J. Plasma Physics (2011), *vol.* 77, *part* 5, *pp.* 589–615. © Cambridge University Press 2011
doi:10.1017/S0022377810000760

Variational formulation for weakly nonlinear perturbations of ideal magnetohydrodynamics

M. HIROTA

Japan Atomic Energy Agency, Naka, Ibaraki, 311-0193, Japan
(hirota.makoto@jaea.go.jp)

(Received 26 November 2010 and accepted 7 December 2010;
first published online 26 January 2011)

Abstract. A new equation of motion that governs weakly nonlinear phenomena in ideal magnetohydrodynamics (MHDs) is derived as a natural extension of the well-known linearized equation of motion for the displacement field. This derivation is made possible by expanding the MHD Lagrangian explicitly up to third order with respect to the displacement of plasma, which necessitates an efficient use of the Lie series expansion. The resultant equation of motion (i.e. the Euler–Lagrange equation) includes a new quadratic force term which is responsible for various mode–mode coupling due to the MHD nonlinearity. The third-order potential energy serves to quantify the coupling coefficient among resonant three modes and its cubic symmetry proves the Manley–Rowe relations. In contrast to earlier works, the coupling coefficient is expressed only by the displacement vector field, which is already familiar in the linear MHD theory, and both the fixed and free boundary cases are treated systematically.

1. Introduction

A Set of magnetohydrodynamic (MHD) equations has been widely known as a fundamental model that describes macroscopic behaviour of plasma. The equilibrium and stability theories for MHD have been successfully applied to various large-scale structures in astrophysical and laboratory plasmas. In the meantime, however, it is hard to say that the nonlinear MHD theory is well established, in comparison to the sophisticated linear theory such as the energy principle [1]. In particular, we are still lacking a general theory that properly analyzes weakly nonlinear phenomena caused by large-amplitude MHD modes in non-uniform and bounded plasma, even though a number of nonlinear analyses have been made for specific issues of concern by taking case-by-case approaches. The goal of this paper is to establish a governing equation for weakly nonlinear perturbations of ideal MHD on the basis of the Lagrangian theory [2] and propose a practical method for analyzing nonlinear mode–mode couplings, so that it can be universally applicable to any ideal MHD mode around any structured equilibrium (like the energy principle).

Historically, three-wave interaction of plane MHD waves in uniform plasma is one of the most well-known nonlinear processes [3, 4], although it is already labouring to prove the corresponding Manley–Rowe relations directly from the MHD equations (see Brodin and Stenflo [5]). This theory of three-wave interaction has extensively been applied to non-uniform plasmas by means of the eikonal (or Wentzel-Kramers-Brillouin (WKB)) approximation (see Webb et al. [6] for the

https://doi.org/10.1017/S0022377810000760 Published online by Cambridge University Press

MHD case), where the dispersion relation for plane waves is locally available for short-wavelength waves. On the other hand, the interaction among global (or long-wavelength) modes has not been explored so much due to its analytical complications. Since the eikonal approximation is no longer valid for global modes, one must perform the weakly nonlinear analysis based upon the eigenvalue equation around non-uniform and bounded equilibria, instead of the local dispersion relation.

The Lagrangian–Hamiltonian theory can facilitate such the formulation of nonlinear mode coupling. Larsson [7, 8] and Axelsson [9] have derived the amplitude equations for resonant three eigenmodes in a quite general manner by increasing the number of field variables twice. This requirement of the same number of auxiliary variables reflects the non-canonical Hamiltonian property of the MHD system [10]. The resultant formulae of the coupling coefficients (which measure the strength of coupling) are, however, highly complicated because of the large number of variables, which force us to solve the accessibility equations [7] in addition to the linearized MHD equations. Moreover, while the satisfaction of the Manley–Rowe relations is guaranteed by the non-canonical Hamiltonian structure (i.e. the Lie–Poisson bracket), the associated symmetry among the coupling coefficients is not explicit from their lengthy expressions, which makes it difficult to extract further physical interpretations analytically. The present work will show that the coupling coefficients can be greatly reduced to a symmetric form in terms of a single intuitive variable, that is, the *displacement* vector field.

In contrast to earlier works [7–9], we shall employ the standard MHD action principle established by Newcomb [2]. It is based on the Lagrangian viewpoint of fluid mechanics and therefore the constrained variation must be taken so that all the Eulerian fields are varied in conjunction with the displacement of the infinitesimal fluid elements (or the fluid *particles*). Although our consideration will be restricted to such a class of perturbations that preserves the topology of the basic state, most ideal MHD oscillations and instabilities of interest occur within this class. Owing to this topological constraint (or the frozen-in condition), the field variables can be reduced to only the displacement field. In the linear MHD theory, the energy principle [1] already takes advantage of the displacement field and Frieman and Rotenberg [11] derived its evolution equation (referred to as the FR equation in this work). The small amplitude expansion of the MHD Lagrangian up to second order (demonstrated by Dewar [12]) can reproduce this FR equation as the Euler–Lagrange equation.

However, the next third-order MHD Lagrangian has not been shown in any literature, while it would enable us to study weakly nonlinear phenomena such as three-wave resonance. Although Brizard and Kaufman [13] proposed a general framework even allowing for non-eikonal perturbations, the naive Taylor expansion of the field Lagrangian in Ref. 13 is not appropriate for the MHD case (and the hydrodynamic case as well).

At the beginning of this work (Sec. 2), we will carefully define the displacement of plasma not as infinitesimal (namely, linear) perturbation but as finite (nonlinear) one, and claim that the constrained Eulerian fields should be subject to the Lie series expansion. The advantageous properties of the Lie series expansion will be remarked in Sec. 3. For the reader's convenience, the detailed procedure of deriving the third-order MHD Lagrangian will be presented in Appendix A.

https://doi.org/10.1017/S0022377810000760 Published online by Cambridge University Press

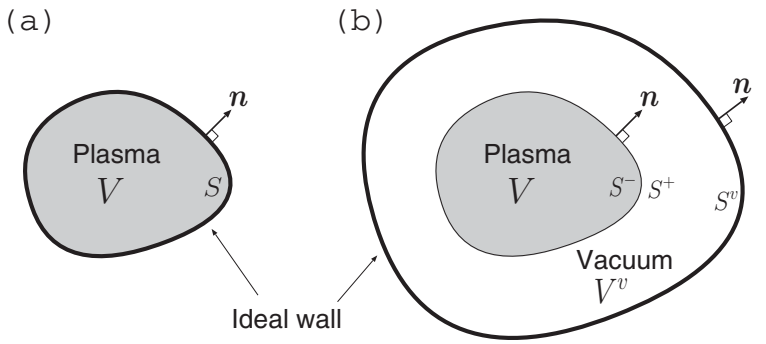


Figure 1. (a) Fixed boundary and (b) free boundary problems.

This paper is intended to deal with the standard set of MHD equations;

$$\rho\frac{\partial \boldsymbol{v}}{\partial t}+\rho(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}=-\nabla p+(\nabla\times\boldsymbol{B})\times\boldsymbol{B}, \tag{1.1}$$

$$\frac{\partial \boldsymbol{B}}{\partial t}=\nabla\times(\boldsymbol{v}\times\boldsymbol{B}), \tag{1.2}$$

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\boldsymbol{v})=0, \tag{1.3}$$

$$\frac{\partial s}{\partial t}+\boldsymbol{v}\cdot\nabla s=0, \tag{1.4}$$

where $\boldsymbol{v}$ is velocity field, $\boldsymbol{B}$ is magnetic field, ρ is mass density and s is specific entropy. The pressure $p(\rho,s)$ is a given function of ρ and s. We will consider nonlinear behaviour of perturbation around any exact solution $(\boldsymbol{v},\boldsymbol{B},\rho,s)$ that is supposed to be known. The boundary of plasma is also one of the significant spatial inhomogeneities. In Sec. 4, the fixed boundary case will be first considered [see Fig. 1(a)], where

$$\boldsymbol{n}\cdot\boldsymbol{v}=\boldsymbol{n}\cdot\boldsymbol{B}=0 \quad \text{on} \quad S(=\partial V). \tag{1.5}$$

The well-known FR equation of the displacement vector field will be rigorously extended to nonlinear regime thanks to the third-order Lagrangian with the cubic symmetry. In Sec. 5, the amplitude equations for resonant three eigenmodes will be formulated by Whitham's averaged-Lagrangian method [14]. The coupling coefficient will be associated with the third-order potential energy, whose cubic symmetry obviously proves the Manley–Rowe relations.

For the sake of practical use, we will also generalize this framework to the free boundary case [see Fig. 1(b)]. We assume that the unperturbed surface of plasma (denoted by S again) is stationary and identical with a magnetic surface, namely, the condition (1.5) holds at the unperturbed state. Such a free boundary problem is important for studying external modes of plasmas in various contexts. In Sec. 6, we will carefully consider the nonlinear displacement of the plasma-vacuum interface and the vacuum magnetic field as well. Then, legitimate third-order expressions of surface and vacuum potential energies will be formulated in Sec. 7. A summary and possible future applications of this work will be presented in Sec. 8.

https://doi.org/10.1017/S0022377810000760 Published online by Cambridge University Press

2. Lie series expansion of the MHD Lagrangian

In this section, we revisit Newcomb's variational principle for MHD [2] based on a little knowledge about differential geometry and show how to mathematically deal with nonlinear displacement of ideal plasma. This consideration will eventually require use of the Lie series expansion for the MHD Lagrangian.

Let $V_0 \subset \mathbb{R}^3$ be any domain filled with ideal plasma at $t = 0$. In the Lagrangian description, the dynamical variable is represented by an one-parameter group of mappings $\varphi_t : V_0 \to V_t$, $t \in \mathbb{R}$, that describes all orbits of fluid particles by $\boldsymbol{x}(t) = \varphi_t(\boldsymbol{X}) \in V_t$, where the initial position $\boldsymbol{X} \in V_0$ distinguishes each fluid particle and $V_t \subset \mathbb{R}^3$ is the plasma domain at time t. Then, the Eulerian velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$ is a vector field on V_t defined by $\partial_t \varphi_t(\boldsymbol{X}) = \boldsymbol{v}(\varphi_t(\boldsymbol{X}), t)$.

Let $\boldsymbol{x} = (x^1, x^2, x^3)$ be any curvilinear coordinates for the domain V_t, and $J(\boldsymbol{x})$ be the corresponding Jacobian. The magnetic field $\boldsymbol{B}(\boldsymbol{x}, t)$, the mass density $\rho(\boldsymbol{x}, t)$ and the specific entropy $s(\boldsymbol{x}, t)$ are, respectively, identified with differential 2-form β_t, 3-form ϱ_t and 0-form s_t as follows:

$$\beta_t = J(\boldsymbol{x})[B^1(\boldsymbol{x}, t)dx^2 \wedge dx^3 + B^2(\boldsymbol{x}, t)dx^3 \wedge dx^1 + B^3(\boldsymbol{x}, t)dx^1 \wedge dx^2], \tag{2.1}$$

$$\varrho_t = \rho(\boldsymbol{x}, t)d^3x = \rho(\boldsymbol{x}, t)J(\boldsymbol{x})dx^1 \wedge dx^2 \wedge dx^3, \tag{2.2}$$

$$s_t = s(\boldsymbol{x}, t), \tag{2.3}$$

where a constraint $d\beta_t = 0$ (implying $\nabla \cdot \boldsymbol{B} = 0$) is imposed on β_t. The MHD equations for $\boldsymbol{B}, \rho$ and s are then interpreted as

$$(1.2) \quad \Leftrightarrow \quad \frac{\partial \beta_t}{\partial t} + \mathscr{L}_{\boldsymbol{v}} \beta_t = 0, \tag{2.4}$$

$$(1.3) \quad \Leftrightarrow \quad \frac{\partial \varrho_t}{\partial t} + \mathscr{L}_{\boldsymbol{v}} \varrho_t = 0, \tag{2.5}$$

$$(1.4) \quad \Leftrightarrow \quad \frac{\partial s_t}{\partial t} + \mathscr{L}_{\boldsymbol{v}} s_t = 0, \tag{2.6}$$

where we have used the Lie derivative $\mathscr{L}_{\boldsymbol{v}}$ of the differential n-forms (a detailed review of the Lie derivative is given by Larsson [15]). If the fluid motion φ_t is given, the solutions of these equations are formally written as

$$\varphi_t^* \beta_t = \beta_0, \quad \varphi_t^* \varrho_t = \varrho_0, \quad \varphi_t^* s_t = s_0, \tag{2.7}$$

where φ_t^* denotes the pull-back of differential forms by the map φ_t and the initial data β_0, ϱ_0, s_0 are supposed to be provided on V_0.

In what follows, we regard φ_t as an unperturbed fluid motion and assume that it is perturbed into $\varphi_t \to \varphi_{\epsilon,t} \circ \varphi_t$, where the map $\varphi_{\epsilon,t} : V_t \to V_{\epsilon,t}$ represents nonlinear displacements of orbits and a parameter $\epsilon \in \mathbb{R}$ scales the amplitude of perturbation, $|\varphi_{\epsilon,t}(\boldsymbol{x}) - \boldsymbol{x}| \sim \epsilon$ (see Fig. 2). The conventional notation for the displacement of orbit would be

$$\boldsymbol{x}(t) \to \boldsymbol{x}(t) + \boldsymbol{\Xi}(\boldsymbol{x}(t), t). \tag{2.8}$$

As is obvious from the correspondence $\varphi_{\epsilon,t}(\boldsymbol{x}) = \boldsymbol{x} + \boldsymbol{\Xi}(\boldsymbol{x}, t)$, this nonlinear displacement $\boldsymbol{\Xi}$ is *not* a vector field, but a mapping. Needless to say, it is generally difficult to solve full-nonlinear evolution of $\boldsymbol{\Xi}$ directly.

https://doi.org/10.1017/S0022377810000760 Published online by Cambridge University Press

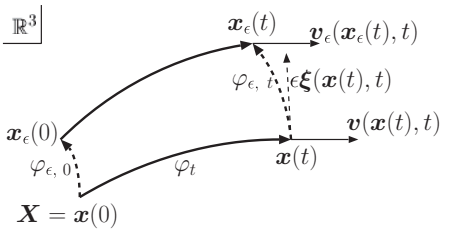


Figure 2. Unperturbed orbit $\boldsymbol{x}(t)$ and perturbed orbit $\boldsymbol{x}_\epsilon(t)$.

As far as the perturbation is not so large ($\epsilon \ll 1$), the perturbation expansion in powers of ϵ provides a systematic way to solve $\boldsymbol{\Xi}$ approximately. Now, we imaginarily regard ϵ as another continuous parameter of $\varphi_{\epsilon,t}$ that coincides with the identity map, $\boldsymbol{x} = \varphi_{0,t}(\boldsymbol{x})$, when $\epsilon = 0$. According to the theory of Lie groups, such a near-identity map $\varphi_{\epsilon,t}$ is represented by the exponential map without loss of generality;

$$\varphi_{\epsilon,t}(\boldsymbol{x}) = (e^{\epsilon\boldsymbol{\xi}\cdot\nabla})(\boldsymbol{x}) = \boldsymbol{x} + \epsilon\boldsymbol{\xi} + \frac{\epsilon^2}{2}\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi} + \frac{\epsilon^3}{6}\boldsymbol{\xi}\cdot\nabla(\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi}) + \ldots, \tag{2.9}$$

where $\boldsymbol{\xi}(\boldsymbol{x},t)$ is a vector field on V_t (i.e. for fixed t, the Lie group $\varphi_{\epsilon,t}$ is identified with the Lie algebra $\boldsymbol{\xi}$). In other words, $\boldsymbol{\xi}$ is understood as the infinitesimal generator of the displacement map $\varphi_{\epsilon,t}$,

$$\frac{\partial\varphi_{\epsilon,t}}{\partial\epsilon}(\boldsymbol{x}) = \boldsymbol{\xi}(\varphi_{\epsilon,t}(\boldsymbol{x}),t). \tag{2.10}$$

Note that there is no distinction between $\boldsymbol{\Xi}$ and $\epsilon\boldsymbol{\xi}$ within the linear perturbation theory $O(\epsilon)$. However, in the nonlinear stage $O(\epsilon^n)$ ($n \geqslant 2$), they need to be treated differently and the vector field $\boldsymbol{\xi}$ is usually more tractable than the map $\boldsymbol{\Xi}$. For instance, even the elementary vector formula is not applicable to $\boldsymbol{\Xi}$.

When the fluid particle orbits are perturbed as above, the perturbed magnetic field $\beta_{\epsilon,t}$ is given by $\varphi^*_{\epsilon,t}\beta_{\epsilon,t} = \beta_t$. Using the property of the exponential map $(\exp\epsilon\boldsymbol{\xi}\cdot\nabla)^* = \exp\epsilon\mathscr{L}_{\boldsymbol{\xi}}$, this $\beta_{\epsilon,t}$ is represented by the Lie series expansion,

$$\beta_{\epsilon,t} = e^{-\epsilon\mathscr{L}_{\boldsymbol{\xi}}}\beta_t = \beta_t - \epsilon\mathscr{L}_{\boldsymbol{\xi}}\beta_t + \frac{\epsilon^2}{2}\mathscr{L}_{\boldsymbol{\xi}}\mathscr{L}_{\boldsymbol{\xi}}\beta_t - \frac{\epsilon^3}{6}\mathscr{L}_{\boldsymbol{\xi}}\mathscr{L}_{\boldsymbol{\xi}}\mathscr{L}_{\boldsymbol{\xi}}\beta_t + \ldots \tag{2.11}$$

Similarly, we can express the perturbed density $\varrho_{\epsilon,t}$ and entropy $s_{\epsilon,t}$ in Lie series.

On the other hand, the perturbed velocity field, say $\boldsymbol{v}_\epsilon(\boldsymbol{x},t)$, is defined by

$$\boldsymbol{v}_\epsilon = \frac{\partial\varphi_{\epsilon,t}\circ\varphi_t}{\partial t}\circ(\varphi_{\epsilon,t}\circ\varphi_t)^{-1} = \frac{\partial\varphi_{\epsilon,t}}{\partial t}\circ\varphi_{\epsilon,t}^{-1} + (\varphi_{\epsilon,t})_*\boldsymbol{v}, \tag{2.12}$$

where $(\varphi_{\epsilon,t})_*\boldsymbol{v}$ denotes the push-forward of the vector field $\boldsymbol{v}$ by $\varphi_{\epsilon,t}$. By differentiating this expression with respect to ϵ, we get

$$\frac{\partial\boldsymbol{v}_\epsilon}{\partial\epsilon} = \frac{\partial\boldsymbol{\xi}}{\partial t} - \mathscr{L}_{\boldsymbol{\xi}}\boldsymbol{v}_\epsilon, \tag{2.13}$$

https://doi.org/10.1017/S0022377810000760 Published online by Cambridge University Press

where the Lie derivative of the vector field is defined by $\mathscr{L}_\xi \boldsymbol{v} = \boldsymbol{\xi}\cdot\nabla\boldsymbol{v} - \boldsymbol{v}\cdot\nabla\boldsymbol{\xi}$. Since $\boldsymbol{v}_\epsilon|_{\epsilon=0} = \boldsymbol{v}$, this solution is formally represented by

$$\begin{aligned}\boldsymbol{v}_\epsilon &= \frac{e^{-\epsilon\mathscr{L}_\xi}-1}{-\mathscr{L}_\xi}\frac{\partial\boldsymbol{\xi}}{\partial t} + e^{-\epsilon\mathscr{L}_\xi}\boldsymbol{v} \\ &= \boldsymbol{v} + \epsilon\left(\frac{\partial\boldsymbol{\xi}}{\partial t} - \mathscr{L}_\xi\boldsymbol{v}\right) - \frac{\epsilon^2}{2}\mathscr{L}_\xi\left(\frac{\partial\boldsymbol{\xi}}{\partial t} - \mathscr{L}_\xi\boldsymbol{v}\right) + \dots \end{aligned} \tag{2.14}$$

(see Sattinger and Weaver [16], Lemma 3.4). If we technically identify $\boldsymbol{v}$ as a four-dimensional vector field $\mathbf{v}_t \overset{\text{def}}{=} \partial_t + \boldsymbol{v}\cdot\nabla$ (and $\boldsymbol{\xi}$ as $0 + \boldsymbol{\xi}\cdot\nabla$), then (2.14) can be also viewed as a Lie series expansion,

$$\mathbf{v}_{\epsilon,t} = \mathbf{v}_t - \epsilon\mathscr{L}_\xi\mathbf{v}_t + \frac{\epsilon^2}{2}\mathscr{L}_\xi\mathscr{L}_\xi\mathbf{v}_t - \frac{\epsilon^3}{6}\mathscr{L}_\xi\mathscr{L}_\xi\mathscr{L}_\xi\mathbf{v}_t + \dots \tag{2.15}$$

Thus, by adopting the expansion (2.9), we have expressed all perturbed Eulerian fields in Lie series. With this point in mind, we introduce a state vector $u = (\boldsymbol{v}, \boldsymbol{B}, \rho, s)^T$ of the Eulerian fields and define a useful operator δ_ξ as

$$\delta_\xi u = \begin{pmatrix}\delta_\xi\boldsymbol{v}\\ \delta_\xi\boldsymbol{B}\\ \delta_\xi\rho\\ \delta_\xi s\end{pmatrix} = \begin{pmatrix}\partial_t\boldsymbol{\xi} + \boldsymbol{v}\cdot\nabla\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\nabla\boldsymbol{v}\\ \nabla\times(\boldsymbol{\xi}\times\boldsymbol{B})\\ -\nabla\cdot(\rho\boldsymbol{\xi})\\ -\boldsymbol{\xi}\cdot\nabla s\end{pmatrix}, \tag{2.16}$$

which acts on each $\boldsymbol{v}, \boldsymbol{B}, \rho$ and s as if the *negative* Lie derivative, $\delta_\xi \Leftrightarrow -\mathscr{L}_\xi$. The perturbed state u_ϵ is then expanded as

$$u_\epsilon = u + \epsilon\delta_\xi u + \frac{\epsilon^2}{2}\delta_\xi\delta_\xi u + \frac{\epsilon^3}{6}\delta_\xi\delta_\xi\delta_\xi u + \dots \tag{2.17}$$

Following Newcomb [2], the MHD Lagrangian is written in terms of the Eulerian fields u at time t,

$$L = \int_{V_t} \mathrm{L}(u)d^3x, \tag{2.18}$$

where the Lagrangian density $\mathrm{L}(u)$ is given by

$$\mathrm{L}(u) = \frac{\rho}{2}|\boldsymbol{v}|^2 - \frac{1}{2}|\boldsymbol{B}|^2 - \rho U(\rho, s), \tag{2.19}$$

and $U(\rho, s)$ denotes the internal energy per unit volume.

The Lagrangian evaluated at the perturbed state is therefore

$$L_\epsilon = \int_{V_{\epsilon,t}} \mathrm{L}(u_\epsilon)d^3x. \tag{2.20}$$

https://doi.org/10.1017/S0022377810000760 Published online by Cambridge University Press

Using the pull-back from the perturbed domain $V_{\epsilon,t}$ to the unperturbed one V_t, this is transformed into

$$\begin{aligned} L_\epsilon &= \int_{V_t} \varphi^*_{\epsilon,t}[\mathrm{L}(u_\epsilon)d^3x] \\ &= \int_{V_t} \left\{ \mathrm{L}(u_\epsilon)d^3x + \epsilon \mathscr{L}_\xi[\mathrm{L}(u_\epsilon)d^3x] + \frac{\epsilon^2}{2}\mathscr{L}_\xi\mathscr{L}_\xi[\mathrm{L}(u_\epsilon)d^3x] + \dots \right\} \\ &= \int_{V_t} \left\{ \mathrm{L}(u_\epsilon) + \epsilon \mathcal{L}_\xi \mathrm{L}(u_\epsilon) + \frac{\epsilon^2}{2}\mathcal{L}_\xi\mathcal{L}_\xi \mathrm{L}(u_\epsilon) + \dots \right\} d^3x, \end{aligned} \tag{2.21}$$

where $\mathcal{L}_\xi$ acts on any functional $f(u)$ as

$$\mathcal{L}_\xi f \stackrel{\text{def}}{=} \nabla \cdot (f\boldsymbol{\xi}), \tag{2.22}$$

and hence $\mathcal{L}_\xi\mathcal{L}_\xi f = \nabla\cdot[\nabla\cdot(f\boldsymbol{\xi})\boldsymbol{\xi}]$. Moreover, the Lagrangian density $\mathrm{L}(u_\epsilon)$ is in itself expanded as

$$\mathrm{L}(u_\epsilon) = \mathrm{L} + \epsilon\delta_\xi \mathrm{L} + \frac{\epsilon^2}{2}\delta_\xi\delta_\xi \mathrm{L} + \frac{\epsilon^3}{6}\delta_\xi\delta_\xi\delta_\xi \mathrm{L} + \dots \tag{2.23}$$

Here, the action of δ_ξ on a functional $f(u)$ means that

$$\delta_\xi f = \frac{\partial}{\partial\epsilon}\bigg|_{\epsilon=0} f(u_\epsilon) = D_u f \cdot \delta_\xi u, \tag{2.24}$$

where D_u denotes the functional derivative with respect to u. In summary, L_ϵ is expanded in powers of ϵ as follows:

$$L_\epsilon = \int_{V_t} \left(\mathrm{L} + \epsilon \mathsf{D}_\xi \mathrm{L} + \frac{\epsilon^2}{2}\mathsf{D}_\xi\mathsf{D}_\xi \mathrm{L} + \frac{\epsilon^3}{6}\mathsf{D}_\xi\mathsf{D}_\xi\mathsf{D}_\xi \mathrm{L} + O(\epsilon^4) \right) d^3x, \tag{2.25}$$

where we have newly introduced an operator D_ξ acting on functionals $f(u)$ as

$$\mathsf{D}_\xi f \stackrel{\text{def}}{=} \delta_\xi f + \mathcal{L}_\xi f. \tag{2.26}$$

For example, a calculation of $\mathsf{D}_\xi|\boldsymbol{B}|^2/2$ goes like

$$\begin{aligned} \mathsf{D}_\xi \frac{|\boldsymbol{B}|^2}{2} &= \boldsymbol{B}\cdot\delta_\xi\boldsymbol{B} + \mathcal{L}_\xi\frac{|\boldsymbol{B}|^2}{2} \\ &= \boldsymbol{B}\cdot\nabla\times(\boldsymbol{\xi}\times\boldsymbol{B}) + \nabla\cdot\left(\frac{|\boldsymbol{B}|^2}{2}\boldsymbol{\xi}\right). \end{aligned} \tag{2.27}$$

3. Properties of Lie derivative

Calculating the above expansion (2.25) for the MHD Lagrangian (2.19) is somewhat labouring especially for the third order. Before working on this task, it is insightful to get accustomed to the general properties of the Lie derivative in advance.

The key formula is only the Jacobi identity; $\mathscr{L}_\xi\mathscr{L}_\eta - \mathscr{L}_\eta\mathscr{L}_\xi = \mathscr{L}_{\xi\cdot\nabla\eta - \eta\cdot\nabla\xi}$ for all $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. It enables us to define an operator

$$\mathscr{L}^2_{\xi,\eta} \stackrel{\text{def}}{=} \mathscr{L}_\xi\mathscr{L}_\eta - \mathscr{L}_{\xi\cdot\nabla\eta} = \mathscr{L}_\eta\mathscr{L}_\xi - \mathscr{L}_{\eta\cdot\nabla\xi}, \tag{3.1}$$

https://doi.org/10.1017/S0022377810000760 Published online by Cambridge University Press

which is symmetric with respect to the subscripts $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. In this way, the second-order Lie derivative $\mathscr{L}_{\boldsymbol{\eta}}\mathscr{L}_{\boldsymbol{\xi}}$ can be made symmetric by subtracting the first-order Lie derivative $\mathscr{L}_{\boldsymbol{\eta}\cdot\nabla\boldsymbol{\xi}}$.

The same idea can be also applied to the third-order Lie derivative. If we define

$$\mathscr{L}^3_{\boldsymbol{\zeta},\boldsymbol{\eta},\boldsymbol{\xi}} \overset{\text{def}}{=} \mathscr{L}_{\boldsymbol{\zeta}}\mathscr{L}_{\boldsymbol{\eta}}\mathscr{L}_{\boldsymbol{\xi}} - \mathscr{L}^2_{\boldsymbol{\zeta},\boldsymbol{\eta}\cdot\nabla\boldsymbol{\xi}} - \mathscr{L}^2_{\boldsymbol{\eta},\boldsymbol{\zeta}\cdot\nabla\boldsymbol{\xi}} - \mathscr{L}^2_{\boldsymbol{\xi},\boldsymbol{\zeta}\cdot\nabla\boldsymbol{\eta}} - \mathscr{L}_{\boldsymbol{\zeta}\cdot\nabla(\boldsymbol{\eta}\cdot\nabla\boldsymbol{\xi})}, \tag{3.2}$$

then the operator $\mathscr{L}^3_{\boldsymbol{\zeta},\boldsymbol{\eta},\boldsymbol{\xi}}$ is found to be symmetric with respect to any permutation of vector fields $\boldsymbol{\xi},\boldsymbol{\eta},\boldsymbol{\zeta}$ (which will be referred to as cubic symmetry). This can be proved as follows: First, the symmetry with respect to $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ is confirmed by finding an alternative expression,

$$\begin{aligned}\mathscr{L}^3_{\boldsymbol{\zeta},\boldsymbol{\eta},\boldsymbol{\xi}} &= \mathscr{L}_{\boldsymbol{\zeta}}\mathscr{L}_{\boldsymbol{\eta}}\mathscr{L}_{\boldsymbol{\xi}} - \mathscr{L}_{\boldsymbol{\zeta}}\mathscr{L}_{\boldsymbol{\eta}\cdot\nabla\boldsymbol{\xi}} - \mathscr{L}^2_{\boldsymbol{\eta},\boldsymbol{\zeta}\cdot\nabla\boldsymbol{\xi}} - \mathscr{L}^2_{\boldsymbol{\xi},\boldsymbol{\zeta}\cdot\nabla\boldsymbol{\eta}} \\ &= \mathscr{L}_{\boldsymbol{\zeta}}\mathscr{L}^2_{\boldsymbol{\eta},\boldsymbol{\xi}} - \mathscr{L}^2_{\boldsymbol{\eta},\boldsymbol{\zeta}\cdot\nabla\boldsymbol{\xi}} - \mathscr{L}^2_{\boldsymbol{\xi},\boldsymbol{\zeta}\cdot\nabla\boldsymbol{\eta}}.\end{aligned} \tag{3.3}$$

Next, the symmetry with respect to $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ follows from yet another expression,

$$\begin{aligned}\mathscr{L}^3_{\boldsymbol{\zeta},\boldsymbol{\eta},\boldsymbol{\xi}} &= \mathscr{L}_{\boldsymbol{\zeta}}\mathscr{L}_{\boldsymbol{\eta}}\mathscr{L}_{\boldsymbol{\xi}} - \mathscr{L}^2_{\boldsymbol{\zeta},\boldsymbol{\eta}\cdot\nabla\boldsymbol{\xi}} - \mathscr{L}^2_{\boldsymbol{\eta},\boldsymbol{\zeta}\cdot\nabla\boldsymbol{\xi}} - \mathscr{L}_{\boldsymbol{\zeta}\cdot\nabla\boldsymbol{\eta}}\mathscr{L}_{\boldsymbol{\xi}} + \mathscr{L}_{(\boldsymbol{\zeta}\cdot\nabla\boldsymbol{\eta})\cdot\nabla\boldsymbol{\xi}} - \mathscr{L}_{\boldsymbol{\zeta}\cdot\nabla(\boldsymbol{\eta}\cdot\nabla\boldsymbol{\xi})} \\ &= \mathscr{L}^2_{\boldsymbol{\zeta},\boldsymbol{\eta}}\mathscr{L}_{\boldsymbol{\xi}} - \mathscr{L}^2_{\boldsymbol{\zeta},\boldsymbol{\eta}\cdot\nabla\boldsymbol{\xi}} - \mathscr{L}^2_{\boldsymbol{\eta},\boldsymbol{\zeta}\cdot\nabla\boldsymbol{\xi}} + \mathscr{L}_{(\boldsymbol{\zeta}\cdot\nabla\boldsymbol{\eta})\cdot\nabla\boldsymbol{\xi}-\boldsymbol{\zeta}\cdot\nabla(\boldsymbol{\eta}\cdot\nabla\boldsymbol{\xi})},\end{aligned} \tag{3.4}$$

where the vector formula,

$$(\boldsymbol{\zeta}\cdot\nabla\boldsymbol{\eta})\cdot\nabla\boldsymbol{\xi} - \boldsymbol{\zeta}\cdot\nabla(\boldsymbol{\eta}\cdot\nabla\boldsymbol{\xi}) = (\boldsymbol{\eta}\cdot\nabla\boldsymbol{\zeta})\cdot\nabla\boldsymbol{\xi} - \boldsymbol{\eta}\cdot\nabla(\boldsymbol{\zeta}\cdot\nabla\boldsymbol{\xi}), \tag{3.5}$$

proves the required symmetry. These symmetrizations of $\mathscr{L}_{\boldsymbol{\eta}}\mathscr{L}_{\boldsymbol{\xi}}$ and $\mathscr{L}_{\boldsymbol{\zeta}}\mathscr{L}_{\boldsymbol{\eta}}\mathscr{L}_{\boldsymbol{\xi}}$ are essential for deriving the perturbed Lagrangian in a symmetric form. In fact, the Lie series are symmetrized as follows:

$$\begin{aligned}e^{\epsilon\mathscr{L}_{\boldsymbol{\xi}}} &= 1 + \epsilon\mathscr{L}_{\boldsymbol{\xi}} + \frac{\epsilon^2}{2}\mathscr{L}_{\boldsymbol{\xi}}\mathscr{L}_{\boldsymbol{\xi}} + \frac{\epsilon^3}{6}\mathscr{L}_{\boldsymbol{\xi}}\mathscr{L}_{\boldsymbol{\xi}}\mathscr{L}_{\boldsymbol{\xi}} + O(\epsilon^4) \\ &= 1 + \mathscr{L}_{\epsilon\boldsymbol{\xi}+\frac{\epsilon^2}{2}\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi}+\frac{\epsilon^3}{6}\boldsymbol{\xi}\cdot\nabla(\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi})} + \frac{\epsilon^2}{2}\mathscr{L}^2_{\boldsymbol{\xi},\boldsymbol{\xi}} + \frac{\epsilon^3}{6}\left(3\mathscr{L}^2_{\boldsymbol{\xi},\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi}} + \mathscr{L}^3_{\boldsymbol{\xi},\boldsymbol{\xi},\boldsymbol{\xi}}\right) + O(\epsilon^4).\end{aligned} \tag{3.6}$$

Note that the variants of the Lie derivative such as $\delta_{\boldsymbol{\xi}}$, $\mathsf{D}_{\boldsymbol{\xi}}$ and $\mathcal{L}_{\boldsymbol{\xi}}$ also satisfy the Jacobi identity and enjoy the same properties. We actually demonstrate calculations of $\mathsf{D}_{\boldsymbol{\xi}}\mathrm{L}$, $\mathsf{D}_{\boldsymbol{\eta}}\mathsf{D}_{\boldsymbol{\xi}}\mathrm{L}$ and $\mathsf{D}_{\boldsymbol{\zeta}}\mathsf{D}_{\boldsymbol{\eta}}\mathsf{D}_{\boldsymbol{\xi}}\mathrm{L}$ in Appendix A.

4. Hamilton's principle (fixed boundary case)

Now, we are ready to derive a weakly nonlinear equation of $\boldsymbol{\xi}$ by invoking Hamilton's principle. In this section, we assume that the plasma domain V_t is surrounded by a rigid ideal wall $V_t = V = \text{const.}$ as shown in Fig. 1(a) (the free boundary case is postponed to Sec. 7). The fixed boundary condition for perturbation is simply given by

$$\boldsymbol{n}\cdot\boldsymbol{\xi} = 0 \quad \text{on } S. \tag{4.1}$$

Note that, if the displacement map $\boldsymbol{\Xi}$ was employed, $\boldsymbol{n}\cdot\boldsymbol{\Xi}\neq 0$ in general and the boundary condition would be more complicated.

https://doi.org/10.1017/S0022377810000760 Published online by Cambridge University Press

Leaving the details of the derivation to Appendix A, the Lie series expansion (2.25) of the MHD Lagrangian is written in the form of

$$L_\epsilon = \int_V \left\{ \mathrm{L} - \Delta_{EL} \cdot \left(\epsilon\boldsymbol{\xi} + \frac{\epsilon^2}{2}\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi} + \frac{\epsilon^3}{6}\boldsymbol{\xi}\cdot\nabla(\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi}) \right) + \frac{\epsilon^2}{2} l^{(2)}(\boldsymbol{\xi},\boldsymbol{\xi}) \right.$$
$$+ \frac{\epsilon^3}{6}[3l^{(2)}(\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi},\boldsymbol{\xi}) - w^{(3)}(\boldsymbol{\xi},\boldsymbol{\xi},\boldsymbol{\xi})] + \partial_t \left[\rho\boldsymbol{v}\cdot\left(\epsilon\boldsymbol{\xi} + \frac{\epsilon^2}{2}\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi} + \frac{\epsilon^3}{6}\boldsymbol{\xi}\cdot\nabla(\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi}) \right) \right]$$
$$\left. + O(\epsilon^4) \right\} d^3x, \tag{4.2}$$

where the definitions of Δ_{EL} and $l^{(2)}$ are given in (A 2) and (A 14), respectively. It is remarkable that we succeed in deriving $w^{(3)}$ which is defined by

$$\begin{aligned} w^{(3)}(\boldsymbol{\xi},\boldsymbol{\eta},\boldsymbol{\zeta}) = {} & \nabla\cdot(\boldsymbol{\zeta}\cdot\nabla\boldsymbol{\xi} - \boldsymbol{\zeta}\nabla\cdot\boldsymbol{\xi})\delta_{\boldsymbol{\eta}}P + \nabla\cdot(\boldsymbol{\zeta}\cdot\nabla\boldsymbol{\eta} - \boldsymbol{\zeta}\nabla\cdot\boldsymbol{\eta})\delta_{\boldsymbol{\xi}}P \\ & + \nabla\cdot(\boldsymbol{\xi}\cdot\nabla\boldsymbol{\eta} - \boldsymbol{\xi}\nabla\cdot\boldsymbol{\eta})\delta_{\boldsymbol{\zeta}}P - (\nabla\cdot\boldsymbol{\xi})\mathscr{B}\boldsymbol{\zeta}\cdot\mathscr{B}\boldsymbol{\eta} - (\nabla\cdot\boldsymbol{\eta})\mathscr{B}\boldsymbol{\zeta}\cdot\mathscr{B}\boldsymbol{\xi} \\ & - (\nabla\cdot\boldsymbol{\zeta})\mathscr{B}\boldsymbol{\xi}\cdot\mathscr{B}\boldsymbol{\eta} + (\nabla\cdot\boldsymbol{\xi})\boldsymbol{\eta}\cdot(\boldsymbol{\zeta}\cdot\nabla)\nabla P + (\nabla\cdot\boldsymbol{\zeta})\boldsymbol{\eta}\cdot(\boldsymbol{\xi}\cdot\nabla)\nabla P \\ & + (\nabla\cdot\boldsymbol{\eta})\boldsymbol{\xi}\cdot(\boldsymbol{\zeta}\cdot\nabla)\nabla P + \boldsymbol{\eta}\boldsymbol{\zeta}\boldsymbol{\xi} : \nabla\nabla\nabla P \\ & - \left(\rho^2\frac{\partial^2 p}{\partial\rho^2} + 2\rho c_s^2 \right)(\nabla\cdot\boldsymbol{\xi})(\nabla\cdot\boldsymbol{\eta})(\nabla\cdot\boldsymbol{\zeta}), \end{aligned} \tag{4.3}$$

for all $\boldsymbol{\xi},\boldsymbol{\eta},\boldsymbol{\zeta}$. This expression is shown to have the cubic symmetry in Appendix B, which assures the resemblance of (4.2) to (3.6).

As the simplest choice of the unperturbed state, we regard $u = (\boldsymbol{v},\boldsymbol{B},\rho,s)^T$ as an exact solution of the MHD equations, namely, $\Delta_{EL} = 0$. Let us investigate the extremum condition $\delta\int_{t_1}^{t_2} L_\epsilon dt = 0$ with respect to any variation $\boldsymbol{\xi} \to \boldsymbol{\xi} + \delta\boldsymbol{\xi}$ satisfying $\boldsymbol{n}\cdot\delta\boldsymbol{\xi} = 0$ on S and $\delta\boldsymbol{\xi}|_{t=t_1,t_2} = 0$. In the above expression (4.2), one may omit the terms inside the time derivative $\partial_t(\circ)$ since it does not affect the Euler–Lagrange equation. In this work, we will occasionally omit them by using the notation $L_\epsilon \simeq L_\epsilon + \partial_t(\circ)$.

The Lagrangian to be varied with respect to $\boldsymbol{\xi}$ is, therefore,

$$L_\epsilon \simeq \int_V \left\{ \mathrm{L} + \frac{\epsilon^2}{2} l^{(2)}(\boldsymbol{\xi},\boldsymbol{\xi}) + \frac{\epsilon^3}{6}[3l^{(2)}(\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi},\boldsymbol{\xi}) - w^{(3)}(\boldsymbol{\xi},\boldsymbol{\xi},\boldsymbol{\xi})] + O(\epsilon^4) \right\} d^3x. \tag{4.4}$$

If we neglect the $O(\epsilon^3)$ terms, the quadratic Lagrangian $(\epsilon^2/2)\int_V l^{(2)}(\boldsymbol{\xi},\boldsymbol{\xi})d^3x$ yields an Euler–Lagrange equation,

$$\Delta'_{EL}(\boldsymbol{\xi}) \overset{\text{def}}{=} \rho\frac{D^2\boldsymbol{\xi}}{Dt^2} - \mathscr{F}\boldsymbol{\xi} = 0, \tag{4.5}$$

which corresponds to the linearized equation of motion derived by Frieman and Rotenberg [11] [use (A 11) in Appendix A].

In (4.4), $w^{(3)}(\boldsymbol{\xi},\boldsymbol{\xi},\boldsymbol{\xi})$ plays the role of third-order potential energy density. Owing to its cubic symmetry, we can define the second-order force field $\mathscr{F}^{(2)}(\boldsymbol{\xi},\boldsymbol{\xi})$ by

$$-\int_V w^{(3)}(\boldsymbol{\xi},\boldsymbol{\xi},\delta\boldsymbol{\xi})d^3x = \int_V \{\delta\boldsymbol{\xi}\cdot\mathscr{F}^{(2)}(\boldsymbol{\xi},\boldsymbol{\xi}) - \nabla\cdot[2(\delta\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi})\delta_{\boldsymbol{\xi}}P]\}d^3x, \tag{4.6}$$

https://doi.org/10.1017/S0022377810000760 Published online by Cambridge University Press

where $\mathscr{F}^{(2)}$ is written as (A 26). The variation of (4.4) then becomes

$$\delta \int_{t_1}^{t_2} L_\epsilon dt = \int_{t_1}^{t_2} \int_V \left\{ -\epsilon^2 \Delta'_{EL}(\xi) \cdot \delta\xi - \frac{\epsilon^3}{2} \Delta'_{EL}(\xi) \cdot (\delta\xi \cdot \nabla\xi + \xi \cdot \nabla\delta\xi) \right.$$
$$\left. - \frac{\epsilon^3}{2} [\Delta'_{EL}(\xi \cdot \nabla\xi) - \mathscr{F}^{(2)}(\xi, \xi)] \cdot \delta\xi + O(\epsilon^4) \right\} d^3x dt, \tag{4.7}$$

where the boundary terms have been eliminated by using $\boldsymbol{n} \cdot (-\delta\xi \cdot \nabla\xi + \xi \cdot \nabla\delta\xi) = 0$. Since the $O(\epsilon^2)$-term is still dominant, the FR equation (4.5) should be satisfied approximately, $\Delta'_{EL}(\xi) = O(\epsilon)$. The deviation from the linearized dynamics is due to the next $O(\epsilon^3)$ terms. Since the second term in the integrand of (4.7) becomes of order $O(\epsilon^4)$, we get the Euler–Lagrange equation of ξ as

$$\Delta'_{EL}(\xi) + \frac{\epsilon}{2} \Delta'_{EL}(\xi \cdot \nabla\xi) - \frac{\epsilon}{2} \mathscr{F}^{(2)}(\xi, \xi) = O(\epsilon^2). \tag{4.8}$$

In terms of the displacement map $\Xi = \epsilon\xi + (\epsilon^2/2)\xi \cdot \nabla\xi + O(\epsilon^3)$, we can rewrite this equation simply as

$$\rho \frac{D^2\Xi}{Dt^2} = \mathscr{F}\Xi + \frac{1}{2} \mathscr{F}^{(2)}(\Xi, \Xi) + O(\epsilon^3), \tag{4.9}$$

where the boundary condition $\boldsymbol{n} \cdot \xi = 0$ must be interpreted as $\boldsymbol{n} \cdot [\Xi - (1/2)\Xi \cdot \nabla\Xi] = O(\epsilon^3)$. Obviously, this is the nonlinear extension of the FR equation (4.5). The new term $\mathscr{F}^{(2)}$ allows for all quadratic nonlinearity (such as $\boldsymbol{v} \cdot \nabla\boldsymbol{v}$ and $\boldsymbol{J} \times \boldsymbol{B}$) of the MHD equations in terms of the single variable ξ.

5. Nonlinear three-mode coupling

The above variational formalism can handle various nonlinear couplings among three modes in a unified way.

Let $u = (\boldsymbol{v}, \boldsymbol{B}, \rho, s)$ be a given *equilibrium* state and consider three eigenmodes labelled by a, b, c;

$$\Delta'_{EL}(\hat{\xi}_j e^{-i\omega_j t}) = 0, \quad j = a, b, c, \tag{5.1}$$

where ω_j and $\hat{\xi}_j$ are, respectively, eigenfrequencies and eigenfunctions of the FR equation. These eigenfrequencies are assumed to be real and not so small, $0 < \epsilon \ll \omega_j \in \mathbb{R}$, and moreover satisfy the resonant condition $\omega_a = \omega_b + \omega_c + \Delta\omega$ with a small mismatch $\Delta\omega \sim \epsilon$.

Then, we seek for an approximate solution of the weakly nonlinear equation (4.8) in the form of

$$\epsilon\xi = \sum_{j=a,b,c} [C_j(t) \hat{\xi}_j e^{-i\omega_j t} + \text{c.c.}], \tag{5.2}$$

where the modal amplitudes $C_j(t) \sim \epsilon$ are slowly varying $dC_j/dt \sim \epsilon^2$ due to the nonlinear coupling (we normalize the eigenfunctions to $\hat{\xi}_j \sim 1$). Since our trial function (5.2) already satisfies $\Delta'_{EL}(\xi) = O(\epsilon)$, we can reduce $l^{(2)}(\xi \cdot \nabla\xi, \xi)$ into divergence terms, $\nabla \cdot (\circ)$, using (A 11). It follows that (4.4) becomes

$$L_\epsilon \simeq \int_V \left[\mathrm{L} + \frac{\epsilon^2}{2} l^{(2)}(\xi, \xi) \right] d^3x - \frac{\epsilon^3}{6} W^{(3p)}(\xi, \xi, \xi) + O(\epsilon^4), \tag{5.3}$$

https://doi.org/10.1017/S0022377810000760 Published online by Cambridge University Press

where we have properly defined the third-order potential energy (for plasma) with the cubic symmetry as

$$W^{(3p)}(\boldsymbol{\xi},\boldsymbol{\eta},\boldsymbol{\zeta}) \overset{\text{def}}{=} \int_V \left[w^{(3)}(\boldsymbol{\xi},\boldsymbol{\eta},\boldsymbol{\zeta}) + \mathcal{L}^2_{\boldsymbol{\zeta},\boldsymbol{\xi}}(\delta_{\boldsymbol{\eta}}P) + \mathcal{L}^2_{\boldsymbol{\zeta},\boldsymbol{\eta}}(\delta_{\boldsymbol{\xi}}P) + \mathcal{L}^2_{\boldsymbol{\xi},\boldsymbol{\eta}}(\delta_{\boldsymbol{\zeta}}P)\right] d^3x. \tag{5.4}$$

While $W^{(3p)}(\boldsymbol{\xi},\boldsymbol{\xi},\boldsymbol{\xi})$ includes the integral of the divergence terms $3\mathcal{L}^2_{\boldsymbol{\xi},\boldsymbol{\xi}}(\delta_{\boldsymbol{\xi}}P)$ [see (A 15)], it is not negligible at the boundary since $\boldsymbol{n}\cdot(\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi})\neq 0$ on S in general.

Let us substitute (5.2) into the action integral $\int_{t_1}^{t_2} L_\epsilon dt$ and average over sufficiently long period ($t_2-t_1\sim\epsilon^{-1}$) such that any fast oscillation ($e^{-i\omega t}$, $\omega\gg\epsilon$) in the integrand would be averaged out (Whitham's method [14]). As for $O(\epsilon^2)$, only the following non-oscillatory terms remain:

$$\begin{aligned}\int_{t_1}^{t_2}\int_V \frac{\epsilon^2}{2} l^{(2)}(\boldsymbol{\xi},\boldsymbol{\xi}) d^3x dt &= \sum_{j=a,b,c}\int_{t_1}^{t_2}\int_V l^{(2)}\left(C_j\hat{\boldsymbol{\xi}}_j e^{-i\omega_j t}, C_j^*\hat{\boldsymbol{\xi}}_j^* e^{i\omega_j t}\right) d^3x dt \\ &= \sum_{j=a,b,c}\int_{t_1}^{t_2}\int_V \left\{ \left|\frac{dC_j}{dt}\right|^2 \rho|\hat{\boldsymbol{\xi}}_j|^2 \right. \\ &\quad + \left(\frac{dC_j^*}{dt} C_j \rho\hat{\boldsymbol{\xi}}_j^*\cdot(-i\omega_j + \boldsymbol{v}\cdot\nabla)\hat{\boldsymbol{\xi}}_j + \text{c.c.}\right) \\ &\quad \left. + |C_j|^2 l^{(2)}\left(\hat{\boldsymbol{\xi}}_j e^{-i\omega_j t}, \hat{\boldsymbol{\xi}}_j^* e^{i\omega_j t}\right)\right\} d^3x dt,\end{aligned} \tag{5.5}$$

in which the first term will be neglected due to $|dC_j/dt|^2\sim\epsilon^4$, and the last term vanishes since $\hat{\boldsymbol{\xi}}_j e^{-i\omega_j t}$ is a linear solution (the superscript $*$ denotes the complex conjugate).

Similarly, when the three modes (5.2) are substituted into $W^{(3p)}$, only 12 terms survive after the time average, in which six terms are identical to $C_a^* C_b C_c W^{(3p)}(\hat{\boldsymbol{\xi}}_a^*,\hat{\boldsymbol{\xi}}_b,\hat{\boldsymbol{\xi}}_c)e^{i\Delta\omega t}$ (owing to the cubic symmetry) and the other six terms are its complex conjugate. In summary, the action integral is reduced to

$$\int_{t_1}^{t_2} L_\epsilon dt = \int_{t_1}^{t_2}\left\{L - \sum_{j=a,b,c}\left(i\frac{\mu_j}{2}\frac{dC_j^*}{dt}C_j + \text{c.c.}\right) - \left(W^{(3)}_{a,b,c} C_a^* C_b C_c e^{i\Delta\omega t} + \text{c.c.}\right) + O(\epsilon^4)\right\} dt, \tag{5.6}$$

where we have defined

$$\mu_j = 2\int_V \hat{\boldsymbol{\xi}}_j^*\cdot\rho(\omega_j + i\boldsymbol{v}\cdot\nabla)\hat{\boldsymbol{\xi}}_j d^3x \in \mathbb{R}, \quad (j=a,b,c) \tag{5.7}$$

$$W^{(3)}_{a,b,c} = W^{(3p)}\left(\hat{\boldsymbol{\xi}}_a^*, \hat{\boldsymbol{\xi}}_b, \hat{\boldsymbol{\xi}}_c\right). \tag{5.8}$$

https://doi.org/10.1017/S0022377810000760 Published online by Cambridge University Press

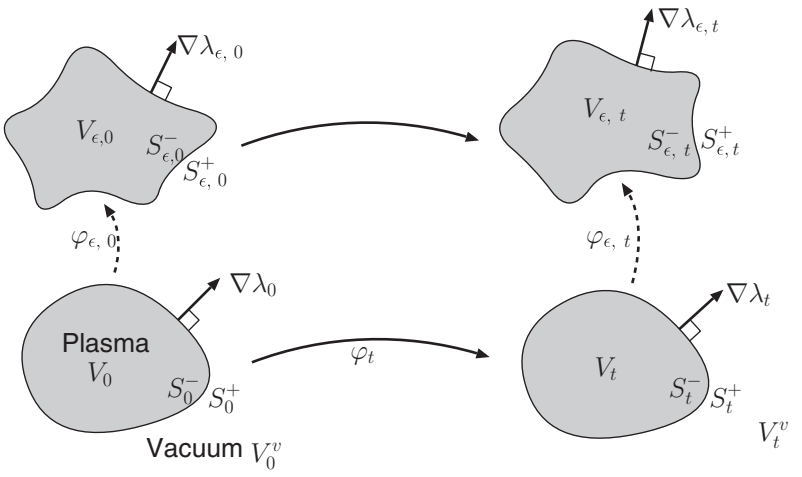


Figure 3. Unperturbed and perturbed motions of plasma domains surrounded by vacuum.

By neglecting $O(\epsilon^4)$, the variational principle $\delta\int_{t_1}^{t_2} L_\epsilon dt = 0$ yields the amplitude equations,

$$\mu_a\frac{dC_a}{dt} = -iW^{(3)}_{a,b,c}C_bC_ce^{i\Delta\omega t}, \quad \mu_b\frac{dC^*_b}{dt} = iW^{(3)}_{a,b,c}C^*_aC_ce^{i\Delta\omega t} \quad \text{and}$$

$$\mu_c\frac{dC^*_c}{dt} = iW^{(3)}_{a,b,c}C^*_aC_be^{i\Delta\omega t}. \tag{5.9}$$

The subsequent analysis of these coupled equations conforms to the classical theory for the three-wave interaction [4, 17]. The quantity $\mu_j|C_j|^2$ $(j = a, b, c)$ represents the wave action (or action variable) of each eigenmode, which agrees with the recent Hamiltonian theory for linear waves [18, 19]. The Manley–Rowe relations are obviously satisfied by the fact that the common coupling coefficient $W^{(3)}_{a,b,c}$ appears in these equations. We have shown that the coupling coefficient is the third-order change in the potential energy caused by the three displacement fields. We emphasize that the formulae (5.7) and (5.8) are widely applicable to any ideal MHD modes under any equilibrium profiles.

6. Free boundary problem

The remaining part of this paper is devoted to the extension of the above weakly nonlinear theory to free boundary problems. Although the original paper on the energy principle by Bernstein et al. [1] has already taken the free boundary into account, we need more careful consideration when dealing with *nonlinear* displacement of boundary.

Let $V_0 \subset \mathbb{R}^3$ be any plasma domain surrounded by vacuum at $t = 0$ (see Fig. 3). In order to define the normal vector $\boldsymbol{n}$ explicitly, we introduce a smooth scalar function $\lambda_0(\boldsymbol{X})$ on $\mathbb{R}^3$ that becomes zero only at the boundary of the plasma domain

https://doi.org/10.1017/S0022377810000760 Published online by Cambridge University Press

V_0;

$$\lambda_0(\boldsymbol{X}) \begin{cases} > 0 & (\boldsymbol{X} \in V_0) \\ = 0 & (\boldsymbol{X} \in \partial V_0) \\ < 0 & (\boldsymbol{X} \notin V_0). \end{cases} \tag{6.1}$$

The normal vector is then given by $\boldsymbol{n} = \nabla\lambda_0/|\nabla\lambda_0|$ on the boundary $S_0 = \partial V_0$, where the gradient field $\nabla\lambda_0$ is also identified with the 1-form $d\lambda_0$.

Let S_0^- and S_0^+, respectively, represent the interior and exterior faces of the plasma-vacuum interface (as shown in Fig. 3). From the divergence-free condition $\nabla \cdot \boldsymbol{B} = 0$, the normal component $\boldsymbol{n} \cdot \boldsymbol{B}_0$ of the initial magnetic field $\boldsymbol{B}_0(\boldsymbol{X}) = \boldsymbol{B}(\boldsymbol{X}, 0)$ must be continuous on S_0, namely, $\boldsymbol{n} \cdot \boldsymbol{B}_0|_{S_0^-} = \boldsymbol{n} \cdot \boldsymbol{B}_0|_{S_0^+}$. In this work, we equate this value to zero, $\boldsymbol{n} \cdot \boldsymbol{B}_0 = 0$ on S_0, by assuming that the plasma-vacuum interface is a magnetic surface, which is commonly the case for the magnetic confinement of plasma.

By identifying the magnetic field $\boldsymbol{B}$ with the 2-form β_t again [see (2.1)], the condition $\boldsymbol{B}_0 \cdot \nabla\lambda_0 = 0$ is equivalent to

$$d\lambda_0 \wedge \beta_0 = 0 \quad \text{on} \quad S_0. \tag{6.2}$$

When the plasma moves according to the map $\varphi_t : V_0 \to V_t$, $\boldsymbol{x} = \varphi_t(\boldsymbol{X})$, we define $\lambda_t(\boldsymbol{x})$ as a passive scalar function,

$$(\varphi_t^* \lambda_t)(\boldsymbol{X}) = \lambda_t(\varphi_t(\boldsymbol{X})) = \lambda_0(\boldsymbol{X}). \tag{6.3}$$

Clearly, $\lambda_t(\boldsymbol{x}) = 0$ holds at all $\boldsymbol{x} \in S_t = \partial V_t$, and thereby $d\lambda_t$ again determines the normal vector of the domain V_t at time t. Recall that, inside the plasma region V_t, the magnetic field $\boldsymbol{B}(\boldsymbol{x}, t)$ is frozen-in to plasmas ($\varphi_t^* \beta_t = \beta_0$), and the relation,

$$d\lambda_t \wedge \beta_t = \left(\varphi_t^{-1}\right)^* (d\lambda_0 \wedge \varphi_t^* \beta_t) = \left(\varphi_t^{-1}\right)^* (d\lambda_0 \wedge \beta_0), \tag{6.4}$$

holds in V_t. It follows from (6.2) that $d\lambda_t \wedge \beta_t = 0$ (or $\boldsymbol{n} \cdot \boldsymbol{B} = 0$) will be automatically satisfied on S_t^-. Since $\boldsymbol{n} \cdot \boldsymbol{B}$ must be continuous across the surface S_t, the boundary condition $\boldsymbol{n} \cdot \boldsymbol{B} = 0$ on S_t^+ will be *imposed* on the vacuum magnetic field.

Now, we perturb the fluid motion, $\varphi_{\epsilon,t} : V_t \to V_{\epsilon,t}$, $\boldsymbol{x}_\epsilon = \varphi_{\epsilon,t}(\boldsymbol{x})$, as before. By defining $\lambda_{\epsilon,t}$ as

$$(\varphi_{\epsilon,t}^* \lambda_{\epsilon,t})(\boldsymbol{x}) = \lambda_{\epsilon,t}(\varphi_{\epsilon,t}(\boldsymbol{x})) = \lambda_t(\boldsymbol{x}), \tag{6.5}$$

the boundary condition $\boldsymbol{n} \cdot \boldsymbol{B}_\epsilon = 0$ (or $d\lambda_{\epsilon,t} \wedge \beta_{\epsilon,t} = 0$) on $S_{\epsilon,t}^+$ is similarly imposed on the perturbed magnetic field $\boldsymbol{B}_\epsilon$ (or $\beta_{\epsilon,t}$) in the outer region. Since

$$d\lambda_{\epsilon,t} \wedge \beta_{\epsilon,t} = \left(\varphi_{\epsilon,t}^{-1}\right)^* [d\lambda_t \wedge \varphi_{\epsilon,t}^* \beta_{\epsilon,t}] \tag{6.6}$$

holds in $V_{\epsilon,t}$, this boundary condition can be translated into the condition $d\lambda_t \wedge \varphi_{\epsilon,t}^* \beta_{\epsilon,t} = 0$ on the *unperturbed* boundary S_t^+, which is moreover expanded as

$$d\lambda_t \wedge \left(\beta_{\epsilon,t} + \epsilon \mathscr{L}_\xi \beta_{\epsilon,t} + \frac{\epsilon^2}{2} \mathscr{L}_\xi \mathscr{L}_\xi \beta_{\epsilon,t} + \dots\right) = 0 \quad \text{on} \quad S_t^+. \tag{6.7}$$

Namely, $\boldsymbol{B}_\epsilon$ must satisfy

$$\boldsymbol{n} \cdot \left(\boldsymbol{B}_\epsilon - \epsilon \delta_\xi \boldsymbol{B}_\epsilon + \frac{\epsilon^2}{2} \delta_\xi \delta_\xi \boldsymbol{B}_\epsilon - \dots\right) = 0 \quad \text{on} \quad S_t^+, \tag{6.8}$$

https://doi.org/10.1017/S0022377810000760 Published online by Cambridge University Press

where $\delta_\xi \boldsymbol{B}_\epsilon = \nabla \times (\boldsymbol{\xi} \times \boldsymbol{B}_\epsilon)$. Note that the vacuum magnetic field $\boldsymbol{B}_\epsilon$ must be also expanded with respect to ϵ;

$$\boldsymbol{B}_\epsilon = \boldsymbol{B} + \epsilon \boldsymbol{B}_1 + \frac{\epsilon^2}{2}\boldsymbol{B}_2 + \frac{\epsilon^3}{6}\boldsymbol{B}_3 + \dots \tag{6.9}$$

Therefore, the boundary conditions for $\boldsymbol{B}_n$, $n = 1, 2, \dots$, are found to be

$$0 = \boldsymbol{n} \cdot (\boldsymbol{B}_1 - \delta_\xi \boldsymbol{B}), \tag{6.10}$$

$$0 = \boldsymbol{n} \cdot (\boldsymbol{B}_2 - 2\delta_\xi \boldsymbol{B}_1 + \delta_\xi \delta_\xi \boldsymbol{B}), \tag{6.11}$$

$$0 = \boldsymbol{n} \cdot (\boldsymbol{B}_3 - 3\delta_\xi \boldsymbol{B}_2 + 3\delta_\xi \delta_\xi \boldsymbol{B}_1 - \delta_\xi \delta_\xi \delta_\xi \boldsymbol{B}), \tag{6.12}$$

and so on.

The Lagrangian for the vacuum magnetic field is well known as

$$L^v = \int_{V_t^v} \left(-\frac{|\boldsymbol{B}|^2}{2} \right) d^3x. \tag{6.13}$$

Since the vector field $\boldsymbol{B}$ is constrained by $\nabla \cdot \boldsymbol{B} = 0$, its variation is usually taken with respect to the vector potential $\boldsymbol{A}$ (where $\boldsymbol{B} = \nabla \times \boldsymbol{A}$), and the Euler–Lagrange equation corresponds to $\nabla \times \boldsymbol{B} = 0$.

When both the plasma surface and the vacuum magnetic field are perturbed, the vacuum Lagrangian at the perturbed state is expanded as

$$L_\epsilon^v = \int_{V_{\epsilon,t}^v} \left(-\frac{|\boldsymbol{B}_\epsilon|^2}{2} \right) d^3x = \int_{V_t^v} \left(1 + \epsilon \mathcal{L}_\xi + \frac{\epsilon^2}{2}\mathcal{L}_\xi \mathcal{L}_\xi + \dots \right) \left(-\frac{|\boldsymbol{B}_\epsilon|^2}{2} \right) d^3x, \tag{6.14}$$

where

$$-\frac{|\boldsymbol{B}_\epsilon|^2}{2} = -\frac{|\boldsymbol{B}|^2}{2} - \epsilon \boldsymbol{B} \cdot \boldsymbol{B}_1 - \frac{\epsilon^2}{2}(\boldsymbol{B} \cdot \boldsymbol{B}_2 + |\boldsymbol{B}_1|^2) - \frac{\epsilon^3}{6}(\boldsymbol{B} \cdot \boldsymbol{B}_3 + 3\boldsymbol{B}_1 \cdot \boldsymbol{B}_2) - O(\epsilon^4). \tag{6.15}$$

It is almost trivial to show that the equations for the perturbations $\boldsymbol{B}_n$, $n = 1, 2, \dots$, are also $\nabla \times \boldsymbol{B}_n = 0$. These equations can be solved recursively under the boundary conditions (6.10)–(6.12). (By putting $\boldsymbol{B}_n = \nabla \Phi_n$, one needs to solve the Laplace equation $\Delta \Phi_n = 0$ under the Neumann boundary condition.)

In this work, for the purpose of simplifying the vacuum Lagrangian, we suppose that $\boldsymbol{B}_n$, $n = 1, 2, \dots$, in the vacuum region V_t^v have been solved for arbitrary displacement ($\boldsymbol{\xi}$) of the plasma surface and we will substitute the solutions *a priori* into the vacuum Lagrangian. More specifically, we define $\boldsymbol{B}_1(\boldsymbol{\xi})$, $\boldsymbol{B}_2(\boldsymbol{\eta}, \boldsymbol{\xi})$ and $\boldsymbol{B}_3(\boldsymbol{\zeta}, \boldsymbol{\eta}, \boldsymbol{\xi})$ as the solutions of

$$\nabla \times \boldsymbol{B}_1(\boldsymbol{\xi}) = 0, \quad \nabla \times \boldsymbol{B}_2(\boldsymbol{\eta}, \boldsymbol{\xi}) = 0, \quad \nabla \times \boldsymbol{B}_3(\boldsymbol{\zeta}, \boldsymbol{\eta}, \boldsymbol{\xi}) = 0, \tag{6.16}$$

$$\nabla \cdot \boldsymbol{B}_1(\boldsymbol{\xi}) = 0, \quad \nabla \cdot \boldsymbol{B}_2(\boldsymbol{\eta}, \boldsymbol{\xi}) = 0, \quad \nabla \cdot \boldsymbol{B}_3(\boldsymbol{\zeta}, \boldsymbol{\eta}, \boldsymbol{\xi}) = 0, \tag{6.17}$$

https://doi.org/10.1017/S0022377810000760 Published online by Cambridge University Press

under the boundary conditions on S_t^+,

$$0 = \boldsymbol{n} \cdot \{\boldsymbol{B}_1(\boldsymbol{\xi}) - \delta_{\boldsymbol{\xi}} \boldsymbol{B}\}, \tag{6.18}$$

$$0 = \boldsymbol{n} \cdot \{\boldsymbol{B}_2(\boldsymbol{\eta}, \boldsymbol{\xi}) - \delta_{\boldsymbol{\xi}} \boldsymbol{B}_1(\boldsymbol{\eta}) - \delta_{\boldsymbol{\eta}} [\boldsymbol{B}_1(\boldsymbol{\xi}) - \delta_{\boldsymbol{\xi}} \boldsymbol{B}]\}, \tag{6.19}$$

$$\begin{aligned} 0 = \boldsymbol{n} \cdot \{&\boldsymbol{B}_3(\boldsymbol{\zeta}, \boldsymbol{\eta}, \boldsymbol{\xi}) - \delta_{\boldsymbol{\xi}} \boldsymbol{B}_2(\boldsymbol{\zeta}, \boldsymbol{\eta}) - \delta_{\boldsymbol{\eta}} [\boldsymbol{B}_2(\boldsymbol{\zeta}, \boldsymbol{\xi}) - \delta_{\boldsymbol{\xi}} \boldsymbol{B}_1(\boldsymbol{\zeta})] \\ &- \delta_{\boldsymbol{\zeta}} [\boldsymbol{B}_2(\boldsymbol{\eta}, \boldsymbol{\xi}) - \delta_{\boldsymbol{\xi}} \boldsymbol{B}_1(\boldsymbol{\eta}) - \delta_{\boldsymbol{\eta}} [\boldsymbol{B}_1(\boldsymbol{\xi}) - \delta_{\boldsymbol{\xi}} \boldsymbol{B}]]\} \end{aligned} \tag{6.20}$$

(if the vacuum region V_t^v is bounded by a rigid ideal wall as shown in Fig. 1, the conditions $\boldsymbol{n} \cdot \boldsymbol{B}_n = 0$ are imposed on S^v). Notice that these boundary conditions on S_t^+ can be rephrased in terms of the vector potentials $\boldsymbol{A}_n$, $n = 1, 2, 3$, as follows:

$$\boldsymbol{n} \times \boldsymbol{A}_1(\boldsymbol{\xi}) = -(\boldsymbol{n} \cdot \boldsymbol{\xi}) \boldsymbol{B}, \tag{6.21}$$

$$\boldsymbol{n} \times \boldsymbol{A}_2(\boldsymbol{\eta}, \boldsymbol{\xi}) = -(\boldsymbol{n} \cdot \boldsymbol{\xi}) \boldsymbol{B}_1(\boldsymbol{\eta}) + (\boldsymbol{n} \cdot \delta_{\boldsymbol{\eta}} \boldsymbol{B}) \boldsymbol{\xi} - (\boldsymbol{n} \cdot \boldsymbol{\eta}) [\boldsymbol{B}_1(\boldsymbol{\xi}) - \delta_{\boldsymbol{\xi}} \boldsymbol{B}], \tag{6.22}$$

$$\begin{aligned} \boldsymbol{n} \times \boldsymbol{A}_3(\boldsymbol{\zeta}, \boldsymbol{\eta}, \boldsymbol{\xi}) = &-(\boldsymbol{n} \cdot \boldsymbol{\xi}) \boldsymbol{B}_2(\boldsymbol{\zeta}, \boldsymbol{\eta}) + \boldsymbol{n} \cdot \{\delta_{\boldsymbol{\eta}} \boldsymbol{B}_1(\boldsymbol{\zeta}) + \delta_{\boldsymbol{\zeta}} [\boldsymbol{B}_1(\boldsymbol{\eta}) - \delta_{\boldsymbol{\eta}} \boldsymbol{B}]\} \boldsymbol{\xi} \\ &- (\boldsymbol{n} \cdot \boldsymbol{\eta}) [\boldsymbol{B}_2(\boldsymbol{\zeta}, \boldsymbol{\xi}) - \delta_{\boldsymbol{\xi}} \boldsymbol{B}_1(\boldsymbol{\zeta})] + \boldsymbol{n} \cdot \{\delta_{\boldsymbol{\zeta}} [\boldsymbol{B}_1(\boldsymbol{\xi}) - \delta_{\boldsymbol{\xi}} \boldsymbol{B}]\} \boldsymbol{\eta} \\ &- (\boldsymbol{n} \cdot \boldsymbol{\zeta}) \{\boldsymbol{B}_2(\boldsymbol{\eta}, \boldsymbol{\xi}) - \delta_{\boldsymbol{\xi}} \boldsymbol{B}_1(\boldsymbol{\eta}) - \delta_{\boldsymbol{\eta}} [\boldsymbol{B}_1(\boldsymbol{\xi}) - \delta_{\boldsymbol{\xi}} \boldsymbol{B}]\}. \end{aligned} \tag{6.23}$$

It should be also remarked that $\boldsymbol{B}_n$ and $\boldsymbol{A}_n$ linearly depend on each argument (such as $\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta}$), but do not have the symmetric property. For instance, $\boldsymbol{B}_2(\boldsymbol{\eta}, \boldsymbol{\xi}) \neq \boldsymbol{B}_2(\boldsymbol{\xi}, \boldsymbol{\eta})$, and therefore the variation of $\boldsymbol{B}_2$ would results in $\delta \boldsymbol{B}_2(\boldsymbol{\xi}, \boldsymbol{\xi}) = \boldsymbol{B}_2(\delta \boldsymbol{\xi}, \boldsymbol{\xi}) + \boldsymbol{B}_2(\boldsymbol{\xi}, \delta \boldsymbol{\xi})$.

When these solutions $\boldsymbol{B}_n$ are substituted, the perturbed vacuum Lagrangian L_ϵ^v can be read as a functional of $\boldsymbol{\xi}$ and written in the form of

$$L_\epsilon^v = L^v + \epsilon L_1^v(\boldsymbol{\xi}) + \frac{\epsilon^2}{2} L_2^v(\boldsymbol{\xi}, \boldsymbol{\xi}) + \frac{\epsilon^3}{6} L_3^v(\boldsymbol{\xi}, \boldsymbol{\xi}, \boldsymbol{\xi}) + \ldots, \tag{6.24}$$

where the explicit expressions of L_n^v, $n = 1, 2, 3$, are derived in Appendix C.

7. Hamilton's principle (free boundary case)

As remarked in the introduction using Fig. 1, we now restrict ourselves to the case that the unperturbed state satisfies (1.5), which implies that the unperturbed plasma-vacuum boundary is immobile, $S_t = S = \text{const.}$

In consequence of Appendixes A and C, the plasma and vacuum Lagrangians are combined into

$$\begin{aligned} L_\epsilon + L_\epsilon^v = L + L^v + L_{\mathrm{I}} &\left(\epsilon \boldsymbol{\xi} + \frac{\epsilon^2}{2} \boldsymbol{\xi} \cdot \nabla \boldsymbol{\xi} + \frac{\epsilon^3}{6} \boldsymbol{\xi} \cdot \nabla \boldsymbol{\xi} \cdot \nabla \boldsymbol{\xi}\right) \\ &+ \frac{\epsilon^2}{2} L_{\mathrm{II}}(\boldsymbol{\xi}, \boldsymbol{\xi}) + \frac{\epsilon^3}{6} [3 L_{\mathrm{II}}(\boldsymbol{\xi} \cdot \nabla \boldsymbol{\xi}, \boldsymbol{\xi}) + L_{\mathrm{III}}(\boldsymbol{\xi}, \boldsymbol{\xi}, \boldsymbol{\xi})] + O(\epsilon^4), \end{aligned} \tag{7.1}$$

https://doi.org/10.1017/S0022377810000760 Published online by Cambridge University Press

where we have put

$$L_{\rm I}(\xi) = \int_V [-\Delta_{EL}\cdot\xi + \nabla\cdot(P\xi) + \partial_t(\rho\boldsymbol{v}\cdot\xi)]\,d^3x + \int_{V^v}\nabla\cdot\left(\frac{|\boldsymbol{B}|^2}{2}\xi\right)d^3x, \quad (7.2)$$

$$L_{\rm II}(\xi,\eta) = \int_V l^{(2)}(\xi,\eta)d^3x - W^{(2s)}(\xi,\eta) - W^{(2v)}(\xi,\eta), \quad (7.3)$$

$$L_{\rm III}(\xi,\eta,\zeta) = -W^{(3p)}(\xi,\eta,\zeta) - W^{(3s)}(\xi,\eta,\zeta) - W^{(3v)}(\xi,\eta,\zeta). \quad (7.4)$$

In the above expressions, the definition of $W^{(3p)}$ is the same as (5.4) and we have also defined

$$W^{(2s)}(\xi,\eta) = -\int_{V+V^v}\mathcal{L}^2_{\xi,\eta}P\,d^3x, \quad (7.5)$$

$$W^{(3s)}(\xi,\eta,\zeta) = -\int_{V+V^v}\left[\mathcal{L}^2_{\eta,\xi}P_1(\zeta) + \mathcal{L}^2_{\zeta,\xi}P_1(\eta) + \mathcal{L}^2_{\zeta,\eta}P_1(\xi) + \mathcal{L}^3_{\xi,\eta,\zeta}P\right]d^3x, \quad (7.6)$$

$$W^{(2v)}(\xi,\eta) = \int_{V^v}\boldsymbol{B}_1(\xi)\cdot\boldsymbol{B}_1(\eta)d^3x, \quad (7.7)$$

$$W^{(3v)}(\xi,\eta,\zeta) = -\int_{V^v}\nabla\cdot[\boldsymbol{B}_1(\zeta)\cdot\boldsymbol{B}_1(\eta)\xi + \boldsymbol{B}_1(\zeta)\cdot\boldsymbol{B}_1(\xi)\eta + \boldsymbol{B}_1(\eta)\cdot\boldsymbol{B}_1(\xi)\zeta]d^3x, \quad (7.8)$$

where the total pressure P is understood as $P = |\boldsymbol{B}|^2/2$ in the vacuum region V^v, and

$$P_1(\xi) \overset{\rm def}{=} \begin{cases}\delta_\xi P & \text{in } V\\ \boldsymbol{B}\cdot\boldsymbol{B}_1(\xi) & \text{in } V^v.\end{cases} \quad (7.9)$$

Note that, by using Gauss' law for the divergence operators (such as $\mathcal{L}^2_{\xi,\eta}$), both $W^{(2s)}$ and $W^{(3s)}$ are transformed into surface integrals on S. They can be regarded as the second- and third-order surface potential energies, respectively, since they vanish $W^{(2s)} = W^{(3s)} = 0$ if P and $P_1(\xi)$ are continuous and smooth across the surface S. Accordingly, $W^{(2v)}$ and $W^{(3v)}$ represent the vacuum potential energies.

It is remarkable that $L_{\rm II}$ and $L_{\rm III}$ are symmetric with respect to any permutation of their arguments ξ,η,ζ. Thus, the series expansion (7.1) again takes the form of (3.6).

As with the fixed boundary case, we now assume that the unperturbed state is an *exact solution* of the MHD equations (1.1)–(1.4) (namely, $\Delta_{EL} = 0$) in V and

$$[[P]] = 0 \quad \text{on } S \quad (7.10)$$

where $[[P]]$ denotes the jump of P across S,

$$[[P]] = P|_{S^+} - P|_{S^-} = \left.\frac{|\boldsymbol{B}|^2}{2}\right|_{S^+} - P|_{S^-}. \quad (7.11)$$

Then, we find that $L_{\rm I}(\xi) = \partial_t\int_V\rho\boldsymbol{v}\cdot\xi d^3x \simeq 0$ for any ξ, and the expansion (7.1) around this unperturbed state is reduced to

$$L_\epsilon + L^v_\epsilon \simeq L + L^v + \frac{\epsilon^2}{2}L_{\rm II}(\xi,\xi) + \frac{\epsilon^3}{6}[3L_{\rm II}(\xi\cdot\nabla\xi,\xi) + L_{\rm III}(\xi,\xi,\xi)] + O(\epsilon^4). \quad (7.12)$$

https://doi.org/10.1017/S0022377810000760 Published online by Cambridge University Press

In addition, using $[[P]] = 0$ (which also implies $[[\nabla P]] \propto \boldsymbol{n}$), the surface potential energy $W^{(2s)}$ agrees with the well-known expression [1],

$$W^{(2s)}(\boldsymbol{\xi}, \boldsymbol{\eta}) = \int_S (\boldsymbol{n} \cdot \boldsymbol{\xi})(\boldsymbol{n} \cdot \boldsymbol{\eta}) \boldsymbol{n} \cdot [[\nabla P]] d^2x. \tag{7.13}$$

Hence, if we extremize the Lagrangian (7.12) to second-order $O(\epsilon^2)$, we recover the linearized system,

$$\Delta'_{EL}(\boldsymbol{\xi}) = \rho \frac{D^2 \boldsymbol{\xi}}{Dt^2} - \mathscr{F} \boldsymbol{\xi} = 0 \quad \text{in } V, \tag{7.14}$$

$$[[P_1(\boldsymbol{\xi}) + \boldsymbol{\xi} \cdot \nabla P]] = 0 \quad \text{on } S. \tag{7.15}$$

In order to extend these equations to the weakly nonlinear regime, let us manipulate the third-order Lagrangian L_{III} as follows: First, by using the definition (A 24) of $\mathcal{L}^3_{\boldsymbol{\zeta},\boldsymbol{\eta},\boldsymbol{\xi}}$ and $[[P]] = 0$, the third-order surface potential energy can be rewritten as

$$\begin{aligned} W^{(3s)}(\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta}) = -\int_{V+V^v} \Big\{ & \mathcal{L}^2_{\boldsymbol{\eta},\boldsymbol{\xi}}[P_1(\boldsymbol{\zeta}) + \boldsymbol{\zeta} \cdot \nabla P] + \mathcal{L}^2_{\boldsymbol{\zeta},\boldsymbol{\xi}}[P_1(\boldsymbol{\eta}) + \boldsymbol{\eta} \cdot \nabla P] \\ & + \mathcal{L}^2_{\boldsymbol{\zeta},\boldsymbol{\eta}}[P_1(\boldsymbol{\xi}) + \boldsymbol{\xi} \cdot \nabla P] - \mathcal{L}_{\boldsymbol{\eta}} \mathcal{L}_{\boldsymbol{\xi}}(\boldsymbol{\zeta} \cdot \nabla P) - \mathcal{L}_{\boldsymbol{\zeta}}[\boldsymbol{\xi} \cdot \nabla (\boldsymbol{\eta} \cdot \nabla P)] \Big\} d^3x. \end{aligned} \tag{7.16}$$

Next, by extensively using $[[P]] = 0$ and the boundary conditions (6.21)–(6.23), one can break the cubic symmetry of (7.4) such that $\boldsymbol{\zeta}$ is extracted as follows:

$$\begin{aligned} L_{\mathrm{III}}(\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta}) = & \int_V \boldsymbol{\zeta} \cdot \mathscr{F}^{(2)}(\boldsymbol{\xi}, \boldsymbol{\eta}) d^3x + \int_S \{ (\boldsymbol{n} \cdot \boldsymbol{\zeta})[[P_1(\boldsymbol{\eta} \cdot \nabla \boldsymbol{\xi}) + (\boldsymbol{\eta} \cdot \nabla \boldsymbol{\xi}) \cdot \nabla P]] \\ & - (\boldsymbol{n} \cdot \boldsymbol{\zeta})[[P_2(\boldsymbol{\eta}, \boldsymbol{\xi}) + \boldsymbol{\xi} \cdot \nabla P_1(\boldsymbol{\eta}) + \boldsymbol{\eta} \cdot \nabla (P_1(\boldsymbol{\xi}) + \boldsymbol{\xi} \cdot \nabla P)]] \\ & + \boldsymbol{n} \cdot (\boldsymbol{\zeta} \cdot \nabla \boldsymbol{\eta})[[P_1(\boldsymbol{\xi}) + \boldsymbol{\xi} \cdot \nabla P]] + \boldsymbol{n} \cdot (\boldsymbol{\zeta} \cdot \nabla \boldsymbol{\xi})[[P_1(\boldsymbol{\eta}) + \boldsymbol{\eta} \cdot \nabla P]] \\ & - (\boldsymbol{n} \cdot \boldsymbol{\zeta})(\nabla \cdot \boldsymbol{\eta})[[P_1(\boldsymbol{\xi}) + \boldsymbol{\xi} \cdot \nabla P]] - (\boldsymbol{n} \cdot \boldsymbol{\zeta})(\nabla \cdot \boldsymbol{\xi})[[P_1(\boldsymbol{\eta}) + \boldsymbol{\eta} \cdot \nabla P]] \} d^2x, \end{aligned} \tag{7.17}$$

where

$$P_2(\boldsymbol{\eta}, \boldsymbol{\xi}) \overset{\text{def}}{=} \begin{cases} \delta_{\boldsymbol{\eta}} \delta_{\boldsymbol{\xi}} P & \text{in } V, \\ \boldsymbol{B} \cdot \boldsymbol{B}_2(\boldsymbol{\eta}, \boldsymbol{\xi}) + \boldsymbol{B}_1(\boldsymbol{\eta}) \cdot \boldsymbol{B}_1(\boldsymbol{\xi}) & \text{in } V^v. \end{cases} \tag{7.18}$$

Therefore, the variation of (7.12) is obtained as

$$\begin{aligned} \delta\left(L_\epsilon + L^v_\epsilon\right) &\simeq \epsilon^2 L_{\mathrm{II}}\left(\boldsymbol{\xi}, \delta\boldsymbol{\xi} + \frac{\epsilon}{2}\delta(\boldsymbol{\xi} \cdot \nabla \boldsymbol{\xi})\right) + \frac{\epsilon^3}{2}\left[L_{\mathrm{II}}(\boldsymbol{\xi} \cdot \nabla \boldsymbol{\xi}, \delta\boldsymbol{\xi}) + L_{\mathrm{III}}(\boldsymbol{\xi}, \boldsymbol{\xi}, \delta\boldsymbol{\xi})\right] \\ &= \int_V \Big[-\epsilon^2 \Delta'_{EL}(\boldsymbol{\xi}) \cdot \left(\delta\boldsymbol{\xi} + \frac{\epsilon}{2}\delta(\boldsymbol{\xi} \cdot \nabla \boldsymbol{\xi})\right) - \frac{\epsilon^3}{2} \Delta'_{EL}(\boldsymbol{\xi} \cdot \nabla \boldsymbol{\xi}) \cdot \delta\boldsymbol{\xi} \end{aligned}$$

https://doi.org/10.1017/S0022377810000760 Published online by Cambridge University Press

$$+\frac{\epsilon^3}{2}\mathscr{F}^{(2)}(\xi,\xi)\cdot\delta\xi\Big]d^3x+\int_S\Big\{-\epsilon^2\boldsymbol{n}\cdot\Big(\delta\xi+\frac{\epsilon}{2}(\xi\cdot\nabla\delta\xi-\delta\xi\cdot\nabla\xi)$$

$$+\epsilon(\nabla\cdot\xi)\delta\xi\Big)[[P_1(\xi)+\xi\cdot\nabla P]]-\frac{\epsilon^3}{2}(\boldsymbol{n}\cdot\delta\xi)[[P_2(\xi,\xi)+2\xi\cdot\nabla P_1(\xi)$$

$$+\xi\cdot\nabla(\xi\cdot\nabla P)]]\Big\}d^2x. \tag{7.19}$$

In the same manner as the fixed boundary case, the linearized system (7.14) and (7.15) are still approximately satisfied; $\Delta'_{EL}(\xi)=O(\epsilon)$ and $[[P_1(\xi)+\xi\cdot\nabla P]]=O(\epsilon)$. The Euler–Lagrange equation is then found to be

$$\Delta'_{EL}(\xi)+\frac{\epsilon}{2}\Delta'_{EL}(\xi\cdot\nabla\xi)-\frac{\epsilon}{2}\mathscr{F}^{(2)}(\xi,\xi)=O(\epsilon^2)\quad\text{in } V, \tag{7.20}$$

$$[[P_1(\xi)+\xi\cdot\nabla P]]+\frac{\epsilon}{2}[[P_2(\xi,\xi)+2\xi\cdot\nabla P_1(\xi)+\xi\cdot\nabla(\xi\cdot\nabla P)]]=O(\epsilon^2)\quad\text{on } S. \tag{7.21}$$

As might be expected, the equation of motion in the plasma region is identical with (4.8). The free boundary condition (7.21) is also reasonable in that it requires the perturbed total pressure to be continuous on the nonlinearly displaced surface.

Finally, let us conclude this section with the resonant three-mode coupling for the free boundary case. We assume that the three eigenmodes $\hat{\xi}_j e^{-i\omega_j t}$, $j=a,b,c$, in (5.2) satisfy the eigenvalue problem for (7.14) and (7.15). From the fact that $\Delta'_{EL}(\xi)=O(\epsilon)$ and $[[P_1(\xi)+\xi\cdot\nabla P]]=0$, one can neglect $L_{\mathrm{II}}(\xi,\xi\cdot\nabla\xi)\sim O(\epsilon)$ in (7.12). The remaining procedure is the same as the fixed boundary case. We find that only the coupling coefficient (5.8) should be replaced by

$$W^{(3)}_{a,b,c}=-L_{\mathrm{III}}(\hat{\xi}^*_a,\hat{\xi}_b,\hat{\xi}_c)$$

$$=W^{(3p)}(\hat{\xi}^*_a,\hat{\xi}_b,\hat{\xi}_c)+W^{(3s)}(\hat{\xi}^*_a,\hat{\xi}_b,\hat{\xi}_c)+W^{(3v)}(\hat{\xi}^*_a,\hat{\xi}_b,\hat{\xi}_c), \tag{7.22}$$

in the free boundary case, which is clearly the sum of the plasma, surface and vacuum potential energies.

For such ξ satisfying $[[P_1(\xi)+\xi\cdot\nabla P]]=0$, we can also utilize the relation,

$$\eta\cdot[[\nabla(P_1(\xi)+\xi\cdot\nabla P)]]=(\boldsymbol{n}\cdot\eta)\boldsymbol{n}\cdot[[\nabla(P_1(\xi)+\xi\cdot\nabla P)]], \tag{7.23}$$

to simplify the expression (7.16), which results in

$$W^{(3s)}(\xi,\eta,\zeta)=\int_S[(\boldsymbol{n}\cdot\xi)(\boldsymbol{n}\cdot\eta)\boldsymbol{n}\cdot[[\nabla(P_1(\zeta)+\zeta\cdot\nabla P)]]$$

$$+(\boldsymbol{n}\cdot\xi)(\boldsymbol{n}\cdot\zeta)\boldsymbol{n}\cdot[[\nabla(P_1(\eta)+\eta\cdot\nabla P)]]$$

$$+(\boldsymbol{n}\cdot\zeta)(\boldsymbol{n}\cdot\eta)\boldsymbol{n}\cdot[[\nabla(P_1(\xi)+\xi\cdot\nabla P)]]$$

$$-(\boldsymbol{n}\cdot\eta)\xi\cdot[[\nabla(\zeta\cdot\nabla P)]]-(\boldsymbol{n}\cdot\zeta)\xi\cdot[[\nabla(\eta\cdot\nabla P)]]$$

$$-(\boldsymbol{n}\cdot\eta)(\boldsymbol{n}\cdot\zeta)(\nabla\cdot\xi)\boldsymbol{n}\cdot[[\nabla P]]]d\mathrm{d}^3x. \tag{7.24}$$

When calculating the coupling coefficient (7.22) among three eigenmodes, the last formula of the surface potential energy would be most easily applied.

https://doi.org/10.1017/S0022377810000760 Published online by Cambridge University Press

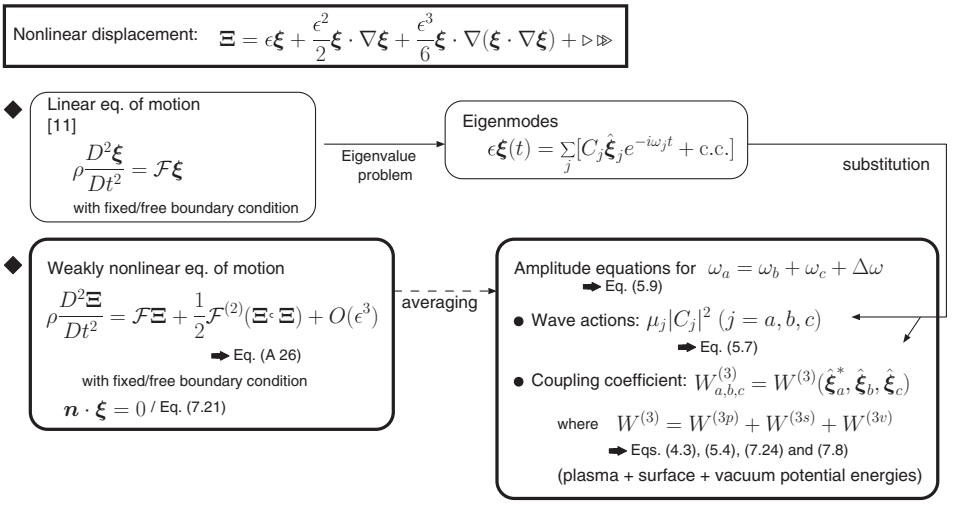


Figure 4. General scheme for weakly nonlinear analysis of three-mode coupling.

8. Summary

In this work, the MHD Lagrangian theory [2] is applied to the formulation of the weakly nonlinear equation of motion in terms of the displacement field. Using the technique of Lie series, the small-amplitude expansion of the MHD Lagrangian is explicitly performed up to third order for both fixed and free boundary cases. The variational principle for the third-order Lagrangian can rigorously extend the well-known linearized equation of motion (Frieman and Rotenberg [11]) to the weakly nonlinear one (see Fig. 4), which includes the quadratic force term $\mathscr{F}^{(2)}$ that brings about nonlinear coupling among resonant three eigenmodes $\omega_a = \omega_b + \omega_c + \Delta\omega$ (and second-harmonic resonance $\omega_a = 2\omega_b + \Delta\omega$ as well). The coupling coefficient $W^{(3)}_{a,b,c}$, which quantifies the effectiveness of the three-mode coupling, can be evaluated by substituting the three eigenfunctions into the third-order potential energy. This framework, illustrated in Fig. 4, enables a systematic analysis of global mode couplings (without relying on the eikonal approximation) and is widely applicable to any equilibrium state (including flow) with fixed/free boundary.

The advantages of our result over the earlier work [9] can be highlighted by the following two points. First, we have formulated the amplitude equations in terms of the familiar displacement vector field $\boldsymbol{\xi}$, which is commonly solved in the linear MHD analysis. Therefore, we can utilize the accumulated knowledge about the linear solutions obtained by the energy principle and the existing eigenvalue solvers. Second, we have shown that the third-order potential energy $W^{(3)}$ has the cubic symmetry and generates a greatly reduced expression of the coupling coefficient. In analogy with the energy principle [where $\delta W(\boldsymbol{\xi}, \boldsymbol{\xi})$ is symmetric], we expect further theoretical and numerical studies based on the formula of $W^{(3)}$ to be developed by assuming a certain class of equilibria and eigenmodes which are related to realistic problems.

Note that, within the present work, we have not taken account of the existence of the Alvén and sound continuum modes. In fact, these continuous spectra would occupy a considerable range of frequencies due to the spatial inhomogeneity of

https://doi.org/10.1017/S0022377810000760 Published online by Cambridge University Press

equilibria. We will show elsewhere that the continuum mode can be incorporated into the three-mode coupling with the help of the profound knowledge of the MHD continuous spectrum [19].

If the equilibrium is uniform in one or two directions (such as the axisymmetric or cylindrical equilibrium), one may use the wavenumber k in those directions. Then, the coupling among three modes satisfying $k_a = k_b + k_c$ can be discussed in the same manner as Sec. 5, by replacing the time average with the spatial one (along the specified direction) in Whitham's method. The coupling coefficient would be further simplified by the use of the wavenumber.

The weakly nonlinear analysis using the displacement *vector field* would also enhance the understanding of the wave-mean field interaction, since Dewar [12], Andrews and McIntyer [20], Hirota and Tokuda [21] have not clearly distinguished the displacement *vector field* from the displacement *map*. Fukumoto and Hirota [22] have recently utilized the expansion (2.9) to calculate the wave-induced mean flow in a vortex tube. The application of the same method to the MHD case is expected, too.

Acknowledgements

The author would like to thank Y. Fukumoto, Z. Yoshida, M. Furukawa, S. Tokuda and M. Yagi for fruitful discussions. This work was partially supported by a grant-in-aid for scientific research from the Japan Society for the Promotion of Science (No. 22740369).

Appendix A. Series expansion of the MHD Lagrangian

A.1. First order

For any $\boldsymbol{\xi}$, the uses of (2.16) and (2.19) leads to

$$\begin{aligned}\delta_\xi \mathrm{L} &= \rho \boldsymbol{v} \cdot \left(\frac{D\boldsymbol{\xi}}{Dt} - \boldsymbol{\xi} \cdot \nabla \boldsymbol{v} \right) - \frac{1}{2} \nabla \cdot (\rho \boldsymbol{\xi}) |\boldsymbol{v}|^2 - \boldsymbol{B} \cdot [\nabla \times (\boldsymbol{\xi} \times \boldsymbol{B})] \\ &\quad + \nabla \cdot (\rho \boldsymbol{\xi}) U + \rho \frac{\partial U}{\partial \rho} \nabla \cdot (\rho \boldsymbol{\xi}) + \rho \frac{\partial U}{\partial s} \boldsymbol{\xi} \cdot \nabla s \\ &= - \left[\partial_t \rho + \nabla \cdot (\rho \boldsymbol{v}) \right] (\boldsymbol{v} \cdot \boldsymbol{\xi}) - \Delta_{EL} \cdot \boldsymbol{\xi} + \partial_t (\rho \boldsymbol{v} \cdot \boldsymbol{\xi}) \\ &\quad + \nabla \cdot \left[\left(\rho U + p - \rho \frac{|\boldsymbol{v}|^2}{2} + |\boldsymbol{B}|^2 \right) \boldsymbol{\xi} + (\rho \boldsymbol{v} \cdot \boldsymbol{\xi}) \boldsymbol{v} - (\boldsymbol{B} \cdot \boldsymbol{\xi}) \boldsymbol{B} \right], \end{aligned} \tag{A 1}$$

where $D/Dt = \partial_t + \boldsymbol{v} \cdot \nabla$ is the convective derivative, and $p = \rho^2 \partial U / \partial \rho$ is the pressure. We have introduced the following abbreviation:

$$\Delta_{EL} = \rho \frac{D\boldsymbol{v}}{Dt} - \boldsymbol{B} \cdot \nabla \boldsymbol{B} + \nabla P, \tag{A 2}$$

where $P = p(\rho, s) + |\boldsymbol{B}|^2/2$ is called the total pressure. Obviously, $\Delta_{EL} = 0$ corresponds to the equation of motion (1.1), namely, the unperturbed Euler–Lagrange equation.

Recall that the continuity equation $\partial_t \rho + \nabla \cdot (\rho \boldsymbol{v}) = 0$ is already assumed in the Lagrangian description of fluid. Hence, the first term on the right-hand side of (A 1)

https://doi.org/10.1017/S0022377810000760 Published online by Cambridge University Press

vanishes and we, therefore, obtain

$$\mathsf{D}_\xi \mathrm{L} = -\Delta_{EL}\cdot\boldsymbol{\xi} + \partial_t(\rho\boldsymbol{v}\cdot\boldsymbol{\xi}) + \nabla\cdot[P\boldsymbol{\xi} + (\rho\boldsymbol{v}\cdot\boldsymbol{\xi})\boldsymbol{v} - (\boldsymbol{B}\cdot\boldsymbol{\xi})\boldsymbol{B}]\,. \tag{A 3}$$

It is also remarkable that the elimination of $\partial_t\rho + \nabla\cdot(\rho\boldsymbol{v})$ at this point will not affect the subsequent calculations of the second- and third-order expansions of L. Indeed, one can confirm that

$$\delta_\xi[\partial_t\rho + \nabla\cdot(\rho\boldsymbol{v})] = \partial_t(\delta_\xi\rho) + \nabla\cdot[(\delta_\xi\rho)\boldsymbol{v} + \rho(\delta_\xi\boldsymbol{v})] = 0 \tag{A 4}$$

holds automatically and, in this way, the perturbed continuity equation is satisfied to any higher order by adopting the Lie series expansion (2.17).

A.2. Second order

Using the same vector formula as (3.5), we get

$$\begin{aligned}\delta_\xi\Delta_{EL} &= \rho[\partial_t(\delta_\xi\boldsymbol{v}) + (\boldsymbol{v}\cdot\nabla)(\delta_\xi\boldsymbol{v}) + (\delta_\xi\boldsymbol{v}\cdot\nabla)\boldsymbol{v}] + (\delta_\xi\rho)[\partial_t\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}]\\ &\quad + \nabla\delta_\xi P - (\boldsymbol{B}\cdot\nabla)(\delta_\xi\boldsymbol{B}) - (\delta_\xi\boldsymbol{B}\cdot\nabla)\boldsymbol{B}\\ &= \rho\frac{D^2\boldsymbol{\xi}}{Dt^2} - \mathscr{F}\boldsymbol{\xi} - (\boldsymbol{\xi}\cdot\nabla)\Delta_{EL} - (\nabla\cdot\boldsymbol{\xi})\Delta_{EL},\end{aligned} \tag{A 5}$$

where the variation of the total pressure is given by

$$\begin{aligned}\delta_\xi P &= \delta_\xi p + \boldsymbol{B}\cdot\delta_\xi\boldsymbol{B}\\ &= \boldsymbol{B}\cdot\mathscr{B}\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\nabla P - \rho c_s^2\nabla\cdot\boldsymbol{\xi},\end{aligned} \tag{A 6}$$

with $c_s^2 = \partial p/\partial\rho$ being the sound speed, and we have defined linear operators $\mathscr{B}$ and $\mathscr{F}$ as

$$\mathscr{B}\boldsymbol{\xi} = \boldsymbol{B}\cdot\nabla\boldsymbol{\xi} - (\nabla\cdot\boldsymbol{\xi})\boldsymbol{B}, \tag{A 7}$$

$$\begin{aligned}\mathscr{F}\boldsymbol{\xi} &= (\boldsymbol{B}\cdot\nabla)[(\boldsymbol{B}\cdot\nabla)\boldsymbol{\xi}] - (\boldsymbol{\xi}\cdot\nabla)\nabla P + \nabla\left[\left(|\boldsymbol{B}|^2 + \rho c_s^2\right)\nabla\cdot\boldsymbol{\xi}\right] - (\nabla\cdot\boldsymbol{\xi})\nabla P + \nabla(\boldsymbol{\xi}\cdot\nabla P)\\ &\quad - (\boldsymbol{B}\cdot\nabla)[(\nabla\cdot\boldsymbol{\xi})\boldsymbol{B}] - \nabla\{\boldsymbol{B}\cdot[(\boldsymbol{B}\cdot\nabla)\boldsymbol{\xi}]\}.\end{aligned} \tag{A 8}$$

Note that (A 5)=0 corresponds to the linearized equation of motion derived by Frieman–Rotenberg [11]. If the unperturbed state is a solution of $\Delta_{EL} = 0$, it is equivalent to

$$\Delta'_{EL}(\boldsymbol{\xi}) \overset{\text{def}}{=} \rho\frac{D^2\boldsymbol{\xi}}{Dt^2} - \mathscr{F}\boldsymbol{\xi} = 0. \tag{A 9}$$

In this work, we employ this $\mathscr{F}$ as the definition of *force* operator, which looks slightly different from the conventional one.

For any $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, let us operate $\mathsf{D}_{\boldsymbol{\eta}}$ on $\mathsf{D}_\xi\mathrm{L}$. First, we obtain

$$\mathsf{D}_{\boldsymbol{\eta}}(\Delta_{EL}\cdot\boldsymbol{\xi}) = \Delta'_{EL}(\boldsymbol{\eta})\cdot\boldsymbol{\xi} + \Delta_{EL}\cdot(\boldsymbol{\eta}\cdot\nabla\boldsymbol{\xi}), \tag{A 10}$$

in which the first term on the right-hand side is further rewritten as

$$\begin{aligned}\Delta'_{EL}(\boldsymbol{\eta})\cdot\boldsymbol{\xi} &= -\rho\frac{D\boldsymbol{\xi}}{Dt}\cdot\frac{D\boldsymbol{\eta}}{Dt} + w^{(2)}(\boldsymbol{\xi},\boldsymbol{\eta}) + \nabla\cdot[(\delta_{\boldsymbol{\eta}}P)\boldsymbol{\xi}]\\ &\quad + \partial_t\left(\rho\frac{D\boldsymbol{\eta}}{Dt}\cdot\boldsymbol{\xi}\right) + \nabla\cdot\left[\left(\rho\frac{D\boldsymbol{\eta}}{Dt}\cdot\boldsymbol{\xi}\right)\boldsymbol{v} - (\mathscr{B}\boldsymbol{\eta}\cdot\boldsymbol{\xi})\boldsymbol{B}\right],\end{aligned} \tag{A 11}$$

https://doi.org/10.1017/S0022377810000760 Published online by Cambridge University Press

where

$$w^{(2)}(\boldsymbol{\xi},\boldsymbol{\eta}) = \mathscr{B}\boldsymbol{\xi}\cdot\mathscr{B}\boldsymbol{\eta} + \boldsymbol{\eta}\cdot[(\boldsymbol{\xi}\cdot\nabla)\nabla P] + \rho c_s^2(\nabla\cdot\boldsymbol{\xi})(\nabla\cdot\boldsymbol{\eta}) + (\nabla\cdot\boldsymbol{\xi})\boldsymbol{\eta}\cdot\nabla P + (\nabla\cdot\boldsymbol{\eta})\boldsymbol{\xi}\cdot\nabla P, \tag{A 12}$$

is symmetric with respect to $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. After some manipulations, we obtain

$$\begin{aligned}\mathsf{D}_{\boldsymbol{\eta}}\mathsf{D}_{\boldsymbol{\xi}}\mathrm{L} &= -\Delta_{EL}\cdot(\boldsymbol{\eta}\cdot\nabla\boldsymbol{\xi}) + l^{(2)}(\boldsymbol{\xi},\boldsymbol{\eta}) + \nabla\cdot[\nabla\cdot(P\boldsymbol{\xi})\boldsymbol{\eta}]\\ &\quad + \partial_t[\rho\boldsymbol{v}\cdot(\boldsymbol{\eta}\cdot\nabla\boldsymbol{\xi})] + \nabla\cdot\{[\rho\boldsymbol{v}\cdot(\boldsymbol{\eta}\cdot\nabla\boldsymbol{\xi})]\boldsymbol{v} - [\boldsymbol{B}\cdot(\boldsymbol{\eta}\cdot\nabla\boldsymbol{\xi})]\boldsymbol{B}\},\\ &= \mathsf{D}_{\boldsymbol{\eta}\cdot\nabla\boldsymbol{\xi}}\mathrm{L} + l^{(2)}(\boldsymbol{\xi},\boldsymbol{\eta}) + \mathcal{L}^2_{\boldsymbol{\eta},\boldsymbol{\xi}}P,\end{aligned} \tag{A 13}$$

where

$$l^{(2)}(\boldsymbol{\xi},\boldsymbol{\eta}) \overset{\text{def}}{=} \rho\frac{D\boldsymbol{\xi}}{Dt}\cdot\frac{D\boldsymbol{\eta}}{Dt} - w^{(2)}(\boldsymbol{\xi},\boldsymbol{\eta}), \tag{A 14}$$

$$\mathcal{L}^2_{\boldsymbol{\xi},\boldsymbol{\eta}} \overset{\text{def}}{=} \mathcal{L}_{\boldsymbol{\xi}}\mathcal{L}_{\boldsymbol{\eta}} - \mathcal{L}_{\boldsymbol{\xi}\cdot\nabla\boldsymbol{\eta}} = \mathcal{L}_{\boldsymbol{\eta}}\mathcal{L}_{\boldsymbol{\xi}} - \mathcal{L}_{\boldsymbol{\eta}\cdot\nabla\boldsymbol{\xi}}. \tag{A 15}$$

Recall that $\mathcal{L}_{\boldsymbol{\xi}}P = \nabla\cdot(P\boldsymbol{\xi})$. This second-order Lagrangian density $l^{(2)}$ agrees with Dewar [12].

A.3. Third order

By operating $\mathsf{D}_{\boldsymbol{\zeta}}$ on the last expression of $\mathsf{D}_{\boldsymbol{\eta}}\mathsf{D}_{\boldsymbol{\xi}}\mathrm{L}$, the similar calculation leads to

$$\begin{aligned}\mathsf{D}_{\boldsymbol{\zeta}}\mathsf{D}_{\boldsymbol{\eta}}\mathsf{D}_{\boldsymbol{\xi}}\mathrm{L} &= -\Delta_{EL}\cdot(\boldsymbol{\zeta}\cdot\nabla(\boldsymbol{\eta}\cdot\nabla\boldsymbol{\xi})) + l^{(2)}(\boldsymbol{\eta}\cdot\nabla\boldsymbol{\xi},\boldsymbol{\zeta}) + \mathsf{D}_{\boldsymbol{\zeta}}l^{(2)}(\boldsymbol{\xi},\boldsymbol{\eta})\\ &\quad + \nabla\cdot\{-(\boldsymbol{\eta}\cdot\nabla\boldsymbol{\xi})\delta_{\boldsymbol{\zeta}}P + \nabla\cdot[(\delta_{\boldsymbol{\zeta}}P)\boldsymbol{\xi}]\boldsymbol{\eta} + \nabla\cdot[\nabla\cdot(P\boldsymbol{\xi})\boldsymbol{\eta}]\boldsymbol{\zeta}\}\\ &\quad + \partial_t[\rho\boldsymbol{v}\cdot(\boldsymbol{\zeta}\cdot\nabla(\boldsymbol{\eta}\cdot\nabla\boldsymbol{\xi}))] + \nabla\cdot\{[\rho\boldsymbol{v}\cdot(\boldsymbol{\zeta}\cdot\nabla(\boldsymbol{\eta}\cdot\nabla\boldsymbol{\xi}))]\boldsymbol{v}\\ &\quad - [\boldsymbol{B}\cdot(\boldsymbol{\zeta}\cdot\nabla(\boldsymbol{\eta}\cdot\nabla\boldsymbol{\xi}))]\boldsymbol{B}\}.\end{aligned} \tag{A 16}$$

As for $\mathsf{D}_{\boldsymbol{\zeta}}l^{(2)}(\boldsymbol{\xi},\boldsymbol{\eta})$, one can prove a relation

$$\mathsf{D}_{\boldsymbol{\zeta}}\left(\rho\frac{D\boldsymbol{\xi}}{Dt}\cdot\frac{D\boldsymbol{\eta}}{Dt}\right) = \rho\frac{D}{Dt}(\boldsymbol{\zeta}\cdot\nabla\boldsymbol{\xi})\cdot\frac{D\boldsymbol{\eta}}{Dt} + \rho\frac{D}{Dt}(\boldsymbol{\zeta}\cdot\nabla\boldsymbol{\eta})\cdot\frac{D\boldsymbol{\xi}}{Dt}, \tag{A 17}$$

whereas the remaining calculation of $\mathsf{D}_{\boldsymbol{\zeta}}w^{(2)}(\boldsymbol{\xi},\boldsymbol{\eta})$ is somewhat labouring. First, by using

$$\delta_{\boldsymbol{\zeta}}(\rho c_s^2) = -\nabla\cdot(\rho c_s^2\boldsymbol{\zeta}) - \rho^2\frac{\partial^2 p}{\partial\rho^2}\nabla\cdot\boldsymbol{\zeta}, \tag{A 18}$$

https://doi.org/10.1017/S0022377810000760 Published online by Cambridge University Press

we directly obtain

$$\begin{aligned}\delta_\zeta w^{(2)}(\xi,\boldsymbol{\eta}) = {}& \mathscr{B}(\zeta\cdot\nabla\xi)\cdot\mathscr{B}\boldsymbol{\eta} + \mathscr{B}(\zeta\cdot\nabla\boldsymbol{\eta})\cdot\mathscr{B}\xi \\ & + \nabla\cdot(\zeta\cdot\nabla\xi - \zeta\nabla\cdot\xi)\boldsymbol{B}\cdot\mathscr{B}\boldsymbol{\eta} + \nabla\cdot(\zeta\cdot\nabla\boldsymbol{\eta} - \zeta\nabla\cdot\boldsymbol{\eta})\boldsymbol{B}\cdot\mathscr{B}\xi \\ & + \nabla\cdot(\xi\cdot\nabla\boldsymbol{\eta} - \xi\nabla\cdot\boldsymbol{\eta})\big(\boldsymbol{B}\cdot\mathscr{B}\zeta - \zeta\cdot\nabla P - \rho c_s^2\nabla\cdot\zeta\big) \\ & - (\nabla\cdot\xi)\mathscr{B}\zeta\cdot\mathscr{B}\boldsymbol{\eta} - (\nabla\cdot\boldsymbol{\eta})\mathscr{B}\zeta\cdot\mathscr{B}\xi - (\nabla\cdot\zeta)\mathscr{B}\xi\cdot\mathscr{B}\boldsymbol{\eta} \\ & + \rho c_s^2\zeta\cdot\nabla[(\nabla\cdot\xi)(\nabla\cdot\boldsymbol{\eta})] - \rho^2\frac{\partial^2 p}{\partial\rho^2}(\nabla\cdot\zeta)(\nabla\cdot\xi)(\nabla\cdot\boldsymbol{\eta}) \\ & - \nabla\cdot[(\mathscr{B}\xi\cdot\mathscr{B}\boldsymbol{\eta})\zeta] - \nabla\cdot[(\xi\cdot\nabla\boldsymbol{\eta} - \xi\nabla\cdot\boldsymbol{\eta})\delta_\zeta P] \\ & - \nabla\cdot\big[(\nabla\cdot\xi)(\nabla\cdot\boldsymbol{\eta})\rho c_s^2\zeta\big] + \nabla\cdot[(\boldsymbol{\eta}\cdot\nabla\delta_\zeta P)\xi], \end{aligned} \tag{A 19}$$

Now, let us heuristically introduce the following notation:

$$\boldsymbol{\eta}\zeta\xi : \nabla\nabla\nabla P \overset{\text{def}}{=} \boldsymbol{\eta}\cdot\{\zeta\cdot\nabla[(\xi\cdot\nabla)\nabla P] - [(\zeta\cdot\nabla\xi)\cdot\nabla]\nabla P\}. \tag{A 20}$$

By transforming this definition into

$$\begin{aligned}\boldsymbol{\eta}\zeta\xi : \nabla\nabla\nabla P = {}& -(\nabla\cdot\zeta)\boldsymbol{\eta}\cdot[(\xi\cdot\nabla)\nabla P] - (\nabla\cdot\xi)\boldsymbol{\eta}\cdot(\zeta\cdot\nabla)\nabla P - (\nabla\cdot\boldsymbol{\eta})\xi\cdot(\zeta\cdot\nabla)\nabla P \\ & + (\nabla\cdot\xi)\zeta\cdot\nabla(\boldsymbol{\eta}\cdot\nabla P) + (\nabla\cdot\boldsymbol{\eta})\zeta\cdot\nabla(\xi\cdot\nabla P) + [\xi\cdot\nabla(\zeta\cdot\nabla\boldsymbol{\eta})]\cdot\nabla P \\ & + [\boldsymbol{\eta}\cdot\nabla(\zeta\cdot\nabla\xi)]\cdot\nabla P - \nabla\cdot\{[(\zeta\cdot\nabla\boldsymbol{\eta})\cdot\nabla P]\xi\} - \nabla\cdot\{[(\zeta\cdot\nabla\xi)\cdot\nabla P]\boldsymbol{\eta}\} \\ & + \nabla\cdot\{[\boldsymbol{\eta}\cdot(\xi\cdot\nabla)\nabla P]\zeta\}, \end{aligned} \tag{A 21}$$

we can rewrite the above $\delta_\zeta w^{(2)}(\xi,\boldsymbol{\eta})$ in the form of

$$\begin{aligned}\delta_\zeta w^{(2)}(\xi,\boldsymbol{\eta}) = {}& w^{(2)}(\zeta\cdot\nabla\xi,\boldsymbol{\eta}) + w^{(2)}(\xi,\zeta\cdot\nabla\boldsymbol{\eta}) + w^{(3)}(\xi,\boldsymbol{\eta},\zeta) \\ & - \nabla\cdot[w^{(2)}(\xi,\boldsymbol{\eta})\zeta] - \nabla\cdot[(\xi\cdot\nabla\boldsymbol{\eta})\delta_\zeta P - \nabla\cdot((\delta_\zeta P)\boldsymbol{\eta})\xi]. \end{aligned} \tag{A 22}$$

The desired expression (4.3) of $w^{(3)}$ has appeared in this way.

Substituting this expression, we arrive at a simple form,

$$\begin{aligned}\mathsf{D}_\zeta\mathsf{D}_{\boldsymbol{\eta}}\mathsf{D}_\xi\mathrm{L} = {}& -\Delta_{EL}\cdot(\zeta\cdot\nabla(\boldsymbol{\eta}\cdot\nabla\xi)) + l^{(2)}(\boldsymbol{\eta}\cdot\nabla\xi,\zeta) + l^{(2)}(\zeta\cdot\nabla\xi,\boldsymbol{\eta}) + l^{(2)}(\xi,\zeta\cdot\nabla\boldsymbol{\eta}) \\ & - w^{(3)}(\xi,\boldsymbol{\eta},\zeta) + \nabla\cdot[\nabla\cdot(\nabla\cdot(P\xi)\boldsymbol{\eta})\zeta] + \partial_t[\rho\boldsymbol{v}\cdot(\zeta\cdot\nabla(\boldsymbol{\eta}\cdot\nabla\xi))] \\ & + \nabla\cdot\{[\rho\boldsymbol{v}\cdot(\zeta\cdot\nabla(\boldsymbol{\eta}\cdot\nabla\xi))]\boldsymbol{v} - [\boldsymbol{B}\cdot(\zeta\cdot\nabla(\boldsymbol{\eta}\cdot\nabla\xi))]\boldsymbol{B}\} \\ = {}& \mathsf{D}_{\zeta\cdot\nabla(\boldsymbol{\eta}\cdot\nabla\xi)}\mathrm{L} + l^{(2)}(\boldsymbol{\eta}\cdot\nabla\xi,\zeta) + l^{(2)}(\zeta\cdot\nabla\xi,\boldsymbol{\eta}) + l^{(2)}(\xi,\zeta\cdot\nabla\boldsymbol{\eta}) \\ & - w^{(3)}(\xi,\boldsymbol{\eta},\zeta) + \mathcal{L}^3_{\zeta,\boldsymbol{\eta},\xi}P + \mathcal{L}^2_{\zeta,\boldsymbol{\eta}\cdot\nabla\xi}P + \mathcal{L}^2_{\boldsymbol{\eta},\zeta\cdot\nabla\xi}P + \mathcal{L}^2_{\xi,\zeta\cdot\nabla\boldsymbol{\eta}}P, \end{aligned} \tag{A 23}$$

where we have defined

$$\mathcal{L}^3_{\zeta,\boldsymbol{\eta},\xi} \overset{\text{def}}{=} \mathcal{L}_\zeta\mathcal{L}_{\boldsymbol{\eta}}\mathcal{L}_\xi - \mathcal{L}^2_{\zeta,\boldsymbol{\eta}\cdot\nabla\xi} - \mathcal{L}^2_{\boldsymbol{\eta},\zeta\cdot\nabla\xi} - \mathcal{L}^2_{\xi,\zeta\cdot\nabla\boldsymbol{\eta}} - \mathcal{L}_{\zeta\cdot\nabla(\boldsymbol{\eta}\cdot\nabla\xi)}. \tag{A 24}$$

https://doi.org/10.1017/S0022377810000760 Published online by Cambridge University Press

Finally, we determine the second-order force field $\mathscr{F}^{(2)}(\boldsymbol{\xi},\boldsymbol{\eta})$ by the relation,

$$w^{(3)}(\boldsymbol{\xi},\boldsymbol{\eta},\boldsymbol{\zeta}) = -\boldsymbol{\zeta}\cdot\mathscr{F}^{(2)}(\boldsymbol{\xi},\boldsymbol{\eta}) - \mathcal{L}^2_{\boldsymbol{\zeta},\boldsymbol{\xi}}\delta_{\boldsymbol{\eta}}P - \mathcal{L}^2_{\boldsymbol{\zeta},\boldsymbol{\eta}}\delta_{\boldsymbol{\xi}}P - \nabla\cdot\{(\delta_{\boldsymbol{\eta}}\delta_{\boldsymbol{\xi}}P - \delta_{\boldsymbol{\eta}\cdot\nabla\boldsymbol{\xi}}P)\boldsymbol{\zeta} - \boldsymbol{\zeta}\cdot[\nabla\cdot(\boldsymbol{\xi}\cdot\nabla\boldsymbol{\eta} - \boldsymbol{\xi}\nabla\cdot\boldsymbol{\eta})\boldsymbol{B} - (\nabla\cdot\boldsymbol{\xi})\mathscr{B}\boldsymbol{\eta} - (\nabla\cdot\boldsymbol{\eta})\mathscr{B}\boldsymbol{\xi}]\boldsymbol{B}\}. \tag{A 25}$$

The explicit form of $\mathscr{F}^{(2)}(\boldsymbol{\xi},\boldsymbol{\eta})$ is, therefore, given by

$$\begin{aligned}\mathscr{F}^{(2)}(\boldsymbol{\xi},\boldsymbol{\eta}) = &-(\boldsymbol{\xi}\cdot\nabla)\nabla\delta_{\boldsymbol{\eta}}P - (\nabla\cdot\boldsymbol{\xi})\nabla\delta_{\boldsymbol{\eta}}P - (\boldsymbol{\eta}\cdot\nabla)\nabla\delta_{\boldsymbol{\xi}}P - (\nabla\cdot\boldsymbol{\eta})\nabla\delta_{\boldsymbol{\xi}}P\\ &+\boldsymbol{B}\cdot\nabla[\nabla\cdot(\boldsymbol{\xi}\cdot\nabla\boldsymbol{\eta} - \boldsymbol{\xi}\nabla\cdot\boldsymbol{\eta})\boldsymbol{B} - (\nabla\cdot\boldsymbol{\xi})\mathscr{B}\boldsymbol{\eta} - (\nabla\cdot\boldsymbol{\eta})\mathscr{B}\boldsymbol{\xi}]\\ &+\nabla\cdot(\boldsymbol{\xi}\cdot\nabla\boldsymbol{\eta} - \boldsymbol{\xi}\nabla\cdot\boldsymbol{\eta})\nabla P - (\nabla\cdot\boldsymbol{\xi})(\boldsymbol{\eta}\cdot\nabla)\nabla P - (\nabla\cdot\boldsymbol{\eta})(\boldsymbol{\xi}\cdot\nabla)\nabla P\\ &-\boldsymbol{\eta}\cdot\nabla[(\boldsymbol{\xi}\cdot\nabla)\nabla P] + [(\boldsymbol{\eta}\cdot\nabla\boldsymbol{\xi})\cdot\nabla]\nabla P - \nabla(\delta_{\boldsymbol{\eta}}\delta_{\boldsymbol{\xi}}P - \delta_{\boldsymbol{\eta}\cdot\nabla\boldsymbol{\xi}}P),\end{aligned} \tag{A 26}$$

where

$$\begin{aligned}\delta_{\boldsymbol{\eta}}\delta_{\boldsymbol{\xi}}P - \delta_{\boldsymbol{\eta}\cdot\nabla\boldsymbol{\xi}}P = &\mathscr{B}\boldsymbol{\eta}\cdot\mathscr{B}\boldsymbol{\xi} - (\nabla\cdot\boldsymbol{\xi})\boldsymbol{B}\cdot\mathscr{B}\boldsymbol{\eta} - (\nabla\cdot\boldsymbol{\eta})\boldsymbol{B}\cdot\mathscr{B}\boldsymbol{\xi} - \boldsymbol{\eta}\cdot\nabla\delta_{\boldsymbol{\xi}}P - \boldsymbol{\xi}\cdot\nabla\delta_{\boldsymbol{\eta}}P\\ &-\boldsymbol{\xi}\cdot(\boldsymbol{\eta}\cdot\nabla)\nabla P + \left(2\rho c_s^2 + \rho^2\frac{\partial^2 p}{\partial\rho^2}\right)(\nabla\cdot\boldsymbol{\xi})(\nabla\cdot\boldsymbol{\eta})\\ &+\left(|\boldsymbol{B}|^2 + \rho c_s^2\right)\nabla\cdot(\boldsymbol{\eta}\cdot\nabla\boldsymbol{\xi} - \boldsymbol{\eta}\nabla\cdot\boldsymbol{\xi}).\end{aligned} \tag{A 27}$$

We can check the symmetry $\mathscr{F}^{(2)}(\boldsymbol{\xi},\boldsymbol{\eta}) = \mathscr{F}^{(2)}(\boldsymbol{\eta},\boldsymbol{\xi})$ according to the result in Appendix B.

Appendix B. Cubic symmetry of the third-order potential energy

The cubic symmetry of $w^{(3)}(\boldsymbol{\xi},\boldsymbol{\eta},\boldsymbol{\zeta})$, given by (4.3), is best observed by expressing it in general curvilinear coordinates.

B.1. Coordinate expression of $\nabla\cdot(\boldsymbol{\xi}\cdot\nabla\boldsymbol{\eta} - \boldsymbol{\xi}\nabla\cdot\boldsymbol{\eta})$

Let us denote the contravariant components of the vector field by $\boldsymbol{\xi} = (\xi^1,\xi^2,\xi^3)$. Then,

$$\begin{aligned}\nabla\cdot(\boldsymbol{\xi}\cdot\nabla\boldsymbol{\eta}) - \boldsymbol{\xi}\cdot\nabla(\nabla\cdot\boldsymbol{\eta}) &= \frac{1}{J}\partial_i\left[J\left(\xi^j\partial_j\eta^i + \xi^j\Gamma^i_{jk}\eta^k\right)\right] - \xi^j\partial_j\left[\frac{1}{J}\partial_i(J\eta^i)\right]\\ &= \left(\partial_i\xi^j + \Gamma^j_{ik}\xi^k\right)\left(\partial_j\eta^i + \Gamma^i_{jl}\eta^l\right) + \xi^j\eta^i R^k_{ikj},\end{aligned} \tag{B 1}$$

where J and $\{\Gamma^k_{ik}\}$ are the Jacobian and the Christoffel symbols, respectively, and we have used $\partial_i J = J\Gamma^k_{ik}$. Now, the Riemann–Christoffel curvature tensor,

$$R^l_{ikj} = \partial_k\Gamma^l_{ji} - \partial_j\Gamma^l_{ik} + \Gamma^l_{nk}\Gamma^n_{ji} - \Gamma^l_{jn}\Gamma^n_{ik}, \tag{B 2}$$

should vanish $R^l_{ikj} = 0$ identically since the curvilinear coordinates are defined on Euclidean space $\mathbb{R}^3$. Therefore, we obtain

$$\nabla\cdot(\boldsymbol{\xi}\cdot\nabla\boldsymbol{\eta} - \boldsymbol{\xi}\nabla\cdot\boldsymbol{\eta}) = \left(\partial_i\xi^j + \Gamma^j_{ik}\xi^k\right)\left(\partial_j\eta^i + \Gamma^i_{jl}\eta^l\right) - (\nabla\cdot\boldsymbol{\xi})(\nabla\cdot\boldsymbol{\eta}), \tag{B 3}$$

which is obviously symmetric with respect to the permutation of $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$.

https://doi.org/10.1017/S0022377810000760 Published online by Cambridge University Press

B.2. Coordinate expression of $\boldsymbol{\eta}\cdot(\boldsymbol{\xi}\cdot\nabla)\nabla P$ *and* $\boldsymbol{\eta}\boldsymbol{\zeta}\boldsymbol{\xi}:\nabla\nabla\nabla P$

First, note that

$$\begin{aligned}\boldsymbol{\eta}\cdot(\boldsymbol{\xi}\cdot\nabla)\nabla P &= \boldsymbol{\xi}\cdot\nabla(\boldsymbol{\eta}\cdot\nabla P)-(\boldsymbol{\xi}\cdot\nabla\boldsymbol{\eta})\cdot\nabla P\\ &= \xi^i\partial_i(\eta^j\partial_j P)-\left(\xi^i\partial_i\eta^j+\xi^i\Gamma^j_{ik}\eta^k\right)\partial_j P\\ &= \xi^i\eta^j\left(\partial_i\partial_j P-\Gamma^k_{ij}\partial_k P\right),\end{aligned}\tag{B 4}$$

which is symmetric with respect to $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. Next, we get

$$\begin{aligned}\boldsymbol{\eta}\boldsymbol{\zeta}\boldsymbol{\xi}:\nabla\nabla\nabla P &= \boldsymbol{\eta}\cdot\{\boldsymbol{\zeta}\cdot\nabla[(\boldsymbol{\xi}\cdot\nabla)\nabla P]-[(\boldsymbol{\zeta}\cdot\nabla\boldsymbol{\xi})\cdot\nabla]\nabla P\}\\ &= \boldsymbol{\zeta}\cdot\nabla\{\boldsymbol{\eta}\cdot(\boldsymbol{\xi}\cdot\nabla)\nabla P\}-(\boldsymbol{\zeta}\cdot\nabla\boldsymbol{\eta})\cdot(\boldsymbol{\xi}\cdot\nabla)\nabla P-(\boldsymbol{\zeta}\cdot\nabla\boldsymbol{\xi})\cdot(\boldsymbol{\eta}\cdot\nabla)\nabla P\\ &= \zeta^l\partial_l\left[\xi^i\eta^j\left(\partial_i\partial_j P-\Gamma^k_{ij}\partial_k P\right)\right]-\left(\zeta^l\partial_l\eta^j+\zeta^l\Gamma^j_{ln}\eta^n\right)\xi^i\left(\partial_i\partial_j P-\Gamma^k_{ij}\partial_k P\right)\\ &\quad-\left(\zeta^l\partial_l\xi^i+\zeta^l\Gamma^i_{ln}\xi^n\right)\eta^j\left(\partial_i\partial_j P-\Gamma^k_{ij}\partial_k P\right)\\ &= \xi^i\eta^j\zeta^l\left[\partial_l\left(\partial_i\partial_j P-\Gamma^k_{ij}\partial_k P\right)-\Gamma^n_{lj}\left(\partial_i\partial_n P-\Gamma^k_{in}\partial_k P\right)\right.\\ &\quad\left.-\Gamma^n_{li}\left(\partial_n\partial_j P-\Gamma^k_{nj}\partial_k P\right)\right]\\ &= \xi^i\eta^j\zeta^l\left[\partial_i\partial_j\partial_l P-\partial_l\left(\Gamma^k_{ij}\partial_k P\right)-\partial_i\left(\Gamma^k_{lj}\partial_k P\right)-\partial_j\left(\Gamma^k_{li}\partial_k P\right)\right.\\ &\quad\left.+\left(\partial_i\Gamma^k_{lj}+\Gamma^n_{lj}\Gamma^k_{in}+\partial_j\Gamma^k_{li}+\Gamma^n_{li}\Gamma^k_{nj}\right)\partial_k P\right].\end{aligned}\tag{B 5}$$

Since the Riemann–Christoffel curvature tensor (B 2) is zero, we find that

$$T^k_{ijl}\overset{\text{def}}{=}\partial_i\Gamma^k_{jl}+\Gamma^k_{mi}\Gamma^n_{jl}=\partial_j\Gamma^k_{il}+\Gamma^k_{jn}\Gamma^n_{il}\tag{B 6}$$

is symmetric with respect to any permutations of (i,j,l). Hence,

$$\boldsymbol{\eta}\boldsymbol{\zeta}\boldsymbol{\xi}:\nabla\nabla\nabla P=\xi^i\eta^j\zeta^l\left[\partial_i\partial_j\partial_l P-\partial_l\left(\Gamma^k_{ij}\partial_k P\right)-\partial_i\left(\Gamma^k_{lj}\partial_k P\right)-\partial_j\left(\Gamma^k_{li}\partial_k P\right)+2T^k_{ijl}\partial_k P\right]\tag{B 7}$$

has the cubic symmetry.

Appendix C. Series expansion of the vacuum Lagrangian

C.1. First order

In (6.24), the first order becomes

$$\begin{aligned}L^v_1(\boldsymbol{\xi}) &= \int_{V^v_t}\left[-\boldsymbol{B}\cdot\boldsymbol{B}_1(\boldsymbol{\xi})-\nabla\cdot\left(\frac{|\boldsymbol{B}|^2}{2}\boldsymbol{\xi}\right)\right]d^3x\\ &= \int_{V^v_t}\left[\nabla\cdot(\boldsymbol{B}\times\boldsymbol{A}_1(\boldsymbol{\xi}))-(\nabla\times\boldsymbol{B})\cdot\boldsymbol{A}_1(\boldsymbol{\xi})-\nabla\cdot\left(\frac{|\boldsymbol{B}|^2}{2}\boldsymbol{\xi}\right)\right]d^3x.\end{aligned}\tag{C 1}$$

https://doi.org/10.1017/S0022377810000760 Published online by Cambridge University Press

Since $\nabla \times \boldsymbol{B} = 0$ holds and $\boldsymbol{A}_1(\xi)$ satisfies the boundary condition (6.21), this is transformed into

$$L_1^v(\xi) = \int_{S_t^+} (-\boldsymbol{n}) \cdot (\boldsymbol{B} \times \boldsymbol{A}_1(\xi)) d^2x + \int_{V_t^v} \left[-\nabla \cdot \left(\frac{|\boldsymbol{B}|^2}{2} \xi \right) \right] d^3x$$

$$= \int_{S_t^+} (-\boldsymbol{n} \cdot \xi) |\boldsymbol{B}|^2 d^2x + \int_{V_t^v} \left[-\nabla \cdot \left(\frac{|\boldsymbol{B}|^2}{2} \xi \right) \right] d^3x$$

$$= \int_{V_t^v} \nabla \cdot \left(\frac{|\boldsymbol{B}|^2}{2} \xi \right) d^3x, \tag{C 2}$$

where $(-\boldsymbol{n})$ denotes the outer normal vector on ∂V_t^v.

C.2. Second order

The second-order Lagrangian is given by

$$L_2^v(\eta, \xi) = \int_{V_t^v} \Big\{ -\boldsymbol{B} \cdot \boldsymbol{B}_2(\eta, \xi) - \boldsymbol{B}_1(\eta) \cdot \boldsymbol{B}_1(\xi)$$

$$- \nabla \cdot \left[(\boldsymbol{B} \cdot \boldsymbol{B}_1(\eta)) \xi + (\boldsymbol{B} \cdot \boldsymbol{B}_1(\xi)) \eta + \nabla \cdot \left(\frac{|\boldsymbol{B}|^2}{2} \xi \right) \eta \right] \Big\} d^3x. \tag{C 3}$$

By using the boundary condition (6.22), we can eliminate $\boldsymbol{B}_2(\eta, \xi) = \nabla \times \boldsymbol{A}_2(\eta, \xi)$ and get

$$L_2^v(\eta, \xi) = \int_{V_t^v} \left[-\boldsymbol{B}_1(\eta) \cdot \boldsymbol{B}_1(\xi) + \mathcal{L}_\eta \mathcal{L}_\xi \left(\frac{|\boldsymbol{B}|^2}{2} \right) \right] d^3x, \tag{C 4}$$

where recall that $\mathcal{L}_\xi \circ = \nabla \cdot (\circ \xi)$.

If we further put $\boldsymbol{B}_1(\eta) = \nabla \times \boldsymbol{A}_1(\eta)$ and use (6.21), we obtain an alternative expression,

$$L_2^v(\eta, \xi) = \int_{V_t^v} \nabla \cdot \left[(\boldsymbol{B} \cdot \boldsymbol{B}_1(\xi)) \eta + \nabla \cdot \left(\frac{|\boldsymbol{B}|^2}{2} \xi \right) \eta \right] d^3x. \tag{C 5}$$

C.3. Third order

The third-order Lagrangian is explicitly written as

$$L_3^v(\zeta, \eta, \xi) = \int_{V_t^v} \Big\{ -\boldsymbol{B} \cdot \boldsymbol{B}_3(\zeta, \eta, \xi) - \boldsymbol{B}_1(\zeta) \cdot \boldsymbol{B}_2(\eta, \xi) - \boldsymbol{B}_1(\eta) \cdot \boldsymbol{B}_2(\zeta, \xi) - \boldsymbol{B}_1(\xi) \cdot \boldsymbol{B}_2(\zeta, \eta)$$

$$- \mathcal{L}_\xi [\boldsymbol{B}_1(\zeta) \cdot \boldsymbol{B}_1(\eta) + \boldsymbol{B} \cdot \boldsymbol{B}_2(\zeta, \eta)] - \mathcal{L}_\eta [\boldsymbol{B}_1(\zeta) \cdot \boldsymbol{B}_1(\xi) + \boldsymbol{B} \cdot \boldsymbol{B}_2(\zeta, \xi)]$$

$$- \mathcal{L}_\eta \mathcal{L}_\xi [\boldsymbol{B} \cdot \boldsymbol{B}_1(\zeta)] - \mathcal{L}_\zeta [\boldsymbol{B}_1(\eta) \cdot \boldsymbol{B}_1(\xi) + \boldsymbol{B} \cdot \boldsymbol{B}_2(\eta, \xi)] - \mathcal{L}_\zeta \mathcal{L}_\xi [\boldsymbol{B} \cdot \boldsymbol{B}_1(\eta)]$$

$$- \mathcal{L}_\zeta \mathcal{L}_\eta [\boldsymbol{B} \cdot \boldsymbol{B}_1(\xi)] - \mathcal{L}_\zeta \mathcal{L}_\eta \mathcal{L}_\xi \left(\frac{|\boldsymbol{B}|^2}{2} \right) \Big\} d^3x. \tag{C 6}$$

As we have done for the first- and second-order calculations, we can eliminate $\boldsymbol{B}_3$ and $\boldsymbol{B}_2$ by putting $\boldsymbol{B}_3 = \nabla \times \boldsymbol{A}_3$ and $\boldsymbol{B}_2 = \nabla \times \boldsymbol{A}_2$ and using the boundary conditions (6.22)

https://doi.org/10.1017/S0022377810000760 Published online by Cambridge University Press

and (6.23). After this lengthy manipulation, we obtain the following expression:

$$L_3^v(\zeta,\boldsymbol{\eta},\xi) = \int_{V_t^v} \Big\{ \nabla\cdot[\boldsymbol{B}_1(\zeta)\cdot\boldsymbol{B}_1(\boldsymbol{\eta})\xi + \boldsymbol{B}_1(\zeta)\cdot\boldsymbol{B}_1(\xi)\boldsymbol{\eta} + \boldsymbol{B}_1(\boldsymbol{\eta})\cdot\boldsymbol{B}_1(\xi)\zeta] + \mathcal{L}_{\boldsymbol{\eta}}\mathcal{L}_{\xi}[\boldsymbol{B}\cdot\boldsymbol{B}_1(\zeta)] + \mathcal{L}_{\zeta}\mathcal{L}_{\xi}[\boldsymbol{B}\cdot\boldsymbol{B}_1(\boldsymbol{\eta})] + \mathcal{L}_{\zeta}\mathcal{L}_{\boldsymbol{\eta}}[\boldsymbol{B}\cdot\boldsymbol{B}_1(\xi)] + \mathcal{L}_{\zeta}\mathcal{L}_{\boldsymbol{\eta}}\mathcal{L}_{\xi}\left(\frac{|\boldsymbol{B}|^2}{2}\right)\Big\} d^3x. \tag{C 7}$$

By making use of the notations (A 15) and (A 24), this result is plugged into the $O(\epsilon^3)$ terms of (7.1).

References

[1] Bernstein, I. B., Frieman, E. A., Kruskal, M. D. and Kulsrud, R. M. 1958 *Proc. Roy. Soc. Lond.* A**244**, 17.

[2] Newcomb, W. A. 1962 *Nucl. Fusion Suppl.* Part 2, 451.

[3] Galeev, A. A. and Oraevski, V. N. 1963 *Sov. Phys. Dokl.* **7**, 988.

[4] Sagdeev, R. Z. and Galeev, A. A. 1969 *Nonlinear Plasma Theory*, New York: Benjamin.

[5] Brodin, G. and Stenflo, L. 1988 *J. Plasma Phys.* **39**, 277.

[6] Webb, G. M., Zakharian, A. R., Brio, M. and Zank G. P. 2000 *J. Plamsa Phys.* **63**, 393.

[7] Larsson, J. 1996 *J. Plasma Phys.* **55**, 261.

[8] Larsson, J. 2000 *J. Plasma Phys.* **64**, 155.

[9] Axelsson, P. 2000 *Nonlinearity* **13**, 19.

[10] Morrison, P. J. and Greene, J. M. 1980 *Phys. Rev. Lett.* **45**, 790. (ERRATA) 1982 *Phys. Rev. Lett.* **48**, 569.

[11] Frieman, E. and Rotenberg, M. 1960 *Rev. Mod. Phys.* **32**, 898.

[12] Dewar, R. L. 1970 *Phys. Fluids* **13**, 2710.

[13] Brizard, A. J. and Kaufman, A. N. 1995 *Phys. Rev. Lett.* **74**, 4567.

[14] Whitham, G. B. 1967 *Proc. Roy. Soc.* A **299**, 6.

[15] Larsson, J. 2003 *J. Plasma Phys.* **69**, 211.

[16] Sattinger, D. H. and Weaver, O. L. 1986 *Lie Groups and Algebras with Applications to Physics, Geometry, and Mechanics*, New York: Springer-Verlag.

[17] Coppi, B., Rosenbluth, M. N. and Sudan, R. N. 1969 *Ann. Phys.* **55**, 207.

[18] Hirota, M. and Fukumoto, Y. 2008 *J. Math. Phys.* **49**, 083101.

[19] Hirota, M. and Fukumoto, Y. 2008 *Phys. Plasmas* **15**, 122101.

[20] Andrews, D. G. and Mcintyre, M. E. 1978 *J. Fluid Mech.* **89**, 609.

[21] Hirota, M. and Tokuda, S. 2010 *Phys. Plasmas* **17**, 082109.

[22] Fukumoto, Y. and Hirota, M. 2008 *Phys. Scr.* T**132**, 014041.

https://doi.org/10.1017/S0022377810000760 Published online by Cambridge University Press