \in A$, $\eta_i a \to \eta$. It now follows from Lemma 4 that $\eta \in AG'$. □

The converse of Theorem 1 is not valid. For example, in any weakly mixing minimal flow (X, T) with T abelian, $Q = X \times X$ but every proximal cell is dense in X.

A natural question is whether Q is an equivalence relation when P is an equivalence relation.

Now, as promised, we show that $E = DG'$. In fact, this will follow from a more general result, namely a dynamical condition which is equivalent to our group theoretic condition $E \subset \mathcal{G}(X)G'$. The minimal flow (X, T) is said to satisfy the *local Bronstein condition* or to be *locally Bronstein* if, whenever (x, y) is an almost periodic point in Q, there is a net $\{(x_n, y_n)\}$ in $X \times X$ with (x_n, y_n) almost periodic, and a net $\{t_n\}$ in T, such that $(x_n, y_n) \to (x, y)$ and $(x_n t_n, y_n t_n) \to \Delta_X$. If the flow satisfies the 'Bronstein condition' (the almost periodic points are dense in $X \times X$) it is certainly locally Bronstein. Since a flow (X, T) is distal if and only if $(X \times X, T)$ is pointwise almost periodic, a distal flow is locally Bronstein.

THEOREM 2. *Let (X, T) be a minimal flow. Then (X, T) is locally Bronstein if and only if $E \subset \mathcal{G}(X)G'$.*

We first require a lemma. Let $H = [\alpha \in G \mid \mathrm{gr}(\alpha) \subset Q]$. H is a closed subset of G but is not in general a subgroup.

LEMMA 6. *E is the closed subgroup of G generated by H.*

Proof. Denote this subgroup by F. Clearly $F \subset E$. Now let (X, T) be a minimal flow with group F, and let $\pi : M \to X$. In order to show that $F = E$ it is sufficient to show that (X, T) is a proximal extension of an equicontinuous flow. For this it is sufficient in turn to show that there are no non-trivial almost periodic points in $Q(X)$ (since in that case $Q(X) = P(X)$ and therefore $P(X)$ is closed and, hence, an equivalence relation [**A**]). Suppose $(x, x') \in Q$ with (x, x') almost periodic and $x \neq x'$. Let (m, m') be an almost periodic point of $Q(M)$ with $\pi(m, m') = (x, x')$. Then $m' = \alpha(m)$ with $\alpha \in H \subset F$, so $x = x'$. □

Proof of Theorem 2. Let $\pi : M \to X$ be a homomorphism and let $A = \mathcal{G}(X)$. Suppose $E \subset AG'$. Let (x, y) be an almost periodic point of $Q(X)$ and let (m, n) be an almost periodic point in $Q(M)$ with $\pi(m, n) = (x, y)$. Then $n = \theta(m)$ where $\theta \in H \subset E \subset AG'$. Therefore $\pi(n) = \pi(\varphi(m))$ where $\varphi \in G'$. Moreover, as we observed in the discussion preceding Lemma 3, $(m, \varphi(m)) \in Q$ and there is a net of

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almost periodic points approaching $(m, \varphi(m))$ for which some translates by elements of T approach the diagonal. Clearly $(x, y) = \pi(m, \varphi(m))$ has the same property. Therefore, (X, T) is locally Bronstein.

Conversely, suppose (X, T) is locally Bronstein. By Lemma 6, it is sufficient to show that $H \subset AG'$. Let $\alpha \in H$ and let $m_0 \in M$. Let u be a minimal idempotent such that $m_0 u = m_0$. Then $(m_0, \alpha(m_0)) \in Q(M)$ so $(x, x') = \pi(m_0, \alpha(m_0)) \in Q(X)$. Since (X, T) is locally Bronstein there are nets $\{(x_i, x_i')\}$ in $X \times X$ with (x_i, x_i') almost periodic and $\{t_i\}$ in T such that $(x_i, x_i') \to (x, x')$ and $(x_i, x_i')t_i \to (x, x)$. Let (m_i, m_i') be almost periodic points of $M \times M$ such that $\pi(m_i, m_i') = (x_i, x_i')$. Let (subnets of) $(m_i, m_i') \to (m, m')$, $(m_i, m_i')t_i \to (n, n')$. Then $\pi(m, m') = (x, x')$ and $\pi(n, n') = (x, x)$.

We may suppose $m, m', n, n' \in Mu$ (see below). Then there are $\theta_i, \theta, \psi, \beta \in G$ such that $m_i' = \theta_i(m_i)$, $m' = \theta(m)$, $n' = \psi(n)$ and $\beta(m) = m_0$. Note that $\beta, \psi \in A$. Also, since $\pi(\theta(m)) = \pi(m') = x' = \pi(\alpha\beta(m))$, it follows that $\alpha\beta\theta^{-1} \in A$.

We have $(m_i, \theta_i(m_i)) \to (m, \theta(m))$ and $(m_i t_i, \theta_i(m_i t_i)) \to (n, \psi(n))$. Therefore $\theta_i \to \theta$, ψ and $\theta\psi^{-1} \in G'$. Then $\alpha = \alpha\beta\theta^{-1}\theta\beta^{-1} \in AG'\psi\beta^{-1} = A\psi\beta^{-1}G' = AG'$.

Finally, we show that we may assume $m, m', n, n' \in Mu$. Let W be a neighborhood of u in $\boldsymbol{\beta}T$ and let $s \in T \cap W$. Then $(m_i s, m_i' s) \to (ms, m's)$ which is near $(mu, m'u)$ and $(m_i s s^{-1} t_i s, m_i' s s^{-1} t_i s) \to (ns, n's)$ which is near $(nu, n'u)$. □

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