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An approximate solution for two-dimensional transonic flow past thin airfoils

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Abstract. The transonic, two-dimensional flow past a thin airfoil is treated by approximating the transonic small perturbation equation by a linear equation valid in the neighbourhood of the airfoil. The solution is derived by representing the body by a line of suitable sources. Pressure distributions are given for airfoils with maximum thickness both forward and aft, and agreement with experiment improves as the position of maximum thickness moves aft.

1. *Introduction.* The small disturbance equations for transonic flow are well established, but owing to their non-linearity, the derivation of exact solutions is a formidable task. A few such solutions have been obtained using the hodograph plane in which the equations become linear, but as the body profile cannot be specified *a priori*, this approach offers little scope for general airfoil shapes. Confronted with this situation, it is natural to ask whether it is possible to approximate the non-linear term, so that the equations become linear, and yet still retain sufficient accuracy in the prediction of surface pressure on typical airfoil shapes to give a useful engineering approximation.

To investigate this possibility, consider a uniform stream, flowing in the positive z direction of a rectangular Cartesian system (x, y, z), past either a slender body lying along the z axis, or a thin airfoil which is everywhere close to the Oxz plane. For steady, isentropic, compressible flow we may introduce a potential

$$U_\infty(z+\phi(x,y,z)),$$

and as U_∞ approaches the speed of sound, the small disturbance equation reduces to the transonic equation

$$\phi_{rr}+(1/r)\,\phi_r \equiv \phi_{xx}+\phi_{yy} = \{(\gamma+1)\,M_\infty^2\,\phi_z-(1-M_\infty^2)\}\,\phi_{zz}. \tag{1·1}$$

For the derivation of this equation, see, for example, ((4)). One has the choice of approximating either ϕ_{zz} or $\{(\gamma+1)\,M_\infty^2\,\phi_z-(1-M_\infty^2)\}$.

The first step was taken by Oswatitsch and Keune ((2)), who considered sonic ($M_\infty = 1$) flow past a slender, parabolic body of revolution, and noted that in practice, the acceleration was nearly constant for fluid near the surface of the forward half of the body. Accordingly the non-linear term $(\gamma+1)\,M_\infty^2\,\phi_z\phi_{zz}$ was replaced by $K\phi_z$, giving the equation

$$\phi_{xx}+\phi_{yy} = K\phi_z, \tag{1·2}$$

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which may be readily solved. The calculated pressure distribution was in promising agreement with experimental results. However, it should be noted that even in this favourable case, the approximate equation is valid only in the immediate neighbourhood of the body. It is this relative insensitivity to poor approximations away from the body which provides the basis for further development. Maeder and Thommen ((7)), employing the same approximation for $(\gamma+1)M_\infty^2\phi_z\phi_{zz}$, retained the $(1-M_\infty^2)\phi_{zz}$ term, thus extending the theory to cover the whole Mach number range. The method was applied to both two-dimensional thin airfoils and slender bodies of revolution, but for bodies with maximum thickness forward of centre, computed values differed considerably from experimental ones. Miles ((10)) made the following criticisms: (1) that at $M_\infty = 1$, a mixed type elliptic-hyperbolic equation had been replaced by a parabolic one; (2) that for bodies of revolution at large distances the solution differed from the asymptotic solution provided by Guderley and Yoshihara ((5)); (3) the method failed to account for a region at some distance from the body in which non-linear effects were of decisive importance. However, Cole pointed out that no finite drag could be associated with the asymptotic field, and suggested that there is no simple relation between the field near the body, and the far field, but that conditions near the body are of primary importance in determining the pressure. It would appear that the region of criticism 3 accounts for criticism 2. At least at the present stage, we must be content to assume this.

In ((3)), Cole and Royce initiated the second type of approximation to the non-linear term, by replacing $(\gamma+1)M_\infty^2\phi_z\phi_{zz}$ by $a^2z\phi_{zz}$. The case considered was that of sonic flow ($M_\infty = 1$) past a slender body of revolution, for which the equation is

$$\phi_{xx}+\phi_{yy} = a^2z\phi_{zz}. \tag{1·3}$$

This approach meets Miles's first criticism, in that it preserves the elliptic-hyperbolic type of the equation. The analysis is not so simple as for the first type of approximation, but gives quite good agreement with experiment for a variety of smooth slender bodies. However, the simpler analytic results of (7) have attracted all further development.

Maeder and Thommen in ((8)) improved their earlier work by showing how the choice of K might best be made assuming the prediction of the sonic position was sufficiently accurate, and they also calculated a correction term. The sonic point z^* was given by $\{(1-M_\infty^2)-(\gamma+1)\phi_{1z}^*\}_{y=0} = 0$, while K was chosen as $\{(\gamma+1)\phi_{1zz}^*\}_{y=0}$. The correction term was simultaneously derived in a different manner by Hosokawa ((6)). He assumed $\phi \simeq \phi_1+g$, where ϕ_1 is the solution given in ((7)). Having obtained the appropriate changing type equation for g from (1·1), he argued that the behaviour of g is largely one-dimensional in the neighbourhood of the body, and that g may be taken as a function of z only. With this assumption, the equation is readily integrated, and a quadratic equation for the velocity g_z results. The constant of integration is chosen so that $g_z = 0$ at z^*, and the choice of sign is made so that $\phi_{1z}+g_z \gtrless 0$ according as the flow is supersonic or subsonic. For the case of a two-dimensional parabolic arc thin airfoil agreement with experiment is excellent, but as the position of maximum thickness moves forward it deteriorates rapidly. In the axisymmetric case at $M_\infty = 1$,

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even for the parabolic arc the results are little better than ((7)). Maeder and Thommen's analysis ((8)) showed that the errors largely stemmed from the failure of ϕ_1 to represent the solution sufficiently accurately, especially near the sonic point. Maeder also considers this point in his article in ((11)).

An alternative improvement on (7) which exploits the insensitivity to poor approximations away from the body has been developed by Spreiter and Alksne. For $M_\infty \doteq 1$, the choice of K in the expression for the velocity at the airfoil surface is made as $K = (\gamma+1) M_\infty^2 \phi_{zz}$ at each point of the airfoil, so giving an ordinary differential equation for $\phi_z|_{y=0}$. For M_∞ away from one, a similar 'local linearization' technique is applied to the classical subsonic and supersonic theories. This method has been applied to two-dimensional thin airfoils ((13)), slender bodies of revolution ((14)) and wings of finite span ((15)). It gives good agreement with experiment over a wide range of shapes, but there are a number of objections. The criticism concerning the use of (1·2) as an approximation to (1·1) has already been included. In addition, the method has the disadvantages that it requires a different treatment for different ranges of Mach number, so that the simple covering of the Mach range provided by ((7)) is lost, and also that for three-dimensional problems it requires considerable numerical work. The principal objection to the method from a mathematical standpoint, however, is that it is basically inconsistent. K is taken as a constant until after the formal solution to the differential equation (1·2) is obtained and then is allowed to vary with z. If one assumes at the outset that K is a function of z to be found, a formal solution may still be derived. For $M_\infty = 1$, the respective formal solutions are:

$$K \text{ constant:} \quad \phi_z|_{y=0} = \frac{U_\infty}{\sqrt{(\pi K)}} \left\{ -\frac{F(0)}{\sqrt{z}} - \int_0^z \frac{F'(\xi)}{\sqrt{(z-\xi)}} d\xi \right\}; \tag{1·4}$$

$$K = K(z): \quad \phi_z|_{y=0} = \frac{U_\infty}{\sqrt{\pi}} \frac{1}{K(z)} \left\{ -\frac{F(0)}{\sqrt{\eta}} - \int_0^z \frac{F'(\xi).d\xi}{\sqrt{[\eta(z)-\eta(\xi)]}} \right\}, \tag{1·5}$$

where

$$\eta(\xi) = \int_0^\xi \frac{dt}{K(t)}.$$

$F(z)$ is the local slope of the airfoil, and $z = 0$ at the leading edge. It is evident that although (1·5) reduces to (1·4) when $K(z)$ is a constant, if we regard them as equations for $u(z)$, where $u(z) = \phi_z|_{y=0}$, and $K(z) = (\gamma+1)\,\partial u/\partial z$, then they are quite different. An investigation into the consequences of this inconsistency is in progress but at present we cannot regard agreement with experiment as a justification for the Spreiter theory. It may have arisen because of the cancellation of the physical error by the mathematical error, and it is desirable to find an alternative model for which a mathematically consistent theory may be developed. That proposed by Cole and Royce in ((3)) has the attractive feature that it preserves the structure of the equation, and for the axisymmetric case, the results of the mathematically consistent theory are as encouraging as those of the Spreiter and Alksne work, particularly when the maximum disturbance occurs towards the rear of the body. In this paper, we extend the Cole and Royce method to include two-dimensional shapes in order to test the accuracy of the method for the extreme case of airfoils with finite aspect ratio.

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The approach in this work is similar to that of Cole and Royce. Their model is one of accelerated flow, so that the flow is subsonic over the forward part of the airfoil, and supersonic behind, and the technique is to represent the body by sources in a manner analogous to subsonic theory. Extension to two dimensions is, in principle, simple. The point sources are integrated to give line sources, and the same technique is applied. However, as the subsonic and supersonic sources have different forms, an additional term is necessary to give a continuous potential, and establishing that the two expressions for the potential join smoothly constitutes the main difficulty of the work. The general results are applied to the particular case of airfoils given by polynomials, and for powers up to the sixth, the tables provided make the calculation of pressure distributions on such airfoils quite easy.

2. *The approximate equation and the source solutions.* The Cole and Royce approximate transonic equation for axisymmetric flow at $M_\infty = 1$ is

$$\phi_{rr} + (1/r)\,\phi_r = a^2 z \phi_{zz}, \tag{2·1}$$

where $z = 0$ at the sonic point, and by use of Hankel transforms, the following point source solutions have been derived:

$$\text{for} \quad \zeta > 0, \quad \Phi(H) = \left\{ \begin{array}{ll} -\dfrac{z+\zeta-\frac{1}{4}a^2r^2}{2\zeta\{[z-\zeta-\frac{1}{4}a^2r^2]^2 - a^2r^2\zeta\}^{\frac{1}{2}}} & (\sqrt{z} > \sqrt{\zeta}+\tfrac{1}{2}ar), \\ 0 & (\sqrt{z} < \sqrt{\zeta}+\tfrac{1}{2}ar), \end{array} \right\} \tag{2·2}$$

$$\text{for} \quad \zeta < 0, \quad \Phi(E) = -\frac{1}{4\zeta} - \frac{z+\zeta-\frac{1}{4}a^2r^2}{4\zeta\{[z-\zeta-\frac{1}{4}a^2r^2]^2 - a^2r^2\zeta\}^{\frac{1}{2}}}. \tag{2·3}$$

The two-dimensional form of equation (2·1),

$$\phi_{yy} - a^2 z \phi_{zz} = 0, \tag{2·4}$$

obtained by regarding ϕ as independent of x, corresponds to choosing ϕ_z as

$$a^2 z/[M_\infty^2(\gamma+1)]$$

in the non-linear term $(\gamma+1)\,M_\infty^2\,\phi_z\phi_{zz}$ of the two-dimensional transonic equation, where $z = 0$ is the sonic point. Since $\Phi(E)$, $\Phi(H) \to 0$ as $x \to \pm\infty$ then

$$\phi(E) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(E)\,dx$$

is a solution of (2·4) for $\zeta < 0$, and

$$\phi(H) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(H)\,.\,dx$$

a solution for $\zeta > 0$. The factors $1/2\pi$ are introduced so that the potentials represent unit sources. Furthermore, it is convenient to define $\phi(E) = 0$ for $\zeta \geqslant 0$, and $\phi(H) = 0$ for $\zeta \leqslant 0$. Now $\phi(E)$ and $\phi(H)$ may be expressed in terms of elliptic integrals. For $\zeta < 0$, the integral for $\phi(E)$ takes two forms according as $z \gtrless 0$.

For $z = -s < 0$,

$$\phi(E) = \frac{-1}{4\pi a\xi}\{(\sqrt{\xi}+\sqrt{s})^2 + \theta^2\}^{\frac{1}{2}}\{2(K-E) - k^2K\}, \tag{2·5}$$

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where $K(k)$, $E(k)$ are complete elliptic integrals, with

$$k^2 = \frac{4\sqrt{(\xi s)}}{(\sqrt{\xi}+\sqrt{s})^2+\theta^2}, \quad \theta = \tfrac{1}{2}ay \quad \text{and} \quad \xi = -\zeta.$$

For $z > 0$,

$$\phi(E) = \frac{4z}{a\pi A^2(2q-A)}\{2E_* - 2\lambda_1 F_* - (1-\lambda_1)\}, \tag{2·6}$$

where E_*, F_* are the incomplete elliptic integrals

$$E_* = E\left(k, \frac{1}{\sqrt{(1+\lambda_1)}}\right) = \int_0^{1/\sqrt{(1+\lambda_1)}} \left\{\frac{1-k^2u^2}{1-u^2}\right\}^{\frac{1}{2}} du,$$

$$F_* = \int_0^{1/\sqrt{(1+\lambda_1)}} \{(1-u^2)(1-k^2u^2)\}^{-\frac{1}{2}} du$$

and
$$k^2 = 1-\lambda_1^2, \quad q = (\Theta^2+\gamma^2)^{\frac{1}{4}}, \quad A = \sqrt{2}\sqrt{(q^2-\Theta)}, \quad \lambda_1 = \frac{2q-A}{2q+A},$$
$$\lambda_1\lambda_2 = 1, \quad \gamma^2 = 4\xi z, \quad \Theta = \xi+\theta^2-z.$$

The integral for $\phi(H)$ takes the simpler forms

$$\phi(H) = \begin{cases} \dfrac{1}{a\pi\zeta}\left[\dfrac{2\sqrt{(\zeta z)}}{\{(\sqrt{z}+\sqrt{\zeta})^2-\theta^2\}^{\frac{1}{2}}}K - \{(\sqrt{z}+\sqrt{\zeta})^2-\theta^2\}^{\frac{1}{2}}E\right] & (\sqrt{z} > \tfrac{1}{2}ay+\sqrt{\zeta}), \\ 0 & (\sqrt{z} < \tfrac{1}{2}ay+\sqrt{\zeta}), \end{cases} \tag{2·7}$$

where K, E are complete elliptic integrals with

$$k^2 = \frac{(\sqrt{z}-\sqrt{\zeta})^2-\theta^2}{(\sqrt{z}+\sqrt{\zeta})^2-\theta^2}, \quad \theta = \tfrac{1}{2}ay.$$

In the Cole and Royce work, it was necessary to introduce a solution singular on the surface $z = \frac{1}{4}a^2r^2$. The significance of this is discussed later. Here, we simply note that the line $z = \frac{1}{4}a^2y^2$ corresponds to the surface $z = \frac{1}{4}a^2r^2$, and investigate the possibility of solutions singular on this line. Putting $h = \frac{1}{4}a^2y^2 - z$, and seeking solutions which are functions of h only, $\sqrt{|h|}$ is found as the solution. Though not singular itself, the derivatives are. If we further seek solutions of the form $(z\zeta)^{\frac{1}{4}}f(\eta)$, where

$$\eta = (z+\zeta-\theta^2)/(z\zeta)^{\frac{1}{2}},$$

of which $\sqrt{|h|}$ is a particular case, then two independent solutions ensue. For $z < 0$, $\zeta < 0$ the first takes the form (2·5), while for $z > 0$, $\zeta > 0$ the second takes the form (2·7).

It will be noted that the first gives a solution for $z > 0$, $\zeta > 0$, and likewise the second for $\zeta < 0$, $z < 0$, but these are of no help in satisfying boundary conditions.

3. *A potential derived by superposition of the two-dimensional sources.* Consider

$$\phi_2 = \begin{cases} \dfrac{2\Omega(0)}{a}\sqrt{(\tfrac{1}{4}a^2y^2-z)} + \displaystyle\int_{-b}^{0} 2\Omega(\zeta)\,\phi(E)\,d\zeta & (\tfrac{1}{4}a^2y^2 > z), \qquad (3·1a) \\ \displaystyle\int_{-b}^{(\sqrt{z}-\frac{1}{2}ay)^2} 2\Omega(\zeta)\{\phi(E)+\phi(H)\}\,d\zeta & (\tfrac{1}{4}a^2y^2 < z), \qquad (3·1b) \end{cases}$$

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where in (3·1b), the Cauchy Principal value is taken at $\zeta = 0$. This potential may be regarded as due to a line of sources along $y = 0$, and by analogy with subsonic theory, we expect $\Omega(\zeta)$ to be the local slope of the airfoil. The additional term

$$[2\Omega(0)]/a\sqrt{(\tfrac{1}{4}a^2y^2 - z)}$$

is introduced, anticipating difficulty in establishing continuity across $z = \frac{1}{4}a^2y^2$. It stems from the effective doubling in strength of sources in passing from the subsonic to supersonic régime. Any discontinuity at $y = 0$, $z = 0$ propagates along the characteristic through the point, i.e. along $z = \frac{1}{4}a^2y^2$.

The proof of the existence of the integrals of (3·1) is straightforward when $z \neq \frac{1}{4}a^2y^2$. For (3·1a) consider the cases $z \geqslant 0$ separately; for $z < 0$, using (2·5), the integrand has an integrable singularity at $\zeta = 0$. For (3·1b) using the expressions (2·6) and (2·7), it may be shown that the integrand behaves like $1/\zeta$ at the origin, so that the principal value is required. As $h = \frac{1}{4}a^2y^2 - z \to 0$, these considerations are no longer applicable. The examination of the behaviour of ϕ and its derivative is quite involved and is not given here.

The conclusion is that ϕ_2, $\partial\phi_2/\partial y$ and $\partial\phi_2/\partial z$ are continuous across $z = \frac{1}{4}a^2y^2$ as required by the physical situation. Thus, the necessity for the addition of the term

$$\begin{cases} \dfrac{2\Omega(0)}{a}\sqrt{(\tfrac{1}{4}a^2y^2 - z)} & (\tfrac{1}{4}a^2y^2 > z), \\ 0 & (\tfrac{1}{4}a^2y^2 < z), \end{cases}$$

to the potential due to the sources alone is established. This is a natural extension of the Cole and Royce work, as it is the x integral of the $\log|z - \frac{1}{4}a^2r^2|$ term which they had to add.

We now examine the velocities, for $z \neq \frac{1}{4}a^2y^2$ beginning with $\partial\phi_2/\partial y$. It is convenient to consider separately the ranges $z < 0$, $\frac{1}{4}a^2y^2 > z \geqslant 0$, and $z > \frac{1}{4}a^2y^2$.

(i) *The range* $z < 0$. Interchanging the order of integration and differentiation, and using expressions for the derivatives of elliptic integrals (see (16) or (1)), it may be shown that

$$\frac{\partial\phi_2}{\partial y} = \Omega(0)\frac{\theta}{\sqrt{(\theta^2+s)}} - \frac{1}{4\pi}\int_0^b \frac{\theta}{\{\theta^2+(\sqrt{\xi}+\sqrt{s})^2\}^{\frac{1}{2}}}\left\{2K - \left(\frac{2-k^2}{1-k^2}\right)E\right\}\frac{\Omega(-\xi)}{\xi}\,d\xi, \qquad (3\cdot2)$$

where

$$k^2 = \frac{4\sqrt{(\xi s)}}{\{\theta^2+(\sqrt{\xi}+\sqrt{s})^2\}}.$$

When $s \neq 0$, as $\xi \to 0$, $k^2 = O(\sqrt{\xi})$ and $2(K-E) - [k^2/(1-k^2)]E = O(\xi)$. Thus, for $s \neq 0$, $y \neq 0$ the integrand in (3·2) is bounded at $\xi = 0$; furthermore, it is continuous for $0 \leqslant \xi \leqslant b$, so that the integral is defined. For $y = 0$, the integrand has a singularity at $\xi = s$, where $k = 1$. Consider the behaviour of (3·2) as $y \to 0$, Since $s \neq 0$, the first term vanishes. In the integrand K has a singularity like $\log y$, but it is multiplied by y, and so makes no contribution: $E \to 1$. Thus

$$\frac{\partial\phi_2}{\partial y} \sim \frac{\Omega(-s)}{2\pi}\int_0^b \frac{ay}{2}\frac{1}{\sqrt{s}}\frac{4s}{4s(\frac{1}{2}ay)^2+(\xi-s)^2}\,d\xi = \frac{\Omega(-s)}{\pi}\left[\tan^{-1}\frac{\xi-s}{ay\sqrt{s}}\right]_0^b,$$

$$\frac{\partial\phi_2}{\partial y} \to \Omega(-s) \quad \text{as} \quad y \to 0.$$

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When $s = 0$, the integral in (3·2) makes no contribution, as $y \to 0$, but the first term gives $\Omega(0)$. Thus for $z \leqslant 0$,

$$\frac{\partial \phi_2}{\partial y} \to \Omega(z) \quad \text{as} \quad y \to 0. \tag{3·3}$$

(ii) *The range* $\frac{1}{4}a^2y^2 > z \geqslant 0$. An expression for $\partial\phi_2/\partial y$ has been derived, by returning to the definition of $\phi(E)$, for use in (3·1), and interchanging orders of integration and differentiation. Like the potential in this range, it may be reduced to a single integral, with incomplete elliptic integrals in the integrand, but it is unwieldy. However, it is fairly easy to show that the only possibility of a singularity in the integrand occurs at $\xi = 0$, and if it occurs, it is certainly integrable.

(iii) *The range* $z - \frac{1}{4}a^2y^2 = h' > 0$. Writing $\phi_2 = A_1(\epsilon) + A_2(\epsilon)$, where

$$A_1(\epsilon) = \int_\epsilon^b 2\Omega(-\xi)\,\phi(E)\,d\xi \quad \text{and} \quad A_2(\epsilon) = \int_\epsilon^{(\sqrt{z}-\theta)^2} 2\Omega(\zeta)\,\phi(H)\,d\zeta,$$

we require

$$\frac{\partial \phi_2}{\partial y} = \lim_{\epsilon \to 0}\left(\frac{\partial A_1(\epsilon)}{\partial y} + \frac{\partial A_2(\epsilon)}{\partial y}\right).$$

Now, by the approach indicated in case (ii), it may be shown that

$$\frac{\partial A_1}{\partial y} = -\frac{1}{\pi}\int_\epsilon^b \Omega(-\xi)\,\frac{2z\theta}{4q^4(1-\lambda_1)\,A}\,[(3-2\lambda_1+3\lambda_1^2)\,2F_* - (6-\lambda_1-\lambda_2)\,(2E_*-1+\lambda_1)]\,d\xi.$$

The only singularity of the integrand is at $\xi = 0$. For small ξ

$$q^2 \sim h', \quad A \sim 2\sqrt{h'}, \quad 2q - A \sim 2\sqrt{h'}\left(\frac{z}{2h'^2}\right)\xi, \quad \lambda_1 \sim \frac{z}{4h'^2}\xi.$$

Thus the integrand $\sim \dfrac{\Omega(0)}{\pi\sqrt{h'}}\theta\dfrac{1}{\xi}$ near $\xi = 0$, and $\dfrac{\partial A_1}{\partial y}$ has the form

$$\frac{\partial A_1}{\partial y} = y\,G_1(y) + \frac{\Omega(0)}{\pi}\frac{ay}{2}\frac{1}{\sqrt{h'}}\log\epsilon + O(\epsilon), \tag{3·4}$$

where $G_1(y)$ is a bounded function of y, independent of ϵ.

Also, using the results for derivatives of elliptic integrals, as in case (i),

$$\frac{\partial A_2}{\partial y} = \frac{\Omega([\sqrt{z}-\theta]^2)}{[1-\theta/\sqrt{z}]^{\frac{1}{2}}} + \frac{1}{2\pi}\int_\epsilon^{(\sqrt{z}-\theta)^2} \frac{\Omega(\zeta)}{\zeta}\,\frac{\theta}{\{(\sqrt{z}+\sqrt{\zeta})^2-\theta^2\}^{\frac{1}{2}}}\left\{\left(\frac{1+k^2}{k^2}\right)E - \left(\frac{1-k^2}{k^2}\right)K\right\}d\zeta,$$

where

$$k^2 = \frac{(\sqrt{z}-\sqrt{\zeta})^2-\theta^2}{(\sqrt{z}+\sqrt{\zeta})^2-\theta^2}.$$

Near $\zeta = 0$, $k^2 \sim 1$, the integrand $\sim \dfrac{1}{2\pi}\dfrac{\Omega(0)}{\zeta}\dfrac{\theta}{\sqrt{(z-\theta^2)}}2$, and thus $\partial A_2/\partial y$ has the form

$$\frac{\partial A_2}{\partial y} = \frac{\Omega([\sqrt{z}-\theta]^2)}{[1-\theta/\sqrt{z}]^{\frac{1}{2}}} + yG_2(y) - \frac{\Omega(0)}{\pi}\frac{ay}{2}\frac{1}{\sqrt{h'}}\log\epsilon + O(\epsilon), \tag{3·5}$$

where $G_2(y)$ is a bounded function of y independent of ϵ.

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Adding (3·4) and (3·5) and taking the limit $\epsilon \to 0$, we obtain a well defined expression for $\partial\phi_2/\partial y$. Further, as $y \to 0$ it follows

$$\frac{\partial \phi_2}{\partial y} \to \Omega(z). \tag{3·6}$$

Equations (3·3) and (3·6) indicate that $\Omega(z)$ must be taken as the local slope of the airfoil, in order to satisfy the boundary condition of tangential flow. The usual approximation of applying the condition as $y \to 0$ rather than at the airfoil surface is made, the error being at most of second order.

However, it should be noticed that although we have established the existence of a continuous velocity field which satisfies the boundary condition on the body, we have failed to satisfy the condition of vanishing disturbances at large distances. For consider $\partial\phi_2/\partial y$ in the region of $\frac{1}{4}a^2y^2 > z$. The additional term in the potential gives a velocity $\Omega(0)\theta/\sqrt{(\theta^2 - z)}$ which is non-zero as $y \to 0$, for z finite, while the integral gives a velocity which makes no contribution as $y \to 0$. This inability to satisfy boundary conditions need not be surprising, if we recall that the equation (2·4) is a valid approximation only in the neighbourhood of the body.

The examination of $\partial\phi_2/\partial z$ follows a similar pattern to that for $\partial\phi_2/\partial y$. However, some of the expressions are lengthy, so it is proposed to give only the values as $y \to 0$, these being the expressions of practical interest.

For $z < 0$,

$$\left.\frac{\partial \phi_2}{\partial z}\right|_{y=0} = -\frac{\Omega(0)}{a\sqrt{s}} + \frac{1}{2\pi a}\int_0^b \frac{\Omega(-\xi)}{\xi}\left\{\frac{E}{(\sqrt{\xi}-\sqrt{s})} + \frac{K}{(\sqrt{\xi}+\sqrt{s})}\right\} d\xi, \tag{3·7}$$

where

$$k^2 = \frac{4\sqrt{(\xi s)}}{(\sqrt{\xi}+\sqrt{s})^2}$$

and the Cauchy principal value is taken at $\xi = s$.

The range $0 < z < \frac{1}{4}a^2y^2$ is not required as $y \to 0$. For $z > 0$, we consider the range $\frac{1}{4}a^2y^2 < z$, as $y \to 0$. $\partial\phi_2/\partial z$ has the form

$$\lim_{\epsilon \to 0}\left\{\frac{\partial A_1(\epsilon)}{\partial z} + \frac{\partial A_2(\epsilon)}{\partial z}\right\},$$

as there are $1/\zeta$ singularities in each of the parts, if we put $\epsilon = 0$. The resulting expression for the velocity is

$$\begin{aligned}\left.\frac{\partial \phi_2}{\partial z}\right|_{y=0} (z > 0) = \lim_{\epsilon \to 0}\Bigg\{&\frac{2}{a\pi}\int_\epsilon^b \frac{\Omega(-\xi)}{8z(\sqrt{(z+\xi)}-\sqrt{z})}\Big[2E_* - 2\lambda_1 F_* - (1-\lambda_1) \\ &+ \frac{\xi}{4(\xi+z)}\{(3-2\lambda_1+3\lambda_1^2)\,2F_* - (6-\lambda_1-\lambda_2)\,(2E_* - 1 + \lambda_1)\}\Big] d\xi \\ &+ \frac{2}{a\pi}\int_\epsilon^z \frac{\Omega(\zeta)}{\zeta}\frac{1}{2(z-\zeta)}\{2\sqrt{\zeta}K - (\sqrt{\zeta}+\sqrt{z})\,E\}\,d\zeta - \frac{\Omega(z)}{a\sqrt{z}}\Bigg\},\end{aligned} \tag{3·8}$$

where it is understood that λ_1 is evaluated at $y = 0$, and in the second integral

$$k^2 = \left(\frac{\sqrt{z}-\sqrt{\zeta}}{\sqrt{z}+\sqrt{\zeta}}\right)^2.$$

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For analytic work (3·8) is rather clumsy, and it is frequently better to return to the double integral implied in the definition. However, if we remove the contribution from $\Omega(0)$, so that the remaining integrands are no longer singular at $\zeta = 0$, it is quite amenable to computation.

Some difficulty is experienced near $z = 0$, as can only be expected from our previous comments about behaviour at the sonic point.

4. *Behaviour in the neighbourhood of the sonic point; its prediction and the choice of the value of* a^2. The aim here is to show that near the origin (3·7) has the form

$$a\frac{\partial\phi_2}{\partial z}\bigg|_{y=0} = \left\{-\frac{\Omega(-b)}{b^{\frac{1}{2}}} - \int_0^b \frac{\Omega'(-\xi)}{\xi^{\frac{1}{2}}}\,d\xi\right\} - \frac{1}{4}\left\{\frac{\Omega(-b)}{b^{\frac{3}{2}}} - \frac{2\Omega'(-b)}{b^{\frac{1}{2}}} - \int_0^b \frac{2\Omega''(-\xi)}{\xi^{\frac{1}{2}}}\,d\xi\right\} s + o(s), \qquad (4{\cdot}1a)$$

and similarly that (3·8) has the form

$$a\frac{\partial\phi_2}{\partial z}\bigg|_{y=0} = \left\{-\frac{\Omega(-b)}{b^{\frac{1}{2}}} - \int_0^b \frac{\Omega'(-\xi)}{\xi^{\frac{1}{2}}}\,d\xi\right\} + \frac{1}{4}\left\{\frac{\Omega(-b)}{b^{\frac{3}{2}}} - \frac{2\Omega'(-b)}{b^{\frac{1}{2}}} - \int_0^b \frac{2\Omega''(-\xi)}{\xi^{\frac{1}{2}}}\,d\xi\right\} z + o(z), \qquad (4{\cdot}1b)$$

so that we establish the continuity of $\partial\phi_2/\partial z|_{y=0}$ at the origin, obtain its value $\partial\phi_2/\partial z|_{y=0,z=0}$, and also the value of $\partial^2\phi_2/\partial z^2|_{y=0,z=0}$.

For consistency, the value of $\partial\phi_2/\partial z|_{y=0,z=0}$ should vanish, as we have taken the origin at the sonic point, and this gives the means of determining b, i.e. the equation

$$\frac{\Omega(-b)}{b^{\frac{1}{2}}} + \int_0^b \frac{\Omega'(-\xi)}{\xi^{\frac{1}{2}}}\,d\xi = 0. \qquad (4{\cdot}2)$$

The value of $\partial^2\phi_2/\partial z^2|_{y=0,z=0}$ is required, because we choose to make $a^2 z$ the best approximation to $(\gamma+1)M_\infty^2\phi_z$ at the sonic point. This is the choice made by Maeder and Thommen in ((8)), to make their correction term minimum, except that they require to choose a point which gives the best value of $(\gamma+1)M_\infty^2\phi_{zz}$, rather than $(\gamma+1)M_\infty^2\phi_z$.

From this choice we obtain the equation for a^2:

$$a^2 = \frac{(\gamma+1)}{4a}\left\{\frac{\Omega(-b)}{b^{\frac{3}{2}}} - \frac{2\Omega'(-b)}{b^{\frac{1}{2}}} - \int_0^b \frac{2\Omega''(-\xi)}{\xi^{\frac{1}{2}}}\,d\xi\right\}. \qquad (4{\cdot}3)$$

To proceed with our stated intention, consider equation (3·7). In the integral, split the range of integration at η. As we are considering $s \to 0$, we take $1 \gg \eta \gg s$.

$$a\frac{\partial\phi_2}{\partial z}\bigg|_{y=0} = -\frac{\Omega(0)}{\sqrt{s}} + \frac{1}{2\pi}\int_0^\eta \frac{\Omega(-\xi)}{\xi}\left\{\frac{E}{(\sqrt{\xi}-\sqrt{s})} + \frac{K}{(\sqrt{\xi}+\sqrt{s})}\right\}d\xi + \frac{1}{2\pi}\int_\eta^b \frac{\Omega(-\xi)}{\xi}\left\{\frac{E}{(\sqrt{\xi}-\sqrt{s})} + \frac{K}{(\sqrt{\xi}+\sqrt{s})}\right\}d\xi. \qquad (4{\cdot}4)$$

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In the second term, for sufficiently small η,

$$\Omega(-\xi) = \sum_{m=0}^{\infty} (-\xi)^m \frac{\Omega^m(0)}{m!}$$

over the range of integration.

Now define

$$\left.\begin{aligned} E_0(s) &= \frac{1}{2\pi}\left[\int_0^1 \left\{\frac{E}{(\sqrt{\xi}-\sqrt{s})} + \frac{K}{(\sqrt{\xi}+\sqrt{s})}\right\} d\xi - \frac{2\pi}{\sqrt{s}}\right], \\ E_n(s) &= \frac{1}{2\pi}\int_0^1 \left\{\frac{E}{(\sqrt{\xi}-\sqrt{s})} + \frac{K}{(\sqrt{\xi}+\sqrt{s})}\right\} \xi^{n-1} d\xi \quad (n > 0), \end{aligned}\right\} \tag{4·5}$$

and note also, that since $\eta \gg s$, the third term of (4·4) may be written

$$\frac{1}{2\pi}\sum_0^{\infty} a_n s^{\frac{1}{2}n} \int_\eta^b \frac{\Omega(-\xi)}{\xi^{\frac{1}{2}(3+n)}} d\xi,$$

where

$$\left\{\frac{E}{(\sqrt{\xi}-\sqrt{s})} + \frac{K}{(\sqrt{\xi}+\sqrt{s})}\right\} = \frac{1}{\sqrt{\xi}} \sum_{n=0}^{\infty} a_n \left(\frac{s}{\xi}\right)^{\frac{1}{2}n},$$

the coefficients a_n being $a_0 = \pi$, $a_1 = 0$, $a_2 = \frac{3}{4}\pi$, ... Thus (4·4) may be written

$$a\frac{\partial \phi_2}{\partial z}\bigg|_{y=0} = \sum_{m=0}^{\infty} (-1)^m \frac{\Omega^m(0)}{m!}\, \eta^{m-\frac{1}{2}} E_m\left(\frac{s}{\eta}\right) + \frac{1}{2\pi}\sum_{n=0}^{\infty} a_n s^{\frac{1}{2}n} \int_\eta^b \frac{\Omega(-\xi)}{\xi^{\frac{1}{2}(3+n)}} d\xi. \tag{4·6}$$

Now it may be shown that

$$E_0\left(\frac{s}{\eta}\right) = -\frac{2}{\pi}\frac{\sqrt{\eta}}{(\sqrt{\eta}+\sqrt{s})} K, \quad E_1\left(\frac{s}{\eta}\right) = \frac{2}{\pi}\left[\left(1+\sqrt{\frac{s}{\eta}}\right) E - \frac{s/\eta}{(1+\sqrt{s/\eta})} K\right],$$

where $k^2 = \dfrac{4\sqrt{(\eta s)}}{(\sqrt{\eta}+\sqrt{s})^2}$, and K, E are complete elliptic integrals with modulus k^2.

By using series expressions for the elliptic integrals, we obtain

$$E_0\left(\frac{s}{\eta}\right) = -1 + \frac{1}{4}\left(\frac{s}{\eta}\right) + O(s^{\frac{3}{2}}),$$

$$E_1\left(\frac{s}{\eta}\right) = 1 - \frac{3}{4}\left(\frac{s}{\eta}\right) + O(s^{\frac{3}{2}}),$$

$$\Delta E_n\left(\frac{s}{\eta}\right) = E_n\left(\frac{s}{\eta}\right) - \left(\frac{s}{\eta}\right) E_{n-1}\left(\frac{s}{\eta}\right) = \frac{1}{2\pi}\int_0^1 \xi^{n-2}\left[\left(\sqrt{\xi}+\sqrt{\frac{s}{\eta}}\right) E + \left(\sqrt{\xi}-\sqrt{\frac{s}{\eta}}\right) K\right] d\xi$$

$$= \frac{1}{(2n-1)} - \frac{s}{2\eta}\frac{1}{(2n-3)} + O(s^{\frac{3}{2}});$$

and hence

$$E_n\left(\frac{s}{\eta}\right) = \frac{1}{2n-1} - \frac{3s}{2\eta}\frac{1}{(2n-3)} + O(s^{\frac{3}{2}}).$$

Using these expansions in (4·6)

$$a\frac{\partial \phi_2}{\partial z}\bigg|_{y=0} = \left[-\frac{\Omega(0)}{\eta^{\frac{1}{2}}} - \Omega'(0)\,\eta^{\frac{1}{2}} + \sum_{m=2}^{\infty} (-1)^m \frac{\Omega^m(0)}{m!}\frac{\eta^{m-\frac{1}{2}}}{(2m-1)} + \frac{1}{2}\int_\eta^b \frac{\Omega(-\xi)}{\xi^{\frac{3}{2}}} d\xi\right]$$

$$+ \frac{s}{4}\left[\frac{\Omega(0)}{\eta^{\frac{3}{2}}} + \frac{3\Omega'(0)}{\eta^{\frac{1}{2}}} + \sum_{m=2}^{\infty} (-1)^m \frac{\Omega^m(0)}{m!}\frac{6\eta^{m-\frac{3}{2}}}{(2m-3)} + \frac{3}{2}\int_\eta^b \frac{\Omega(-\xi)}{\xi^{\frac{5}{2}}} d\xi\right] + O(s^{\frac{3}{2}}).$$

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Now, η was chosen arbitrarily, so the right-hand side must be independent of η, and, in particular, the coefficients of the powers of s must be independent of η. Thus

$$a\frac{\partial\phi_2}{\partial z}\bigg|_{\substack{y=0\\z=0-}} = \mathscr{F}\frac{1}{2}\int_0^b \frac{\Omega(-\xi)}{\xi^{\frac{3}{2}}}d\xi = -\frac{\Omega(-b)}{b^{\frac{1}{2}}} - \int_0^b \frac{\Omega'(-\xi)}{\xi^{\frac{1}{2}}}d\xi,$$

$$a\frac{\partial^2\phi_2}{\partial z^2}\bigg|_{\substack{y=0\\z=0-}} = -\tfrac{3}{8}\mathscr{F}\int_0^b \frac{\Omega(-\xi)}{\xi^{\frac{5}{2}}}d\xi = \frac{1}{4}\left\{\frac{\Omega(-b)}{b^{\frac{3}{2}}} - \frac{2\Omega'(-b)}{b^{\frac{1}{2}}} - \int_0^b \frac{2\Omega''(-\xi)}{\xi^{\frac{1}{2}}}d\xi\right\},$$

where $\mathscr{F}$ denotes the finite part.

Having established the result for $z \to 0^-$, we now consider the behaviour of (3·8), as $z \to 0^+$:

$$\frac{\partial\phi_2}{\partial z}\bigg|_{y=0} = \lim_{\epsilon\to 0}\left\{\frac{\partial A_1(\epsilon)}{\partial z} + \frac{\partial A_2(\epsilon)}{\partial z}\right\}.$$

In this case, for $\partial A_1(\epsilon)/\partial z$ it is convenient to return to the form provided in the definition of $A_1(\epsilon)$, and from this we obtain

$$\frac{\partial A_1(\epsilon)}{\partial z} = \frac{2}{a\pi}\int_\epsilon^b d\xi\,\Omega(-\xi)\int_0^\infty \frac{z+\xi+x^2}{\{(z-\xi-x^2)^2+4\xi z\}^{\frac{3}{2}}}dx.$$

Also
$$\frac{\partial A_2(\epsilon)}{\partial z} = \frac{2}{a\pi}\int_\epsilon^z \frac{\Omega(\zeta)}{2\zeta(z-\zeta)}\{2\sqrt{\zeta}K - (\sqrt{\zeta}+\sqrt{z})E\}d\zeta - \frac{\Omega(z)}{a\sqrt{z}}.$$

The method is essentially the same as that for $z < 0$, in that we divide the range of integration (in $\partial A_1(\epsilon)/\partial z$) so that we may employ a Taylor expansion of $\Omega(\zeta)$ near the origin, although the ϵ limit alters the form a little.

For $1 \gg \eta \gg z$,

$$\frac{2}{\pi}\int_\epsilon^\eta d\xi\,\Omega(-\xi)\int_0^\infty \frac{(z+\xi+x^2)}{\{(z-\xi-x^2)^2+4\xi z\}^{\frac{3}{2}}}dx$$

$$= \sum_{m=0}^\infty (-1)^m\frac{\Omega^m(0)}{m!}\frac{2}{\pi}\int_\epsilon^\eta d\xi\,\xi^m\int_0^\infty \frac{(z+\xi+x^2)}{\{(z-\xi-x^2)^2+4\xi z\}^{\frac{3}{2}}}dx$$

and
$$\frac{2}{\pi}\int_\epsilon^z \frac{\Omega(\zeta)}{2\zeta(z-\zeta)}\{2\sqrt{\zeta}K - (\sqrt{\zeta}+\sqrt{z})E\}d\zeta - \frac{\Omega(z)}{\sqrt{z}}$$

$$= \sum_{n=0}^\infty \frac{\Omega^n(0)}{n!}\left\{\frac{2}{\pi}\int_\epsilon^z \frac{\{2\sqrt{\zeta}K - (\sqrt{\zeta}+\sqrt{z})E\}}{2(z-\zeta)}\zeta^{m-1}d\zeta - \frac{z^n}{\sqrt{z}}\right\}.$$

Now putting

$$\left.\begin{aligned} V_0(z) &= \lim_{\epsilon\to 0}\left\{\frac{2}{\pi}\int_\epsilon^1 d\xi\int_0^\infty \frac{(z+\xi+x^2)\,dx}{\{(z-\xi-x^2)^2+4\xi z\}^{\frac{3}{2}}} + \frac{2}{\pi}\int_\epsilon^z \frac{2\sqrt{\zeta}K-(\sqrt{\zeta}+\sqrt{z})E}{2\zeta(z-\zeta)}d\zeta - \frac{1}{\sqrt{z}}\right\},\\ V_n(z) &= \frac{2}{\pi}\int_0^1 d\xi\,\xi^n\int_0^\infty \frac{(z+\xi+x^2)\,dx}{\{(z-\xi-x^2)^2+4\xi z\}^{\frac{3}{2}}} \quad (n \geqslant 1), \end{aligned}\right\} \tag{4·7}$$

$$\left.\begin{aligned} U_m &= \frac{2}{\pi}\int_0^1 \frac{\{2\sqrt{\zeta}K-(\sqrt{\zeta}+1)E\}}{2(1-\zeta)}\zeta^{m-1}d\zeta - 1 \quad (m \geqslant 0),\\ U_0 &= 0, \end{aligned}\right\} \tag{4·8}$$

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we may write

$$a\frac{\partial\phi_2}{\partial z}\bigg|_{y=0} = \frac{2}{\pi}\int_\eta^b d\xi\,\Omega(-\xi)\int_0^\infty \frac{(z+\xi+x^2)\,dx}{\{(z-\xi-x^2)^2+4\xi z\}^{\frac{3}{2}}} + \sum_{m=0}^{\infty}(-1)^m\frac{\Omega^m(0)}{m!}\eta^{m-\frac{1}{2}}V_m\left(\frac{z}{\eta}\right)+\sum_{n=0}^{\infty}\frac{\Omega^n(0)}{n!}z^{n-\frac{1}{2}}U_n. \tag{4·9}$$

Now it may be shown

$$V_0\left(\frac{z}{\eta}\right) = -\frac{2}{\pi}\frac{2\sqrt{\eta}}{(\sqrt{(\eta+z)}+\sqrt{z})}F_*,$$

$$V_1\left(\frac{z}{\eta}\right) = \frac{2}{\pi}\frac{1}{\sqrt{\eta}}\{(\sqrt{(z+\eta)}+\sqrt{z})\,2E_*-4\sqrt{z}\},$$

$$U_1 = -4/\pi,$$

where F_*, E_* are the incomplete elliptic integrals, defined in (2·6), with

$$\lambda_1 = \frac{\sqrt{(\eta+z)}-\sqrt{z}}{\sqrt{(\eta+z)}+\sqrt{z}}.$$

As $\eta > z$, by expanding the integrand it is easily established that

$$\frac{2}{\pi}\int_\eta^b d\xi\,\Omega(-\xi)\int_0^\infty \frac{(z+\xi+x^2)\,dx}{\{(z-\xi-x^2)^2+4\xi z\}^{\frac{3}{2}}} = \left[\frac{1}{2}\int_\eta^b\frac{\Omega(-\xi)}{\xi^{\frac{3}{2}}}d\xi\right]-\left[\frac{3}{8}\int_\eta^b\frac{\Omega(-\xi)}{\xi^{\frac{5}{2}}}d\xi\right]z+O(z^2). \tag{4·10}$$

Further, by employing expansions for incomplete elliptic integrals ((1)) when

$$\lambda_1 = \frac{\sqrt{(\eta+z)}-\sqrt{z}}{\sqrt{(\eta+z)}+\sqrt{z}},$$

$$E_* = \frac{\pi}{4}+\left(\frac{z}{\eta}\right)^{\frac{1}{2}}\left(1-\frac{\pi}{4}\right)+\left(\frac{z}{\eta}\right)\left(\frac{5\pi}{16}-1\right)+O\left(\left(\frac{z}{\eta}\right)^{\frac{3}{2}}\right),$$

$$F_* = \frac{\pi}{4}+\left(\frac{z}{\eta}\right)^{\frac{1}{2}}\frac{\pi}{4}+\left(\frac{z}{\eta}\right)\frac{\pi}{16}+O\left(\left(\frac{z}{\eta}\right)^{\frac{3}{2}}\right),$$

and hence

$$V_0\left(\frac{z}{\eta}\right) = -1+\frac{z}{4\eta}+O\left(\left(\frac{z}{\eta}\right)^{\frac{3}{2}}\right),$$

$$V_1\left(\frac{z}{\eta}\right) = 1-\frac{4}{\pi}\left(\frac{z}{\eta}\right)^{\frac{1}{2}}+\frac{3}{4}\left(\frac{z}{\eta}\right)+O\left(\left(\frac{z}{\eta}\right)^{\frac{3}{2}}\right).$$

If in (4·9), we now take the two infinite series, and write them as

$$\sum_{m=0}^{\infty}(-1)^m\frac{\Omega^m(0)}{m!}I_m, \quad \text{where} \quad I_m = \left(\eta^{m-\frac{1}{2}}V_m\left(\frac{z}{\eta}\right)+(-1)^m z^{m-\frac{1}{2}}U_m\right),$$

then using the above approximations

$$\left.\begin{aligned} I_0 &= -\frac{1}{\eta^{\frac{1}{2}}}+\frac{z}{4\eta^{\frac{3}{2}}}+O(z^{\frac{3}{2}}),\\ I_1 &= \eta^{\frac{1}{2}}+\frac{3}{4}\frac{z}{\eta^{\frac{1}{2}}}+O(z^{\frac{3}{2}}). \end{aligned}\right\} \tag{4·11}$$

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Now consider $I_m\,(m>1)$. Expanding the integrand in $V_m(z/\eta)$, for small z, for $m>1$ it may be shown

$$I_m = \frac{1}{2m-1}\eta^{m-\frac{1}{2}} - \frac{3}{4}\frac{1}{2m-3}\eta^{m-\frac{3}{2}}z + o(z). \tag{4·12}$$

Using the expansions (4·10), (4·11), (4·12) in (4·9)

$$a\frac{\partial\phi_2}{\partial z}\bigg|_{y=0} = \left[-\frac{\Omega(0)}{\eta^{\frac{1}{2}}} - \Omega'(0)\,\eta^{\frac{1}{2}} + \sum_{m=2}^{\infty}(-1)^m\frac{\Omega^m(0)}{m!}\frac{\eta^{m-\frac{1}{2}}}{(2m-1)} + \frac{1}{2}\int_\eta^b \frac{\Omega(-\xi)}{\xi^{\frac{3}{2}}}d\xi\right]$$
$$+\frac{z}{4}\left[\frac{\Omega(0)}{\eta^{\frac{3}{2}}} - \frac{3\Omega'(0)}{\eta^{\frac{1}{2}}} + 3\sum_{m=2}^{\infty}(-1)^m\frac{\Omega^m(0)}{m!}\frac{\eta^{m-\frac{3}{2}}}{(2m-3)} - \frac{3}{2}\int_\eta^b\frac{\Omega(-\xi)}{\xi^{\frac{5}{2}}}d\xi\right] + o(z).$$

Now η was chosen arbitrarily, so the right-hand side must be independent of η and in particular the coefficients of the powers of z must be independent of η. Thus

$$\frac{\partial\phi_2}{\partial z}\bigg|_{\substack{y=0\\z=0+}} = \mathscr{F}\frac{1}{2}\int_0^b\frac{\Omega(-\xi)}{\xi^{\frac{3}{2}}}d\xi = -\frac{\Omega(-b)}{b^{\frac{1}{2}}} - \int_0^b\frac{\Omega'(-\xi)}{\xi^{\frac{1}{2}}}d\xi,$$

$$\frac{\partial^2\phi_2}{\partial z^2}\bigg|_{\substack{y=0\\z=0+}} = -\mathscr{F}\frac{3}{8}\int_0^b\frac{\Omega(-\xi)}{\xi^{\frac{5}{2}}}d\xi = \frac{1}{4}\left\{\frac{\Omega(-b)}{b^{\frac{3}{2}}} - \frac{2\Omega'(-b)}{b^{\frac{1}{2}}} - \int_0^b\frac{2\Omega''(-\xi)}{\xi^{\frac{1}{2}}}d\xi\right\},$$

which completes the proof of equations (4·1 *a*), (4·1 *b*), and consequently of (4·2) which gives b, and (4·3) which gives a.

5. *The calculation of pressure distribution on the airfoil.* In this section we collect the results required for the calculation of pressure distributions on thin airfoils having small slope, and then apply the method to the particular class of airfoils whose slopes are given by polynomials.

The non-dimensional pressure distribution $C_p = (p-p_\infty)/\frac{1}{2}\rho_\infty U_\infty^2$ is given by $C_p = -2(\partial\phi_2/\partial z)|_{y=0}$, to the usual linear approximation.

According to the transonic similarity rules, the quantity

$$\tilde{C}_p = \frac{(\gamma+1)^{\frac{1}{3}}}{\tau^{\frac{2}{3}}}C_p$$

should be independent of the thickness ratio τ, and in our approximation this is the case. For the purpose of comparison with other work, results are given in terms of $\tilde{C}_p$.

Suppose the local slope of the airfoil $\Omega(z) = \tau\tilde{A}w(z)$, then equation (4·2) becomes

$$\frac{w(-b)}{b^{\frac{1}{2}}} + \int_0^b\frac{w'(-\xi)}{\xi^{\frac{1}{2}}}d\xi = 0, \tag{5·1}$$

and the value of a, from (4·3), is given by

$$a^3 = (\gamma+1)\tau B^3,$$

where
$$B^3 = \frac{\tilde{A}}{4}\left\{\frac{w(-b)}{b^{\frac{3}{2}}} - \frac{2w'(-b)}{b^{\frac{1}{2}}} - \int_0^b\frac{2w''(-\xi)}{\xi^{\frac{1}{2}}}d\xi\right\}. \tag{5·2}$$

Using these results in (3·7) and (3·8), we obtain

$$\tilde{C}_p(z<0) = -2\frac{\tilde{A}}{B}\left\{-\frac{w(0)}{\sqrt{s}} + \frac{1}{2\pi}\int_0^b\frac{w(-\xi)}{\xi}\left\{\frac{E}{(\sqrt{\xi}-\sqrt{s})} + \frac{K}{(\sqrt{\xi}+\sqrt{s})}\right\}d\xi\right., \tag{5·3}$$

$$\tilde{C}_p(z>0) = -2\frac{\tilde{A}}{B}\left\{D(\epsilon) - \frac{w(z)}{\sqrt{z}} + \frac{2}{\pi}\int_\epsilon^z\frac{w(\zeta)}{2\zeta(z-\zeta)}\{2\sqrt{\zeta}K - (\sqrt{\zeta}+\sqrt{z})E\}d\zeta\right\}, \tag{5·4}$$

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where $\lim \epsilon \to 0$ is understood and $D(\epsilon)$ takes the alternative forms

$$\frac{2}{\pi}\int_\epsilon^b w(-\xi)\int_0^\infty \frac{(z+\xi+x^2)\,dx}{\{(z-\xi-x^2)^2+4\xi z\}^{\frac{3}{2}}}, \tag{5·5a}$$

$$\frac{2}{\pi}\int_\epsilon^b \frac{w(-\xi)}{8z(\sqrt{(z+\xi)}-\sqrt{z})}\Big[2E_*-2\lambda_1 F_*-(1-\lambda_1)+\frac{\xi}{4(\xi+z)}\{(3-2\lambda_1+3\lambda_1^2)\,2F_* - (6-\lambda_1-\lambda_2)\,(2E^*-1+\lambda_1)\}\Big]d\xi. \tag{5·5b}$$

The forms (5·3), (5·4) are not immediately useful for numerical calculation. However, if the case $w(z) = z^m\,(0 \leqslant m < 1)$ can be treated, then the remaining numerical integrations are straightforward. We illustrate this by the example of a polynomial for $w(z)$, the case $w(z) =$ constant being the extreme case from the point of view of singular integrands.

Suppose then, that

$$w(z) = \sum_{n=0}^{N} w^n(0)\frac{z^n}{n!}.$$

First consider (5·3):

$$\tilde{C}_p(z<0) = -\frac{2\tilde{A}}{B}\left\{-\frac{w(0)}{\sqrt{s}}+\frac{1}{2\pi}\sum_{n=0}^{N}(-1)^n\frac{w^n(0)}{n!}\int_0^b\left\{\frac{E}{(\sqrt{\xi}-\sqrt{s})}+\frac{K}{(\sqrt{\xi}+\sqrt{s})}\right\}\xi^{n-1}d\xi\right\}.$$

This may be written

$$\tilde{C}_p(z<0) = -\frac{2\tilde{A}}{B}\sum_{n=0}^{N}\frac{w^n(0)}{n!}(-1)^n b^{n-\frac{1}{2}}E_n\left(\frac{s}{b}\right), \tag{5·6}$$

where

$$E_0(s) = -\frac{1}{\sqrt{s}}+\frac{1}{2\pi}\int_0^1\left\{\frac{E}{(\sqrt{\xi}-\sqrt{s})}+\frac{K}{(\sqrt{\xi}+\sqrt{s})}\right\}\frac{d\xi}{\xi},$$

$$E_n(s) = \frac{1}{2\pi}\int_0^1\left\{\frac{E}{(\sqrt{\xi}-\sqrt{s})}+\frac{K}{(\sqrt{\xi}+\sqrt{s})}\right\}\xi^{n-1}d\xi. \tag{5·7}$$

Next consider (5·4):

$$\frac{2}{\pi}\int_\epsilon^z \frac{w(\zeta)}{2\zeta(z-\zeta)}\{2\sqrt{\zeta}K-(\sqrt{\zeta}+\sqrt{z})\,E\}\,d\zeta-\frac{w(z)}{\sqrt{z}}$$
$$= w(0)\left[\frac{2}{\pi}\int_\epsilon^z\frac{\{2\sqrt{\zeta}K-(\sqrt{\zeta}+\sqrt{z})\,E\}}{2\zeta_0(z-\zeta)}d\zeta-\frac{1}{\sqrt{z}}\right]$$
$$+\sum_{n=1}^{\infty}\frac{w^n(0)}{n!}(-1)^n\left[\frac{2}{\pi}\int_0^z\frac{\{2\sqrt{\zeta}K-(\sqrt{\zeta}+\sqrt{z})\,E\}}{2(z-\zeta)}\zeta^{n-1}d\zeta-z^{n-\frac{1}{2}}\right],$$

$$D(\epsilon) = w(0)\frac{2}{\pi}\int_\epsilon^b d\xi\int_0^\infty dx\,\frac{z+\xi+x^2}{\{(z-\xi-x^2)^2+4\xi z\}^{\frac{3}{2}}}$$
$$+\sum_{n=1}^{N}\frac{w^n(0)}{n!}(-1)^n\int_0^b d\xi\,\xi^n\int_0^\infty dx\,\frac{z+\xi+x^2}{\{(z-\xi-x^2)^2+4\xi z\}^{\frac{3}{2}}}.$$

Define

$$b^{-\frac{1}{2}}V_0\left(\frac{z}{b}\right) = \lim_{\epsilon\to 0}\left[\frac{2}{\pi}\int_\epsilon^b d\xi\int_0^\infty dx\,\frac{z+\xi+x^2}{\{(z-\xi-x^2)^2+4\xi z\}^{\frac{3}{2}}}+\frac{2}{\pi}\int_\epsilon^z\frac{\{2\sqrt{\zeta}K-(\sqrt{\zeta}+\sqrt{z})\,E\}}{2\zeta(z-\zeta)}d\zeta-\frac{1}{\sqrt{z}}\right], \tag{5·8}$$

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$$V_n(z) = \frac{2}{\pi}\int_0^1 \frac{\xi^n}{8z(\sqrt{(z+\xi)}-\sqrt{z})}\left[2E_* - 2\lambda_1 F_* - (1-\lambda_1)\right.$$
$$\left. + \frac{\xi}{4(\xi+z)}\{(3-2\lambda_1+3\lambda_1^2)\,2F_* - (6-\lambda_1-\lambda_2)\,(2E_*-1+\lambda_1)\}\right]d\xi, \qquad (5{\cdot}9)$$

$$\left.\begin{aligned} U_n &= \left[-1+\frac{2}{\pi}\int_0^1 \frac{t^{2n-1}}{(1-t^2)}\{2tK-(t+1)\,E\}\,dt\right] \quad (n>0),\\ U_0 &= 0. \end{aligned}\right\} \qquad (5{\cdot}10)$$

Then, substituting in (5·4),

$$\tilde{C}_p(z>0) = -\frac{2\tilde{A}}{B}\sum_{n=0}^{N}\frac{w^n(0)}{n!}\left[(-1)^n b^{n-\frac{1}{2}} V_n\left(\frac{z}{b}\right) + z^{n-\frac{1}{2}} U_n\right]. \qquad (5{\cdot}11)$$

Table 1. *U_n integrals*

U_1	$-1{\cdot}27324$
U_2	$-1{\cdot}13177$
U_3	$-1{\cdot}08650$
U_4	$-1{\cdot}06433$
U_5	$-1{\cdot}05119$

Table 2. *$E_n(s)$ integrals*

s	$E_0(s)$	$E_1(s)$	$E_2(s)$	$E_3(s)$	$E_4(s)$	$E_5(s)$
0·0	−1·00000	1·00000	0·33333	0·20000	0·14286	0·11111
0·1	−1·02651	0·92251	0·40103	0·23131	0·16082	0·12356
0·2	−1·05655	0·83935	0·45311	0·27223	0·18655	0·14085
0·3	−1·09110	0·74920	0·48750	0·31759	0·22130	0·16570
0·4	−1·13160	0·65016	0·50137	0·36107	0·26384	0·20026
0·5	−1·18034	0·53935	0·49073	0·39471	0·30963	0·24449
0·6	−1·24113	0·41208	0·44927	0·40753	0·34916	0·29356
0·7	−1·32122	0·25973	0·36615	0·38294	0·36470	0·33343
0·8	−1·43698	0·06352	0·21898	0·29078	0·32106	0·32859
0·9	−1·64126	−0·23462	−0·05734	0·05356	0·12859	0·18091

Now $E_0(s)$, $V_0(z)$ may be evaluated analytically. The results have been quoted in section 4 (near (4·6), (4·9)), together with those for $E_1(s)$, $V_1(z)$, U_1. The actual evaluation is straightforward except for U_1 which requires a little ingenuity, but it is tedious and is not included.

$V_n(z)$, $n \geqslant 1$, may be readily computed, and the analytic evaluation of $V_1(z)$ serves as a check on accuracy. In (5·11), the argument is (z/b), so that for accurate work, $V_n(z)$ must be generated for each particular case. However, a table is provided for $n = 1, \ldots, 5$, and for typical values of the argument.

U_n may be derived by considering $\Delta U_n = U_n - U_{n-1}$, which may be readily computed. Values of $U_1, \ldots, U_5$ are given.

$E_n(s)$ may be derived by considering $\Delta E_n(s) = E_n(s) - sE_{n-1}(s)$. Then

$$\Delta E_n(s) = \frac{1}{2\pi}\int_0^1 \xi^{n-2}\{(\sqrt{\xi}+\sqrt{s})\,E + (\sqrt{\xi}-\sqrt{s})\,K\}\,d\xi$$

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Table 3. $V_n(z)$ *integrals*

	Analytic integration		Numerical integration				
z	$V_0(z)$	$V_1(z)$	$V_1(z)$	$V_2(z)$	$V_3(z)$	$V_4(z)$	$V_5(z)$
0·2	−0·95493	0·57219	0·57130	0·25809	0·16387	0·11956	0·09403
0·4	−0·91769	0·46410	0·46384	0·21906	0·14165	0·10428	0·08244
0·6	−0·88611	0·40031	0·40018	0·19309	0·12605	0·09333	0·07404
0·8	−0·85867	0·35657	0·35651	0·17420	0·11440	0·08499	0·06755
1·0	−0·83461	0·32417	0·32410	0·15960	0·10523	0·07837	0·06239
1·2	−0·81322	0·29883	0·29883	0·14795	0·09785	0·07302	0·05819
1·4	−0·79393	0·27839	0·27839	0·13840	0·09174	0·06850	0·05462
1·6	−0·77642	0·26152	0·26152	0·13038	0·08652	0·06468	0·05163
1·8	−0·76044	0·24720	0·24720	0·12350	0·08212	0·06143	0·04908
2·0	−0·74574	0·23491	0·23491	0·11758	0·07824	0·05857	0·04679
2·2	−0·73218	Identical	0·22422	0·11236	0·07480	0·05602	0·04482
2·4	−0·71957	with	0·21480	0·10772	0·07181	0·05379	0·04304
2·6	−0·70779	numerical	0·20646	0·10364	0·06907	0·05182	0·04144
2·8	−0·69678	result to	0·19894	0·09994	0·06665	0·04997	0·04000
3·0	−0·68647	5 decimal	0·19220	0·09658	0·06443	0·04832	0·03864
3·2	−0·67673	places	0·18602	0·09352	0·06245	0·04686	0·03750
3·4	−0·66750		0·18035	0·09072	0·06061	0·04545	0·03635
3·6	−0·65877		0·17526	0·08817	0·05889	0·04418	0·03533
3·8	−0·65050		0·17049	0·08582	0·05730	0·04304	0·03444
4·0	−0·64267		0·16609	0·08359	0·05583	0·04195	0·03355
4·2	−0·63516		0·16202	0·08155	0·05449	0·04093	0·03272
4·4	−0·62796		0·15820	0·07964	0·05322	0·03998	0·03196
4·6	−0·62115		0·15463	0·07786	0·05201	0·03909	0·03126
4·8	−0·61459		0·15132	0·07620	0·05093	0·03826	0·03062
5·0	−0·60829		0·14821	0·07461	0·04985	0·03743	0·02998
5·2	−0·60224		0·14521	0·07315	0·04889	0·03673	0·02941
5·4	−0·59645		0·14241	0·07175	0·04794	0·03603	0·02884
5·6	−0·59085		0·13980	0·07041	0·04705	0·03533	0·02833
5·8	−0·58550		0·13732	0·06914	0·04622	0·03470	0·02782
6·0	−0·58028		0·13490	0·06793	0·04545	0·03412	0·02731
6·2	−0·57525		0·13267	0·06685	0·04469	0·03355	0·02687
6·4	−0·57041		0·13051	0·06576	0·04393	0·03298	0·02642
6·6	−0·56570		0·12847	0·06468	0·04323	0·03247	0·02604
6·8	−0·56118		0·12650	0·06373	0·04259	0·03202	0·02559
7·0	−0·55672		0·12459	0·06277	0·04195	0·03151	0·02521
7·2	−0·55246		0·12280	0·06190	0·04138	0·03107	0·02489
7·4	−0·54832		0·12109	0·06099	0·04074	0·03062	0·02451
7·6	−0·54431		0·11943	0·06016	0·04023	0·03018	0·02419
7·8	−0·54043		0·11784	0·05933	0·03966	0·02979	0·02387
8·0	−0·53661		0·11631	0·05857	0·03915	0·02941	0·02355
8·2	−0·53291		0·11485	0·05781	0·03864	0·02903	0·02324
8·4	−0·52929		0·11345	0·05710	0·03820	0·02871	0·02298
8·6	−0·52578		0·11205	0·05640	0·03775	0·02833	0·02266
8·8	−0·52235		0·11071	0·05577	0·03731	0·02801	0·02241
9·0	−0·51904		0·10944	0·05513	0·03686	0·02769	0·02215
9·2	−0·51579		0·10823	0·05449	0·03641	0·02737	0·02190
9·4	−0·51261		0·10702	0·05389	0·03603	0·02706	0·02165
9·6	−0·50949		0·10587	0·05329	0·03565	0·02674	0·02145
9·8	−0·50643		0·10479	0·05271	0·03527	0·02648	0·02120

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and it is convenient to consider the form

$$\Delta E_n(s) = s^{n-\frac{1}{2}}\left\{\frac{1}{2\pi}\int_0^1 \{(\sqrt{\xi}+1)E+(\sqrt{\xi}-1)K\}\xi^{n-2}\,d\xi + \frac{1}{2\pi}\int_0^{1/s} \{(\sqrt{\xi}+1)E+(\sqrt{\xi}-1)K\}\xi^{n-2}\,d\xi\right\}.$$

The situation for $E_n(s)$ is the same as that for $V_n(z)$. For accurate work it must be generated for each particular case, but for $n = 1, \ldots, 5$, and typical values of (s/b), a table is provided.

Finally we use the expressions for $\tilde{C}_p$ in (5·6), (5·11), and the tables to consider members of a class of airfoils investigated experimentally by Michel, Marchaud and Le Gallo ((9)).

The airfoils are given by

$$y = \tilde{A}2l\left[\left(\frac{\tilde{z}}{2l}\right)-\left(\frac{\tilde{z}}{2l}\right)^n\right], \tag{5·12}$$

where $\tilde{z} = 0$ at the leading edge, and also for the reversed form

$$y = \tilde{A}2l\left[\left(1-\frac{\tilde{z}}{2l}\right)-\left(1-\frac{\tilde{z}}{2l}\right)^n\right].$$

The chord is $2l$, and the thickness ratio τ is given by

$$\tilde{A} = \tau\frac{n^{n/(n-1)}}{2(n-1)}.$$

In terms of the notation of this work

$$\Omega(z) = \tau\left(\frac{n^{n/(n-1)}}{2(n-1)}\right)\begin{cases}\left[1-n\left(\dfrac{z+b}{2l}\right)^{n-1}\right] & \text{for maximum thickness aft,}\\ \left[n\left(1-\dfrac{z+b}{2l}\right)^{n-1}-1\right] & \text{for maximum thickness forward.}\end{cases}$$

The experimental work was performed for values of $n = 2$, (3·38), (6·05) which give maximum thickness at 50 %, 60 %, 70 % chord respectively, and 40 %, 30 % chord when reversed. Spreiter ((13)) and Thommen ((11)) also considered these airfoils. For simplicity, so that we may use the tables, we only consider integer values of n. The cases $n = 2, 3, 6$ have been evaluated and the resulting pressure distributions are plotted in Figures 1–5, see p. 590.

6. *Conclusion.* This work considers the possibility of using the equation

$$\phi_{yy} = a^2 z\phi_{zz}$$

as an approximation to the transonic small disturbance equation for a sonic stream flowing past two-dimensional thin wings.

It has been shown that it is possible to derive a unique ϕ simply by satisfying the equation, and the boundary conditions on the body. The value of a^2 is chosen so that $a^2 z$ constitutes the best linear approximation to $\{(\gamma+1)M_\infty^2\phi_z\}_{y=0}$ in the neighbourhood of the sonic point. However, this potential, unlike the three-dimensional solution

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of which it is the integral, does not give vanishing disturbances at large distances. This feature draws attention to the assumption implicit in all the linearized approximations, that it is sufficient to approximate the equation correctly in the immediate neighbourhood of the body to derive surface pressures. It is not regarded as a serious defect. Physically, in the transonic régime, the influence of a body does extend to large distances in the direction normal to the flow.

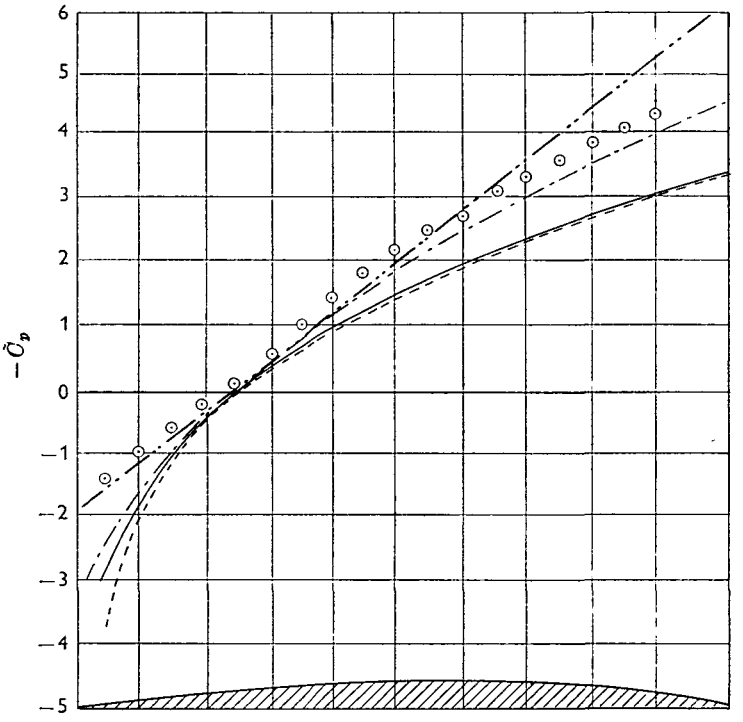


Fig. 1. Pressure distribution $\tilde{C}_p$. Airfoil maximum thickness 50% chord. ———, Present method; —··—··, $\tilde{C}_p$ corresponding to a velocity $a^2/(\gamma+1)\,z$; – – – –, Maeder and Thommen ((7)); —·—·—, Spreiter ((13)); ⊙, Michel ((9)), $\tau = 0{\cdot}1$.

The pressure distributions obtained by the present method are given in Figures 1–5, together with the theoretical distributions of Spreiter ((13)) and Maeder and Thommen ((7)), and the experimental distributions of Michel ((9)). Agreement with experiment is not as good as that of Spreiter's work ((13)) for maximum thickness forward, but for maximum thickness towards the rear, it is at least equal if not better. In ((2)) and in Spreiter's article in ((11)) some doubts have been raised about the experimental results but they do not seem to justify regarding differences between theory and experiment for maximum thickness aft as due to experimental error, while ignoring the error in other cases. However, as explained in the introduction, the theoretical basis of Spreiter's method is suspect. The present results are better than those of the mathematically consistent theory of Maeder and Thommen ((7)), though the method lacks the ease of application of ((7)). The improvements on ((7)) given in ((8)) and ((6)) provide better agreement with experiment for the parabolic-arc thin airfoil and

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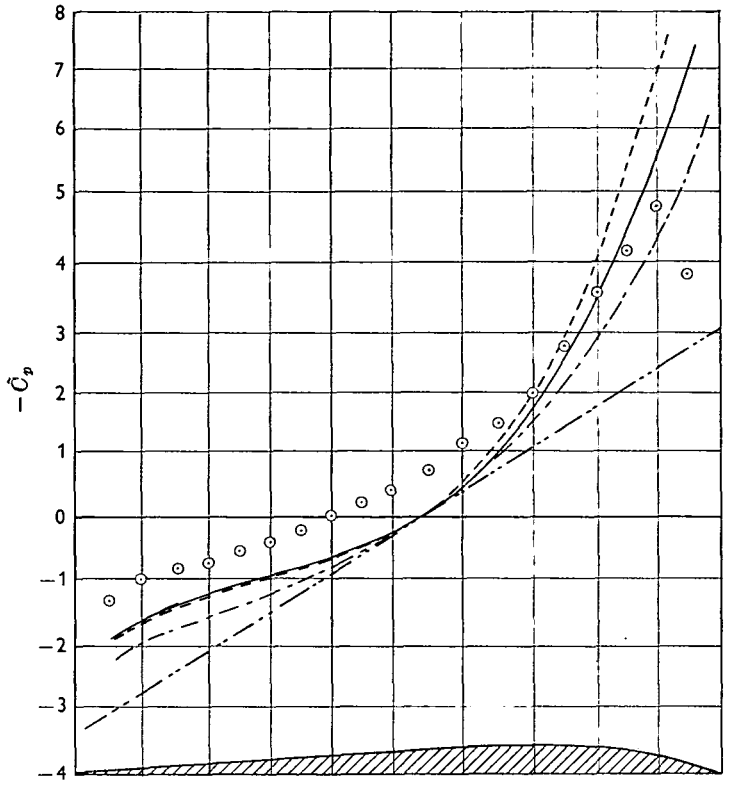


Fig. 2. Pressure distribution $\tilde{C}_p$. Airfoil maximum thickness = 70% chord. ———, Present method; —··—··—, $\tilde{C}_p$ corresponding to a velocity $a^2/(\gamma+1)\,z$; – – – –, Maeder and Thommen ((7)); —·—·—, Spreiter ((13)); N.B. In the notation of (5·12), theoretical results are for $n = 6$, and experimental results are for $n = 6{\cdot}05$. ⊙, Michel ((9)), $\tau = 0{\cdot}08$.

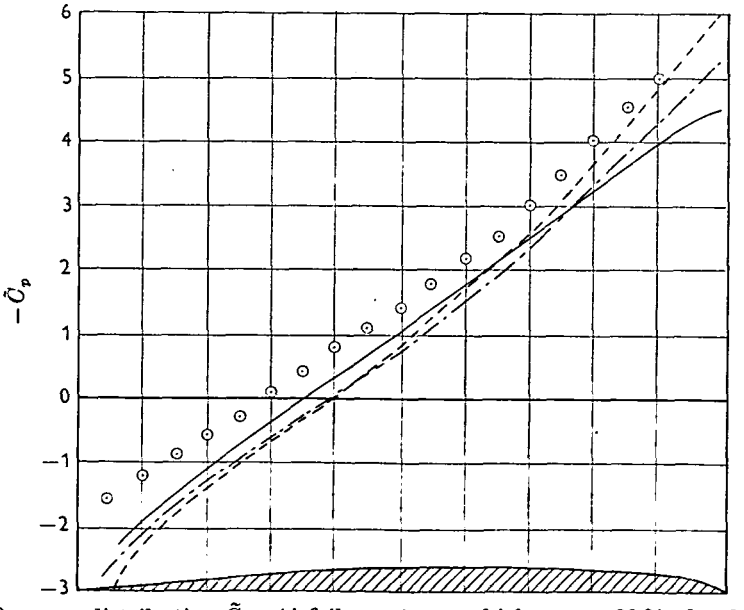


Fig. 3. Pressure distribution $\tilde{C}_p$. Airfoil maximum thickness = 60% chord. ———, Present method; – – – –, Maeder and Thommen ((7)); —·—·—, Spreiter ((13)); N.B. In the notation of (5·12), theoretical results are for $n = 3$, and experimental results are for $n = 3{\cdot}38$. ⊙, Michel ((9)), $\tau = 0{\cdot}08$.

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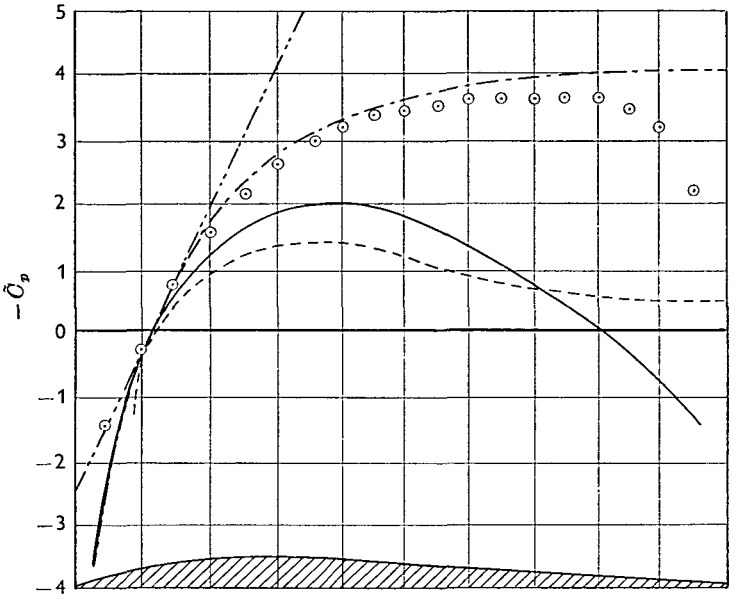


Fig. 4. Pressure distribution $\tilde{C}_p$. Airfoil maximum thickness = 30 % chord. ———, Present method; —··—··, $\tilde{C}_p$ corresponding to a velocity $a^2/(\gamma+1)\,z$; – – – –, Maeder and Thommen ((7)); —·—·—, Spreiter ((13)); N.B. In the notation of (5·12), theoretical results are for $n = 6$, and experimental results are for $n = 6{\cdot}05$. ⊙, Michel ((9)), $\tau = 0{\cdot}08$.

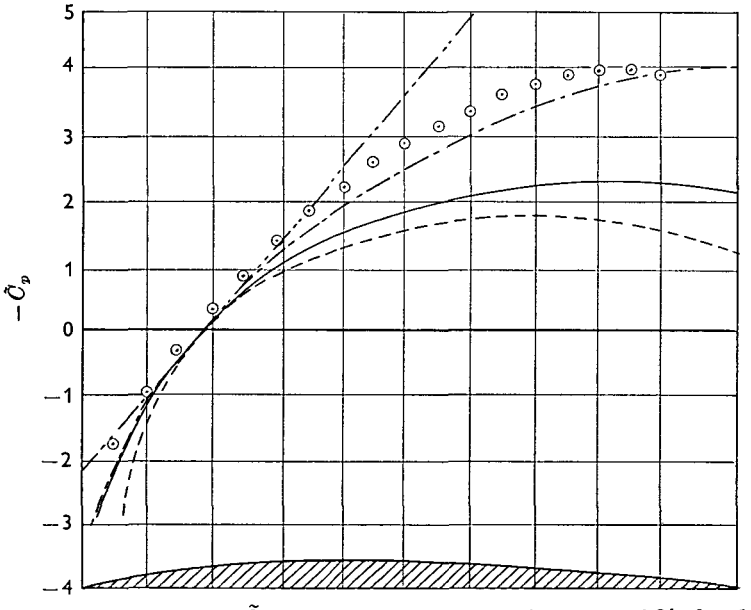


Fig. 5. Pressure distribution $\tilde{C}_p$. Airfoil maximum thickness = 40 % chord. ———, Present method; —··—··—, $\tilde{C}_p$ corresponding to a velocity $a^2/(\gamma+1)\,z$; – – – –, Maeder and Thommen ((7)); —·—·—, Spreiter ((13)); N.B. In the notation of (5·12), theoretical results are for $n = 3$, and experimental results are for $n = 3{\cdot}38$. ⊙, Michel ((9)), $\tau = 0{\cdot}08$.

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neighbouring profiles. However, in the axisymmetric case ((6)), the results constitute little improvement on ((7)), and are inferior to those of Cole and Royce in ((3)).

It is considered that, in the two-dimensional case, in view of the ease of application of ((7)) and its improvements, the advantage in using the present method is perhaps marginal. However, the reasonable agreement with experiment when the maximum disturbance (i.e. maximum thickness) is towards the rear, together with the excellent results of ((6)) for the axisymmetric case with maximum disturbance towards the rear, suggest that for three-dimensional shapes satisfying this condition, fairly accurate results may be obtained. One such shape is the delta, and it is hoped to consider in a later paper the application of the present approach to this shape.

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