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On the Ashkin–Teller Model and Tutte–Whitney Functions

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The partition functions of the Ising and Potts models in statistical mechanics are well known to be partial evaluations of the Tutte–Whitney polynomial of the appropriate graph. The Ashkin–Teller model generalizes the Ising model and the four-state Potts model, and has been extensively studied since its introduction in 1943. However, its partition function (even in the symmetric case) is not a partial evaluation of the Tutte–Whitney polynomial. In this paper, we show that the symmetric Ashkin–Teller partition function can be obtained from a generalized Tutte–Whitney function which is intermediate in a precise sense between the usual Tutte–Whitney polynomial of the graph and that of its dual.

1. Introduction

The Ising and Potts models are fundamental objects of study in statistical mechanics (see, *e.g.*, [2]), and are now frequently studied in the combinatorial literature (see, *e.g.*, [19, 25]). In the *Ising model* [11] on a graph $G = (V, E)$, each vertex v receives a *state* or *spin*, $\sigma(v) \in \{\pm 1\}$, and the edges are divided by σ into two subsets: those whose endpoints receive the same spin, and those whose endpoints receive different spins. We call these sets $E^+(\sigma)$ and $E^-(\sigma)$, respectively. Much information of physical and combinatorial interest is contained in the *partition function*

$$Z_I(G) = Z_I(G;K) = \sum_\sigma e^{K(|E^+(\sigma)-E^-(\sigma)|)} = \sum_\sigma e^{K|E|-2K|E^-(\sigma)|},$$

where the sum is over all $\sigma : V \to \{-1, 1\}$. In physical terms, K is proportional to the size of the *interaction energy* associated with an edge, with the sign of that energy depending only on whether or not the spins on the edge's endpoints are different.

The *Potts model* [17] extends the Ising model to q states $\{0, \ldots, q-1\}$, and its partition function extends Z_I (modulo some simple algebra).

Much of the combinatorial interest in these partition functions arises from the fact that they can both be obtained, by appropriate algebraic substitutions, from the *Tutte*

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polynomial [22] or *Whitney rank generating function* (explicit in [21], implicit in [26]). In the latter form, this polynomial is defined for any graph $G = (V, E)$ by

$$R(G; x, y) = \sum_{X \subseteq E} x^{\rho(E)-\rho(X)} y^{|X|-\rho(X)},$$

where the *rank* $\rho(X)$ of a set of edges X is the number of vertices these edges meet minus the number of components of the subgraph they define (excluding isolated vertices from such subgraphs). The Tutte polynomial is $T(G; x, y) = R(G; x-1, y-1)$. For a history of these polynomials (which we collectively refer to as *Tutte–Whitney polynomials*), see [8]. The connection with Ising and Potts model partition functions is due to Fortuin and Kasteleyn [9] (see also [25, 28]). Other information contained in the Tutte–Whitney polynomial includes the numbers of spanning trees, forests, spanning subgraphs and acyclic orientations of a graph, the chromatic and flow polynomials, the percolation or all-terminal reliability polynomial, the weight enumerator of a linear code and the Jones polynomial of an alternating link [3, 12, 25].

The interface between combinatorics and statistical mechanics provided by these polynomials has yielded progress in both areas. On the one hand, for example, algorithmic and complexity-theoretic techniques often used in combinatorics have been applied to give results on the complexity of evaluating or approximating partition functions [12, 13, 25]. On the other hand, insights from statistical mechanics have recently proved instrumental in finding new results and proofs relating to roots of chromatic and reliability polynomials (see, *e.g.*, [19]).

The purpose of this paper is to expand this interface so that it embraces one of the oldest statistical mechanical models on graphs, namely the *symmetric Ashkin–Teller model* [1]. This was introduced in 1943, and extends both the Ising model and the $q = 4$ case of the Potts model (though it predates the latter). Its partition function cannot, in general, be obtained from the Tutte–Whitney polynomial. Our main result is that it can be obtained from a recent generalization of the Tutte–Whitney polynomial [7] that is, in a precise sense, intermediate between the usual Tutte–Whitney polynomial of a graph and that of its (matroid-theoretic) dual. We review this generalization in Section 2.

In the Ashkin–Teller model, each vertex of $G = (V, E)$ is assigned a pair of spins, each ± 1, and three kinds of interaction between adjacent vertices are considered: two interactions due to the component Ising spins, and an additional interaction due to the product of the spins. Let $\sigma, \tau : V \to \{-1, 1\}$ be the two functions assigning spins to vertices, and put $\sigma\tau(v) = \sigma(v)\tau(v)$ for all $v \in V$. For any function φ defined on V, put $E^-(\varphi) = \{uv \in E \mid \varphi(u) \neq \varphi(v)\}$. The partition function of the Ashkin–Teller model on G is

$$\begin{aligned} Z_{AT}(G) &= Z_{AT}(G; K_\sigma, K_\tau, K_{\sigma\tau}) \\ &= \sum_{\substack{\sigma: V \to \{-1,1\} \\ \tau: V \to \{-1,1\}}} e^{(K_\sigma + K_\tau + K_{\sigma\tau})|E| - 2K_\sigma |E^-(\sigma)| - 2K_\tau |E^-(\tau)| - 2K_{\sigma\tau}|E^-(\sigma\tau)|}. \end{aligned} \tag{1.1}$$

For further information on the Ashkin–Teller model and its partition function, see, *e.g.*, [2, 18].

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If $K_{\sigma\tau} = 0$ then Z_{AT} is just a product of two Ising model partition functions. If $K_\sigma = K_\tau = K_{\sigma\tau} = K$, we have (up to a factor depending only on K and $|E|$) a four-state Potts model partition function.

In this paper, we concentrate on the *symmetric* Ashkin–Teller model, in which $K_\sigma = K_\tau = K$. It is in this case that we show that the partition function, which we denote by $Z_{AT}(G; K, K_{\sigma\tau})$, can be obtained from a generalization of the Tutte–Whitney polynomial.

Throughout the paper, $G = (V, E)$ is a graph with vertex set V and edge set E. $k(G)$ denotes the number of connected components (or clusters) of G. If $\rho : 2^E \to \mathbb{R}$ then its *dual* $\rho^* : 2^E \to \mathbb{R}$ is defined by

$$\rho^*(X) = |X| + \rho(E \setminus X) - \rho(E) + \rho(\emptyset). \tag{1.2}$$

This is just matroid-theoretic duality, in the sense that if ρ is the rank function of a matroid then ρ^* is the rank function of the dual matroid. However, we allow this dual operation to be applied to functions ρ that are not necessarily the rank function of any matroid. The cutset space of G is denoted by $\mathscr{C}$. Note that $E^-(\sigma), E^-(\tau) \in \mathscr{C}$ for all σ, τ, and that, for all $W \in \mathscr{C}$,

$$|\{\sigma \mid E^-(\sigma) = W\}| = 2^{k(G)}. \tag{1.3}$$

Unless indicated otherwise, $f : 2^E \to \{0, 1\}$ is the indicator function of $\mathscr{C}$:

$$f(X) = \begin{cases} 1, & \text{if } X \in \mathscr{C}, \\ 0, & \text{if } X \notin \mathscr{C}. \end{cases}$$

It will be convenient to put

$$A_f(z) = \sum_{X \subseteq E} z^{|X|} f(X).$$

If f is the indicator function of $\mathscr{C}$, then $A_f(z)$ is just the weight enumerator of $\mathscr{C}$.

2. A continuum of generalized Tutte–Whitney functions

This section briefly reviews the λ-Tutte–Whitney functions introduced in [7] and describes their relationship to ordinary Tutte–Whitney polynomials.

The philosophy is to consider how the concepts of deletion, contraction and rank look when they are expressed in terms of the cutset space indicator function f of a graph G, and generalize from there.

We write $f /\!/ e$ and $f \setminus\!\setminus e$ for the cutset space indicator functions of the graphs G/e and $G \setminus e$, obtained by contracting or deleting e, respectively. It is routine to show that their values for any $X \subseteq E \setminus \{e\}$ are given by

$$(f /\!/ e)(X) = \frac{f(X)}{f(\emptyset)}, \qquad (f \setminus\!\setminus e)(X) = \frac{f(X) + f(X \cup \{e\})}{f(\emptyset) + f(\{e\})}. \tag{2.1}$$

The rank function ρ of G is given by

$$\rho(X) = \log_2\left(\frac{\sum_{W \subseteq E} f(W)}{\sum_{W \subseteq E \setminus X} f(W)}\right), \tag{2.2}$$

as shown in [6, Corollary 2.2] (see also [8, Section 6]).

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Let $\lambda \in \mathbb{C}$, though we will mostly be interested in $\lambda \in \mathbb{R}$, and especially $\lambda \in [0,1]$. For any $\lambda \in \mathbb{C} \setminus \{-1\}$, put

$$\lambda^* = \frac{1-\lambda}{1+\lambda}.$$

The fixed points of this map are of interest at times, and we denote them by $\alpha_1 = -1+\sqrt{2}$ and $\alpha_2 = -1-\sqrt{2}$.

The starting point for the λ-Tutte–Whitney functions of [7] is to use a parameter λ to interpolate between the contraction and deletion operations of (2.1).

This gives an intermediate operation $\|_\lambda$ for each $\lambda \in [0,1]$, and indeed new operations for all $\lambda \in \mathbb{C}$:

$$(f \|_\lambda e)(X) = \frac{f(X) + \lambda f(X \cup \{e\})}{f(\emptyset) + \lambda f(\{e\})}.$$

Ordinary contraction and deletion correspond to the cases $\lambda = 0$ and $\lambda = 1$, respectively. The theory of these intermediate operations is developed at length in [7], where it is shown that the operations $\|_\lambda$ and $\|_{\lambda^*}$ are dual, extending the duality between contraction and deletion. One part of that theory deals with extending the rank function of (2.2) so that it is compatible with the operations $\|_\lambda$, in the same way in which the ordinary contraction and deletion operations of (2.1) are compatible with the ordinary rank function of (2.2).

To do this, the *λ-rank function* $Q^{(\lambda)}f$ is defined for each λ and any cutset space indicator function f as follows [7]:

$$(Q^{(\lambda)}f)(X) = \log_2\left(\frac{(1+\lambda^*)^{|V|}\sum_{W\subseteq E}\lambda^{|W|}f(W)}{\sum_{W\subseteq E}\lambda^{|W\cap(E\setminus X)|}(\lambda^*)^{|W\cap X|}f(W)}\right),$$

where $X \subseteq E$. The function $Q^{(\lambda)}f$ may be regarded as interpolating between the ordinary rank function and its dual: these correspond to the cases $\lambda = 1$ and $\lambda = 0$, respectively [6, 7]. If $\lambda = \lambda^*$, then we are in the exact middle, and the distinction between deletion and contraction is completely blurred: in that case, $Q^{(\alpha_1)}f(V) = |V|/2$ and $Q^{(\alpha_2)}f(V) = |V|i\pi/2$, so that all information about G and f is lost [7, p. 250]. The map $\lambda \mapsto \lambda^*$ describes duality: the function $Q^{(\lambda^*)}f$ is the dual $(Q^{(\lambda)}f)^*$ of $Q^{(\lambda)}f$, in the sense of (1.2) [7, Theorem 9]. Duality is also described by applying the $\mathbb{Z}_2^{|E|}$ Fourier transform to f, as discussed in [6, 7]. This transform is defined by

$$\hat{f}(X) = 2^{-|E|}\sum_{W\subseteq E}(-1)^{|X\cap W|}f(W).$$

We can use this rank-like function $Q^{(\lambda)}f$ to define a *λ-Tutte–Whitney function* as follows:

$$R^{(\lambda)}(G;x,y) = R^{(\lambda)}(f;x,y) = \sum_{X\subseteq E} x^{Q^{(\lambda)}f(E)-Q^{(\lambda)}f(X)}y^{(Q^{(\lambda)}f)^*(E)-(Q^{(\lambda)}f)^*(E\setminus X)},$$

$$R_1^{(\lambda)}(G;x,y) = R_1^{(\lambda)}(f;x,y) = y^{-Q^{(\lambda)}f(E)}\sum_{X\subseteq E}(xy)^{Q^{(\lambda)}f(E)-Q^{(\lambda)}f(X)}y^{|X|}.$$

If $\lambda = 1$, then we have the ordinary Tutte–Whitney function of G, and if $\lambda = 0$ then we have the Tutte–Whitney function of its dual (using matroid duality if G is not planar). Properties of the functions $R^{(\lambda)}$ and $R_1^{(\lambda)}$ are studied in [7]. It is shown that $R^{(\lambda)}(G;x,y)$

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obeys a deletion–contraction-type relation, using the operations $\|_\lambda$ and $\|_{\lambda^*}$ described above. If a general Boolean function $f : 2^E \to \{0,1\}$ is used, instead of an indicator function of a linear space, then it is shown that $R^{(\lambda)}(f;x,y)$ contains the weight enumerator of a general (not necessarily linear) code, giving a generalization of Greene's theorem [10]. (See also [6].) Note that $R^{(\lambda)}$ and $R_1^{(\lambda)}$ are identical if $Q^{(\lambda)}f$ is integer valued or at least one of x, y is positive. In the degenerate case $\lambda = \lambda^* = \alpha_1$, we find that

$$R^{(\alpha_1)}(f;x,y) = \left(x^{1/2} + y^{1/2}\right)^{|E|}, \qquad \text{and}$$
$$R_1^{(\alpha_1)}(f;x,y) = y^{-|E|/2}\left((xy)^{1/2} + y\right)^{|E|}.$$

For further information on the transform $Q^{(\lambda)}$, λ-Tutte–Whitney functions, and so on, see [7]. For a theory of Tutte–Whitney functions of arbitrary Boolean functions (not just those that are indicator functions of cutset spaces), see [6] or [7]. Another introduction to the topic may be found in [8]. For earlier work by Kung on chromatic polynomials of Boolean functions, see [15]. Connections with the work of Kung are discussed in [7].

3. The symmetric Ashkin–Teller model partition function

The partition function of the symmetric Ashkin–Teller model cannot be obtained from the Tutte polynomial. This can be seen, for example, by considering two graphs due to M. C. Gray: see [23] or [24, exercise 15.4.11]. These have the same Tutte polynomial (though their cycle matroids are not isomorphic), yet direct calculation shows that the partition functions of the symmetric Ashkin–Teller models on these graphs are different.

In contrast to this observation, our main result is as follows.

Theorem 3.1. *For each K and $K_{\sigma\tau}$ there exist λ, s and κ such that the partition function $Z(G;K,K_{\sigma\tau})$ of the symmetric Ashkin–Teller model on a graph G can be obtained from the λ-Tutte–Whitney function as follows.*

(a) *If $K_{\sigma\tau} \neq 0$,*

$$Z_{AT}(G;K,K_{\sigma\tau}) = 4^{k(G)}e^{(2K+K_{\sigma\tau})|E|}\left(\frac{(1+\lambda)^2}{4(1+s)}\right)^{|E|} A_f(\lambda^*)^2(e^\kappa - 1)^{Q^{(\lambda)}f(E)}R_1^{(\lambda)}\left(G;\frac{4}{e^\kappa - 1}, e^\kappa - 1\right).$$

The possible values of λ are given by

$$\lambda = \frac{1 + AB \pm \sqrt{(1+AB)^2 - 4A(1-B)(1+A)}}{2(1+A)},$$

where $A = e^{-2(K+K_{\sigma\tau})}$ and $B = e^{-2(K-K_{\sigma\tau})}$. The values of s and κ are given by

$$s = \frac{\lambda - A}{A - \lambda^*} = \frac{(\lambda - B)\lambda}{(B - \lambda^*)\lambda^*},$$
$$\kappa = \ln(1 + s(1+\lambda^*)^2).$$

(i) *If $K < K_{\sigma\tau}$, there is such a λ in each of $(-1,0)$ and $(1,\infty)$.*

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(ii) *If* $\cosh(2K + \ln\alpha_1) > e^{-2K_{\sigma\tau}}$ *and* $K \geqslant K_{\sigma\tau}$, *then there is such a* λ *in each of* $[0, \alpha_1)$ *and* $(\alpha_1, 1]$, *where* $\alpha_1 = -1 + \sqrt{2}$.

(iii) *If* $\cosh(2K + \ln\alpha_1) = e^{-2K_{\sigma\tau}}$, *then* $\lambda = \alpha_1$.

(iv) *Otherwise,* $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

(b) *If* $K_{\sigma\tau} = 0$, *we must put* $\lambda = \alpha_1$ *and the above relation is modified by replacing* $A_f(\lambda^*)$ *by* $A_f(e^{-2K})$. *In this case,* κ *is arbitrary and* $s = (e^{\kappa} - 1)/2$.

Proof.

$$\begin{aligned} e^{-(2K+K_{\sigma\tau})|E|} & Z_{AT}(G) \\ &= \sum_{\sigma,\tau} e^{-2K|E^-(\sigma)|-2K|E^-(\tau)|-2K_{\sigma\tau}|E^-(\sigma\tau)|} \\ &= \sum_{W_1,W_2\in\mathscr{C}} e^{-2K|W_1|-2K|W_2|-2K_{\sigma\tau}|(W_1\setminus W_2)\cup(W_2\setminus W_1)|}|\{\sigma,\tau \mid E^-(\sigma) = W_1, E^-(\tau) = W_2\}| \\ &= 4^{k(G)} \sum_{W_1,W_2\in\mathscr{C}} e^{-2(K+K_{\sigma\tau})|W_1\cup W_2|-2(K-K_{\sigma\tau})|(W_1\cap W_2)|}, \end{aligned} \tag{3.1}$$

using (1.3).

(a) Suppose $K_{\sigma\tau} \neq 0$. We find it convenient to put

$$A = e^{-2(K+K_{\sigma\tau})},$$
$$B = e^{-2(K-K_{\sigma\tau})},$$

noting that $A \neq B$ by assumption. Let λ and $s \neq -1$ be such that

$$\frac{\lambda + \lambda^* s}{1 + s} = A,$$
$$\frac{\lambda^2 + (\lambda^*)^2 s}{\lambda + \lambda^* s} = B.$$

We will now work out how to express λ and s in terms of A and B.

Observe that if $\lambda = \lambda^*$ then these equations can only be solved if $A = B = \lambda$, but this contradicts $K_{\sigma\tau} \neq 0$. Suppose then that $\lambda \neq \lambda^*$. Eliminating s from the equations, we have

$$(\lambda^2 + 2\lambda - 1)((1 + A)\lambda^2 - (1 + AB)\lambda + A(1 - B)) = 0.$$

Since $\lambda \neq \lambda^*$, we know that $\lambda^2 + 2\lambda - 1 \neq 0$. We therefore find

$$\lambda = \frac{1 + AB \pm \sqrt{(1 + AB)^2 - 4A(1 - B)(1 + A)}}{2(1 + A)}. \tag{3.2}$$

The discriminant here may be rewritten $\Delta = 1 - 4A - 4A^2 + 6AB + 4A^2B + A^2B^2$, and (since $B \geqslant 0$) is nonnegative if and only if $\alpha_1/A + B/\alpha_1 \geqslant 2$. The equivalent condition in terms of K and $K_{\sigma\tau}$ is $\cosh(2K + \ln\alpha_1) \geqslant e^{-2K_{\sigma\tau}}$. Note also that if the roots (3.2) are λ_1, λ_2, then $\lambda_1 = \lambda_2^*$.

Consider the possibilities for λ.

(i) If $K < K_{\sigma\tau}$, then $B > 1$ and $\Delta > 0$, and it is easily verified that the larger root $\lambda_2 > 1$ and hence that $-1 < \lambda_1 < 0$.

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(ii) Suppose then that $K \geqslant K_{\sigma\tau}$. This implies $B \leqslant 1$, which in turn implies that $\lambda \in [0,1]$ if $\Delta \geqslant 0$. It follows that $0 \leqslant \lambda_1 \leqslant \alpha_1 \leqslant \lambda_2 \leqslant 1$ (using $\lambda_1 = \lambda_2^*$).

(iii) It is routine to show that if $\Delta = 0$ then $\lambda = \alpha_1$.

(iv) In situations not covered by (i)–(iii), $\Delta < 0$ so $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

The value of s corresponding to λ is given by

$$s = \frac{\lambda - A}{A - \lambda^*} = \frac{(\lambda - B)\lambda}{(B - \lambda^*)\lambda^*}.$$

Put $\kappa = \ln(1 + s(1+\lambda^*)^2)$, so that $s = (e^\kappa - 1)/(1+\lambda^*)^2$.

Resuming at (3.1), we have

$$\begin{aligned}
&4^{-k(G)} e^{-(2K+K_{\sigma\tau})|E|} Z_{AT}(G) \\
&\quad = \sum_{W_1, W_2 \in \mathscr{C}} \left(\frac{\lambda + \lambda^* s}{1+s}\right)^{|W_1 \cup W_2|} \left(\frac{\lambda^2 + (\lambda^*)^2 s}{\lambda + \lambda^* s}\right)^{|(W_1 \cap W_2)|} \\
&\quad = (1+s)^{-|E|} \\
&\qquad \times \sum_{W_1, W_2 \in \mathscr{C}} (\lambda + \lambda^* s)^{|W_1 \setminus W_2|} (\lambda + \lambda^* s)^{|W_2 \setminus W_1|} (\lambda^2 + (\lambda^*)^2 s)^{|(W_1 \cap W_2)|} (1+s)^{|E \setminus (W_1 \cup W_2)|} \\
&\quad = (1+s)^{-|E|} \\
&\qquad \times \sum_{W_1, W_2 \in \mathscr{C}} \sum_{X_1 \subseteq W_1 \setminus W_2} \lambda^{|(W_1 \setminus W_2) \setminus X_1|} (\lambda^* s)^{|X_1|} \sum_{X_2 \subseteq W_2 \setminus W_1} \lambda^{|(W_2 \setminus W_1) \setminus X_2|} (\lambda^* s)^{|X_2|} \\
&\qquad \times \sum_{X_3 \subseteq W_1 \cap W_2} \lambda^{2|(W_1 \cap W_2) \setminus X_3|} ((\lambda^*)^2 s)^{|X_3|} \sum_{X_4 \subseteq E \setminus (W_1 \cup W_2)} s^{|X_4|} \\
&\quad = (1+s)^{-|E|} \sum_{W_1, W_2 \in \mathscr{C}} \sum_{X \subseteq E} \lambda^{|W_1 \cap (E \setminus X)| + |W_2 \cap (E \setminus X)|} (\lambda^*)^{|W_1 \cap X| + |W_2 \cap X|} s^{|X|} \\
&\quad = (1+s)^{-|E|} \sum_{X \subseteq E} s^{|X|} \sum_{W_1, W_2 \in \mathscr{C}} \lambda^{|W_1 \cap (E \setminus X)| + |W_2 \cap (E \setminus X)|} (\lambda^*)^{|W_1 \cap X| + |W_2 \cap X|} \\
&\quad = (1+s)^{-|E|} \sum_{X \subseteq E} s^{|X|} \left(\sum_{W \subseteq E} \lambda^{|W \cap (E \setminus X)|} (\lambda^*)^{|W \cap X|} f(W) \right)^2 \\
&\quad = (1+s)^{-|E|} \cdot \frac{\left(\sum_{W \subseteq E} (\lambda^*)^{|W|} f(W)\right)^2}{(1+\lambda^*)^{2|E|}} \left(\frac{(1+\lambda^*)^{|E|} \sum_{W \subseteq E} \lambda^{|W|} f(W)}{\sum_{W \subseteq E} (\lambda^*)^{|W|} f(W)} \right)^2 \\
&\qquad \times \sum_{X \subseteq E} (e^\kappa - 1)^{|X|} \left(\frac{\sum_{W \subseteq E} \lambda^{|W \cap (E \setminus X)|} (\lambda^*)^{|W \cap X|} f(W)}{(1+\lambda^*)^{|X|} \sum_{W \subseteq E} \lambda^{|W|} f(W)} \right)^2 \\
&\quad = (1+s)^{-|E|} (1+\lambda^*)^{-2|E|} A_f(\lambda^*)^2 \sum_{X \subseteq E} (e^\kappa - 1)^{|X|} 2^{2(Q^{(\lambda)} f(E) - Q^{(\lambda)} f(X))} \\
&\quad = (1+s)^{-|E|} (1+\lambda^*)^{-2|E|} A_f(\lambda^*)^2 (e^\kappa - 1)^{Q^{(\lambda)} f(E)} R_1^{(\lambda)} \left(G; \frac{4}{e^\kappa - 1}, e^\kappa - 1 \right).
\end{aligned}$$

(b) Now suppose that $K_{\sigma\tau} = 0$. In this case it is routine to show, from (3.1), that

$$Z_{AT}(G) = 4^{k(G)} e^{2K|E|} A_f(e^{-2K})^2.$$

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This is essentially the standard argument for showing that Z_{AT} is just a product of Ising model partition functions when $K_{\sigma\tau} = 0$. The result for this case then follows, with $\lambda = \alpha_1$, $s = (e^{\kappa} - 1)/2$ and κ arbitrary, using the facts from Section 2. Note that in this case all the partition function information is in the quantity $A_f(e^{-2K})$, and $R^{(\alpha_1)}$ is degenerate (in that it contains no information about G except $|E|$). □

Remarks. **(1)** If $\kappa > 0$, then the arguments of $R_1^{(\lambda)}$ are both positive, so $R_1^{(\lambda)} = R^{(\lambda)}$.

(2) The λ-Tutte–Whitney function of the dual of a connected graph G is obtained by just swapping x and y. (Instead of the dual of G, we can use the circuit space indicator function $\hat{f}(\emptyset)^{-1} \cdot \hat{f}$, where $\hat{f}$ is the $\mathbb{Z}_2^{|E|}$ Fourier transform of f.) Theorem 3.1 thus yields a specialization (to the symmetric case) of a generalization (to arbitrary connected graphs) of Wu and Wang's result [27, 29] on Ashkin–Teller model duality for planar graphs.

(3) It is straightforward to extend the Fortuin–Kasteleyn random cluster model partition function [9, 14] using the rank-like functions $Q^{(\lambda)}f$ instead of the usual rank function (given by the case $\lambda = 1$):

$$Z_{RC}^{(\lambda)}(G; p, q) = Z_{RC}^{(\lambda)}(f; p, q) = \sum_{X \subseteq E} p^{|X|}(1-p)^{|E\setminus X|} q^{Q^{(\lambda)}f(E) - Q^{(\lambda)}f(X)}$$
$$= \left(\frac{p}{1-p}\right)^{Q^{(\lambda)}f(E)} R_1^{(\lambda)}\left(f; \frac{q(1-p)}{p}, \frac{p}{1-p}\right).$$

In this language, Theorem 3.1 expresses the symmetric Ashkin–Teller model partition function in a random cluster form. Generalized random cluster representations of the Ashkin–Teller model partition function have also been given implicitly for general graphs by Salas and Sokal [18] and explicitly for induced subgraphs of square grids by Pfister and Velenik [16] and Chayes and Machta [4]. Those representations differ from the one used here in that they involve sums over pairs $X, Y \subseteq E$ rather than single edge sets, and are not Tutte–Whitney functions of the form considered here.

One application of random cluster representations is to the design of Swendsen–Wang-type algorithms [20, 5] for giving randomized approximations to random cluster model partition functions. Salas and Sokal [18] gave a Swendsen–Wang-type algorithm for the Ashkin–Teller model, essentially based on the generalized random cluster representations of this model cited above. Having developed here a different kind of generalized random cluster representation for the symmetric Ashkin–Teller model, it might be worth exploring the possibility of a new kind of Swendsen–Wang-type algorithm for the symmetric Ashkin–Teller model.

(4) Theorem 3.1 and subsequent remarks have a natural generalization to arbitrary Boolean (and other) functions. Suppose $f : 2^E \to \mathbb{R}$ and define

$$Z_{AT}(f; K, K_{\sigma\tau}) = 4e^{(2K + K_{\sigma\tau})|E|} \sum_{W_1, W_2 \subseteq E} e^{-2(K + K_{\sigma\tau})|W_1 \cup W_2| - 2(K - K_{\sigma\tau})|(W_1 \cap W_2)|} f(W_1) f(W_2).$$

Then the proof of Theorem 3.1 (with appropriate modifications) shows that $Z_{AT}(f; K, K_{\sigma\tau})$ can be obtained from $R_1^{(\lambda)}(f; x, y)$. The main equation of Theorem 3.1(a) still holds if we put $k(G) = 1$ and include an extra factor $f(\emptyset)^{-2}$ on the right. The duality relation now

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has the form

$$2^{-Q^{(1)}\hat{f}(E)}Z_{AT}(\hat{f};K^*,K^*_{\sigma\tau}) = 2^{-Q^{(1)}f(E)}Z_{AT}(f;K,K_{\sigma\tau}),$$

where the constants $K^*, K^*_{\sigma\tau}$ used on the left are dual to those used on the right, as in [27, 29].

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