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Generalized $\mathcal{D}$-Einstein Real Hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$

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Abstract. In this paper we obtain some new characterizations of pseudo-Einstein real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$. More precisely, we prove that a real hypersurface in $\mathbb{C}P^2$ or $\mathbb{C}H^2$ with constant mean curvature is generalized $\mathcal{D}$-Einstein with constant coefficient if and only if it is pseudo-Einstein. We prove that a real hypersurface in $\mathbb{C}P^2$ with constant scalar curvature is generalized $\mathcal{D}$-Einstein with constant coefficient if and only if it is pseudo-Einstein.

1 Introduction

Let $M^n(c)$ be a complex space form of complex dimension n and let M be a real hypersurface of real dimension $2n-1$, $n \geq 2$. On M there exists a natural almost contact metric structure (ϕ, ξ, η, g) induced from the complex structure on the ambient space and the normal vector field. In this paper, we denote by $\mathcal{D}$ the holomorphic distribution determined by all tangent vector fields orthogonal to the structure vector field ξ. If ξ is principal, the hypersurface is said to be Hopf.

A real hypersurface M in a nonflat complex space form $M^n(c)$, $c \neq 0$, is said to be Einstein if its Ricci tensor Ric satisfies

$$\mathrm{Ric}(X, Y) = \mathrm{ag}(X, Y) \text{ for any } X, Y \in \mathfrak{X}(M), \tag{1.1}$$

where a is a constant and $\mathfrak{X}(M)$ is the Lie algebra of all tangent vector fields on M. It is well known that there are no Einstein real hypersurfaces in complex projective space $\mathbb{C}P^n$ (for $n = 2$, see [15] and for $n > 2$, see [13]) and complex hyperbolic space $\mathbb{C}H^n$ (for $n = 2$ see [15] and for $n > 2$ see [14]). In view of this, in 1979, Masahiro Kon [9] introduced the so-called pseudo-Einstein condition, *i.e.*,

$$\mathrm{Ric}(X, Y) = \mathrm{ag}(X, Y) + b\eta(X)\eta(Y) \text{ for any } X, Y \in \mathfrak{X}(M), \tag{1.2}$$

where both a and b are constants and the 1-form η is defined by $\eta(\cdot) := g(\xi, \cdot)$. Such a condition was also said to be quasi or η-Einstein in the literature. Real hypersurfaces of dimension at least five in complex projective space $\mathbb{C}P^n$ satisfying (1.2) were classified in [9]. Also, the pseudo-Einstein real hypersurfaces of dimension at least five in complex hyperbolic space $\mathbb{C}H^n$ were classified by S. Montiel [14] (see also Cecil and Ryan [1]). We remark that pseudo-Einstein real hypersurfaces in a nonflat complex space form of complex dimension two were classified by Kim and Ryan [8] in case of

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$\mathbb{C}P^2$ and also by Ivey and Ryan [3] in case of $\mathbb{C}H^2$. Since then, some conditions for characterizing pseudo-Einstein real hypersurfaces were extensively investigated in recent literature. For example, it was proved in [5, Main theorem] that a real hypersurface in a nonflat complex space form of complex dimension two is pseudo-Einstein if and only if the Ricci operator Q commutes with the structure operator ϕ. This improves corresponding results in [6]. Mayuko Kon [11] proved that a real hypersurface in a nonflat complex space form of complex dimension two is pseudo-Einstein if and only if the Ricci operator is parallel along the structure vector field and the structure vector field is an eigenvector field of the Ricci operator. Some other characterizations for pseudo-Einstein real hypersurfaces can be seen in [4,7,16].

In 2000, Ortega and Pérez [17] introduced the $\mathcal{D}$-Einstein condition; *i.e.,*

$$QX = aX \text{ for any } X \in \mathcal{D}, \tag{1.3}$$

where a is a constant and Q is the associated Ricci operator of the Ricci tensor. Comparing (1.3) with (1.2), we see that every pseudo-Einstein hypersurface is $\mathcal{D}$-Einstein. Conversely, a $\mathcal{D}$-Einstein hypersurface will be pseudo-Einstein if and only if ξ is an eigenvector of the Ricci operator. $\mathcal{D}$-Einstein real hypersurfaces in nonflat quaternionic space forms were classified in [17, 20]. But, as far as we know, $\mathcal{D}$-Einstein real hypersurfaces in nonflat complex space forms have not been studied extensively.

Recently, a new generalization of the pseudo-Einstein condition was introduced by Mayuko Kon [10], that is,

$$\mathrm{Ric}(X, Y) = \mathrm{ag}(X, Y) \text{ for any } X, Y \in \mathcal{D}, \tag{1.4}$$

where a is a constant. It is obvious that (1.3) is a special case of (1.4) and the converse is also true only when ξ is an eigenvector field of the Ricci operator. In view of this, we give the following definition.

Definition 1.1 *A real hypersurface in a nonflat complex space form is said to be generalized $\mathcal{D}$-Einstein if it satisfies* (1.4) *with a a function. In particular, if $a \in \mathbb{R}$, we say that the hypersurface is generalized $\mathcal{D}$-Einstein with constant coefficient.*

Mayuko Kon in [10] studied such a condition and obtained that a generalized $\mathcal{D}$-Einstein real hypersurface in a nonflat complex space form $M^n(c)$, $n \geq 3$, having constant principal curvatures is pseudo-Einstein. Note that the above result was extended by Mayuko Kon in [12] with $a \in \mathbb{R}$. In fact, if all principal curvatures of a generalized $\mathcal{D}$-Einstein real hypersurface are constants, then the mean curvature is a constant and hence a is also a constant. Applying Definition 1.1, we rewrite Kon's result as follows.

Theorem 1.2 ([12]) *A generalized $\mathcal{D}$-Einstein real hypersurface with constant coefficient in a nonflat complex space form $M^n(c)$, $n \geq 3$, is pseudo-Einstein.*

In this paper, in order to present some new characterizations for pseudo-Einstein real hypersurfaces of dimension three, we continue to study the generalized $\mathcal{D}$-Einstein condition. First, we have the following theorem.

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Theorem 1.3 *A non-Hopf hypersurface in $\mathbb{C}P^2$ or $\mathbb{C}H^2$ cannot satisfy (1.3) even if a is allowed to be a function.*

Extending Theorem 1.2 to dimension three, we obtain the following theorem.

Theorem 1.4 *A real hypersurface in a nonflat complex space form of complex dimension two with constant mean curvature is generalized $\mathcal{D}$-Einstein with constant coefficient if and only if it is pseudo-Einstein.*

Replacing the constant mean curvature assumption by constant scalar curvature, we also obtain the following theorem.

Theorem 1.5 *A real hypersurface in $\mathbb{C}P^2$ with constant scalar curvature is generalized $\mathcal{D}$-Einstein with constant coefficient if and only if it is pseudo-Einstein.*

Some remarks and an open question raised by the above theorems are given in the last section.

2 Preliminaries

A complex n-dimensional Kählerian manifold of constant holomorphic sectional curvature c is said to be a *complex space form* and is denoted by $M^n(c)$. A complete and simply connected complex space form is complex analytically isometric to a complex projective space $\mathbb{C}P^n(c)$, a complex Euclidean space $\mathbb{C}^n$ or a complex hyperbolic space $\mathbb{C}H^n(c)$ according to $c > 0$, $c = 0$, or $c < 0$, respectively. Let M be a real hypersurface immersed in a complex space form $M^n(c)$ and let N be a unit normal vector field of M. We denote by $\overline{\nabla}$ the Levi–Civita connection of the metric $\overline{g}$ of $M^n(c)$ and J the complex structure. Let g and ∇ be the induced metric from the ambient space and the Levi–Civita connection of g respectively. Then the Gauss and Weingarten formulas are given respectively as the following:

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \overline{\nabla}_X N = -AX \tag{2.1}$$

for any vector fields X and Y tangent to M, where A denotes the shape operator of M in $M^n(c)$. For any vector field X tangent to M, we put

$$JX = \phi X + \eta(X)N, \quad JN = -\xi. \tag{2.2}$$

We can define on M an almost contact metric structure (ϕ, ξ, η, g) satisfying

$$\phi^2 = -\mathrm{id} + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \tag{2.3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi) \tag{2.4}$$

for any vector fields X and Y on M. If the structure vector field ξ is *principal*, that is, $A\xi = \alpha\xi$, where $\alpha = \eta(A\xi)$, then M is called a *Hopf hypersurface* and α is called *Hopf principal curvature*.

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Moreover, applying the parallelism of the complex structure (*i.e.*, $\overline{\nabla} J = 0$) of $M^n(c)$ and using (2.1), (2.2), we have

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{2.5}$$

$$\nabla_X \xi = \phi AX \tag{2.6}$$

for any vector fields X and Y. We denote by R the Riemannian curvature tensor of M. Since $M^n(c)$ is assumed to be of constant holomorphic sectional curvature c, then the Gauss and Codazzi equations of M in $M^n(c)$ are given respectively as the following:

$$\begin{aligned} R(X, Y)Z = \frac{c}{4}\{&g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X \\ &- g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\} \\ &+ g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{2.7}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\} \tag{2.8}$$

for any vector fields X, Y on M.

From (2.7), we see that the Ricci operator Q is given by

$$QX = \frac{c}{4}\big((2n+1)X - 3\eta(X)\xi\big) + mAX - A^2X \tag{2.9}$$

for any vector field X tangent to the hypersurface, where $m := \mathrm{trace}A$ is the mean curvature.

In this paper, all manifolds are assumed to be connected and of class C^∞.

3 Generalized $\mathcal{D}$-Einstein Real Hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$

Because the pseudo-Einstein hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$ are well known and they are all Hopf, we separate our arguments on the generalized $\mathcal{D}$-Einstein condition into the following two cases.

3.1 The Hopf Case

In this case, we give the following theorem directly.

Theorem 3.1 *For a Hopf hypersurface M in $\mathbb{C}P^2$ or $\mathbb{C}H^2$, the following conditions are equivalent:*

(i) *M is pseudo-Einstein;*
(ii) *M is $\mathcal{D}$-Einstein;*
(iii) *M is generalized $\mathcal{D}$-Einstein.*

Proof The fact that ((i) implies (ii) implies (iii)) is clear from their definitions. Now, at any point in the Hopf hypersurface M, we can use [2, Corollary 6.18] to write the matrix forms of A and Q with respect to a suitable orthonormal basis $\{\xi, U, \phi U\}$ as

$$A = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \mu \end{pmatrix}$$

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and

$$Q = \begin{pmatrix} \frac{1}{2}c + m\alpha - \alpha^2 & 0 & 0 \\ 0 & \frac{5}{4}c + m\gamma - \gamma^2 & 0 \\ 0 & 0 & \frac{5}{4}c + m\mu - \mu^2 \end{pmatrix},$$

where $m = \alpha + \gamma + \mu$.

Because Q leaves $\mathcal{D}$ invariant, it is immediate that (1.3) and (1.4) are equivalent at the point in question and that they are satisfied if and only if

$$(m - \gamma - \mu)(\gamma - \mu) = \alpha(\gamma - \mu) = 0.$$

Recall that for a Hopf hypersurface, α is constant (see [16]). If $\alpha = 0$, then M is generalized $\mathcal{D}$-Einstein with constant coefficient $a = \frac{5}{4}c + \gamma\mu$. But, again by [2, Corollary 6.18], $\gamma\mu = \frac{1}{4}c$, whether or not $\gamma = \mu$. Thus, $a = \frac{3}{2}c$, M is also $\mathcal{D}$-Einstein and Q simplifies as follows:

$$Q = \begin{pmatrix} \frac{1}{2}c & 0 & 0 \\ 0 & \frac{3}{2}c & 0 \\ 0 & 0 & \frac{3}{2}c \end{pmatrix}.$$

In the format of (1.2),

$$\mathrm{Ric}(X, Y) = \frac{3}{2}cg(X, Y) - c\eta(X)\eta(Y)$$

so that M is pseudo-Einstein as well.

In case $\alpha \neq 0$, assuming that M is generalized $\mathcal{D}$-Einstein, we have $\gamma = \mu$. Applying this, according to case (ii) of [16, Corollary 2.3] or [2, Corollary 6.18], we have

$$\gamma^2 = \alpha\gamma + \frac{c}{4}.$$

Hence, by continuity, γ is constant on M. Further,

$$a = \frac{5}{4}c + \alpha\gamma + \gamma^2 = \frac{3}{2}c + 2\alpha\gamma.$$

Consequently, M is $\mathcal{D}$-Einstein and since

$$\mathrm{Ric}(X, Y) = \left(\frac{3}{2}c + 2\alpha\gamma\right)g(X, Y) - c\eta(X)\eta(Y),$$

M is also pseudo-Einstein. This completes the proof that ((iii) implies (ii) implies (i)). ∎

3.2 The Non-Hopf Case

Let M be a non-Hopf hypersurface in a nonflat complex space form $M^n(c)$, $n \geq 2$. Let Ω be a connected open set on which ξ is not principal. By (2.6) and (2.3), $\nabla_\xi\xi = \phi A\xi \neq 0$. We denote by β the length of the nonzero vector field $-\phi\nabla_\xi\xi = A\xi - \eta(A\xi)\xi$. Let U be a unit vector field with the same direction as $-\phi\nabla_\xi\xi$, so that $\beta U = -\phi\nabla_\xi\xi$ and

$$A\xi = \alpha\xi + \beta U \tag{3.1}$$

on Ω, where $\alpha = \eta(A\xi)$. In particular, when $n = 2$, $\{\xi, U, \phi U\}$ is an orthonormal frame on Ω and we have the following lemma.

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Lemma 3.2 ([19, Lemma 1]) *The following relations hold on* Ω*:*

$$
\begin{aligned}
&AU = \gamma U + \delta\phi U + \beta\xi, && A\phi U = \delta U + \mu\phi U, && \\
&\nabla_U\xi = -\delta U + \gamma\phi U, && \nabla_{\phi U}\xi = -\mu U + \delta\phi U, && \nabla_\xi\xi = \beta\phi U,\\
&\nabla_U U = \kappa_1\phi U + \delta\xi, && \nabla_{\phi U}U = \kappa_2\phi U + \mu\xi, && \nabla_\xi U = \kappa_3\phi U,\\
&\nabla_U\phi U = -\kappa_1 U - \gamma\xi, && \nabla_{\phi U}\phi U = -\kappa_2 U - \delta\xi, && \nabla_\xi\phi U = -\kappa_3 U - \beta\xi,
\end{aligned} \tag{3.2}
$$

where γ*,* δ*,* μ*,* κ_i*,* $i = \{1, 2, 3\}$*, are smooth functions on* Ω*.*

Applying this lemma, from the Codazzi equation (2.8) for $X = U$ or $X = \phi U$ and $Y = \xi$, we have

$$U(\beta) - \xi(\gamma) = \alpha\delta - 2\delta\kappa_3, \tag{3.3}$$

$$\xi(\delta) = \alpha\gamma + \beta\kappa_1 + \delta^2 + \mu\kappa_3 + \frac{c}{4} - \gamma\mu - \gamma\kappa_3 - \beta^2, \tag{3.4}$$

$$U(\alpha) - \xi(\beta) = -3\beta\delta, \tag{3.5}$$

$$\xi(\mu) = \alpha\delta + \beta\kappa_2 - 2\delta\kappa_3, \tag{3.6}$$

$$\phi U(\alpha) = \alpha\beta + \beta\kappa_3 - 3\beta\mu, \tag{3.7}$$

$$\phi U(\beta) = \alpha\mu - 2\gamma\mu + 2\delta^2 + \frac{c}{2} + \alpha\gamma + \beta\kappa_1. \tag{3.8}$$

Similarly, from the Codazzi equation for $X = U$ and $Y = \phi U$, we have

$$U(\delta) - \phi U(\gamma) = \mu\kappa_1 - \gamma\kappa_1 - \beta\gamma - 2\delta\kappa_2 - 2\beta\mu, \tag{3.9}$$

$$U(\mu) - \phi U(\delta) = \gamma\kappa_2 + \beta\delta - \kappa_2\mu - 2\delta\kappa_1. \tag{3.10}$$

Moreover, applying Lemma 3.2 again, from the Gauss equation (2.7) and the definition of the Riemannian curvature tensor $R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$, we have

$$U(\kappa_2) - \phi U(\kappa_1) = 2\delta^2 - 2\gamma\mu - \kappa_1^2 - \gamma\kappa_3 - \kappa_2^2 - \mu\kappa_3 - c. \tag{3.11}$$

$$\phi U(\kappa_3) - \xi(\kappa_2) = 2\beta\mu - \mu\kappa_1 + \delta\kappa_2 + \kappa_3\kappa_1 + \beta\kappa_3. \tag{3.12}$$

The above relations can also be seen in [18].

Lemma 3.3 *Under the conditions of Lemma 3.2,* Ω *is generalized* $\mathcal{D}$*-Einstein if and only if* (3.17)–(3.21) *hold.*

Proof First, a simple computation implies that the generalized $\mathcal{D}$-Einstein condition (1.4) is equivalent to

$$QX = aX + (\eta(QX) - a\eta(X) - \mathrm{Ric}(\xi, \xi)\eta(X))\xi + \eta(X)Q\xi \tag{3.13}$$

for any vector field X. One the one hand, the applications of (3.1) and (3.2) in (2.9) for $n = 2$ yield

$$Q\xi = \Big(\frac{c}{2} + \alpha(\gamma + \mu) - \beta^2\Big)\xi + \beta\mu U - \beta\delta\phi U, \tag{3.14}$$

$$QU = \beta\mu\xi + \Big(\frac{5}{4}c + \gamma(\alpha + \mu) - \beta^2 - \delta^2\Big)U + \alpha\delta\phi U, \tag{3.15}$$

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and

$$Q\phi U = -\beta\delta\xi + \alpha\delta U + \Big(\frac{5}{4}c + \mu(\alpha+\gamma) - \delta^2\Big)\phi U. \tag{3.16}$$

On the other hand, it follows directly from (3.13) that

$$QU = \eta(QU)\xi + aU \quad \text{and} \quad Q\phi U = \eta(Q\phi U)\xi + a\phi U.$$

Comparing the previous two relations with (3.15) and (3.16), respectively, we have that the hypersurface is generalized $\mathcal{D}$-Einstein if and only if the following five equations hold:

$$\alpha\delta = 0, \tag{3.17}$$

$$\eta(QU) = \beta\mu, \tag{3.18}$$

$$\eta(Q\phi U) = -\beta\delta, \tag{3.19}$$

$$a = \frac{5}{4}c + \mu(\alpha+\gamma) - \delta^2, \tag{3.20}$$

$$a = \frac{5}{4}c + \gamma(\alpha+\mu) - \beta^2 - \delta^2. \tag{3.21}$$

This completes the proof. ∎

For a real hypersurface in a nonflat complex space form of dimension > 3, if (1.3) is true for certain $a \in \mathbb{R}$, it has been proved in [12] that the hypersurface is pseudo-Einstein. Thus, next we consider real hypersurfaces of dimension three satisfying (1.3) in which a is not necessarily a constant.

Proof of Theorem 1.3 Assume that Ω, U, etc. are defined as above. If (1.3) holds with a a function, then according to (3.15) and (3.16), we have

$$\beta\mu = 0, \quad \alpha\delta = 0, \quad a = \frac{5}{4}c + \gamma(\alpha+\mu) - \beta^2 - \delta^2,$$
$$\beta\delta = 0, \quad a = \frac{5}{4}c + \mu(\alpha+\gamma) - \delta^2.$$

Applying this in (3.14)–(3.16), we see that ξ, U, and ϕU are eigenvectors of the Ricci operator. In matrix form,

$$Q = \begin{pmatrix} \frac{1}{2}c & 0 & 0 \\ 0 & \frac{5}{4}c & 0 \\ 0 & 0 & \frac{5}{4}c \end{pmatrix}.$$

Hence, Ω is pseudo-Einstein, a contradiction, because all pseudo-Einstein hypersurfaces are Hopf. (See discussion surrounding [2, formula (8.85)]). ∎

Since the Hopf case was dealt with earlier, Theorem 1.4 follows immediately from the following lemma.

Lemma 3.4 *If a non-Hopf real hypersurface in $\mathbb{C}P^2$ or $\mathbb{C}H^2$ has constant mean curvature and satisfies (1.4), a cannot be constant.*

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Proof Assume that Ω, U, etc. are defined as above and that a is constant. Now, from the first term of (3.17), we obtain $\delta = 0$, because of $\alpha \neq 0$ on Ω. In fact, if $\alpha = 0$ on some small open set, the subtraction of (3.20) from (3.21) gives $\beta = 0$, a contradiction. Now the shape operator is given by

$$A = \begin{pmatrix} \alpha & \beta & 0 \\ \beta & \gamma & 0 \\ 0 & 0 & \mu \end{pmatrix}$$

with respect to the local orthonormal basis $\{\xi, U, \phi U\}$, and thus the assumption of constancy of mean curvature implies that

$$\alpha + \gamma + \mu \text{ is a constant.} \tag{3.22}$$

Combining (3.22) with (3.20) we observe that μ is also a constant. Using $\mu \in \mathbb{R}$ in (3.6), with the help of $\delta = 0$ and $\beta \neq 0$, we have $k_2 = 0$.

Because $a, \mu \in \mathbb{R}$ and $\delta = 0$, taking the differentiation of relation (3.21) along ξ, we obtain

$$(\alpha + \mu)\xi(\gamma) + \gamma\xi(\alpha) - 2\beta\xi(\beta) = 0,$$

where we have employed (3.6) with $\delta = k_2 = 0$. As μ is a constant, from (3.22) we know $\alpha + \gamma$ is also a constant. Applying this in the previous relation, we obtain

$$(\alpha + \mu - \gamma)\xi(\gamma) - 2\beta\xi(\beta) = 0. \tag{3.23}$$

As $a \in \mathbb{R}$, $\delta = 0$, taking the differentiation of relation (3.21) along U, we obtain

$$(\alpha + \mu)U(\gamma) + \gamma U(\alpha) + \gamma U(\mu) - 2\beta U(\beta) = 0.$$

Because of $\delta = 0$ and $k_2 = 0$, from (3.5), (3.10) and (3.3) we obtain $U(\alpha) = \xi(\beta)$, $U(\mu) = 0$, and $U(\beta) = \xi(\gamma)$, respectively. Applying these in the previous relation, we obtain

$$(\alpha + \mu)U(\gamma) + \gamma\xi(\beta) - 2\beta\xi(\gamma) = 0. \tag{3.24}$$

Moreover, taking the differentiation of (3.20) along U gives $\mu(\xi(\beta) + U(\gamma)) = 0$, where we have applied $\delta = k_2 = 0$, $\mu \in \mathbb{R}$ and (3.5). If $\mu = 0$, in view of $\delta = 0$, (3.18), and (3.19), we see that ξ is an eigenvector field of the Ricci operator and hence the hypersurface is pseudo-Einstein, a contradiction. Therefore, we obtain $\mu \neq 0$ on Ω, and hence $U(\gamma) = -\xi(\beta)$. Using this in (3.24), we obtain

$$(\gamma - \alpha - \mu)\xi(\beta) - 2\beta\xi(\gamma) = 0.$$

Since $\beta \neq 0$ on Ω, combining the above equation with (3.23), we obtain

$$\xi(\beta) = \xi(\gamma) = 0.$$

Now, with the help of $\delta = 0$ and $\xi(\gamma) = 0$, it follows from (3.3) that $U(\beta) = 0$. Applying again $\delta = 0$, from Lemma 3.2, we have $[\xi, U] = (k_3 - \gamma)\phi U$ and taking the differentiation of β along $[\xi, U]$, we obtain

$$(k_3 - \gamma)\phi U(\beta) = 0.$$

If $\phi U(\beta)$ is identically zero in some small open set, β must be a constant. From (3.21) and (3.22) we observe that γ is a constant. Note that if $\gamma \in \mathbb{R}$, from (3.22) we

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observe that α is also a constant. Because of $\gamma \in \mathbb{R}$, $\delta = k_2 = 0$, from (3.9) we obtain $(\mu - \gamma)k_1 = \beta(\gamma + 2\mu)$. In view of $\delta = 0$, subtracting (3.21) from (3.20) implies

$$\alpha(\mu - \gamma) + \beta^2 = 0 \tag{3.25}$$

and applying this in the previous relation, we obtain $\beta k_1 = -\alpha(\gamma + 2\mu)$. Now, k_1 is also a constant. As $\alpha \in \mathbb{R}$ and $\beta \neq 0$ on Ω, from (3.7), we have $\alpha + k_3 = 3\mu$. Applying this in (3.4) we get $\alpha\gamma + \beta k_1 + 3\mu^2 - 4\mu\gamma + \frac{1}{4}c = 0$, where we have used $\delta = 0$ and (3.25). Subtracting $\beta k_1 = -\alpha(\gamma + 2\mu)$ from the previous relation, we get

$$3\mu^2 - 2\alpha\mu - 4\gamma\mu + \frac{1}{4}c = 0. \tag{3.26}$$

On the other hand, as $\beta \in \mathbb{R}$ and $\delta = 0$, from (3.8), we have $\alpha\mu - 2\gamma\mu + \alpha\gamma + \beta k_1 + \frac{1}{2}c = 0$, which is simplified with $\beta k_1 = -\alpha(\gamma + 2\mu)$ implying $\alpha\mu + 2\gamma\mu - \frac{1}{2}c = 0$. Combining this with (3.26) we obtain $c = 4\mu^2$, and hence $\alpha = 2\mu - 2\gamma$, due to $\mu \neq 0$. Finally, applying this and putting $\delta = k_2 = 0$, $k_3 = 3\mu - \alpha = \mu + 2\gamma$ into (3.11) we obtain $5\gamma\mu + k_1^2 + 2\gamma^2 + 5\mu^2 = 0$, or equivalently,

$$\left(\gamma + \frac{5}{4}\mu\right)^2 + \frac{15}{16}\mu^2 + \frac{1}{2}k_1^2 = 0.$$

It follows directly that $\mu = 0$, and hence $c = 4\mu^2 = 0$, a contradiction.

Thus, we have the existence of a point, and therefore a small open set, where $\phi U(\beta) \neq 0$ and hence $k_3 = \gamma$. Because $\alpha + \gamma$ is a constant, taking the differentiation of $\alpha + \gamma$ along ϕU, with the help of (3.7), (3.9), and $\delta = k_2 = 0$, we obtain $\beta(\alpha + 2\gamma - \mu) = (\mu - \gamma)k_1$, which is simplified with (3.25) (note that this relation is still true in this situation) implying

$$\alpha(\alpha + 2\gamma - \mu) + \beta k_1 = 0.$$

With the aid of $\delta = 0$, from (3.4) and $k_3 = \gamma$, we have $\alpha\gamma + \beta k_1 - \gamma^2 - \beta^2 + \frac{1}{4}c = 0$. Subtracting this from the above equation, we get $\alpha^2 + \alpha\gamma - \alpha\mu + \gamma^2 + \beta^2 - \frac{1}{4}c = 0$, which is simplified with (3.25) implying $\alpha^2 + \gamma^2 + 2\beta^2 = \frac{1}{4}c$. Because $\alpha + \gamma$ and μ are constants, applying (3.25) in this relation, we obtain $(\alpha + \gamma)^2 - 2\alpha\mu = \frac{1}{4}c$ and hence α, γ are constants. Consequently, it follows immediately from (3.21) that β is a constant, and hence $\phi U(\beta) = 0$, a contradiction. ∎

We now consider Theorem 1.5, which deals with the constant scalar curvature case. This theorem follows immediately from the following lemma.

Lemma 3.5 *If a non-Hopf real hypersurface in $\mathbb{C}P^2$ has constant scalar curvature and satisfies* (1.4)*, then a cannot be constant.*

Proof Assume that Ω, U, etc. are defined as above and that a is constant. Because (1.4) is true with $a \in \mathbb{R}$, according to (3.14), the scalar curvature is a constant if and only if $\alpha(\gamma + \mu) - \beta^2$ is a constant. As proved at beginning of the proof of Lemma 3.4, we have $\alpha \neq 0$ and hence $\delta = 0$ on Ω. The addition of (3.20) to (3.21) implies $2a - \frac{5}{2}c = \alpha(\gamma + \mu) - \beta^2 + 2\gamma\mu$, and hence $\gamma\mu$ is also a constant, because of $a \in \mathbb{R}$ and $\delta = 0$. Applying again $\delta = 0$ and $\gamma\mu \in \mathbb{R}$ in (3.20), we obtain $\alpha\mu \in \mathbb{R}$. Similarly, applying $a, \gamma\mu \in \mathbb{R}$ and $\delta = 0$ in (3.21) implies that $\alpha\gamma - \beta^2 \in \mathbb{R}$.

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In view of $\gamma\mu \in \mathbb{R}$, taking the differentiation of this equation along ξ and U respectively, we obtain

$$\beta\gamma k_2 + \mu\xi(\gamma) = 0 \quad \text{and} \quad \gamma(\gamma - \mu)k_2 + \mu U(\gamma) = 0, \tag{3.27}$$

where we have used $\delta = 0$, (3.6) and (3.10). Similarly, taking the differentiation of $\alpha\mu \in \mathbb{R}$ along ξ and U respectively, we obtain

$$\alpha\beta k_2 + \mu\xi(\alpha) = 0 \quad \text{and} \quad \alpha(\gamma - \mu)k_2 + \mu\xi(\beta) = 0, \tag{3.28}$$

where we have employed $\delta = 0$, (3.5), (3.6), and (3.10). Similarly, taking the differentiation of $\alpha\gamma - \beta^2 \in \mathbb{R}$ along ξ and U respectively, we obtain

$$\alpha\xi(\gamma) + \gamma\xi(\alpha) - 2\beta\xi(\beta) = 0 \quad \text{and} \quad \alpha U(\gamma) + \gamma\xi(\beta) - 2\beta\xi(\gamma) = 0, \tag{3.29}$$

where we have used $\delta = 0$, (3.3) and (3.5).

We remark that in this case we can (reducing the size of Ω if necessary) assert that $\mu \neq 0$ on Ω. This is because if μ were identically zero, (3.14)–(3.16) would imply that Ω was pseudo-Einstein, a contradiction. Thus, a point exists where $\mu \neq 0$ and we can redefine Ω to be a sufficiently small neighborhood of that point. In view of $\mu \neq 0$, $\alpha \neq 0$, and $\beta \neq 0$, putting the first term of (3.27) and (3.28) into the first term of (3.29) gives $k_2 = 0$, which is used in (3.27) and (3.28) implying $\xi(\alpha) = 0$, $\xi(\beta) = 0$, $\xi(\gamma) = 0$, and $U(\gamma) = 0$. Applying $\delta = 0$, from Lemma 3.2 we have $[\xi, U] = (k_3 - \gamma)\phi U$, and taking the differentiation of γ along $[\xi, U]$, we obtain

$$(k_3 - \gamma)\phi U(\gamma) = 0.$$

If $\phi U(\gamma) = 0$ in small open set, we have $\gamma \in \mathbb{R}$ and hence $\mu \in \mathbb{R}$ due to $\gamma\mu \in \mathbb{R}$. Consequently, we also have $\alpha \in \mathbb{R}$ due to $\alpha\mu \in \mathbb{R}$. Now the mean curvature of the hypersurface is a constant and hence the proof follows immediately from Lemma 3.4.

Thus, we have the existence of a point, and therefore a small open set, where $\phi U(\gamma) \neq 0$ and hence $k_3 = \gamma$. Applying this and taking the differentiation of $\alpha\gamma - \beta^2 \in \mathbb{R}$ along ϕU gives $\alpha\phi U(\gamma) + \gamma\phi U(\alpha) - 2\beta\phi U(\beta) = 0$, which is simplified with (3.7), (3.8) and (3.9) implying $\alpha k_1(\gamma - \mu) + \beta\gamma^2 + \beta\gamma\mu - c\beta - 2\beta^2 k_1 = 0$. In view of $\delta = 0$, subtracting (3.21) from (3.20) implies (3.25). For simplicity and later use, we rewrite it again as follows:

$$\alpha(\gamma - \mu) = \beta^2. \tag{3.30}$$

In view of $\beta \neq 0$ on Ω, putting (3.30) into $\alpha k_1(\gamma - \mu) + \beta\gamma^2 + \beta\gamma\mu - c\beta - 2\beta^2 k_1 = 0$ we obtain

$$\gamma^2 + \gamma\mu - c - \beta k_1 = 0. \tag{3.31}$$

On the other hand, with the aid of $\delta = k_2 = 0$ and $k_3 = \gamma$, (3.4) becomes

$$\alpha\gamma + \beta k_1 + \frac{1}{4}c - \gamma^2 - \beta^2 = 0. \tag{3.32}$$

The addition of (3.31) to (3.32) reduces to $\alpha\gamma - \frac{3}{4}c - \beta^2 + \gamma\mu = 0$. With the help of $\delta = 0$, combining this with (3.21) gives $a = 2c$. Furthermore, using this and $\delta = 0$ in (3.20), we obtain

$$\alpha\mu + \gamma\mu = \frac{3}{4}c. \tag{3.33}$$

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With the aid of $\delta = k_2 = 0$, taking the differentiation of $\gamma\mu \in \mathbb{R}$ along ϕU gives

$$\gamma\phi U(\mu) + \mu(\gamma k_1 - \mu k_1 + \beta\gamma + 2\beta\mu) = 0,$$

where we have used (3.9). Similarly, with the help of $k_3 = \gamma$ and (3.7), taking the differentiation of $\alpha\mu \in \mathbb{R}$ along ϕU gives

$$\alpha\phi U(\mu) + \beta\mu(\alpha + \gamma - 3\mu) = 0.$$

Combining the above equation with the previous one implies $\alpha k_1(\gamma-\mu)+\alpha\beta(\gamma+2\mu) = \beta\gamma(\alpha + \gamma - 3\mu)$, because of $\mu \neq 0$, which is simplified with (3.30) implying

$$k_1\beta = \gamma^2 - 3\gamma\mu - 2\alpha\mu, \tag{3.34}$$

because of $\beta \neq 0$.

Adding (3.34) to (3.31), we obtain $\alpha\mu + 2\gamma\mu = \frac{1}{2}c$ which is compared with (3.33) giving

$$\gamma\mu = -\frac{1}{4}c, \quad \alpha\mu = c, \quad \alpha = -4\gamma. \tag{3.35}$$

Finally, substituting the last two relations of (3.35) in (3.30), we obtain

$$4\gamma^2 + \beta^2 + c = 0. \tag{3.36}$$

Notice that in the complex projective space, $c > 0$, from (3.36), we arrive at a contradiction. This completes the proof. ∎

The classification for pseudo-Einstein real hypersurfaces of dimension three is more difficult than that case of higher dimension. Such a situation still occurs for the classification of the generalized $\mathcal{D}$-Einstein real hypersurfaces. As seen in [12], the generalized $\mathcal{D}$-Einstein condition with a constant coefficient for a real hypersurface of dimension greater than three is equivalent to the pseudo-Einstein condition. However, the corresponding version of this result with a a function has not yet been established. Furthermore, we do not know whether a generalized $\mathcal{D}$-Einstein real hypersurface of dimension three is pseudo-Einstein or not, even when a is assumed to be a constant. In this paper, we classify generalized $\mathcal{D}$-Einstein real hypersurfaces of dimension three with constant coefficient under some additional conditions (*cf.* Theorems 1.4, 1.5). This leads to an interesting open question.

Problem 3.6 *Classify generalized $\mathcal{D}$-Einstein real hypersurfaces in a nonflat complex space form.*

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