Proc. Camb. Phil. Soc. (1966), **62**, 73
PCPS 62-14
Printed in Great Britain

Fourier transforms and the operator H_α

BY G. O. OKIKIOLU
The University of Sussex

(*Received* 24 *June* 1965)

Abstract. This is one of a number of papers in which the author studies the properties of the transform $H_\alpha(f)$ and allied operators. In this case, identities involving these operators and the Fourier transform are obtained, and these are applied in giving alternative proofs to certain inversion processes derived by the author elsewhere.

1. *Introduction.* By applying a particular form of the product formula for the Fourier transform, we derived an inversion process for the transform $H_\alpha(f)$ (defined below) in (2). Properties of $H_\alpha(f)$ have been studied elsewhere by the author without the use of Fourier transforms. In this paper we obtain relations connecting the two transforms. In doing this we shall also consider operators allied to H_α which have been studied in the other cases. We also give alternative proofs to two additional inversion processes involving the operators.

The Fourier transform of a function $f \in L^p(-\infty, \infty)$ is defined by

$$\hat{f}(x) \equiv \mathscr{F}(f)(x) = \frac{1}{\sqrt{(2\pi)}} \lim_{(\text{order } p')} \int f(t)\, e^{ixt}\, dt,$$

where $1/p + 1/p' = 1$, and we also have

$$\int_0^x \hat{f}(t)\, dt = \frac{i}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f(t) \frac{1 - e^{ixt}}{t}\, dt. \tag{1}$$

We define the following integrals:

$$H_\alpha(f)(x) = \frac{1}{\phi(\alpha)} \int_{-\infty}^{\infty} \frac{|t-x|^\alpha}{t-x} f(t)\, dt,$$

$$K_\alpha(f)(x) = \frac{1}{\phi(\alpha)} \int_{-\infty}^{\infty} |t-x|^{\alpha-1} f(t)\, dt,$$

$$L_\alpha(f)(x) = \frac{1}{\phi(\alpha)} \int_{-\infty}^{\infty} \left\{ \frac{|t-x|^\alpha}{t-x} - \frac{|t|^\alpha}{t} \right\} f(t)\, dt,$$

$$M_\alpha(f)(x) = \frac{1}{\phi(\alpha)} \int_{-\infty}^{\infty} \{|t-x|^{\alpha-1} - |t|^{\alpha-1}\} f(t)\, dt,$$

where $\phi(\alpha) = 2\Gamma(\alpha) \sin \frac{1}{2}\pi\alpha$.

Throughout this paper, the pair of numbers p and p' will be connected by the equation $1/p + 1/p' = 1$, the space $L^p(-\infty, \infty)$ will be denoted by L^p, and the norm $\left(\int_{-\infty}^{\infty} |f(t)|^p\, dt \right)^{1/p}$ will be denoted by $\|f\|_p$.

https://doi.org/10.1017/S0305004100039554 Published online by Cambridge University Press

2. *Preliminary results.* The following results will be applied later in obtaining some of the identities.

LEMMA 1. *Let y be a fixed real number and let*

$$m(t) = \frac{|t-y|^\beta}{t-y} - \frac{|t|^\beta}{t}, \quad n(t) = |t-y|^{\beta-1} - |t|^{\beta-1}.$$

Then we have

(a) $m \in L^q$ *and* $n \in L^q$ *for* $1-(1/q) < \beta < 2-(1/q)$,

(b) $$\mathscr{F}(m)(x) = -\frac{i\phi(\beta)}{\sqrt{(2\pi)}} |x|^{1-\beta} \frac{1-e^{ixy}}{x}$$

and $$\mathscr{F}(n)(x) = -\frac{\phi(\beta)\cot \frac{1}{2}\pi\beta}{\sqrt{(2\pi)}} \frac{1-e^{ixy}}{|x|^\beta}$$

for $0 < \beta < 1$.

Proof. It is clearly sufficient to show that either of the results in (a) is true, and to take $y = 1$. It is obvious that the integral

$$\int_{-1}^{+1} \big||t-1|^{\beta-1} - |t|^{\beta-1}\big|^q \, dt$$

converges if and only if $(\beta-1)q+1 > 0$. By making an obvious change in variables we have

$$\int_1^\infty |(t-1)^{\beta-1} - t^{\beta-1}|^q \, dt = \int_0^1 \left|\frac{(1-t)^{\beta-1}-1}{t^{\beta-1+2/q}}\right|^q dt.$$

The right-hand side integral converges if its integrand is of $O(t^\gamma)$ as $t \to 0$, where $\gamma+1 > 0$. Now

$$\lim_{t\to 0} \frac{(1-t)^{\beta-1}-1}{t^{\beta-1+(2/q)}} = \lim_{t\to 0} \frac{(1-\beta)(1-t)^{\beta-2}}{\{\beta+(2/q)-1\}t^{\beta+(2/q)-2}}.$$

Hence $\gamma = q\{2-(2/q)-\beta\}$, and Lemma 1 (a) follows. By using expressions for well-known integrals, we have

$$\begin{aligned}\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} \{|t-y|^{\beta-1} - |t|^{\beta-1}\} e^{ixt}\, dt &= \frac{1}{\sqrt{(2\pi)}}(e^{ixy}-1)\int_{-\infty}^{\infty} |t|^{\beta-1} e^{ixt}\, dt \\ &= \sqrt{\frac{2}{\pi}}(e^{ixy}-1)\int_0^\infty t^{\beta-1}\cos xt\, dt \\ &= \sqrt{\frac{2}{\pi}}\frac{e^{ixy}-1}{|x|^\beta}\,\Gamma(\beta)\cos \tfrac{1}{2}\pi\beta.\end{aligned}$$

This gives $\mathscr{F}(n)$. The expression for $\mathscr{F}(m)$ follows by proceeding similarly.

LEMMA 2. *Let*

$$m_1(t) = |t|^{1-\beta}\frac{1-e^{ity}}{t}, \quad n_1(t) = \frac{1-e^{ity}}{|t|^\beta}.$$

https://doi.org/10.1017/S0305004100039554 Published online by Cambridge University Press

Then

(*a*) $m_1 \in L^q$ *and* $n_1 \in L^q$ *for* $1/q < \beta < 1 + (1/q)$,

(*b*) $$\mathscr{F}(m_1)(-x) = \frac{i\sqrt{(2\pi)}}{\phi(\beta)}\left\{\frac{|x-y|^\beta}{x-y} - \frac{|x|^\beta}{x}\right\}$$

and $$\mathscr{F}(n_1)(-x) = -\frac{\sqrt{(2\pi)}\tan\frac{1}{2}\pi\beta}{\phi(\beta)}\{|x-y|^{\beta-1} - |x|^{\beta-1}\}$$

for $0 < \beta < 1$.

Proof. The results $m_1 \in L^q$ and $n_1 \in L^q$ for $1/q < \beta < 1 + 1/q$ follow from the behaviour of the functions as $t \to 0$ and as $t \to \infty$. The result (*b*) is obtained as in the proof of Lemma 1 (*b*) by applying expressions for well-known integrals. This result may also be obtained by applying the inversion process for Fourier transforms to Lemma 1 (*b*).

LEMMA 3. *For* $0 < \beta < 1$, *we have*

(*a*) $$\frac{i}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}\frac{|t-y|^\beta}{t-y}\frac{1-e^{ixt}}{t}dt = \frac{i\phi(\beta)}{\sqrt{(2\pi)}}\int_0^x \frac{|t|^{1-\beta}}{t}e^{iyt}dt,$$

(*b*) $$\frac{i}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}|t-y|^{\beta-1}\frac{1-e^{ixt}}{t}dt = \frac{\phi(\beta)\cot\frac{1}{2}(\pi\beta)}{\sqrt{(2\pi)}}\int_0^x |t|^{-\beta}e^{iyt}dt.$$

Proof. By writing the results of Lemma 1 (*b*) in the form given in (1), we have

$$\frac{i}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}\left\{\frac{|t-y|^\beta}{t-y} - \frac{|t|^\beta}{t}\right\}\frac{1-e^{ixt}}{t}dt = -\frac{i\phi(\beta)}{\sqrt{(2\pi)}}\int_0^x |t|^{1-\beta}\frac{1-e^{ity}}{t}dt$$

and $$\frac{i}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}\{|t-y|^{\beta-1} - |t|^{\beta-1}\}\frac{1-e^{ixt}}{t}dt = -\frac{\phi(\beta)\cot\frac{1}{2}\pi\beta}{\sqrt{(2\pi)}}\int_0^x \frac{1-e^{iyt}}{|t|^\beta}dt.$$

The result follows by using the known integral formulae

$$\int_{-\infty}^{\infty}\frac{|t|^\beta}{t}\frac{1-e^{ixt}}{t}dt = \frac{\phi(\beta)}{1-\beta}|x|^{1-\beta}$$

and $$\int_{-\infty}^{\infty}|t|^{\beta-1}\frac{1-e^{ixt}}{t}dt = \frac{\phi(\beta)\cot\frac{1}{2}\pi\beta}{1-\beta}\frac{|x|^{2-\beta}}{x}.$$

THEOREM 1. *Let* $f \in L^p$ $(p > 1)$, $0 < \alpha < 1/p$ *and let* $1/r = (1/p) - \alpha$. *Then* $H_\alpha(f) \in L^r$ *and* $K_\alpha(f) \in L^r$. *Also, for* $g \in L^{r'}$, *we have*

(*a*) $$\int_{-\infty}^{\infty} g(t)\,H_\alpha(f)(t)\,dt = -\int_{-\infty}^{\infty} f(t)\,H_\alpha(g)(t)\,dt,$$

(*b*) $$\int_{-\infty}^{\infty} g(t)\,K_\alpha(f)(t)\,dt = \int_{-\infty}^{\infty} f(t)\,K_\alpha(g)(t)\,dt.$$

Proof. The results $H_\alpha(f) \in L^r$ and $K_\alpha(f) \in L^r$ follow as in the proof of a corresponding result for the Weyl and Rieman–Liouville fractional integrals (see Theorem 383 of (1)).

It is clearly sufficient to prove either (*a*) or (*b*). Since $1/p' = (1/r') - \alpha$, it is clear that $K_\alpha(g) \in L^{p'}$.

https://doi.org/10.1017/S0305004100039554 Published online by Cambridge University Press

Hence by the absolute convergence of the integrals involved, we have

$$\phi(\alpha)\int_{-\infty}^{\infty} g(t)\,K_\alpha(f)\,(t)\,dt = \int_{-\infty}^{\infty} g(t)\,dt\int_{-\infty}^{\infty} |x-t|^{\alpha-1}f(x)\,dx$$

$$= \int_{-\infty}^{\infty} f(x)\,dx\int_{-\infty}^{\infty} |x-t|^{\alpha-1}g(t)\,dt = \phi(\alpha)\int_{-\infty}^{\infty} f(x)\,K_\alpha(g)\,(x)\,dx.$$

3. *Identities involving the operators.* The results of this section will be valid for $f\in L^p$ $(1<p\leqslant 2)$, since it is known that $\hat{f}$ exists and belongs to $L^{p'}$ in this case. For $p>2$, however, the results will hold only if it is known that $\mathscr{F}(f)$ exists.

THEOREM 2. *Let $f\in L^p$ $(p>1)$, let $\hat{f}\in L^{p'}$ and let $1-1/p<\beta<1$. Then*

(a) $$L_\beta\{\mathscr{F}(f)\}\,(y) = -\frac{i}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} |t|^{1-\beta}f(t)\frac{1-e^{iyt}}{t}\,dt,$$

(b) $$M_\beta\{\mathscr{F}(f)\}\,(y) = -\frac{\cot\frac{1}{2}\pi\beta}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} |t|^{1-\beta}f(t)\frac{1-e^{iyt}}{|t|}\,dt.$$

Proof. Let $m(x)$ and $n(x)$ be defined as in Lemma 1. Then for $1-1/p<\beta<2-1/p$, we have $m\in L^p$ and $n\in L^p$. If we also have $\beta<1$, then $\mathscr{F}(m)$ and $\mathscr{F}(n)$ are given as in Lemma 1. Hence by applying the product formula for Fourier transforms (Theorems 49 and 75 of (3)), we have

$$\int_{-\infty}^{\infty} f(t)\,\mathscr{F}(m)\,(t)\,dt = \int_{-\infty}^{\infty} \hat{f}(t)\,m(t)\,dt$$

and

$$\int_{-\infty}^{\infty} f(t)\,\mathscr{F}(n)\,(t)\,dt = \int_{-\infty}^{\infty} \hat{f}(t)\,n(t)\,dt.$$

The required result clearly follows.

THEOREM 3. *Let $f\in L^p$ $(p>1)$, let $\hat{f}\in L^{p'}$ and let $1/p<\beta<1$. Then*

(a) $$L_\beta(f)\,(y) = \frac{i}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} |t|^{1-\beta}\hat{f}(t)\frac{1-e^{-iyt}}{t}\,dt,$$

(b) $$M_\beta(f)\,(y) = -\frac{\cot\frac{1}{2}\pi\beta}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} |t|^{1-\beta}\hat{f}(t)\frac{1-e^{-iyt}}{|t|}\,dt.$$

Proof. This result follows from the product formula for the Fourier transform by proceeding as in the proof of Theorem 2, and using Lemma 2.

Next we have the following result from which we derive expressions for the Fourier transform of $H_\alpha(f)$ and $K_\alpha(f)$.

THEOREM 4. *Let $f\in L^p$ where $p>1$, and let $0<\alpha<1/p$. Then*

(a) $$\int_0^x \mathscr{F}\{H_\alpha(f)\}\,(t)\,dt = -\frac{i}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} f(y)\,dy\int_0^x \frac{|t|^{1-\alpha}}{t}\,e^{iyt}\,dt,$$

(b) $$\int_0^x \mathscr{F}\{K_\alpha(f)\}\,(t)\,dt = \frac{\cot\frac{1}{2}\pi\alpha}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} f(y)\,dy\int_0^x |t|^{-\alpha}\,e^{iyt}\,dt.$$

https://doi.org/10.1017/S0305004100039554 Published online by Cambridge University Press

Proof. Since $i/\sqrt{(2\pi)}\,(1-\epsilon^{ixt})/t$ belongs to L^q for $q > 1$, it clearly belongs to L^r. Hence by applying the product formulae of Theorem 1 with Lemma 3, we have

$$\frac{i}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}\frac{1-\epsilon^{ixt}}{t}H_\alpha(f)(t)\,dt = -\frac{i}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}f(y)\,dy\int_0^x\frac{|t|^{1-\alpha}}{t}\,\epsilon^{iyt}\,dt$$

and

$$\frac{i}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}\frac{1-\epsilon^{ixt}}{t}K_\alpha(f)(t)\,dt = \frac{\cot\frac{1}{2}\pi\alpha}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}f(y)\,dy\int_0^x|t|^{-\alpha}\,\epsilon^{iyt}\,dt.$$

The theorem follows by applying (1) to the left-hand side of these equations.

THEOREM 5. *Let $f\in L^p$ where $p > 1$, let $\hat{f}\in L^{p'}$ and let $0 < \alpha < 1/p$. Then*

(*a*) $$\mathscr{F}\{H_\alpha(f)\}(x) = -i\frac{|x|^{1-\alpha}}{x}\mathscr{F}(f)(x)$$ *for almost all values of x,*

(*b*) $$\mathscr{F}\{K_\alpha(f)\}(x) = \cot\tfrac{1}{2}\pi\alpha\,|x|^{-\alpha}\mathscr{F}(f)(x)$$ *for almost all values of x.*

Proof. Let $-\infty < a < b < \infty$. Then it is obvious that

$$\int_a^b f(y)\,dy\int_0^x\frac{|t|^{1-\alpha}}{t}\,\epsilon^{iyt}\,dt = \int_0^x\frac{|t|^{1-\alpha}}{t}\,dt\int_a^b f(y)\,\epsilon^{iyt}\,dy.$$

Let
$$w(t) = \begin{cases}|t|^{1-\alpha}t^{-1} & \text{for} \quad t\in(0,x),\\ 0 & \text{for} \quad t\notin(0,x).\end{cases}$$

Then for all finite values of x, we have $w\in L^p$, since $0 < \alpha < 1/p$. Also, we have

$$\lim_{\substack{a\to-\infty\\ b\to\infty}}\left\|\hat{f}(t)-\frac{1}{\sqrt{(2\pi)}}\int_a^b f(y)\,\epsilon^{iyt}\,dy\right\|_{p'} = 0.$$

Hence on letting $a\to-\infty$ and $b\to\infty$, we have

$$\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}f(y)\,dy\int_0^x\frac{|t|^{1-\alpha}}{t}\,\epsilon^{iyt}\,dt = \int_0^x\frac{|t|^{1-\alpha}}{t}\hat{f}(t)\,dt.$$

Similarly,
$$\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}f(y)\,dy\int_0^x|t|^{-\alpha}\,\epsilon^{iyt}\,dt = \int_0^x|t|^{-\alpha}\hat{f}(t)\,dt.$$

Hence from Theorem 4, we have

$$\int_0^x\mathscr{F}\{H_\alpha(f)\}(t)\,dt = -i\int_0^x\frac{|t|^{1-\alpha}}{t}\mathscr{F}(f)(t)\,dt$$

and
$$\int_0^x\mathscr{F}\{K_\alpha(f)\}(t)\,dt = \cot\tfrac{1}{2}\pi\alpha\int_0^x|t|^{-\alpha}\mathscr{F}(f)(t)\,dt.$$

Theorem 5 clearly follows.

Since the left-hand side of the identities of Theorem 5 belong to L^r, we have incidentally proved the following result.

COROLLARY 5·1. *Let $f\in L^p$ $(p > 1)$, let $\hat{f}\in L^{p'}$ and let $0 < \alpha < 1/p$. Then*

$$|x|^{-\alpha}\mathscr{F}(f)(x)\in L^r,$$

where $1/r = (1/p)-\alpha$.

https://doi.org/10.1017/S0305004100039554 Published online by Cambridge University Press

4. *Remarks on the inversion processes involving H_α and K_α.* In the inversion process obtained in (2), the Hilbert transform of f is expressed in terms of $H_\alpha(f)$. It has been established elsewhere that a similar process holds for $K_\alpha(f)$. It is possible, however, to derive inversion processes by which f may be expressed in terms of $H_\alpha(f)$ and $K_\alpha(f)$. In this section, we give alternative proofs to the latter results. In the proofs given here, we require the existence of the Fourier transform (for the case $p > 2$). This extra restriction is unnecessary, however, since the results hold generally.

THEOREM 6. *Let $f \in L^p$ $(p > 1)$ and let $0 < \alpha < 1/p$. Then*

$$(a) \qquad M_{1-\alpha}\{H_\alpha(f)\}(x) = -\tan \tfrac{1}{2}\pi\alpha \int_0^x f(y)\,dy,$$

$$(b) \qquad L_{1-\alpha}\{K_\alpha(f)\}(x) = -\cot \tfrac{1}{2}\pi\alpha \int_0^x f(y)\,dy.$$

Proof. Suppose that $0 < \alpha < 1/p$. Then the Fourier transforms of $H_\alpha(f)$ and $K_\alpha(f)$ are given in Theorem 5. Hence by substituting in Theorem 3, we have

$$M_\beta\{H_\alpha(f)\}(y) = \cot \tfrac{1}{2}\pi\beta \frac{i}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} |t|^{1-\beta-\alpha} \hat{f}(t) \frac{1-e^{-iyt}}{t}\,dt$$

and

$$L_\beta\{K_\alpha(f)\}(y) = \cot \tfrac{1}{2}\pi\alpha \frac{i}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} |t|^{1-\beta-\alpha} \hat{f}(t) \frac{1-e^{-iyt}}{t}\,dt,$$

where $(1/p) - \alpha = 1/r < \beta < 1$ (since $H_\alpha(f)$ and $K_\alpha(f)$ belong to L^r). In particular, for $\alpha + \beta = 1$, we have

$$M_{1-\alpha}\{H_\alpha(f)\}(y) = \tan \tfrac{1}{2}\pi\alpha \frac{i}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \hat{f}(t) \frac{1-e^{-iyt}}{t}\,dt,$$

$$L_{1-\alpha}\{K_\alpha(f)\}(y) = \cot \tfrac{1}{2}\pi\alpha \frac{i}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \hat{f}(t) \frac{1-e^{-iyt}}{t}\,dt.$$

The result now follows from the inversion formula for Fourier transforms (Theorem 74 of (3)).

REFERENCES

(1) HARDY, G. H., LITTLEWOOD, J. E. and PÓLYA, G. *Inequalities* (Cambridge, 1934).
(2) OKIKIOLU, G. O. A generalisation of the Hilbert transform. *London J. Math. Soc.* **40** (1965), 27–30.
(3) TITCHMARSH, E. C. *The theory of Fourier integrals* (Oxford, 1937).

https://doi.org/10.1017/S0305004100039554 Published online by Cambridge University Press