Combinatorics, Probability and Computing (2002) 11, 343–347. © 2002 Cambridge University Press
DOI: 10.1017/S0963548302005102 Printed in the United Kingdom

Constructing Trees in Graphs whose Complement has no $K_{2,s}$

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Received 27 August 2000; revised 14 November 2001

We show that, if G is a graph of order n with maximal degree $\Delta(G)$ and minimal degree $\delta(G)$ whose complement contains no $K_{2,s}$, $s \geqslant 2$, then G contains every tree T of order $n-s+1$ whose maximal degree is at most $\Delta(G)$ and whose vertex of second-largest degree is at most $\delta(G)$. We then show that this result implies that special cases of two conjectures are true. We verify that the Erdős–Sós conjecture, which states that a graph whose average degree is larger than $k-1$ contains every tree of order $k+1$, is true for graphs whose complement does not contain a $K_{2,4}$, and the Komlós–Sós conjecture, which states that every graph of median degree at least k contains every tree of order $k+1$, is true for graphs whose complement does not contain a $K_{2,3}$.

1. Introduction

In 1963, Erdős and Sós [6] conjectured that every graph of order n with more than $n(k-1)/2$ edges contains every tree of order $k+1$. Recently, there has been a variety of partial solutions to this conjecture showing that the conjecture is true for various classes of graph. Brandt and the author [5] showed that the conjecture holds for graphs with girth at least 5. Saclé and Woźniak [9] then improved this result and showed that the Erdős–Sós conjecture holds for graphs without a $K_{2,2}$. Haxell [7] showed that the conjecture holds for graphs that do not contain a $K_{2,s}$, where $s = \lfloor k/18 \rfloor$. Finally, Li, Liu and Wang [8] showed that the Erdős–Sós conjecture holds for graphs whose complement has girth at least 5. In this paper, we will extend the result of Li, Liu and Wang, and show that the Erdős–Sós conjecture holds for graphs whose complement does not contain a $K_{2,4}$.

Komlós and Sós [1] conjectured that every graph such that at least half of its vertices have degree at least k contains every tree of order $k+1$. Soffer [10] has shown that the Komlós–Sós conjecture holds for graphs with girth at least 7. In this paper, we will verify the Komlós–Sós conjecture (also referred to in the literature as the Loebl–Komlós–Sós conjecture) for graphs whose complement does not contain a $K_{2,3}$.

https://doi.org/10.1017/S0963548302005102 Published online by Cambridge University Press

Notation is relatively standard: see [4] for standard terms not defined here. For a graph G, $\Delta(G)$, and $\delta(G)$ are the maximal and minimal degree, respectively. By $\Delta_2(G)$, we mean the vertex of second-largest degree. That is, if G has degree sequence $d_1, \dots, d_n$ with $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n$, then $\Delta_2(G) = d_2$. An *inclusion* $f : H \to G$ is an injective function from $V(H)$ to $V(G)$ such that if $e \in E(H)$, then $f(e) \in E(G)$. Hence, if there is an inclusion from H to G, then G contains a subgraph isomorphic to H, namely, $f(H)$. The *complement* of a graph G, denoted $\bar{G}$, has vertex set $V(G)$ and edge set $E(\bar{G}) = \{xy : xy \notin E(G)\}$. A *leaf* of a tree T is a vertex v such that $\deg_T(v) = 1$.

A form of the following result is well known, and the result is straightforward to prove using the greedy algorithm.

Lemma 1.1. *Let G be a graph with $\delta(G) \geqslant k$, and $u \in V(G)$. Let T be a tree of order $k+1$ with $v \in V(T)$. Then there exists an inclusion $f : T \to G$ such that $f(v) = u$.*

2. The main result

Theorem 2.1. *Let G be a graph of order n such that $\bar{G}$ contains no $K_{2,s}$, $s \geqslant 2$. Then G contains every tree T of order $n - s + 1$ such that $\Delta(T) \leqslant \Delta(G)$ and $\Delta_2(T) \leqslant \delta(G)$.*

Proof. Assume that T is a tree of minimal order $r \leqslant n - s + 1$ such that $\Delta(T) \leqslant \Delta(G)$, $\Delta_2(T) \leqslant \delta(G)$, and T is not contained in G. Then G contains a subgraph isomorphic to $T - x$ for every leaf $x \in V(T)$. Observe that G clearly contains the star U on $\Delta(T) + 1$ vertices. Let $h : V(U) \to V(G)$ be an inclusion. Thus, if T is a star we are finished. If not, then let $u \in V(T)$ be such that u is the fixed vertex of maximal degree in T. Let $v_1, \dots, v_{\Delta(T)} \in V(T)$ be such that $uv_i \in E(T)$, $1 \leqslant i \leqslant \Delta(T)$. As T has order $r \leqslant n - s + 1$, there are at least s vertices of G that are not contained in $V(h(U))$. If no v_i is adjacent to a vertex of $V(G) - V(h(U))$, then, as $\Delta(T) \geqslant 2$, we have that $\bar{G}$ contains a $K_{2,s}$, a contradiction. Thus some v_i is adjacent to $w \in V(G) - V(h(U))$ and so G contains a subtree isomorphic to the subtree of T formed by removing every edge not incident with u, except one edge incident with a neighbour of u that is not incident with u. Hence such an r exists and $r \geqslant \Delta(T) + 2$. Let x be a leaf of T not adjacent to the fixed vertex of maximal degree so that G contains a subgraph isomorphic to $T - x$. Let $f : V(T - x) \to V(G)$ be an inclusion. Let $v \in V(T)$ be such that $vx \in E(T)$. We need only show that there exists a subgraph of G isomorphic to T.

Let $S = V(G) - V(f(T - x))$. As $|V(T)| \leqslant n - s + 1$, $|V(T - x)| \leqslant n - s$. Hence $|S| \geqslant s$. If $f(v)$ is adjacent in G to a vertex of S, then clearly G contains a subgraph isomorphic to T, so we assume without loss of generality that $f(v)$ is adjacent to no element of S. As $\Delta_2(T) \leqslant \delta(G)$, $\deg_{T-x}(v) < \Delta_2(T) \leqslant \delta(G)$ so that $f(v)$ is adjacent in G to at least one vertex of $f(T - x)$ that is not adjacent in $f(T - x)$ to v. Choose $u \in V(f(T - x))$ such that u is adjacent to $f(v)$ in G, u is not adjacent to $f(v)$ in $f(T - x)$, and $\mathrm{dist}_{T-x}(u, v)$ is maximal. Suppose $y \in V(f(T - x)$ is such that y is not in the component of $f((T - x) - u)$ that contains $f(v)$. Then u is on the unique $f(v)y$ path in $f(T - x)$ so that $\mathrm{dist}_{f(T-x)}(f(v), y) > \mathrm{dist}_{f(T-x)}(f(v), u)$. By choice of u, we have that $f(v)$

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is not adjacent in G to any vertex of any component of $f((T-x)-u)$ that does contain $f(v)$.

Let T_1 be the component of $f(T-x)-u$ that contains $f(v)$ and $T_2 = T_1 \cup \{f(v)u\}$. Let z be the unique neighbour of u (in $f(T-x)$) that is contained in T_1, let T_3 be the subtree of $f(T-x)$ induced by $(V(f(T-x))-V(T_1))\cup\{z\}$, and $R = S \cup \{w \in V(f(T-x)-u) : w$ is not in the same component of $f(T-x)-u$ as $f(v)\}$. We need only show that there exists an inclusion $g : T_3 \to G[R\cup\{z\}]$ such that $g(z) = z$.

As $f(v)$ is not adjacent to any vertex of S and $f(v)$ is not adjacent to any vertex in $\{w \in V(f(T-x)-u) : w$ is not in the same component of $f(T-x)-u$ as $f(v)\}$, we have that $f(v)$ is not adjacent to any vertex of R. As $\bar{G}$ does not contain a subgraph isomorphic to $K_{2,s}$, we have that, if $y \in V(G)$, $y \neq f(v)$, then y must be adjacent to at least $|R-\{y\}|-(s-1) \geqslant |R|-s$ vertices of $R-\{y\}$. Hence, for every $y \in R$, $\deg_{G[R]}(y) \geqslant |R|-(s-1)-1 = |R|-s$. Thus $G[R]$ has minimal degree at least $|R|-s$. As $|R| \geqslant |V(T_3)|-2+s$, we have that $\delta(G[R]) \geqslant |V(T_3)|-2$. Hence, by Lemma 1.1, for every $r \in R$ there exists an inclusion $g_r : (T_3-z) \to G[R]$, with $g_r(u) = r$. Note that z is adjacent to at least $|R-\{z\}|-(s-1) > 0$ vertices of R. Let $r \in R$ such that $zr \in E(G)$. Define $g : V(T_3) \to G[R\cup\{z\}]$ by $g(x) = g_r(x)$ if $x \in R$ and $g(z) = z$. Then g is an inclusion and the result follows. □

3. The Erdős–Sós conjecture

Before proceeding to the results, we note that it is straightforward to show, by inductively deleting vertices of degree less than $\lceil k/2\rceil$, that every graph G of order n with $|E(G)| > n(k-1)/2$ contains a subgraph H order m such that $\Delta(H) \geqslant k$, $\delta(H) \geqslant \lceil k/2\rceil$, and $|E(H)| > m(k-1)/2$.

Corollary 3.1. *Let G be a graph of order n such that $\bar{G}$ contains no $K_{2,s}$, and let k be an integer such that $|E(G)| > n(k-1)/2$. Let $H \subseteq G$ be of order m such that $\Delta(H) \geqslant k$, $\delta(H) \geqslant \lceil k/2\rceil$, and $|E(H)| > m(k-1)/2$. If $k \leqslant m-s$, then G contains every tree T of order $k+1$.*

Proof. If T is a tree of order $k+1$, then clearly $\Delta(T) \leqslant k$. If $\Delta_2(T) > \lceil k/2\rceil$, then $\Delta(T) > \lceil k/2\rceil$. Hence T contains at least $\Delta(T)+1+\Delta_2(T) \geqslant k+2$ vertices. We conclude that $\Delta_2(T) \leqslant \lceil k/2\rceil$. The result then follows by Theorem 2.1 applied to H. □

We now verify the Erdős–Sós conjecture for graphs whose complement contains no $K_{2,4}$.

Corollary 3.2. *Let G be a graph of order n such that $\bar{G}$ contains no $K_{2,s}$, and let k be an integer such that $|E(G)| > n(k-1)/2$. If $s = 4$ or $k \leqslant n-s-2$, then G contains every tree of order $k+1$.*

Proof. Let H be as in Corollary 3.1. Well-known packing theorems (see [11, Theorems 2, 3]) show that the Erdős–Sós conjecture is true if $k = m-1$ or $m-2$, and Woźniak

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[11, Theorem 4] has shown that the result holds if $k = m - 3$. Thus, if $s = 4$, the result holds.

If $k \leqslant n - s - 2$, then we claim that there is at most one vertex v in G whose deletion results in a graph H such that $\Delta(H) \geqslant k$ and $\delta(H) \geqslant \lceil k/2 \rceil$, in which case the result will follow from Corollary 3.1. We may thus assume that G contains some v such that $\deg_G(v) < \lceil k/2 \rceil$. Note that $\Delta(G - v) \geqslant k$. Assume that $\delta(G - v) < \lceil k/2 \rceil$, and let $u \in V(G - v)$ such that $\deg_{G-v}(u) < \lceil k/2 \rceil$. Then $\deg_G(u) \leqslant \lceil k/2 \rceil$ so that $\deg_G(u) + \deg_G(v) \leqslant \lceil k/2 \rceil - 1 + \lceil k/2 \rceil \leqslant k$. As $|V(G) - \{u, v\}| \geqslant k + s$, by the pigeonhole principle there are at least s vertices of $G - \{u, v\}$ that both u and v are not adjacent to in G. Hence $\bar{G}$ contains a subgraph isomorphic to $K_{2,s}$, a contradiction. □

4. The Komlós–Sós conjecture

Corollary 4.1. *Let G be a graph of order n such that $\bar{G}$ does not contain a subgraph isomorphic to $K_{2,s}$, $s \geqslant 2$, and at least half of the vertices of G have degree at least k. If $k \leqslant n - s - 1$, then G contains every tree of order $k + 1$.*

Proof. If $\delta(G) \geqslant \lceil k/2 \rceil$, then by Theorem 2.1 the result follows. If $\delta(G) \leqslant \lceil k/2 \rceil - 1$, then will show that G contains a subgraph H such that $\Delta(H) \geqslant k$, $\delta(H) \geqslant \lceil k/2 \rceil$, and $|V(H)| = |V(G)| - 1$. The result then follows by Theorem 2.1.

Let $v \in V(G)$ such that $\deg_G(v) = \delta(G) \leqslant \lceil k/2 \rceil - 1$, $R = \{u : uv \in E(G)\}$, $S = V(G) - (R \cup \{v\})$, and $H = G - v$. Since at least half of the vertices of G have degree at least k and $k \leqslant n - 3$, there is at least $\lceil n/2 \rceil - \lceil k/2 \rceil \geqslant 1$ vertex of S that has degree at least k in G. Thus there is at least one vertex of S that has degree at least k in H so that $\Delta(H) \geqslant k$. Clearly $|R| = \deg_G(v) \leqslant \lceil k/2 \rceil - 1$ and

$$|S| = n - |R| - 1 \geqslant n - \lceil k/2 \rceil \geqslant k + s + 1 - \lceil k/2 \rceil + 1 > \lceil k/2 \rceil + s.$$

Let $u \in V(H)$. If u is not adjacent in G to at least s vertices of S, then, as v is not adjacent to any element of S, we have that $\bar{G}$ contains a $K_{2,s}$, a contradiction. Thus u is not adjacent in G to at most $s - 1$ vertices of S so that $\deg_H(u) \geqslant |S - \{u\}| - (s - 1) > \lceil k/2 \rceil$, and the result follows. □

Corollary 4.2. *Let G be a graph of order n such that $\bar{G}$ does not contain a subgraph isomorphic to $K_{2,3}$, and at least half of the vertices of G have degree at least k. Then G contains every tree T of order $k + 1$.*

Proof. Bazgan, Li and Woźniak [3] have shown that the Komlós–Sós conjecture holds for $k \geqslant n - 3$ (see [2] for a proof of this result). We may thus assume that $k \leqslant n - 4$. It then follows by Corollary 4.1 that the result holds provided that $n - 4 \leqslant n - s - 1$, so we can take $s = 3$ and the result follows. □

Acknowledgement

The author wishes to acknowledge the anonymous referee for many constructive suggestions, and in particular, for correcting an error in Theorem 2.1.

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References

[1] Ajtai, M., Komlós, J. and Szemerédi, E. (1995) On a conjecture of Loebl. In *Graph Theory, Combinatorics, and Algorithms*, Vol. 1, 2 (Kalamazoo, MI, 1992), Wiley-Interscience, New York, pp. 1135–1146.
[2] Bazgan, C., Li, H. and Woźniak, M. (1997) On the Loebl–Komlós–Sós conjecture. Rapport de Recherche, LRI, Université Paris Sud, no. 1094.
[3] Bazgan, C., Li, H. and Woźniak, M. (2000) On the Loebl–Komlós–Sós conjecture. *J. Graph Theory* **34** 269–276.
[4] Bollobás, B. (1979) *Graph Theory*, Springer, New York.
[5] Brandt, S. and Dobson, E. (1996) The Erdős–Sós conjecture for graphs of girth 5. *Selected Papers in Honour of Paul Erdős on the Occasion of his 80th Birthday* (Keszthely, 1993), *Discrete Math.* **150** 411–414.
[6] Erdős, P. (1965) Extremal problems in graph theory. In *Theory of Graphs and its Applications* (M. Fiedler, ed.), Academic Press, New York, pp. 29–36.
[7] Haxell, P. E. (2001) Tree embeddings. *J. Graph Theory* **36** 121–130.
[8] Li, G., Liu, A. and Wang, M. (2000) A result of Erdős–Sós conjecture. *Ars Combin.* **55** 123–127.
[9] Saclé, J. F. and Woźniak, M. (1997) The Erdős–Sós conjecture for graphs without C_4. *J. Combin. Theory Ser. B* **70** 367–372.
[10] Soffer, S. N. (2000) The Komlós–Sós conjecture for graphs of girth 7. *Discrete Math.* **214** 279–283.
[11] Woźniak, M. (1996) On the Erdős–Sós conjecture. *J. Graph Theory* **25** 229–234.

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