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OPTIMAL GROWTH, DEBT DYNAMICS, AND WELFARE UNDER GDP-BASED COLLATERALS

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We analyze the consequences of external debt collaterals on the optimal growth path of a country. We develop a small open economy model of endogenous growth where public spending can be financed by borrowing on imperfect international financial markets, where the country's borrowing capacity is limited. In contrast to the existing literature, which assumes that debt is constrained by the stock of capital, we investigate the consequences of gross domestic product (GDP)-based collaterals. First, we demonstrate that the economy may converge in a finite time, which is determined endogenously, to the regime with binding collaterals. Second, in such regime the steady-state public expenditures-to-GDP ratio is greater than that of the existing literature's models. Finally, we show that the degree of financial openness rises welfare if the collateral constraint is nonbinding and reduces welfare if the constraint binds. The first effect prevails always over the second and total intertemporal welfare increases.

Keywords: External Debt, GDP-Based Collaterals, Multistage Optimal Control, Welfare

1. INTRODUCTION

The rapid development of financial globalization started in the mid-1980s allowed developing countries to open to capital inflows and many of them accumulated high levels of external debt. This exposed them to credit risk and solvency problems, so as international financial markets reacted by limiting their external borrowing.

This paper is a contribution to the growth literature in open economy and studies the effects of external debt collaterals on the optimal growth dynamics of a country. We extend Barro's (1990) and Turnovsky's (1996) models of productive public expenditures. Notably, in Barro, public spending is enhanced by the government only by levying income taxes, whereas, in Turnovsky, the country can also borrow without limits from the rest of the world. In our model, the country's external debt is constrained by gross domestic product (GDP) that, being the tax base, also sets an

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upper bound to the government's fiscal capacity. In this context, financial openness means that the country is allowed to borrow on international financial markets. The literature on growth, collateral constraints, and financial markets is very limited and assumes that a country's debt is constrained by physical capital [see, for example, Eaton and Gersovitz (1981), Cohen and Sachs (1986), Turnovsky (1997), Brock (2009), Boucekkine and Pintus (2012)]. Our main assumption on the borrowing constraint is that the country's collateral is GDP, rather than (private or public) physical capital. This assumption is as appropriate as that of physical capital since public capital usually takes the form of public patrimony, portions of territory, monuments, infrastructures, and so on, which are of difficult appropriability for foreign creditors.

We focus on the consequences of a collateral constraint, based on GDP, for the optimal growth path of a country that borrows on international financial markets. In this perspective, an increase in the economy's growth rate corresponds to a slackening of the constraint, as the borrowing capacity of the country increases.[1]

We also introduce some technical novelties. In order to verify whether the debt stock actually reaches the constraint's upper bound and to endogenously determine the date at which it can happen, we solve the model by using a multistage optimal control technique. Here, the standard optimal control problem is constructed as a dynamic programming problem, which allows us to find the instant at which the economy changes regime. Optimal switching problems that can be found in the literature typically involve regimes that are characterized by different technological or ecological states, and each regime is defined as a certain set of values of the parameters [see Tomiyama (1985), Boucekkine et al. (2004)]. A more recent branch of research uses this technique to analyze the problem of irreversibility: When the state variable reaches a certain threshold, the economy switches to a different regime in which the parameters of the economy are different [see Prieur et al. (2011), Boucekkine et al. (2013a, 2013b, 2013c)]. In this paper, we focus on another type of optimal switching time problem. The fact that at the initial date the constraint is not binding and then, if switching is optimal, it becomes binding modifies one of the state equations, thus, defining two possible states of the economy.

We obtain a number of novel results and some broad policy implications. In the regime where the constraint is not binding (nonbinding regime), the optimal public expenditures-to-GDP ratio is equal to the income share of public expenditures as in Barro's and Turnovsky's models. However, this phase may be transitory: under certain conditions, the economy will switch to the regime in which the constraint is binding (binding regime). Here, we show that the economy admits a balanced growth path (BGP hereafter), which is not immediately reached because of the presence of investment adjustment costs and we determine analytically the conditions for the stability of the BGP. This state of the economy displays an important difference with respect to Barro's and Turnovsky' models: the public expenditures-to-GDP ratio is greater than the income share of public expenditures. This happens because a higher investment in public goods relaxes the borrowing

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constraint. The increased borrowing capacity allows to finance greater public expenditures, which in turns boosts growth.

We also determine the endogenous switching date. In the papers previously mentioned, the endogenous switching date can be found because all of the regimes admit closed form solutions and the date can consequently be determined by replacing these solutions in the matching conditions between the regimes. Our binding dynamic system is nonlinear and cannot be solved in a closed form. Nevertheless, we are able to give analytical conditions for corner solutions and then simulate the interregime transition. First, we find that if there are no adjustment costs of capital, the economy immediately jumps to the regime where the constraint is binding. In other words, the presence of investment adjustment costs makes the interregime transition slower. Second, there exist some threshold conditions that must be satisfied by the relevant parameters, for the economy to switch to the binding regime. This happens when the interest rate and the adjustment costs parameter are low and the productivity is high. Otherwise, the economy starting with a positive stock of net debt will end up accumulating foreign assets, without hitting the constraint. We also perform a comparative statics analysis of the effects of the relevant parameters on the switching date. This date turns out to be an increasing function of the degree of financial openness, the international interest rate, the adjustment costs, and it is a decreasing function of the productivity parameter.[2]

The paper is close to Turnovsky's (1997) model. Here, it is assumed that limited borrowing is represented by an upward-sloping supply schedule of debt, where the cost of borrowing increases with the level of foreign debt, relative to the stock of capital, which is equivalent to set an always binding capital-based collateral constraint. Although our results are similar to Turnovsky's (1997) ones in what concern the BGP (along the BGP, which is not immediately reached, all the variables grow at the same rate), the model presented in this paper shows some important differences. First, in Turnovsky's the constraint binds at all the dates, whereas in this article we ask if a country, starting with a level of debt smaller than the collaterals, moves to the limited borrowing regime at a finite date. Second, when the stock of debt is constrained, the optimal public expenditures-to-GDP ratio is greater than that of Turnovsky's model, which is equal to the income share of public spending. On the contrary, in the present paper, this ratio depends on all the parameters of the model, including the terms of borrowing.

Finally, we study how the welfare level of the economy varies when the degree of financial liberalization, which modifies the borrowing limits, changes. We find that, as the degree of financial openness increases, welfare in the nonbinding regime rises and welfare in the binding regime reduces. The first result can be explained by the positive effect of financial openness on the initial level of consumption and on the switching date. On the contrary, welfare in the binding regime is decreasing in the degree of financial liberalization. This derives by the fact that the time spent in the binding regime reduces with financial openness and that the debt burden will be greater in the long run, given the relaxation of the borrowing limits.

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The paper is structured as follow. In Section 2, we present and solve our model. Section 3 analyzes the BGP. Section 4 discusses the interregime transition and the determination of the optimal switching date. Section 5 presents the welfare analysis. Finally, Section 6 concludes. The appendix contains technical details.

2. THE MODEL

In this section, we first describe the economy's framework under the assumption of unlimited borrowing capacity, as in Turnovsky's (1996) model,[3] and subsequently discuss the case of GDP-based collateral constraints.

All the variables that will be introduced refer to time t. We will drop the time index t to ease the notation ($x(t) = x$) in the equations and we will indicate it in the relevant cases. The production function is the same as in Barro's (1990) model: Government's purchases of goods and services enter the production function as pure public goods. The technology is Cobb–Douglas

$$Y = AG^{\alpha}K^{1-\alpha}, A > 0, 0 < \alpha < 1, \tag{1}$$

where K is the firm's stock of physical capital, A is the technology parameter, and G is the flow of productive government expenditures. α represents the income share of public services. For a given G, the economy faces diminishing returns to scale in the stock of capital and constant returns to scale in G and K. We normalize the population to one throughout the paper: Labor is supplied inelastically and the growth rate of the population is exogenous. Capital can be interpreted in a broad sense by including human capital.

We define public expenditures as a flow variable. This assumption is usually adopted in the literature [see, e.g., Futagami et al. (2008)].[4]

Since we are interested in analyzing the dynamics of a net debtor country, the economy's resource constraint (current account) can be expressed, from the standpoint of a debtor, as

$$\dot{B} = C + I + G + \frac{h}{2}\frac{I^2}{K} + \hat{r}B - Y, \tag{2}$$

which states that the economy accumulates debt to finance its total expenditures on consumption, C, public expenditures, private capital, interest payments on debt B ($\hat{r}$ is the exogenous international interest) net of produced output. Since we assume here that the country is a net borrower on international financial markets, the variable $B \geq 0, \forall t$ represents the country's stock of net debt. Capital accumulation involves convex investment adjustment costs of the form $\Phi(I, K) = I(1 + \frac{h}{2}\frac{I}{K}), h > 0$. This is the traditional Hayashi's (1982) formulation, where the adjustment costs are proportional to the rate of investment in terms of units of installed capital. Their presence is a necessary assumption of small open economy models for two reasons. First, in a world where the country is not constrained by internal savings and has access to international financial

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markets, the investment rate would be infinity in the absence of adjustment costs. Second, without adjustment costs, this model would display the strong knife-edge condition, according to which the internal marginal productivity of capital would be constrained to be equal to the exogenous interest rate [see Turnovsky (2002)].

The agent's intertemporal utility is given by $U(C) = \int_0^{+\infty} \ln C e^{-\rho t} dt$, where $0 < \rho < 1$ is the discount factor.[5] Finally, we assume that capital does not depreciate for the sake of simplicity,[6] so as

$$\dot{K} = I. \tag{3}$$

2.1. Unlimited Borrowing Capacity: Turnovsky's Model

Here, the economy has unconstrained access to international financial markets. For coherency, we also assume that the country is a net borrower on world capital markets at the initial date.[7]

The planner has to choose consumption, public expenditures, investment, debt, and capital in order to maximize the intertemporal utility subject to (1)–(3). The optimal public expenditures-to-GDP ratio is equal to the income share of public goods α, as in Barro's model. The growth rate of capital is $\dot{K}/K = (q-1)/h$, where $q \equiv -\lambda_2/\lambda_1$ is the market value of capital in terms of the price of bonds (Tobin's q),[8] and that of consumption is

$$\Psi = \hat{r} - \rho. \tag{4}$$

In order for this growth rate to be positive, we keep the following assumption throughout the paper.

Assumption. We set $\hat{r} > \rho$.

The growth rate of capital is determined by q, whose dynamics is given by $\dot{q}/q + (q-1)^2/(2hq) + (1-\alpha)/qA^{1/(1-\alpha)}\alpha^{\alpha/(1-\alpha)} = \hat{r}$. By setting $\dot{q} = 0$, we obtain two solutions:

$$q_1 = (1+\hat{r}h) - \sqrt{(1+\hat{r}h)^2 - \left[1 + 2h\,(1-\alpha)\,A^{1/(1-\alpha)}\alpha^{\alpha/(1-\alpha)}\right]}, \tag{5}$$

$$q_2 = (1+\hat{r}h) + \sqrt{(1+\hat{r}h)^2 - \left[1 + 2h\,(1-\alpha)\,A^{1/(1-\alpha)}\alpha^{\alpha/(1-\alpha)}\right]}.$$

As it is shown in Turnovsky (1996), the smaller equilibrium value, q_1, is an unstable root, whereas q_2 is stable. However, any stable arm converging to q_2 violates the transversality condition, $\lim_{t\to\infty} \lambda_2(t)K(t) = \lim_{t\to\infty} -q(t)\lambda_1(t)K(t) = 0$, where $K(t) = K(0)e^{\int_0^t \frac{q(s)-1}{h} ds}$ and $\lambda_1(t) = \lambda_1(0)e^{-\hat{r}t}$. In fact, the stable adjustment path is $q(t) = q_2 + (q(0) - q_2)e^{\mu t}$, where $\mu < 0$ is the stable eigenvalue . Hence, $\int_0^t \frac{q(s)-1}{h} ds = \frac{q_2-1}{h} t + \frac{q(0)-q_2}{\mu}(e^{\mu t} - 1)$. By substituting this expression into the transversality condition, it is possible to check that it is violated. In the case of the unstable root q_1, there is no transitional dynamics and the transversality condition

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is satisfied for $q = q_1$. Hence, q is always at its steady state value q_1. This implies that capital is always on its balanced growth path, with a growth rate given by

$$\Lambda = \frac{q_1 - 1}{h}. \tag{6}$$

By substituting (4)–(6) into the economy's current account, we can solve for the stock of net debt:

$$B(t) = -\frac{C(0)}{\rho} e^{(\hat{r}-\rho)t} - \theta K(0)\, e^{\frac{q_1-1}{h}t} + \left(B(0) + \frac{C(0)}{\rho} + \theta K(0) \right) e^{\hat{r}t},$$

where $\theta \equiv \frac{[\frac{q_1-1}{h} + A^{1/(1-\alpha)}\alpha^{1/(1-\alpha)} + \frac{(q_1-1)^2}{2h} - A^{1/(1-\alpha)}\alpha^{\alpha/(1-\alpha)}]}{(\hat{r} - \frac{q_1-1}{h})}$. In order to ensure intertemporal solvency, the transversality condition $\lim_{t\to\infty} \lambda_1(t)B(t) = \lim_{t\to\infty} \lambda_1(0)e^{-\hat{r}t}B(t) = 0$ must be satisfied and this holds if and only if

$$C(0) = -\rho\,(B(0) + \theta K(0)).$$

If $C(0) > 0$, it must be $B(0) + \theta K(0) < 0$. Given the definition of θ and given that the transversality condition implies $\hat{r} > \frac{q_1-1}{h}$, a necessary condition is $\frac{q_1-1}{h} + A^{1/(1-\alpha)}\alpha^{1/(1-\alpha)} + \frac{(q_1-1)^2}{2h} - A^{1/(1-\alpha)}\alpha^{\alpha/(1-\alpha)} < 0$.

The equilibrium stock of net debt is hence given by

$$B(t) = (B(0) + \theta K(0))\, e^{(\hat{r}-\rho)t} - \theta K(0)\, e^{\frac{q_1-1}{h}t}. \tag{7}$$

Turnovsky's model has some important implications. First, the small open economy assumption implies that the growth rate of domestic consumption can be different from the growth rate of domestic output (capital), which is in sharp contrast with the closed economy version of the model, where all the variables grow at the same rate. Second, the growth rate of the stock of debt varies over time and asymptotically converges to $[\frac{q_1-1}{h}, \hat{r} - \rho]$. In particular if $\frac{q_1-1}{h} < \hat{r} - \rho$, $\lim_{t\to+\infty} B(t) = -\infty$. On the contrary, if $\frac{q_1-1}{h} > \hat{r} - \rho$, $\lim_{t\to+\infty} B(t) = +\infty$ and the country would accumulate a forever increasing debt. Finally, the fact that the economy can borrow from the rest of the world does not influence the part of domestic output invested in public expenditures, which is equal to the income share of public goods, α, as in Barro's (1990) model.

2.2. The Model with GDP-Based Collaterals

We assume now that the country's debt capacity is bounded by GDP. The external debt constraint is

$$0 \le B \le \phi Y, \phi \ge 0. \tag{8}$$

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The case $\phi = 0$ corresponds to autarky,[9] whereas $\phi > 0$ describes international financial integration. Hence, the parameter ϕ represents the degree of financial liberalization. An increase in ϕ relaxes the borrowing constraint.[10]

The social planner must choose the level of consumption, public expenditures, investment, external net debt, and capital that solve the problem

$$\begin{aligned} \max_{\{C,G,I,B,K\}} \quad & U(C) = \int_0^{+\infty} \ln C e^{-\rho t} dt \\ \text{s.t.} \quad & \dot{K} = I \\ & \dot{B} = C + I + G + \frac{h}{2}\frac{I^2}{K} + \hat{r}B - Y \\ & B \leq \phi Y,\ B \geq 0, \forall t \\ & K(0),\ B(0) \text{ given,} \end{aligned} \tag{9}$$

where Y is given by (1).

In order to endogenously find the date at which the constraint becomes binding, we solve the model by using a two-stage optimal control technique, because this date becomes a control variable. In this model, the economy starts from a situation in which the stock of debt is such that the collateral constraint is not binding and it may become binding at a certain date (Proposition 6 of section *The convergence to the binding regime* provides the conditions under which this happens). This modifies one of the state equations. Hence, we can define two regimes, where the two possible states of the economy can be summarized as follows[11]:

$$\begin{cases} \dot{B} = C + I + G + \frac{h}{2}\frac{I^2}{K} + \hat{r}B - AG^{\alpha}K^{1-\alpha} & \text{if } B < \phi Y \quad t \in [0, T] \\ \dot{B} = C + I + \left(\frac{1}{\phi}\right)^{1/\alpha} A^{-1/\alpha} B^{1/\alpha} K^{(\alpha-1)/\alpha} + \hat{r}B + \frac{h}{2}\frac{I^2}{K} - \frac{1}{\phi}B & \text{if } B = \phi Y \quad t \in (T, +\infty). \end{cases} \tag{10}$$

Given (10), a natural approach is to decompose the problem into two subproblems, solve them, and finally identify the optimal timing T. In our case since

$$V(K(0), B(0)) = \int_0^T \ln C(t) e^{-\rho t} dt + \int_T^{+\infty} \ln C(t) e^{-\rho t} dt, \tag{11}$$

one can view the problem sequentially starting from the second regime, for $t \in (T, +\infty)$, that we call *binding regime* and then solving for the first regime, which

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we define as the *nonbinding regime*. In the binding regime, the problem becomes

$$\begin{aligned} \max_{\{C,I,B,K\}} \quad & V_{\mathrm{B}} = \int_{T}^{+\infty} \ln C e^{-\rho t} dt \\ \text{s.t.} \quad & \dot{K} = I \\ & \dot{B} = C + I + \left(\frac{1}{\phi}\right)^{1/\alpha} A^{-1/\alpha} B^{1/\alpha} K^{(\alpha-1)/\alpha} + \hat{r} B + \frac{h}{2}\frac{I^2}{K} - \frac{1}{\phi} B, \\ & \text{for a given } T,\ K(T),\ B(T). \end{aligned} \tag{12}$$

The corresponding Hamiltonian is

$$\begin{aligned} H^{\mathrm{B}} &= \ln C e^{-\rho t} \\ &+ \lambda_1^{\mathrm{B}} \left[C + I + \left(\frac{1}{\phi}\right)^{1/\alpha} A^{-1/\alpha} B^{1/\alpha} K^{(\alpha-1)/\alpha} + \hat{r} B + \frac{h}{2}\frac{I^2}{K} - \frac{1}{\phi} B \right] + \lambda_2^{\mathrm{B}} I, \end{aligned} \tag{13}$$

where the superscript "B" means that the constraint binds. $\lambda_1^{\mathrm{B}} < 0$ is the co-state variable associated with the stock of net debt and $\lambda_2^{\mathrm{B}} > 0$ is the co-state variable associated with capital when the economy is in the second regime.

Then, we solve

$$\begin{aligned} \max_{\{C,G,I,B,K\}} \quad & V_{\mathrm{NB}} = \int_{0}^{T} \ln C e^{-\rho t} dt + V_{\mathrm{B}}^{*}(T, B(T), K(T)) \\ \text{s.t.} \quad & \dot{K} = I \\ & \dot{B} = C + I + G + \frac{h}{2}\frac{I^2}{K} + \hat{r} B - A G^{\alpha} K^{1-\alpha} \\ & B \geq 0 \\ & K(0),\ B(0) \text{ given}, T \text{ free}, \end{aligned} \tag{14}$$

and the corresponding Hamiltonian is

$$H^{\mathrm{NB}} = \ln C e^{-\rho t} + \lambda_1^{\mathrm{NB}} \left(C + I + G + \frac{h}{2}\frac{I^2}{K} + \hat{r} B - A G^{\alpha} K^{1-\alpha} \right) + \lambda_2^{\mathrm{NB}} I + \psi B, \tag{15}$$

where the superscript "NB" means that the regime is nonbinding and ψ is the multiplier attached to the nonnegativity constraint on the stock of net debt.

This recursive scheme works exactly as a dynamic programming problem: The Bellman principle applies to the two subintervals implied by the timing problem.

We will discuss the Pontryagin optimality condition in the subsequent sections.[12] Here, we shall focus on the optimality conditions with respect to the timing variable T. They are summarized by the following theorem.

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THEOREM 1. *Let* $0 < T < +\infty$ *the optimal timing. Then,*

$$\lambda_1^B(T) = \lambda_1^{NB}(T), \tag{16}$$

$$\lambda_2^B(T) = \lambda_2^{NB}(T), \tag{17}$$

$$H^B(T) = H^{NB}(T). \tag{18}$$

A detailed proof is contained in Appendix A. Some comments are in order here. First, the above conditions can be read as continuity or matching conditions at the junction point: Condition (18) imposes the continuity of the Hamiltonian at the optimal junction time and conditions (16) and (17) impose the continuity of the co-state variables. Here, both the co-state variables are continuous because the state variables do not jump at the switching time: The state variables do not have to satisfy any pure state constraint and both can be freely chosen. Second, one can interpret the matching conditions (16)–(18) as first-order optimal timing conditions with respect to the switching date T. In fact, condition (18) can be rewritten as $H^{\mathrm{B}}(T) - H^{\mathrm{NB}}(T) = 0$. The left-hand side can be interpreted as the marginal gain from extending the first regime at the expense of the regime associated to the next interval. The right-hand side represents the marginal cost. Since there is no direct switching cost, this cost is zero. Hence, this condition equalizes the marginal benefit to the marginal cost of delaying the switching time.

2.3. Preliminary Analysis of Economic Regimes

In this section, we will solve the model by starting with the Pontryagin problem of the binding regime and then by proceeding backward to the nonbinding case.

The binding regime. For a given T, $B(T)$, and $K(T)$, the Pontryagin necessary conditions of problem (12) are

$$\frac{1}{C}e^{-\rho t} + \lambda_1 = 0, \tag{19}$$

$$\lambda_1\left(1 + h\frac{I}{K}\right) + \lambda_2 = 0, \tag{20}$$

$$\lambda_1\left[\frac{1}{\alpha}\left(\frac{1}{\phi}\right)^{1/\alpha} A^{-1/\alpha}\left(\frac{B}{K}\right)^{(1-\alpha)/\alpha} + \hat{r} - \frac{1}{\phi}\right] = -\dot{\lambda}_1, \tag{21}$$

$$-\lambda_1\left[\frac{h}{2}\frac{I^2}{K^2} + \frac{1-\alpha}{\alpha}\left(\frac{1}{\phi}\right)^{1/\alpha} A^{-1/\alpha}\left(\frac{B}{K}\right)^{1/\alpha}\right] = -\dot{\lambda}_2, \tag{22}$$

$$\lim_{t\to\infty} \lambda_1 B = 0 \quad \lim_{t\to\infty} \lambda_2 K = 0.$$

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By using the shadow value of debt λ_1 as the numeraire and defining $q = -\frac{\lambda_2}{\lambda_1}$, we find the growth rate of capital $\frac{I}{K} = \frac{q-1}{h}$ from equation (20), and by combining it with (21) and (22), we obtain the arbitrage equation:

$$\frac{\dot{q}}{q} = \frac{1}{\alpha}\left(\frac{1}{\phi}\right)^{1/\alpha} A^{-\frac{1}{\alpha}} \left(\frac{B}{K}\right)^{(1-\alpha)/\alpha} + \hat{r} - \frac{1}{\phi} - \frac{1}{q}\frac{1-\alpha}{\alpha}\left(\frac{1}{\phi}\right)^{1/\alpha} A^{-\frac{1}{\alpha}} \left(\frac{B}{K}\right)^{1/\alpha} - \frac{(q-1)^2}{2hq}.$$

We find it convenient to rewrite the arbitrage equation as a function of $(\frac{G}{Y})$ in order to compare the results with Barro's (1990) and Turnovsky's (1996) models. From the constraint, we know that the ratio $(\frac{B}{K})$ is linked to the ratio $(\frac{G}{K})$ in the following way: $\frac{B}{K} = \phi A(\frac{G}{K})^{\alpha}$. Moreover, by using the production function we know that $\frac{G}{K} = A^{1/(1-\alpha)}(\frac{G}{Y})^{1/(1-\alpha)}$. Hence, the arbitrage equation becomes

$$\frac{\dot{q}}{q} = \frac{1}{\alpha}\left(\frac{1}{\phi}\right)\left(\frac{G}{Y}\right) + \hat{r} - \frac{1}{\phi} - \frac{1}{q}\frac{1-\alpha}{\alpha} A^{1/(1-\alpha)} \left(\frac{G}{Y}\right)^{1/(1-\alpha)} - \frac{(q-1)^2}{2hq}. \quad \textbf{(23)}$$

Log-differentiating equation (19) and using equation (21), we derive the growth rate of consumption as a function of $(\frac{G}{Y})$:

$$\frac{\dot{C}}{C} = \hat{r} - \frac{1}{\phi} + \frac{1}{\alpha}\frac{1}{\phi}\left(\frac{G}{Y}\right) - \rho. \quad \textbf{(24)}$$

We find it convenient to rewrite the dynamical system made up of the aggregate resource constraint, (23), and of (24) by defining $b \equiv \frac{B}{K}$ and $c \equiv \frac{C}{K}$, so that

$$\begin{cases} \dfrac{\dot{c}}{c} = \hat{r} - \dfrac{1}{\phi} + \dfrac{1}{\alpha}\dfrac{1}{\phi}\left(\dfrac{G}{Y}\right) - \rho - \dfrac{q-1}{h} \\ \dfrac{\dot{q}}{q} = \dfrac{1}{\alpha}\left(\dfrac{1}{\phi}\right)\left(\dfrac{G}{Y}\right) + \hat{r} - \dfrac{1}{\phi} - \dfrac{1}{q}\dfrac{1-\alpha}{\alpha} A^{1/(1-\alpha)} \left(\dfrac{G}{Y}\right)^{1/(1-\alpha)} - \dfrac{(q-1)^2}{2hq} \\ \dfrac{\dot{b}}{b} = \dfrac{1}{b}\left[c + \dfrac{q-1}{h} + A^{1/(1-\alpha)}\left(\dfrac{G}{Y}\right)^{1/(1-\alpha)} + \dfrac{(q-1)^2}{2h} + \hat{r}b - A^{1/(1-\alpha)}\left(\dfrac{G}{Y}\right)^{\alpha/(1-\alpha)}\right] - \dfrac{q-1}{h}. \end{cases} \quad \textbf{(25)}$$

The nonbinding regime. For a given T, $B(T)$, and $K(T)$, the first-order necessary conditions of problem (14) are[13]

$$\frac{1}{C}e^{-\rho t} + \lambda_1 = 0, \quad \textbf{(26)}$$

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$$\lambda_1\left(1+h\frac{I}{K}\right)+\lambda_2=0, \tag{27}$$

$$\frac{G}{K}=(A\alpha)^{1/(1-\alpha)} \iff \frac{G}{Y}=\alpha, \tag{28}$$

$$\lambda_1\hat{r}+\psi=-\dot{\lambda}_1, \tag{29}$$

$$-\lambda_1\left[\frac{h}{2}\frac{I^2}{K^2}+A\left(1-\alpha\right)\left(\frac{G}{K}\right)^{\alpha}\right]=-\dot{\lambda}_2, \tag{30}$$

$$\psi\geq 0, \quad \psi B=0, \tag{31}$$

where $q=-\frac{\lambda_2}{\lambda_1}$ and $\frac{I}{K}=\frac{q-1}{h}$. The public expenditures-to-GDP ratio is equal to α.

By using (29), (30), and the definition of q, we get the arbitrage equation:

$$\frac{\dot{q}}{q}=\hat{r}-\frac{(q-1)^2}{2hq}-(1-\alpha)\,A^{1/(1-\alpha)}\alpha^{\alpha/(1-\alpha)}\frac{1}{q}+\frac{\psi}{\lambda_1}. \tag{32}$$

Combining equation (26) with equation (29), we obtain the growth rate of consumption:

$$\frac{\dot{C}}{C}=\hat{r}-\rho+\psi. \tag{33}$$

By defining $c\equiv\frac{C}{K}$, $b\equiv\frac{B}{K}$, we can write the reduced system as

$$\begin{cases} \dfrac{\dot{c}}{c}=(\hat{r}-\rho)+\dfrac{\psi}{\lambda_1}-\dfrac{q-1}{h} \\ \dfrac{\dot{q}}{q}=\hat{r}-\dfrac{(q-1)^2}{2hq}-(1-\alpha)\,A^{1/(1-\alpha)}\alpha^{\alpha/(1-\alpha)}\dfrac{1}{q}+\dfrac{\psi}{\lambda_1} \\ \dfrac{\dot{b}}{b}=\dfrac{1}{b}\left[c+\dfrac{q-1}{h}+A^{1/(1-\alpha)}\alpha^{1/(1-\alpha)}+\dfrac{(q-1)^2}{2h}+\hat{r}b-A^{1/(1-\alpha)}\alpha^{\alpha/(1-\alpha)}\right] \\ \quad -\dfrac{q-1}{h}. \end{cases} \tag{34}$$

3. BALANCED GROWTH PATH ANALYSIS

In this section, we study the BGP of the economy in the case where the economy ends up in the binding regime. The conditions for optimal switching will be given in Section 4. Along the BGP, all the endogenous variables grow at the same, positive, and constant growth rate, given by equation (24), and the endogenous variables (c, G/Y, q) of system (25) grow at a zero rate. This result can be verified by considering that when the collateral constraint binds, Y and B grow at the same rate. From the production function, it can be seen that capital grows at the same rate of output. Finally, the terms of equation (10) are all proportional to K or B,[14] so that also C grows at the same rate.

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TABLE 1. Baseline calibration

A	α	h	$\hat{r}$	ρ	ϕ
0.4	0.5	15	0.035	0.03	0.8

By setting $\dot{c} = 0$ in (25), we obtain $q = h[\hat{r} - \frac{1}{\phi} + \frac{1}{\alpha}\frac{1}{\phi}(\frac{G}{Y}) - \rho] + 1$. By plugging it into the equation $\dot{q} = 0$ and rearranging terms, we end up with the equation determining the equilibrium (G/Y) ratio:

$$-\frac{1-\alpha}{\alpha}A^{1/(1-\alpha)}\phi\left(\frac{G}{Y}\right)^{1/(1-\alpha)} + \frac{h}{2}\left(\frac{1}{\phi}\right)\frac{1}{\alpha^2}\left(\frac{G}{Y}\right)^2 + [h(\hat{r}\phi - 1) + \phi]\frac{1}{\phi}\frac{1}{\alpha}\left(\frac{G}{Y}\right) + \frac{1}{\phi}\left[\frac{h}{2}(\hat{r}\phi - 1)^2 + \phi(\hat{r}\phi - 1) - \frac{h}{2}(\rho\phi)^2\right] = 0. \tag{35}$$

Note that by setting $\phi = 0$, we obtain Barro's (1990) and Turnovsky's (1996) optimal public expenditures-to-GDP ratio: $(G/Y) = \alpha$. All the analyses are conducted for the empirically interesting case $\alpha \leq \frac{1}{2}$.[15] Moreover, whether it is not possible to find analytical solutions, we provide the results of simulations based on the benchmark calibration (see Table 1) for the total factor productivity, A, the income share of public goods, α, the adjustment cost parameter, h, the foreign interest rate, $\hat{r}$, the discount rate, ρ, and the financial openness parameter, ϕ.

The technological parameters have been set consistently with Barro and Sala-i-Martin (2004) and Chatterjee et al. (2003). The chosen values for $\hat{r}$ and ρ allow to satisfy the conditions needed to move from the nonbinding regime to the binding regime which will be introduced later in Section 4.1.[16] A glossary with the definitions is contained in Table F.1.

The following proposition proves the existence of a BGP in the binding regime.

PROPOSITION 2. *The dynamical system (25) admits one steady state.*

The proof is given in Appendix B, where we also show that it satisfies the transversality condition.

In order to obtain analytical solutions for the ratio (G/Y) and to study analytically the local stability of the steady state, we focus on the $\alpha = \frac{1}{2}$ case.[17] This case is perfectly equivalent to the case $\alpha < 1/2$; hence, the conditions for the existence of a BGP do not change.

Equation (35) becomes

$$\left(\frac{2h}{\phi} - A^2\phi\right)\left(\frac{G}{Y}\right)^2 + 2[h(\hat{r}\phi - 1) + \phi]\frac{1}{\phi}\left(\frac{G}{Y}\right) + \frac{1}{\phi}\left[\frac{h}{2}(\hat{r}\phi - 1)^2 + \phi(\hat{r}\phi - 1) - \frac{h}{2}(\rho\phi)^2\right] = 0. \tag{36}$$

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The unique admissible solution is

$$\left(\frac{G}{Y}\right)_{ss} = \frac{-[h(\hat{r}\phi - 1) + \phi]\frac{1}{\phi} + \sqrt{\frac{1}{2}\{2 + 2h^2\rho^2 + A^2\{h - 2(1 + h\hat{r})\phi + [r(2 + h\hat{r}) - h\rho^2]\phi^2\}\}}}{\left(\frac{2h}{\phi} - A^2\phi\right)}. \tag{37}$$

The next step is to study the stability of the BGP. System (25) allows transitional dynamics. This is due to the presence of investment adjustment costs: The economy does not reach immediately its steady state because the adjustment costs do not allow capital to immediately jump to its BGP. Proposition 3 describes the local stability properties of $(\frac{G}{Y})_{ss}$.

PROPOSITION 3. *If* $A^2 > -\frac{2}{h-2(1+\hat{r}h)\phi+[\hat{r}(2+h\hat{r})-h\rho^2]\phi^2}$, *the balanced growth path solution is saddle-point stable.*

The proof is given in Appendix C.

Numerical analysis shows that the BGP is stable under the $\alpha \leq 1/2$ case, for standard calibrations of the parameters.

To conclude this section, we will compare the optimal (G/Y) of our model with those of Barro's (1990) and Turnovsky's (1996, 1997) models.

PROPOSITION 4. *The optimal public expenditures-to-GDP ratio of the model with GDP-based collaterals is greater than Barro's (1990) and Turnovsky's (1996) solutions.*

In order to prove this result, consider the first-order condition with respect to public goods of the maximization problem (9), solved by attaching a Lagrange multiplier $\eta \geq 0$ to the inequality constraint (8)[18]:

$$\frac{\partial Y}{\partial G} - \frac{\eta}{\lambda_1}\frac{\partial Y}{\partial G} = 1, \tag{38}$$

where $\eta = \frac{1}{\phi}[1 - \frac{1}{\alpha}(\frac{G}{Y})]\lambda_1$. Being $\eta > 0$ when the constraint binds, and $\lambda_1 < 0$, equation (38) implies that in the binding regime (G/Y) must be greater than α, i.e., greater than Barro's (1990) and Turnovsky's (1996, 1997) values. Equation (38) states that the social planner will choose the optimal (G/Y) by equating the marginal benefit of increasing G to the marginal cost, which, by the economy's resource constraint, is equal to one. In the literature's models, the corresponding expression is $\frac{\partial Y}{\partial G} = 1$. Hence, we can conclude that in the model with a GDP-based collateral constraint, the overall marginal benefit is equal to the marginal benefit of increasing G on production augmented by the term $-\frac{\eta}{\lambda_1}\frac{\partial Y}{\partial G}$, which represents the marginal benefit of relaxing the collateral constraint, since a greater public

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spending increases the country's borrowing capacity. This is the reason why the public expenditures-to-GDP ratio is greater than that of the literature's models.

We conclude this section with the comparative statics analysis. The results are included in the following proposition.

PROPOSITION 5. *The optimal public expenditures-to-GDP ratio is*

(*i*) *decreasing in the world interest rate* $\hat{r}$*;*
(*ii*) *increasing in the technology parameter* A*;*
(*iii*) *the long-run growth rate is increasing in the total factor productivity (TFP) parameter.*

Proof. The proof is contained in Appendix D.

An increase in the international interest rate makes borrowing more costly, and given that public expenditures are financed by debt, the optimal (G/Y) ratio decreases. Results (ii) and (iii) stem from the double role played by the TFP in our model. On the one hand, A increases the economy's productivity and this boosts growth. On the other hand, an increase in A, by making the economy richer, has the effect of increasing the collaterals, allowing the economy to borrow more from the rest of the world. The possibility to increase the financing capacity enhances growth.

We also obtain the following results:

(*iv*) *The optimal* (G/Y) *and the economy's growth rate are increasing in* ϕ. A higher ϕ implies higher growth because a greater financial openness relaxes the borrowing constraint (growth enhancing effect of foreign borrowing).
(*v*) *The growth rate is decreasing in the world interest rate* $\hat{r}$. Since (G/Y) is decreasing in $\hat{r}$, the same happens for the long-run growth rate.
(*vi*) *The optimal* (G/Y) *and the economy's growth rate are decreasing functions of the installation cost parameter* h. Higher capital adjustment costs reduce the (G/Y) ratio and the growth rate because more resources are spent in installation.

The numerical analysis also shows that the growth rate of GDP is lower than that of Turnovsky's (1996). Recall that in his model the country can borrow as much as it wants because there is no collateral constraint and the growth rate of GDP coincides with the growth rate of capital, which can be different from the growth rate of consumption. This latter is set by the difference between the exogenous international interest rate and the discount rate, and it is independent from any technological parameter. The growth rate in Turnovsky turns out to be greater than that of our model due to the absence of growth-limiting borrowing constraints.

4. DETERMINATION OF THE OPTIMAL SWITCHING DATE

4.1. Optimal Switching

The convergence to the binding regime. In this section, we study the conditions for having the convergence to the binding regime. We suppose that the economy

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is in the nonbinding regime and we assume that the system will be unconstrained forever, that is, $T = +\infty$. Hence, the model turns out to be the same as Turnovsky's (1996) model. Here, we extend his analysis in order to show when the corner solution $T = +\infty$ is possible.

In Section 2.1, we have shown that the dynamics of the stock of net debt, $B(t)$, as given by equation (7), depends on the difference between $(\hat{r} - \rho)$ and $\frac{q_1-1}{h}$, whose sign is determined by q_1, which is given by (5). These expressions represent the growth rates of consumption and capital (output), respectively, in Turnovsky's model. The growth rate of capital (and output) is positive if $q_1 > 1$, which happens when $\hat{r} < (1-\alpha)A^{1/(1-\alpha)}\alpha^{\alpha/(1-\alpha)}$. We shall concentrate on this case.

PROPOSITION 6. *Suppose that* $\hat{r} - \rho < \frac{q_1-1}{h}$. *Then, there exists a date* $T < +\infty$ *at which the economy hits the constraint.*

In the case $\hat{r} - \rho < \frac{q_1-1}{h}$,[19] $B(t)$ is an increasing function of t and $\lim_{t\to+\infty} B(t) = +\infty$. In other words, under the conditions of Proposition 6, for the country, it is optimal to increase the stock of net debt and to hit the constraint at a finite time.

Otherwise, we have the following proposition.

PROPOSITION 7. *When* $\hat{r} - \rho \geq \frac{q_1-1}{h}$, $\lim_{t\to+\infty} B(t) = -\infty$ *and the economy will not hit the constraint.*

The case $\hat{r} - \rho \geq \frac{q_1-1}{h}$ requires a deeper discussion. When this condition is satisfied, $\lim_{t\to+\infty} B(t) = -\infty$, but $B(t)$ can be either always decreasing or first increasing and then decreasing, depending on the values of $\frac{B(0)}{K(0)}$. At $t = 0$, $B'(0)$ is equal to

$$B'(0) = \theta K(0)\left(\hat{r} - \rho - \frac{q_1 - 1}{h}\right) + (\hat{r} - \rho) B(0),$$

which is negative if $\frac{B(0)}{K(0)} < -\theta[1 - \frac{(q_1-1)/h}{\hat{r}-\rho}]$.

The other condition on $\frac{B(0)}{K(0)}$ is given by the collateral constraint: $\frac{B(0)}{K(0)} < \phi A^{1/(1-\alpha)}\alpha^{\alpha/(1-\alpha)}$.

If $\phi A^{1/(1-\alpha)}\alpha^{\alpha/(1-\alpha)} < -\theta[1 - \frac{(q_1-1)/h}{\hat{r}-\rho}]$, then $B(t)$ is always decreasing and never hits the constraint. If $\phi A^{1/(1-\alpha)}\alpha^{\alpha/(1-\alpha)} \geq -\theta[1 - \frac{(q_1-1)/h}{\hat{r}-\rho}]$, then when $\frac{B(0)}{K(0)} < -\theta[1 - \frac{(q_1-1)/h}{\hat{r}-\rho}]$, $B(t)$ is decreasing and never hits the constraint. When $\frac{B(0)}{K(0)} \geq -\theta[1 - \frac{(q_1-1)/h}{\hat{r}-\rho}]$ and is close to the collateral constraint, $B(t)$ is first increasing then decreasing.

Nevertheless, the possibility to hit the constraint is ruled out by the consideration that, under the conditions of Proposition 7, the binding regime is never optimal: The numerical simulations show that the value of (G/Y) is smaller than or equal to α, and this is at odds with the Kuhn–Tucker slackness condition (38) which imposes that it must be greater than α in the binding regime (see Proposition 4).

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To conclude, the country's net stock of debt will become negative and the model behaves exactly as Turnovsky's (1996) model where, under the same conditions, the economy accumulates a forever increasing stock of foreign bonds.

Since we are interested in studying the case of a net debtor country, especially in the long run, we shall assume that $\hat{r} - \rho < \frac{q_1 - 1}{h}$. This translates into some threshold conditions on the relevant parameters that the economy has to satisfy in order to hit the constraint: The interest rate and the investment adjustment cost parameter have to be smaller than a certain threshold and the TFP higher than a certain value.[20]

The existence of the nonbinding regime: the role of investment adjustment costs. As previously shown, the assumption of investment adjustment costs in open economy models is necessary to rule out the unrealistic knife-edge condition on the marginal return of domestic capital. But this assumption has another important implication in our two-regimes model. In the absence of capital adjustment costs, if the knife-edge condition is not satisfied, the regime with slack constraint does not exist, because the country would choose to invest or disinvest at an infinite rate if the internal productivity is, respectively, greater or smaller than the interest rate [see Cohen and Sachs (1986)]. Since we assume that the country is a net debtor, the internal return must be greater than the world interest rate, so that the economy switches to the binding regime since the initial date. Hence, the presence of adjustment costs of investment enriches the interregime transitional dynamics.

4.2. The Interregime Transitional Dynamics

In this section, we simulate the two-regimes model in order to find the endogenous switching date for a net debtor country. The existence of transitional dynamics in the binding regime does not allow us to find analytical solutions for the date T. Our strategy is to apply the matching conditions of Theorem 1 in order to link two two-points boundary value problems, one for each regime. Final boundary problems are commonly used for the simulation of the dynamics of growth models. We follow the method proposed by Trimborn et al. (2008a, 2008b) applied separately to the two regimes, whose reduced form is given by systems (25) and (34). The definition of the variables involved in the simulations is reported in Table 2.

We start by defining the matching conditions in our case. Conditions (16) and (17) of Theorem 1 require the equality of the co-state variables between the two

TABLE 2. Definition of the variables

Variable	Nonbinding regime	Binding regime	Whole interval
Consumption-to-capital ratio	$c^{\mathrm{NB}}(t)$	$c^{\mathrm{B}}(t)$	$c(t)$
Tobin's q	$q^{\mathrm{NB}}(t)$	$q^{\mathrm{B}}(t)$	$q(t)$
External net debt-to-capital ratio	$b^{\mathrm{NB}}(t)$	$b^{\mathrm{B}}(t)$	$b(t)$

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regimes. Hence, at T the following condition must hold:

$$q^{\mathrm{NB}}(T) = q^{\mathrm{B}}(T) = q(T). \tag{39}$$

We replace (18), which is difficult to implement when a regime cannot be solved in the closed form, with two other conditions, which emerge from our model. Given that at T, the state variables cannot jump, it must be $b^{\mathrm{NB}}(T) = b^{\mathrm{B}}(T) = b(T)$. This value can be obtained by the collateral constraint $\frac{B}{K} = b = \phi A(\frac{G}{K})^{\alpha} = \phi A^{1/(1-\alpha)}(\frac{G}{Y})^{\alpha/(1-\alpha)}$. Given that b must be continuous at T, the same applies for the (G/Y)-ratio. In the nonbinding regime, $(\frac{G}{Y}) = \alpha$; hence, when $b^{\mathrm{NB}}(t)$ hits the constraint, it must be

$$b^{\mathrm{NB}}(T) = b^{\mathrm{B}}(T) = b(T) = \phi A^{1/(1-\alpha)}\alpha^{\alpha/(1-\alpha)}. \tag{40}$$

As a consequence, the value of $b(T)$ is known.

The last condition stems from equation (16), the equality between the shadow values of external net debt across regimes, and from the first-order conditions (19) and (26). These together imply that consumption is continuous at T, so that

$$c^{\mathrm{NB}}(T) = c^{\mathrm{B}}(T) = c(T). \tag{41}$$

Equation (41) also derives from the continuity of $(\frac{G}{Y})$ at T and from the expression of the growth rate of consumption in the binding regime: $\frac{\dot{C}}{C} = \hat{r} - \frac{1}{\phi}(1 - \frac{1}{\alpha}\frac{G}{Y}) - \rho$.

The algorithm adopted to simulate the two regimes separately is the relaxation algorithm proposed by Trimborn et al. (2008a, 2008b). Our numerical procedure starts backward with the simulation of the binding regime and then of the nonbinding one. The goal of this exercise is to find the switching date that satisfies the matching conditions (39)–(41). The method employed to solve the interregime transitional dynamics is described in Appendix E.

We assume the same values of the parameters reported in Table 1, contained in Section 3, for the total factor productivity, A, the income share of public goods, α, the adjustment cost parameter, h, the foreign interest rate, $\hat{r}$, and the discount rate, ρ. We present the numerical simulations for different values of the financial openness parameter, ϕ. We set the debt-to-capital ratio, $b(0)$, equal to 0.145.[21,22] The parameterization of Table 1 satisfies the convergence conditions stated in Proposition 3, so as $b(t) \geq 0, \forall t$.

Figure 1 shows the transition of the economy from the nonbinding to the binding regime for three levels of the financial integration parameter: This gives rise to a small, an intermediate, and a large switching date, T, as indicated by the dashed vertical line. Figure 1 plots the paths of the consumption-to-capital ratio, Tobin's q, and, finally, of the debt-to-capital ratio, for $\phi = 0.2, 0.8, 2$.

These graphics show that the transition occurs in a continuous way: The country starts in the nonbinding regime and moves to the binding one at T. Namely, the endogenous variables do not jump: As stated in Section 2.2, the borrowing constraint is not a pure state constraint.

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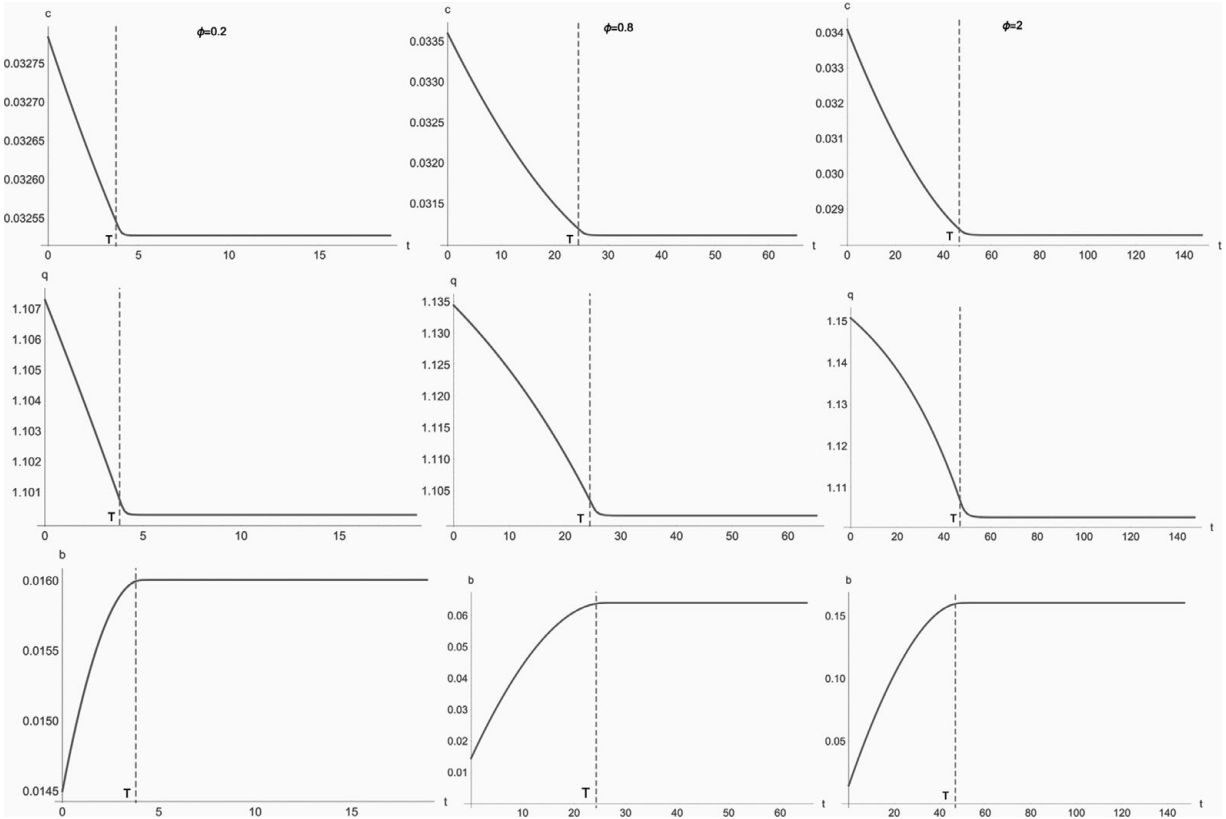


FIGURE 1. The transition interregime: the consumption-to-capital ratio, Tobin's q and the debt-to-capital ratio for $\phi = 0.2, 0.8, 2$.

Note also that the nonbinding regime, described by system (34), does not admit a steady state. As shown in Section 2, the growth rates of the endogenous variables in level are different if the constraint does not bind.

During the transition, the debt-to-capital ratio follows an increasing path, whereas the consumption-to-capital ratio and the Tobin's q are decreasing, until the economy attains its BGP. In order to interpret this result, we refer to the economy's current account, as given by equation (2). Throughout the transition, the stock of external debt is growing more than the stock of capital. The increasing international capital inflows are spent in physical capital, installation costs, public goods, and increasing interest payments. This diverts resources from consumption, thereby reducing the consumption-to-capital ratio. The dynamics of the Tobin's q is consistent with that of capital. Once the switch to the binding regime occurs, the stock of net debt is constrained,[23] forcing the economy to reach an equilibrium where all the variables in level grow at the same rate.

The economy's dynamics is consistent with Turnovsky's (1997) model, where financial market imperfections are introduced by assuming that the cost of borrowing is an increasing function of debt, relative to capital. In particular, a small open economy with unlimited access to international financial markets can sustain differential growth rates of consumption and output along the BGP, in sharp contrast to a closed economy where the economy's resources constraint imposes to all the variables to grow at the same rate. For a net debtor country, consumption grows less than capital (output). When a collateral constraint is added, the growth rates differ along the transition. However, once the steady state is reached, all the variables grow at the same rate.

The degree of financial liberalization, ϕ, has some important consequences on both the dynamics and the long-run equilibrium. From the comparisons of the three scenarios of Figure 1, one can immediately notice that the higher the ϕ is, the higher is the switching date: For $\phi = 0.2$, $T = 3.91$, for $\phi = 0.8$, $T = 25.1136$, and finally for $\phi = 2$, $T = 47.398$. The degree of financial openness influences also the speed of convergence to the steady state: The higher the ϕ is, the slower is the transition to the steady state in the binding regime.[24] Financial liberalization has an impact on the long-run equilibrium: An increase in ϕ reduces the steady-state consumption-to-capital ratio and rises the steady-state Tobin's q as well as the external debt-to-capital ratio. The relaxation of the country's borrowing capacity allows the economy to borrow more. The increasing borrowing costs divert resources from long-run consumption. Moreover, the higher public expenditures-to-capital ratio increases the productivity of capital and hence its price q.

We have also investigated the effects of some relevant parameters on the optimal switching date.[25] The main results are summarized in the findings reported below:

The optimal switching date T is increasing with respect to financial openness (ϕ), the international interest rate ($\hat{r}$), and the investment adjustment costs (h); it is decreasing with respect to the total factor productivity (A).

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An increase of ϕ relaxes the collaterals; hence, the initial value of the net debt-to-capital ratio,[26] $b(0)$, will be farther away from the constraint and the economy will spend more time in the nonbinding regime. A rise in the international interest rate and adjustment costs makes the transition slower. The first result is linked to the amount of interests to be paid: Since the cost of debt is high, the country will not push the stock of net debt until its upper bound immediately. Higher installation costs slow down the accumulation rate of external debt (see section *The existence of the nonbinding regime: the role of investment adjustment costs*). Finally, for a greater productivity, the economy moves faster to the binding regime. When A becomes very large, the optimal switching date approaches zero. In our model, the parameter A plays a double role. A greater TFP increases the economy's productivity making the country richer. This has in turns the effect of increasing the collaterals, allowing the country to borrow more from the rest of the world. Hence, the economy will spend less time in the nonbinding regime, reaching faster to the binding regime.

5. FINANCIAL OPENNESS AND INTERTEMPORAL WELFARE

In this section, we study how intertemporal welfare varies as a function of the degree of financial openness, ϕ. An increase in ϕ relaxes the borrowing constraint.

Since the model admits interregime and intraregime transitional dynamics, the present value of intertemporal welfare (11) cannot be computed in the closed form. Hence, we proceed as follows.

Given that intertemporal welfare is the sum of the nonbinding regime welfare and the binding regime welfare, first we compute them separately and after we sum them, according to equation (11).

For the nonbinding regime, where $t \in [0, T]$, it is possible to write an analytical expression for welfare:

$$\begin{aligned} V_{\text{NB}} &= \int_0^T \ln C\,(t)\, e^{-\rho t} dt = \int_0^T \ln\left[C\,(0)\, e^{(\hat{r}-\rho)t}\right] e^{-\rho t} dt \\ &= \frac{1}{\rho}\left[\ln C\,(0)\left(1 - e^{-\rho T}\right) + \frac{1}{\rho}\left(\hat{r} - \rho\right)\left(1 - (1+\rho T)\, e^{-\rho T}\right)\right]. \end{aligned} \tag{42}$$

For $C(0)$ and T known, the expression of the nonbinding regime welfare (42) is computable. From the numerical simulations, we compute the switching date T as well as the initial value of the consumption-to-capital ratio, $c(0) = C(0)/K(0)$. By considering that the stock of initial capital $K(0)$ is given, we can find the initial consumption $C(0)$. We normalize the initial capital to 100.

In the binding regime, where $t \in (T, +\infty)$, welfare is given by

$$V_{\text{B}} = \int_T^{+\infty} \ln C\,(t)\, e^{-\rho t} dt = \int_T^{+\infty} \ln\left[C(T) e^{\int_T^t \frac{\dot{C}(s)}{C(s)} ds}\right] e^{-\rho t} dt, \tag{43}$$

where $C(T)$ is the value of consumption at the switching time and $\dot{C}/C$ is given by (24). In order to find $C(T)$, we use the continuity of consumption at

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TABLE 3. Financial openness and welfare

ϕ	T	$C(0)$	V_{NB}	V_B	V
0.2	3.9100	3.2784	4.4160	42.3737	46.7896
0.5	17.4395	3.3352	16.8972	29.9219	46.8191
0.8	25.1136	3.3598	22.3490	24.4928	46.8418
2	47.3980	3.4088	33.3262	13.5782	46.9044
3	63.2045	3.4297	38.0537	8.8842	46.9380
10	201.0131	3.46755	46.8102	0.1985	47.0087

T [cf. equation (41)]. Hence, $C(T) = C(0)e^{(\hat{r}-\rho)T}$. The series of $\dot{C}(s)/C(s)$ can be derived by interpolating the series of the growth rate of consumption in the binding regime, as given by equation (24), rewritten in terms of the net debt-to-capital ratio $b(s)$:

$$\frac{\dot{C}(s)}{C(s)} = \hat{r} - \frac{1}{\phi} + 2\left(\frac{1}{\phi}\right)^2 A^{-2} b(s) - \rho,$$

where $b(s)$ corresponds to the time series of the net debt-to-capital ratio derived in the simulations and α has been set equal to 1/2. The exercise has been performed by applying the Hermite interpolation technique of order 3.[27] The number of mesh point in the relaxation algorithm has been set to 10,000 in order to ensure a sufficiently accurate interpolation of the series. The simulation horizon goes from T to $T_{\max} = 700$.[28]

Table 3 presents the results for the switching date, T, the level of initial consumption, $C(0)$, welfare in the nonbinding regime, V_{NB}, welfare in the binding regime, V_B and, finally, total welfare, V, corresponding to different values of the financial openness parameter. The baseline calibration is that of Table 1.

Table 3 shows that as the degree of financial liberalization rises, welfare in the nonbinding regime increases, whereas welfare in the binding regime reduces. As a result, total intertemporal welfare increases.

In order to make the interpretation of these results easier, Table 3 presents also the values of the initial consumption and the switching date. The degree of financial liberalization has a positive impact on T and $C(0)$. From the expression of welfare in the nonbinding regime, equation (42), one can notice that the only terms which depend on financial openness are T and $C(0)$, which we can identify as the *duration effect* and the *initial consumption effect*. They are both positive. An increase in financial integration increases the time spent in the nonbinding regime as well as the level of initial consumption and this increases welfare in the nonbinding regime. The duration effect can be explained by the fact that a higher ϕ increases the borrowing limits; hence, the economy will reach more slowly to the binding regime. The initial consumption effect suggests that an increase in foreign capital inflows allows the country to consume more in the

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short run. This effect has already been analyzed by a few authors in the literature: Our positive initial consumption effect is consistent with the increase of current consumption in Gourinchas and Jeanne (2006) and at odds with Boucekkine et al.'s (2014) negative short run pain effect of financial openness. The duration effect has instead a negative impact on welfare in the binding regime since a higher degree of financial liberalization reduces the time span of the binding regime.

To conclude, an increase in financial liberalization rises welfare in the nonbinding regime and reduces welfare in the binding regime. The sum of these two contrasting effects results in an increase of total intertemporal welfare. This is consistent with the literature [see Gourinchas and Jeanne (2006), Boucekkine et al. (2014)]. Our results show also that current generations' welfare increases with financial integration and future generations' welfare decreases. This is explained by the higher cost of debt that the country will bear in the future.

6. CONCLUSIONS

This study analyzes the consequences of external debt collaterals on the optimal growth path of a country. We depart from the existing growth literature with collateral constraints which assumes that the collaterals are capital stock. We extended Turnovsky's (1996) small open economy model by assuming that public expenditures can be financed by borrowing on imperfect international financial markets, where external debt is bounded by GDP.

The collateral constraint defines two possible states of the economy: The nonbinding regime and the binding regime, where a balanced growth path exists. We determine the conditions that the economy must satisfy in order to switch from the nonbinding to the binding regime. First, in the absence of investment adjustment costs, the economy switches to the binding regime at the initial date. Second, if installation costs are positive, for the economy to switch to the binding regime the TFP should be sufficiently high and/or the adjustment costs and the world interest rate should be low enough. Otherwise, the country will not hit the constraint and the stock of net debt will become negative, meaning that the economy will accumulate foreign assets.

We then compare our results with those of the existing endogenous growth literature. The optimal public expenditures-to-GDP ratio is greater than that of Barro's and Turnovsky's models: The planner has incentive to finance public expenditures by debt because this relaxes the collateral constraint.

Finally, we show that an increase in financial openness, which relaxes the borrowing constraint, has a positive impact on welfare in the nonbinding regime and a negative influence on welfare in the binding regime. The main explanation of these two opposite effects is the presence of a duration effect, which is positive in the nonbinding regime and negative in the binding regime. As the first effect is greater than the second, total intertemporal welfare increases.

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NOTES

1. Domestic revenue is clearly nonseizable. One should think that, in case of default, a government can seize whatever he wants in the country. Hence, the collateral condition should be interpreted as a signal of solvency of the debtor that can be compared to the situation of an individual filing for a loan with a bank.

2. Aizenman et al. (2007) build a model similar to ours: Countries are characterized by limited tax and debt capacity. However, they do not tackle the problem of collaterals and imperfect financial markets. Furthermore, the results and the technical analysis are different. First, the economy is closed. Second, they do not take into account any problem of endogenous switching.

3. The results of this model will turn out to be useful to study the emergence of corner solutions in our case. See Section 4.

4. The consideration of the public stock of capital adds further difficulties to the solution of the model.

5. We assume logarithmic utility throughout the paper in order to make the model tractable.

6. We will maintain this assumption throughout the paper, but the results do not change if the depreciation rate is taken positive.

7. Turnovsky (1996) assumes that the country may accumulate foreign bonds. This difference does not affect the properties of the model.

8. λ_1 and λ_2 are the two co-state variables of the optimization problem attached to debt (negative) and capital (positive), respectively.

9. In this case, the constraint (8) is binding.

10. Imposing a flow variable as a constraint on a stock variable is reasonable in an endogenous growth model because GDP tends to infinity in the long run.

11. Of course, the problem can be solved by using the traditional method of Lagrange, but this does not allow us to determine the endogenous switching time in our case.

12. The problem is concave, hence each of the two subinterval problems is well behaved.

13. This regime is the same as Turnovsky's (1996) model, with exception of the transversality conditions: In our case, the boundary conditions are given by the continuity conditions of Theorem 1.

14. Equation (10) can be rewritten as $\frac{\dot{B}}{B}B = C + \frac{\dot{K}}{K}K + (\frac{1}{\phi})^{1/\alpha}A^{-1/\alpha}(\frac{B}{K})^{1/\alpha}K + \hat{r}B + \frac{h}{2}\frac{I^2}{K^2}K - \frac{1}{\phi}B$.

15. This range of values is consistent with the empirical estimations existing in the literature: Eberts's (1986) estimation for the income share of public expenditures is around 0.43, Aschauer (1989) finds that it is 0.39, Ai and Cassou (1995) find estimations that range between 0.15 and 0.26, and finally Chatterjee et al. (2003) assume a value of 0.2.

16. We repeated the exercise for other values of the parameters and the results are robust.

17. We analyze this case separately since the closed-form solutions of the steady-state variables are necessary to compute the Jacobian matrix for the stability analysis.

18. This allows us to carry out an economic interpretation.

19. Since $(\hat{r} - \rho)$ is the growth rate of consumption in the nonbinding regime, we set $\hat{r} > \rho$. It turns out that $\hat{r} - \rho < \frac{q_1 - 1}{h}$ is possible only if $q_1 > 1$.

20. In other words, the country must be productive enough in order to afford international borrowing at a given interest rate and the cost of borrowing must be small enough.

21. In this numerical exercise, we set $\alpha = 1/2$ because in this case we know what conditions the parameters need to satisfy for the BGP to be stable (cf. Proposition 3).

22. The chosen value of initial conditions affects the switching date T: The smaller the $b(0)$ is, the higher the T is.

23. The transitional dynamics of the variable $b(t)$ is short, so as it less visible in the graphics than other variables transition.

24. The linearization around the steady state shows that the stable eigenvalue is equal to -5.872 when $\phi = 0.2$, -1.402 for $\phi = 0.8$, and -0.508 for $\phi = 2$.

25. The assumed parameterization is the baseline calibration of Table 1.

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26. Given the baseline calibration of Table 1, we can accept values for ϕ until 13.7 in order for the BGP to exist.

27. This is the technique chosen by Mathematica 10.1. This procedure is commonly used in the literature, see, for example, Rendahl (2015). A robustness check has been conducted by using the linear approximation of system (25), as it is suggested by Minea and Villieu (2009), and the results are similar.

28. This guaranties the convergence of the integral in (43).

29. $l(G/Y)$ crosses $g(G/Y)$ before it crosses $f(G/Y)$.

30. Numerical analysis has shown that the result is valid for values of A commonly used in the simulations.

31. The sign of the eigenvalues refers to their real part.

32. Condition (40) shows also that the values of the variables at T do not depend on the choice of T. This choice will only affect the duration of the transition to the steady state.

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APPENDIX A: PROOF OF THEOREM 1

In this section, we prove Theorem 1. We make use of the calculus of variations techniques applied to the sequence of the two control subproblems shown in the main text.

Binding regime. The problem is (12) and the associated Hamiltonian is (13). We denote by the superscript "*" the paths that satisfy the first-order conditions. The corresponding value function depends on T, $B(T)$, and $K(T)$: $V_{\mathrm{B}}^{*}(T, B(T), K(T))$. Moreover, since the problem is concave, the following envelope conditions apply:

$$\frac{\partial V_{\mathrm{B}}^{*}}{\partial T} = -H^{\mathrm{B}}(T), \tag{A.1}$$

$$\frac{\partial V_{\mathrm{B}}^{*}}{\partial B(T)} = \lambda_{1}^{\mathrm{B}}(T), \tag{A.2}$$

$$\frac{\partial V_{\mathrm{B}}^{*}}{\partial K(T)} = \lambda_{2}^{\mathrm{B}}(T). \tag{A.3}$$

Nonbinding regime. The corresponding control subproblem is (14) and the corresponding Hamiltonian is (15).

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The value function can be rewritten in terms of the Hamiltonian

$$V_{\mathrm{NB}} = \int_0^T \left(H^{\mathrm{NB}} - \lambda_1^{\mathrm{NB}}\dot{B} - \lambda_2^{\mathrm{NB}}\dot{K}\right) dt + V_{\mathrm{B}}^{*}\left(T, B(T), K(T)\right)$$

Standard integrations by parts yield

$$\int_0^T \lambda_1^{\mathrm{NB}}\dot{B} = \lambda_1^{\mathrm{NB}}(T)B(T) - \lambda_1^{\mathrm{NB}}(0)\,B(0) - \int_0^T \dot{\lambda}_1^{\mathrm{NB}} B dt,$$

$$\int_0^T \lambda_2^{\mathrm{NB}}\dot{K} = \lambda_2^{\mathrm{NB}}(T)K(T) - \lambda_2^{\mathrm{NB}}(0)\,K(0) - \int_0^T \dot{\lambda}_2^{\mathrm{NB}} K dt,$$

which allow to rewrite the value function as

$$V_{\mathrm{NB}} = \int_0^T \left(H^{\mathrm{NB}} + \dot{\lambda}_1^{\mathrm{NB}}B + \dot{\lambda}_2^{\mathrm{NB}}K\right) dt + V_{\mathrm{B}}^{*}\left(T, B(T), K(T)\right) - \lambda_1^{\mathrm{NB}}(T)B(T)$$
$$+ \lambda_1^{\mathrm{NB}}(0)\,B(0) - \lambda_2^{\mathrm{NB}}(T)K(T) + \lambda_2^{\mathrm{NB}}(0)\,K(0).$$

The first-order variation of V_{NB} with respect to the state and the control variables' paths yields

$$\partial V_{\mathrm{NB}} = \int_0^T \Bigg(\frac{\partial H^{\mathrm{NB}}}{\partial B}\partial B + \frac{\partial H^{\mathrm{NB}}}{\partial K}\partial K + \frac{\partial H^{\mathrm{NB}}}{\partial C}\partial C + \frac{\partial H^{\mathrm{NB}}}{\partial G}\partial G + \frac{\partial H^{\mathrm{NB}}}{\partial I}\partial I$$
$$+ \dot{\lambda}_1^{\mathrm{NB}}\partial B + \dot{\lambda}_2^{\mathrm{NB}}\partial K\Bigg) dt + \left(H^{\mathrm{NB}}(T) + \dot{\lambda}_1^{\mathrm{NB}}(T)B(T) + \dot{\lambda}_2^{\mathrm{NB}}(T)K(T)\right)\partial T$$
$$+ \frac{\partial V_{\mathrm{B}}^{*}\left(T, B(T), K(T)\right)}{\partial T}\partial T + \frac{\partial V_{\mathrm{B}}^{*}\left(T, B(T), K(T)\right)}{\partial B(T)}\partial B(T)$$
$$+ \frac{\partial V_{\mathrm{B}}^{*}\left(T, B(T), K(T)\right)}{\partial K(T)}\partial K(T) - \dot{\lambda}_1^{\mathrm{NB}}(T)B(T)\partial T - \lambda_1^{\mathrm{NB}}(T)\partial B(T)$$
$$- \dot{\lambda}_2^{\mathrm{NB}}(T)K(T)\partial T - \lambda_2^{\mathrm{NB}}(T)\partial K(T).$$

Reorganizing the terms

$$\partial V_{\mathrm{NB}} = \int_0^T \Bigg[\left(\frac{\partial H^{\mathrm{NB}}}{\partial B} + \dot{\lambda}_1^{\mathrm{NB}}\right)\partial B + \left(\frac{\partial H^{\mathrm{NB}}}{\partial K} + \dot{\lambda}_2^{\mathrm{NB}}\right)\partial K + \frac{\partial H^{\mathrm{NB}}}{\partial C}\partial C$$
$$+ \frac{\partial H^{\mathrm{NB}}}{\partial G}\partial G + \frac{\partial H^{\mathrm{NB}}}{\partial I}\partial I\Bigg] dt + \left(H^{\mathrm{NB}}(T) + \frac{\partial V_{\mathrm{B}}^{*}\left(T, B(T), K(T)\right)}{\partial T}\right)\partial T$$
$$+ \left(\frac{\partial V_{\mathrm{B}}^{*}\left(T, B(T), K(T)\right)}{\partial B(T)} - \lambda_1^{\mathrm{NB}}(T)\right)\partial B(T)$$
$$+ \left(\frac{\partial V_{\mathrm{B}}^{*}\left(T, B(T), K(T)\right)}{\partial K(T)} - \lambda_2^{\mathrm{NB}}(T)\right)\partial K(T).$$

A trajectory is (locally) optimal if any (local) departure from it decreases the value function: $\partial V_{\mathrm{NB}} \leq 0$ for any $\partial B(t)$ and $\partial K(t), t \in (0, T)$, for any $\partial C(t)$ and $\partial G(t), t \in [0, T]$, and for any ∂T, $\partial B(T)$, and $\partial K(T)$. This gives the following necessary conditions for an interior

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maximizer:

$$\begin{cases} \dfrac{\partial H^{\mathrm{NB}}}{\partial C} = 0 \\ \dfrac{\partial H^{\mathrm{NB}}}{\partial G} = 0 \\ \dfrac{\partial H^{\mathrm{NB}}}{\partial I} = 0 \\ \dfrac{\partial H^{\mathrm{NB}}}{\partial B} = -\dot{\lambda}_1^{\mathrm{NB}} \\ \dfrac{\partial H^{\mathrm{NB}}}{\partial K} = -\dot{\lambda}_2^{\mathrm{NB}} \\ H^{\mathrm{NB}}(T) + \dfrac{\partial V_{\mathrm{B}}^{*}(T, B(T), K(T))}{\partial T} = 0 \\ \dfrac{\partial V_{\mathrm{B}}^{*}(T, B(T), K(T))}{\partial B(T)} - \lambda_1^{\mathrm{NB}}(T) = 0 \\ \dfrac{\partial V_{\mathrm{B}}^{*}(T, B(T), K(T))}{\partial K(T)} - \lambda_2^{\mathrm{NB}}(T) = 0. \end{cases}$$

The first five equations are the standard Pontryagin conditions, the last three may be interpreted as the optimality conditions with respect to the switching time T and the free state values $B(T)$ and $K(T)$. Combining these last three equations with equations (A.1)–(A.3) obtained in the first subproblem, we get the matching conditions stated in Theorem 1. ■

APPENDIX B: PROOF OF PROPOSITION 2

Consider equation (35) and define

$$f(G/Y) \equiv \frac{h}{2}\left(\frac{1}{\phi}\right)\frac{1}{\alpha^2}(G/Y)^2 + [h(\hat{r}\phi - 1) + \phi]\frac{1}{\phi}\frac{1}{\alpha}(G/Y) + \frac{1}{\phi}\left[\frac{h}{2}(\hat{r}\phi - 1)^2 + \phi(\hat{r}\phi - 1) - \frac{h}{2}(\rho\phi)^2\right]$$

and $g(G/Y) \equiv \frac{1-\alpha}{\alpha}A^{1/(1-\alpha)}\phi(G/Y)^{1/(1-\alpha)}$. $f(G/Y)$ is a convex parabola, whereas $g(G/Y)$ is a convex increasing function such that $g(0) = 0$ and $\lim_{\frac{G}{Y}\to+\infty} g(G/Y) = +\infty$. The assumption $\alpha \leq 1/2$ guarantees that $\lim_{\frac{G}{Y}\to+\infty} \frac{f(G/Y)}{g(G/Y)} = 0$.

We study the intersections of $f(G/Y)$ with the positive (G/Y)-axis because this helps in determining the number of intersections between $g(G/Y)$ and $f(G/Y)$ (the assumption $\alpha \leq 1/2$ limits the number of possible intersections to two). The discriminant of the second degree equation defined by $f(G/Y) = 0$ is positive; hence, it can have either two negative solutions, one positive and one negative solution or two positive solutions, as reported in Table B.1.

We define $D \equiv [h(\hat{r}\phi - 1) + \phi]\frac{1}{\phi}\frac{1}{\alpha}$ and $E \equiv \frac{1}{\phi}\left[\frac{h}{2}(\hat{r}\phi - 1)^2 + \phi(\hat{r}\phi - 1) - \frac{h}{2}(\rho\phi)^2\right]$. The following cases are possible:

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TABLE B.1. Number of possible solutions

	$\frac{h}{2}\left(\frac{1}{\phi}\right)\frac{1}{\alpha^2}$	$[h(\hat{r}\phi-1)+\phi\tau]\frac{1}{\phi}\frac{1}{\alpha}$	$\frac{1}{\phi\tau}\left[\frac{h}{2}(\hat{r}\phi-1)^2+\phi(\hat{r}\phi-1)-\frac{h}{2}(\rho\phi)^2\right]$	$f\left(\frac{G}{Y}\right)=0$
1	+	+	+	Two negative solutions
2	+	+	−	One positive and one negative
3	+	−	−	One positive and one negative
4	+	−	+	Two positive solutions

(1) $D \geq 0$ if $h \leq \frac{\phi}{1-\hat{r}\phi}$ and $E > 0$ if $h > \frac{2\phi(1-\hat{r}\phi)}{(\hat{r}^2-\rho^2)(\phi)^2+1-2\hat{r}\phi}$. Since $\frac{\phi}{1-\hat{r}\phi} < \frac{2\phi(1-\hat{r}\phi)}{(\hat{r}^2-\rho^2)(\phi)^2+1-2\hat{r}\phi}$, the two conditions are incompatible.

(2) $D \geq 0$ if $h \leq \frac{\phi}{1-\hat{r}\phi}$, $E \leq 0$ if $h \leq \frac{2\phi(1-\hat{r}\phi)}{(\hat{r}^2-\rho^2)(\phi)^2+1-2\hat{r}\phi}$. Hence, to have case 2 we must require $h \leq \frac{\phi}{1-\hat{r}\phi}$.

(3) $D < 0$ if $h > \frac{\phi}{1-\hat{r}\phi}$, $E \leq 0$ if $h \leq \frac{2\phi(1-\hat{r}\phi)}{(\hat{r}^2-\rho^2)(\phi)^2+1-2\hat{r}\phi}$. Hence, $\frac{\phi}{1-\hat{r}\phi} < h \leq \frac{2\phi(1-\hat{r}\phi)}{(\hat{r}^2-\rho^2)(\phi)^2+1-2\hat{r}\phi}$.

(4) $D < 0$ if $h > \frac{\phi}{1-\hat{r}\phi}$, $E > 0$ if $h > \frac{2\phi(1-\hat{r}\phi)}{(\hat{r}^2-\rho^2)(\phi)^2+1-2\hat{r}\phi}$. This implies $h > \frac{2\phi(1-\hat{r}\phi)}{(\hat{r}^2-\rho^2)(\phi)^2+1-2\hat{r}\phi}$.

Cases 2 and 3 can be summarized by requiring $h \leq \frac{2\phi(1-\hat{r}\phi)}{(\hat{r}^2-\rho^2)(\phi)^2+1-2\hat{r}\phi}$.

Hence, the dynamical system admits:

(a) one solution iff $h \leq \frac{2\phi(1-\hat{r}\phi)}{(\hat{r}^2-\rho^2)(\phi)^2+1-2\hat{r}\phi}$;

(b) two solutions iff $h > \frac{2\phi(1-\hat{r}\phi)}{(\hat{r}^2-\rho^2)(\phi)^2+1-2\hat{r}\phi}$.

Both the solutions satisfy the transversality condition $\lim_{t\to+\infty} -q_{ss}\lambda_1^B(T)\exp[-\rho t] = 0$, where q_{ss} is the BGP value of the Tobin's q.

We can show that case (b) implies just a false multiplicity.

A solution of equation (35) is a balanced growth path if the growth rate (24) is positive, that is,

$$\hat{r}\phi - 1 + \frac{1}{\alpha}\left(\frac{G}{Y}\right) \geq \rho\phi.$$

This allows us to write the following inequality:

$$\begin{aligned}
&\frac{h}{2}\left(\frac{1}{\phi}\right)\frac{1}{\alpha^2}\left(\frac{G}{Y}\right)^2 + h(\hat{r}\phi-1)\frac{1}{\phi}\frac{1}{\alpha}\left(\frac{G}{Y}\right) + \frac{1}{\phi}\left[\frac{h}{2}(\hat{r}\phi-1)^2 - \frac{h}{2}(\rho\phi)^2\right] + \rho\phi \\
&\leq \frac{h}{2}\left(\frac{1}{\phi}\right)\frac{1}{\alpha^2}\left(\frac{G}{Y}\right)^2 + [h(\hat{r}\phi-1)+\phi]\frac{1}{\phi}\frac{1}{\alpha}\left(\frac{G}{Y}\right) \\
&+\frac{1}{\phi}\left[\frac{h}{2}(\hat{r}\phi-1)^2 + \phi(\hat{r}\phi-1) - \frac{h}{2}(\rho\phi)^2\right] \\
&= \frac{1-\alpha}{\alpha}A^{1/(1-\alpha)}\phi\left(\frac{G}{Y}\right)^{1/(1-\alpha)}.
\end{aligned}$$

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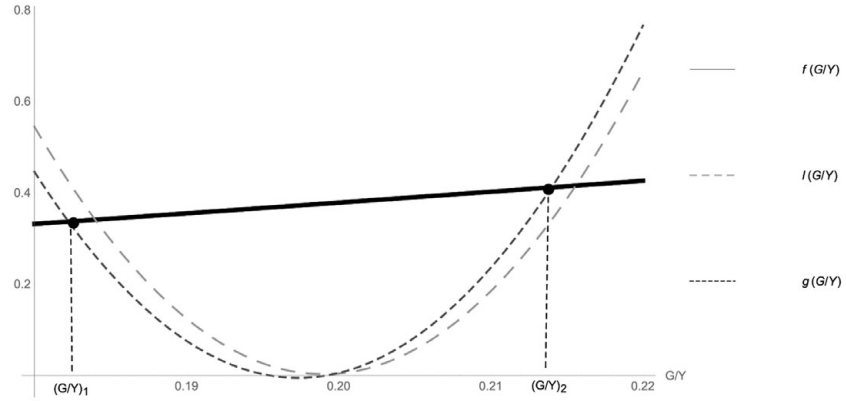


FIGURE B.1. Existence and uniqueness of the BGP.

By defining $l\left(\frac{G}{Y}\right)$ the first function, $f\left(\frac{G}{Y}\right)$ the second one, and $g\left(\frac{G}{Y}\right)$ the third one, each solution of equation (35) must be such that

$$l\left(\frac{G}{Y}\right) \leq f\left(\frac{G}{Y}\right) = g\left(\frac{G}{Y}\right),$$

in order to be a BGP. In what follows, we are going to give a geometrical proof. For $\phi \leq 1$, we can show that

- $l(0) > g(0)$;
- $l\left(\frac{G}{Y}\right)$ and $g\left(\frac{G}{Y}\right)$ cross just once, $\left(\frac{G}{Y}\right)|_{l\left(\frac{G}{Y}\right)=g\left(\frac{G}{Y}\right)} = \alpha\left(\rho\phi - \hat{r}\phi + 1\right) > 0$;
- $\left(\frac{G}{Y}\right)|_{l\left(\frac{G}{Y}\right)=g\left(\frac{G}{Y}\right)} > \left(\frac{G}{Y}\right)_1|_{g\left(\frac{G}{Y}\right)=0} = \frac{h\alpha-\alpha\phi-h\hat{r}\phi-\sqrt{\alpha^2\phi^2+h^2\alpha^2\rho^2\phi^2}}{h}$;
- $\left(\frac{G}{Y}\right)|_{l\left(\frac{G}{Y}\right)=g\left(\frac{G}{Y}\right)} > \left(\frac{G}{Y}\right)_2|_{g\left(\frac{G}{Y}\right)=0} = \frac{h\alpha-\alpha\phi-h\hat{r}\phi+\sqrt{\alpha^2\phi^2+h^2\alpha^2\rho^2\phi^2}}{h}$.

These results exclude the smallest solution of case 2, as it is shown by Figure B.1.

In the figure, $(G/Y)_1$ and $(G/Y)_2$ are the solutions of equation (35). For $(G/Y)_1$, $l\left((G/Y)_1\right) > g\left((G/Y)_1\right) = f\left(G/Y_1\right)$; hence, the growth rate is negative. $(G/Y)_1$ is not a BGP. For $(G/Y)_2$, since $l\left((G/Y)_2\right) < g\left((G/Y)_2\right) = f\left((G/Y)_2\right)$, the second solution of equation (35) constitutes a BGP.[29] It should be remarked that the parameter A, which rules the position of $f(G/Y)$ on the graphic, should not be too low, otherwise we should reject also the larger solution of case 2).[30]

The previous discussion is valid more generally for $\phi < \frac{1}{\hat{r}-\rho}$. Unfortunately, the complexity of the steady-state solutions makes it difficult to find analytical conditions. Nevertheless, it is possible to show by simulations that, as ϕ increases, the steady-state consumption-to-capital ratio becomes negative. Since we cannot accept such negative values, we will consider in the analysis a restricted set of values for ϕ. To give an example, following the baseline parameterization of Table 1, ϕ must be smaller than 13.7. ■

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APPENDIX C: PROOF OF PROPOSITION 3

The dynamical system in the binding regime (25), obtained by writing G/Y as a function of b and by setting $\alpha = 1/2$, is

$$\begin{cases} \dot{c} = \left[\hat{r} - \dfrac{1}{\phi} + 2\left(\dfrac{1}{\phi}\right)^2 A^{-2} b - \rho - \dfrac{q-1}{h}\right] c \\ \dot{q} = \left(\hat{r} - \dfrac{1}{\phi}\right) q + 2\left(\dfrac{1}{\phi}\right)^2 A^{-2} bq - \left(\dfrac{1}{\phi}\right)^2 A^{-2} b^2 - \dfrac{(q-1)^2}{2h} \\ \dot{b} = c + \dfrac{q-1}{h} + \left(\dfrac{1}{\phi}\right)^2 A^{-2} b^2 + \hat{r} b + \dfrac{(q-1)^2}{2h} - \dfrac{1}{\phi} b - \dfrac{q-1}{h} b, \end{cases}$$

and the Jacobian matrix evaluated at the steady state, J_{ss}, is given by

$$\begin{bmatrix} \hat{r} - \frac{1}{\phi} + 2\left(\frac{1}{\phi}\right)^2 A^{-2} b_{ss} - \rho - \frac{q_{ss}-1}{h} & -\frac{1}{h} c_{ss} & 2\left(\frac{1}{\phi}\right)^2 A^{-2} c_{ss} \\ 0 & 2\left(\frac{1}{\phi}\right)^2 A^{-2} b_{ss} + \hat{r} - \frac{1}{\phi} - \frac{q_1 - 1}{h} & 2\left(\frac{1}{\phi}\right)^2 A^{-2} q_{ss} - 2\left(\frac{1}{\phi}\right)^2 A^{-2} b_{ss} \\ 1 & \frac{1}{h} + \frac{q_{ss}-1}{h} - \frac{1}{h} b_{ss} & 2\left(\frac{1}{\phi}\right)^2 A^{-2} b_{ss} + \hat{r} - \frac{1}{\phi} - \frac{q_{ss}-1}{h} \end{bmatrix},$$

where b_{ss}, q_{ss}, and c_{ss} are the steady-state values of debt-to-capital, Tobin's q, and consumption-to-capital, respectively:

$$b_{ss} = \phi A^2 \frac{-\left[h\left(\hat{r}\phi - 1\right) + \phi\right] \frac{1}{\phi} + \sqrt{\frac{1}{2}\{2 + 2h^2\rho^2 + A^2\{h - 2(1 + h\hat{r})\phi + [r(2 + h\hat{r}) - h\rho^2]\phi^2\}}}{\left(\frac{2h}{\phi} - A^2\phi\right)},$$

$$q_{ss} = h\left[\hat{r} - \frac{1}{\phi} + 2\left(\frac{1}{\phi}\right)^2 A^{-2} b_{ss}^2 - \rho\right] + 1,$$

$$c_{ss} = \frac{q_{ss}-1}{h} b_{ss} + \frac{1}{\phi} b_{ss} - \frac{q_{ss}-1}{h} - \left(\frac{1}{\phi}\right)^2 A^{-2} b_{ss}^2 - \hat{r} b_{ss} - \frac{(q_{ss}-1)^2}{2h},$$

$$\det(J_{ss}) = \begin{array}{l} -\frac{1}{\sqrt{2}A^2 h(\phi)^2} \left\{\rho\sqrt{2 + 2h^2\rho^2 + A^2\left\{h - 2\left(1 + \hat{r}h\right)\phi + \left[\hat{r}\left(2 + h\hat{r}\right) - h\rho^2\right](\phi)^2\right\}}\right. \\ \left.\left\{-2h\rho + \sqrt{2}\sqrt{2 + 2h^2\rho^2 + A^2\left\{h - 2\left(1 + \hat{r}h\right)\phi + \left[\hat{r}\left(2 + h\hat{r}\right) - h\rho^2\right](\phi)^2\right\}}\right\}\right\}, \end{array}$$

$$\mathrm{tr}(J_{ss}) = 2\rho.$$

The trace is always positive and the determinant is negative if $A^2 > -\frac{2}{h - 2(1+\hat{r}h)\phi + [\hat{r}(2+h\hat{r}) - h\rho^2]\phi^2}$.

The negative determinant implies that the product of the three eigenvalues is negative, implying that the number of negative eigenvalues[31] is either one or three. However, the positive trace, being the sum of the three eigenvalues, rules out the latter case, thus, implying one negative eigenvalue. Hence, the steady state is saddle-point stable. ■

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APPENDIX D: PROOF OF PROPOSITION 5

In order to prove this proposition, note that an increase in $\hat{r}$ moves the function $g(G/Y)$ of Figure B.1 up in the graph, so that $(G/Y)_2$ decreases while an increase in A moves the $f(G/Y)$ curve of Figure B.1 up in the graph, so that $(G/Y)_2$ increases. Given that (24) is increasing in (G/Y), the growth rate increases as well.

The numerical simulations show that these results apply for all the considered values of ϕ. ■

APPENDIX E: THE INTERREGIME TRANSITIONAL DYNAMICS: METHODOLOGY

In this section, we illustrate the method that has been employed to simulate the interregime transitional dynamics. As we anticipated in Section 4.2, the aim of our numerical exercise is to link two two-points boundary problems [see Trimborn et al. (2008a, 2008b)], in order to find the switching date T.

The simulation proceeds as follows:

Step_1 *Guess a final date T*
Step_2 *Run the binding regime*. Set:

(1) The initial conditions. In our model, they are given by the value of $b(T)$ given by condition (40).
(2) The boundary conditions. The transversality conditions are replaced by the steady-state values of the variables.
(3) Run the simulations and write down the values of $c(T)$ and $q(T)$.[32]

Step_3 *Run the nonbinding regime*. Set:

(1) The initial conditions. In our case, they are represented by the initial value of the state variable, $b(0)$.
(2) The boundary conditions, that is, the values of $c(T)$ and $q(T)$ found in Step_2, as well as the value of $b(T)$ given by condition (40).
(3) Run the simulations. If the variables $c(T)$, $q(T)$, and $b(T)$ of the nonbinding regime converge to the final conditions, as set in (2)-Step_3, the algorithm stops. Otherwise, it comes back to Step_1.

APPENDIX F: GLOSSARY

Table F.1 contains the definition of the parameters of the model.

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TABLE F.1. Definition of the parameters

Parameter	Definition
A	Total factor productivity
α	Income share of public expenditures
h	Investment adjustment costs parameter
$\hat{r}$	Foreign interest rate
ρ	Utility discount rate
ϕ	Financial openness degree

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