J. Appl. Prob. **53,** 688–699 (2016)
*doi:*10.1017/jpr.2016.34
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ERGODICITY OF AGE-DEPENDENT INVENTORY CONTROL SYSTEMS

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Abstract

We consider continuous review inventory systems with general doubly stochastic Poisson demand. In this specific case the demand rate, experienced by the system, varies as a function of the age of the oldest unit in the system. It is known that the stationary distributions of the ages in such models often have a strikingly simple form. In particular, they exhibit a typical feature of a Poisson process: given the age of the oldest unit the remaining ages are uniform. The model we treat here generalizes some known inventory models dealing with partial backorders, perishable items, and emergency replenishment. We derive the limiting joint density of the ages of the units in the system by solving partial differential equations. We also answer the question of the uniqueness of the stationary distributions which was not addressed in the related literature.

Keywords: Inventory; base-stock policy; ergodicity; doubly stochastic Poisson process

2010 Mathematics Subject Classification: Primary 60K10
Secondary 60K25; 90B05

1. Introduction

1.1. Motivation of the model

The model we treat here is inspired by discussions with companies Tetra Pak TS and Volvo Parts Corporation. Both of these companies provide their customers with spare parts when needed. These customers are typically spread worldwide and the replenishment leadtime for normal replenishments may be quite substantial. For example, the production can continue without any waste of production only if an outstanding spare part arrives within a certain time. Thus, the information about the pipeline situation in the supply chain is of great value. Typically, in practice, a possible and popular way of tracking outstanding orders is by using RFID technology (radio-frequency identification); see, e.g. [5] for more information concerning the application of RFID technology.

Here we derive the limiting joint distribution for the ages of the units in the system for an inventory system with a Poisson demand process with age-dependent arrival rates, i.e. a doubly stochastic Poisson process (see, e.g. Kingman [9]). Specifically, the demand rate here is a (rather general) function of the age of the oldest unit in the system. See Kingman [10] for other natural and more complex examples when the interarrival times in the queueing systems are not independent and identically distributed.

Received 19 June 2014; revision received 10 July 2015.
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https://doi.org/10.1017/jpr.2016.34 Published online by Cambridge University Press

1.2. The model and results

We consider a base-stock inventory system with doubly stochastic Poisson demand and a positive leadtime. A base-stock policy implies that a new item is immediately ordered from a supplier, with ample capacity, when a demand occurs. Moreover, initially there are n units in stock and no units outstanding. This means that the inventory position (equal to the stock on hand minus backorders plus outstanding orders) is constant and equal to n. The age of an item is assumed to begin when the item leaves the supplier. It is always the oldest item in the system (not yet assigned for a waiting customer) which is used when satisfying a customer demand, i.e. a first-in–first-out policy.

Fix $n \geq 1$ arbitrarily and let $X_1(t) > \cdots > X_n(t) \geq 0$ denote the ages of the items (not yet assigned to waiting customers) at time $t > 0$ with a given initial state $X_1(0) > \cdots > X_n(0) \geq 0$. The trajectories of the vector $\boldsymbol{X}(t) = (X_1(t), \ldots, X_n(t))$, $t \geq 0$, are right-continuous and piecewise linear on the space

$$\mathcal{A} := \{\boldsymbol{x} = (x_1, \ldots, x_n) \colon 0 \leq x_n < \cdots < x_1\}.$$

Given the initial state $\boldsymbol{X}(0) = \boldsymbol{a} = (a_1, \ldots, a_n) \in \mathcal{A}$, all the components of the vector $\boldsymbol{X}$ are linearly deterministically increasing with speed 1 between random moments T_i, $i \geq 1$, which represent time instances for demand events in the model. We assume that T_i, $i \geq 1$, form a (doubly-stochastic) Poisson process with state-dependent intensity $\Lambda(X_1(t))$ which depends on the largest component only, and we assume that the function $\Lambda(x)$, $x \geq 0$, is a right-continuous positive bounded function with at most a countable number of discontinuities. (The function Λ will be discussed in more detail below.) The deterministic behavior between the moments of demand merely represents the ageing of the items.

For any $a \geq 0$, let $Y_a(t)$, $t \geq 0$, denote a Poisson process with varying intensity $\Lambda(a + t)$. Let $\tau(a)$ be the first point (or 'jump') of this Poisson process, i.e. the density $f_{\tau(a)}$ of $\tau(a)$ is given by

$$f_{\tau(a)}(t) = \Lambda(a + t) \exp\biggl(-\int_a^{a+t} \Lambda(y)\,\mathrm{d}y\biggr), \qquad t > 0. \tag{1.1}$$

Then we define, for any $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathcal{A}$,

$$T_1 = T_1(\boldsymbol{a}) \stackrel{\mathrm{D}}{=} \tau(a_1), \tag{1.2}$$

where '$\stackrel{\mathrm{D}}{=}$' denotes equality in distribution. At time T_1, the state of the process $\boldsymbol{X}$ has a discontinuity: from the state

$$\boldsymbol{X}(T_1-) = (a_1 + T_1-, a_2 + T_1-, \ldots, a_n + T_1-)$$

it jumps down to the state $\boldsymbol{X}(T_1) = (a_2 + T_1, \ldots, a_n + T_1, 0) = (X_2(T_1-), \ldots, X_n(T_1-), 0)$, so that the second largest component becomes the largest, and so on, and the smallest one becomes 0 (see Figure 1).

Recursively, we define, for all $i \geq 1$,

$$T_{i+1} = T_i + \tau(X_1(T_i)) \tag{1.3}$$

(where the increments τ are independent conditionally on the argument), and then at each moment T_i the trajectory of $\boldsymbol{X}$ has a jump from $\boldsymbol{X}(T_i-)$ to $\boldsymbol{X}(T_i)$ defined by

$$X_k(T_i) = X_{k+1}(T_i-), \quad k = 1, \ldots, n-1, \qquad X_n(T_i) = 0.$$

The exact formula for the stationary distribution of the process $\boldsymbol{X}$ is derived in the following theorem.

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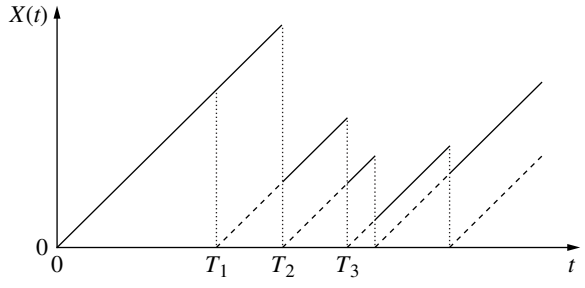


FIGURE 1: A representation of $\boldsymbol{X}(t), t \geq 0$, for $n = 2$ with $X_1(0) = X_2(0) = 0$. The solid trajectories describe the age of the oldest unit, while the dashed trajectories represent the age of the second oldest unit.

Theorem 1.1. *Let $\lambda > 0$ and $L \geq 0$ be arbitrary constants, including the $\lambda = \infty$ case. Assume that the intensity function Λ defined above is on $[0, L)$ a right-continuous positive bounded function with at most a countable number of discontinuities, and such that*

$$\Lambda(x) = \lambda \quad \text{if } x \geq L. \tag{1.4}$$

Assume also that the derivative $\Lambda'(x)$ exists for all points of continuity. Then for any initial state $\boldsymbol{X}(0) = \boldsymbol{a} \in \mathcal{A}$, the distribution of the process $(\boldsymbol{X}(t),\ t \geq 0)$ converges as $t \to \infty$ to the stationary distribution with density

$$f(x_1, \ldots, x_n) = \exp\left(-\int_0^{x_1} \Lambda(y)\,\mathrm{d}y\right) C_n, \qquad (x_1, \ldots, x_n) \in \mathcal{A}, \tag{1.5}$$

where C_n is the normalizing constant defined as

$$C_n = \left(\int_0^{\infty} \frac{x_1^{n-1}}{(n-1)!} \exp\left(-\int_0^{x_1} \Lambda(y)\,\mathrm{d}y\right) \mathrm{d}x_1\right)^{-1}.$$

Observe that the limiting density (1.5) obtained in Theorem 1.1 is unique and positive on the entire space $\mathcal{A}$. This fact also allows us to establish the ergodicity of the process $\boldsymbol{X}$.

Corollary 1.1. *For any initial state $\boldsymbol{X}(0) = \boldsymbol{a} \in \mathcal{A}$, the process $\boldsymbol{X}$ is ergodic.*

One particular characteristic of the limiting distribution in (1.5) follows straight from its form: conditionally on the age of the oldest item (variable x_1) the distribution of the remaining ages is uniform, which is a property of a Poisson process.

Remark 1.1. Observe that when $\lambda = \infty$, the state space $\mathcal{A}$ in Theorem 1.1 can be simply replaced by

$$\mathcal{A}_L := \{\boldsymbol{x} = (x_1, \ldots, x_n) \colon 0 \leq x_n < \cdots < x_1 < L\}. \tag{1.6}$$

This case is used for modelling a finite lifetime of the items in a system.

2. Examples and relations to the previous models

The base-stock model we treat here generalizes some known inventory models dealing with partial backorders, perishable items, and emergency replenishment. Here are some examples.

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Piecewise constant Λ. Assume that μ, λ, L are positive constants, and

$$\Lambda(x) = \begin{cases} \mu & \text{if } x < L, \\ \lambda & \text{if } L \le x < m, \\ \infty & \text{if } x \ge m, \end{cases} \tag{2.1}$$

which satisfies the condition of Theorem 1.1 (see also Remark 1.1). The constant L can be interpreted as the leadtime or any other break-point in time where the demand rate switches for some reason, and the constant $m > 0$ can be interpreted as the lifetime of an item; we can also set $m = \infty$. In the later case, Theorem 1.1 yields the following limiting density:

$$f(x_1, \dots, x_n) = \begin{cases} \exp(-\mu x_1) C_n & \text{if } x_1 < L, \\ \exp(-\lambda x_1) \exp((\lambda - \mu)L) C_n & \text{if } x_1 \ge L, \end{cases} \quad \text{for all } (x_1, \dots, x_n) \in \mathcal{A},$$

where C_n is a normalizing constant defined as

$$C_n = \left(\int_0^L \frac{x_1^{n-1}}{(n-1)!} \exp(-\mu x_1)\, \mathrm{d}x_1 + \int_L^\infty \frac{x_1^{n-1}}{(n-1)!} \exp(-\lambda x_1) \exp((\lambda - \mu)L)\, \mathrm{d}x_1 \right)^{-1}.$$

Note that if $L = 0$ (and $m = \infty$) then the parameter μ does not play any role, and the system is reduced to a trivial case of a renewal process with independent and identically distributed $\mathrm{Exp}(1/\lambda)$ interarrival distributions. If $\lambda = \mu$ then L does not play any role, and the system is equivalent again to the above renewal process. If $\lambda > 0$ and $\mu = 0$ then the model is equivalent to the one studied by Schmidt and Nahmias [18], who dealt with the case of deterministic leadtimes and finite lifetimes of items.

A random threshold L. Let $L > 0$ be fixed arbitrarily, and $\mathcal{L}$ be a positive random variable distributed on $[0, L]$. Consider a model with state-dependent intensity $\Lambda(X_1(t))$, where the demand rate switches between μ and λ independently for the different oldest items. More precisely, we associate with any, say kth, item an independent copy $\mathcal{L}_k$ of a random variable $\mathcal{L}$, and then we set the demand rate for this kth item to be

$$\Lambda_k(X_1(t)) = \begin{cases} \mu & \text{if } X_1(t) < \mathcal{L}_k, \\ \lambda & \text{if } X_1(t) \ge \mathcal{L}_k. \end{cases}$$

Then it is easy to see that the stationary distribution of this model with a *random* rate is described by Theorem 1.1 with the following (deterministic!) rate function:

$$\Lambda(x) = \begin{cases} (1 - \mathbb{P}\{\mathcal{L} \le x\})\mu + \mathbb{P}\{\mathcal{L} \le x\}\lambda & \text{if } x < L, \\ \lambda & \text{if } x \ge L, \end{cases}$$

which satisfies the conditions of Theorem 1.1.

In the proof of Theorem 1.1 we use the method of supplementary variables, already used by Cox [2] and Gnedenko and Kovalenko [6]. Namely, we derive the limiting joint density of the ages of the units in the system by solving partial differential equations (PDEs). The method of PDEs is rather standard in this area. However, the question of the uniqueness of solution, which we address here, seems to be missing in the related applied literature. We prove here that the solution to the PDE related to our Markov process is unique in the class of continuous densities (which is our case). In fact, having the unique stationary distribution helps us to deduce the ergodicity for the considered process.

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Schmidt and Nahmias [18] derived the stationary densities of the ages of the units in the system in the (2.1) model, as commented above. This paper became a source of many following results in related inventory and production control models. The method in [18] follows the classical approach described, e.g. in [6]. However, Schmidt and Nahmias [18] explicitly stated in their paper, 'We assume a steady state distribution exists …', avoiding the question of uniqueness of the stationary distribution (or ergodicity for that matter).

Random acceptable customer waiting time. Perry and Posner [15] introduced to the above (2.1) model a parameter specifying the behavior of each customer. Namely, each customer is willing to wait a random time Y (distributed independently of anything else in the model) before receiving the item. If a customer arrives at time t_0 when there is no stock on hand, i.e. when $X_1(t_0) < L$, then the customer will wait $L - X_1(t_0)$ amount of time with probability $\mathbb{P}\{L - X_1(t_0) \le Y\}$, in which case a new item is ordered at time t_0. Otherwise, with probability $1 - \mathbb{P}\{L - X_1(t_0) \le Y\}$ the customer is lost. Hence, this model is once more a particular case of Theorem 1.1 with the following demand rate:

$$\Lambda(x) = \begin{cases} \mu\mathbb{P}\{L - x \le Y\} & \text{if } x < L, \\ \lambda & \text{if } L \le x \le m, \\ \infty & \text{if } x > m. \end{cases}$$

Note that Theorem 1.1 yields an explicit solution even for general distributions for Y, which Perry and Posner [15] could not treat analytically by their level-crossing method. Perry and Posner [15] do not discuss uniqueness of the stationary solution either, but perhaps one could argue that in their case of a bounded state space (see (1.6)) the question of uniqueness becomes trivial.

Other examples of inventory models with perishable items which are related to our model include, e.g. [8], [13], [14], and [16]. A similar feature of these studies (which is not the case in this paper) is the assumption that the item arrival process and the customer arrival process are independent Poisson processes (or renewal processes in some cases). As mentioned in the introduction, our model also shares similar features with inventory models dealing with emergency replenishments (see, e.g. [12]), and partial backordering (see, e.g. [11] and [21]).

The inventory model considered here is closely related to some queueing models. For example, if $\Lambda(x) = \lambda$, for $x \ge 0$, and the leadtime L is constant, the inventory model can be translated into the M/D/∞ queueing model (where the number of outstanding orders corresponds to the number of customers in service, and the leadtime corresponds to the service time). More related to our model are M/G/∞ models which focus on analyzing ages and residuals of the service time. (For examples of such models, see [3], [4], [17] (the most recent study), and [20, Chapter 3]; for a general overview of queueing models, see [1].)

3. Proofs

Proof of Theorem 1.1. Throughout the proof we assume that $\lambda < \infty$. The case in which $\lambda = \infty$ is treated similarly but on the bounded space $\mathcal{A}_L$; see Remark 1.1.

Density of $\boldsymbol{X}(t)$. The introduced Markov process $\boldsymbol{X}$ is deterministic between the moments of jumps. Therefore, we first derive a conditional distribution of an interarrival interval of the process introduced above. Consider $\tau_i := T_i - T_{i-1}$, $i \ge 1$, where $T_0 = 0$ and $T_1, T_2, \ldots$, are the consecutive moments of the discontinuities of the trajectory of $\boldsymbol{X}$ defined in (1.2) and (1.3). Given $\boldsymbol{X}(0) = \boldsymbol{a} = (a_1, \ldots, a_n)$, we have, by (1.2), $\tau_1 = T_1 \stackrel{\mathrm{D}}{=} \tau(a_1)$. Recall that $\tau(a)$ is the first occurrence of a Poisson process $Y_a(t)$, $t \ge 0$, with varying intensity $\Lambda(a + t)$, which

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satisfies (1.4). Hence, we can construct on the same probability space an exponential random variable $\mathrm{Exp}(\lambda)$ with mean $1/\lambda$ so that $\tau(a)$ almost surely admits uniformly in a the following bound:

$$\tau(a) \le L + \mathrm{Exp}(\lambda). \tag{3.1}$$

Conditioning recursively on the values $\tau_1, \dots, \tau_{n-1}$, we derive

$$\tau_2 \stackrel{\mathrm{D}}{=} \tau(a_2 + \tau_1), \qquad \tau_3 \stackrel{\mathrm{D}}{=} \tau(a_3 + \tau_1 + \tau_2), \dots, \tau_n \stackrel{\mathrm{D}}{=} \tau(a_n + \tau_1 + \cdots + \tau_{n-1}). \tag{3.2}$$

Observe that given the values of the parameters in the brackets, the above random variables are independent (due to the definition of a doubly-stochastic Poisson process). Then at the moment $T_n = \tau_1 + \cdots + \tau_n$, we have

$$X_1(T_n) = \tau_2 + \cdots + \tau_n, \qquad X_2(T_n) = \tau_3 + \cdots + \tau_n, \dots, X_{n-1}(T_n) = \tau_n, \qquad X_n(T_n) = 0, \tag{3.3}$$

and from the state $\boldsymbol{X}(T_n)$ the process evolves in the same manner. In words, this represents the fact that any new item in the system goes through n stages being the youngest, the second youngest, and so on, until it is the eldest, and its age is accumulated by summing up n interarrival intervals. Then the bound (3.1) yields, for any $k \ge n$, existence of a random variable $\Gamma(n, \lambda)$ with gamma distribution such that almost surely

$$X_1(T_k) \le nL + \Gamma(n, \lambda), \tag{3.4}$$

while for $k < n$, we have $X_1(T_k) \le a_1 + kL + \Gamma(n, \lambda)$.

Consider now the sequence $\{X_1(T_k)\}_{k\ge1}$, which is formed by the consecutive maximas of the process. The bound (3.4) immediately implies that this sequence is tight, i.e. for any ε there is a compact set $K \subset \boldsymbol{R}$ such that

$$\sup_{k\ge1} \mathbb{P}\{X_1(T_k) \notin K\} \le \varepsilon$$

for any fixed initial state $\boldsymbol{a}$. This yields, in turn, that $\{\boldsymbol{X}(T_k)\}_{k\ge1}$ is tight as well. Then by Prokhorov's theorem (see, e.g. [19]), the sequence $\{\boldsymbol{X}(T_k)\}_{k\ge1}$ is relatively compact, and, hence, it contains a weakly convergent subsequence. The limiting distribution of this subsequence gives us a stationary distribution for $\{\boldsymbol{X}(T_k)\}_{k\ge1}$. This distribution defines, in turn, a stationary distribution for the process $\boldsymbol{X}$, since it is deterministic between the moments T_k.

On the other hand, below, we prove that the stationary distribution of process $\boldsymbol{X}$ is unique. Therefore, the sequence $\{\boldsymbol{X}(T_k)\}_{k\ge1}$ converges from any initial state $\boldsymbol{a}$ to the unique stationary distribution. This immediately implies that due to the deterministic evolution between the moments T_k, the process $\boldsymbol{X}$ converges weakly as well when $t \to \infty$. It only remains to prove the uniqueness of the stationary distribution of $\boldsymbol{X}$.

Note that if a random variable ξ has an absolutely continuous distribution with bounded density $f_\xi(x)$, which has an almost everywhere (with respect to the Lebesgue measure) derivative (just as $f_{\tau(x)}$ does for a fixed value $x > 0$ by the assumptions of Theorem 1.1), then it is straightforward to check that $\tau(\xi)$ has a continuous density $f_{\tau(\xi)}$. Indeed, we have, by (1.1),

$$\begin{aligned} f_{\tau(\xi)}(t) &= \int_0^\infty f_\xi(x) f_{\tau(x)}(t)\,\mathrm{d}x \\ &= \int_0^\infty f_\xi(x) \Lambda(x+t) \exp\left(-\int_x^{x+t} \Lambda(y)\,\mathrm{d}y\right) \mathrm{d}x \end{aligned}$$

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$$= -\int_0^\infty f_\xi(x)\,\mathrm{d}\exp\Big(-\int_x^{x+t}\Lambda(y)\,\mathrm{d}y\Big) + \int_0^\infty f_\xi(x)\Lambda(x)\exp\Big(-\int_x^{x+t}\Lambda(y)\,\mathrm{d}y\Big)\,\mathrm{d}x,$$

where $\int_x^{x+t}\Lambda(y)\,\mathrm{d}y$ is a continuous function of t. This confirms the continuity of $f_{\tau(\xi)}(t)$ for all t. Therefore, $\boldsymbol{X}(T_n)$, defined by (3.2) and (3.3), has an absolutely continuous distribution, which yields, in turn, that $\boldsymbol{X}(t)$, for all $t > T_n$, has an absolutely continuous distribution also, with positive density in $\mathcal{A}$. Observe that the distribution of $T_n = \tau_1 + \cdots + \tau_n$ is defined by (3.2). In particular, it is stochastically bounded from above by $nL + \Gamma(n, \lambda)$, implying that $T_n < \infty$ almost surely. Since we are interested in the weak limit as $t \to \infty$, we may consider $t \geq T_n$. (Alternatively, we can consider a process $\boldsymbol{X}(t + T_n)$, $t \geq 0$, which evolves from the state $\boldsymbol{X}(T_n)$ defined by (3.2) and (3.3).)

Consider now (for large enough t) the density of the distribution of $\boldsymbol{X}(t)$, which we denote by

$$f(t; x_1, \ldots, x_n) = \frac{\partial^n}{\partial x_1 \cdots \partial x_n}\mathbb{P}\{X_1(t) \leq x_1, \ldots, X_n(t) \leq x_n\}.$$

This function is continuous in all variables for all $t > 0$ (given (3.2) and (3.3) the argument is similar to the one above on the continuity of $f_{\tau(\xi)}$). Furthermore, it has almost everywhere with respect to the Lebesgue measure all partial derivatives. A standard method to derive a PDE for the density $f(t; x_1, \ldots, x_n)$ is as follows (see, e.g. [6]). Observe that

$$\begin{aligned}\boldsymbol{X}(t+h) = {} & (\boldsymbol{X}(t) + (h, \ldots, h))\,\mathbf{1}\{\text{no jump in } [t, t+h]\} \\ & + \boldsymbol{X}(t+h)\,\mathbf{1}\{\text{there is a jump in } [t, t+h]\},\end{aligned} \tag{3.5}$$

where $\mathbf{1}\{B\}$ denotes the indicator of event B. Then, for all $\boldsymbol{x} \in \mathcal{A}$ with $x_n > h > 0$, we have

$$f(t+h; x_1, \ldots, x_n) = f(t; x_1 - h, \ldots, x_n - h)(1 - \Lambda(x_1)h + o(h)),$$

where we use the fact that given that the process $\boldsymbol{X}(t+h)$ is at state $\boldsymbol{x} \in \mathcal{A}$ with $x_n > h > 0$, there could not be a jump within $[t, t+h]$. Writing this as

$$\begin{aligned}& f(t+h; x_1, \ldots, x_n) - f(t; x_1, \ldots, x_n) \\ & \quad = -(f(t; x_1, x_2, \ldots, x_n) - f(t; x_1 - h, x_2, \ldots, x_n)) \\ & \qquad - \cdots - (f(t; x_1 - h, \ldots, x_{n-1} - h, x_n) - f(t; x_1 - h, \ldots, x_n - h)) \\ & \qquad - (\Lambda(x_1)h + o(h))f(t; x_1 - h, \ldots, x_n - h),\end{aligned}$$

yields

$$\frac{\partial}{\partial t}f(t; x_1, \ldots, x_n) = -\sum_{i=1}^n \frac{\partial}{\partial x_i}f(t; x_1, \ldots, x_n) - \Lambda(x_1)f(t; x_1, \ldots, x_n) \tag{3.6}$$

for all $0 < x_n < \cdots < x_1$ and $t > T_n$. (One may view (3.6) in the general context of transport equations and use, for example, the methods of [3] and [4].)

Stationary density. It follows from (3.6) that if there is a stationary distribution for the process $\boldsymbol{X}$, then it has a density, say $f(x_1, \ldots, x_n)$, which is continuous (it even has partial derivatives in all variables) and satisfies the following equation:

$$\sum_{i=1}^n \frac{\partial}{\partial x_i}f(x_1, \ldots, x_n) = -\Lambda(x_1)f(x_1, \ldots, x_n) \quad \text{for all } 0 < x_n < \cdots < x_1. \tag{3.7}$$

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Observe now that the function $f(\boldsymbol{x})$, as a continuous solution to (3.7), can be written as

$$f(x_1, \dots, x_n) = \exp\left(-\int_0^{x_1} \Lambda(y)\,\mathrm{d}y\right) g(x_1, \dots, x_n) \tag{3.8}$$

for some positive function g such that

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} g(x_1, \dots, x_n) = 0 \quad \text{for all } 0 < x_n < \cdots < x_1.$$

This yields, for any $0 < y \le x_n < \cdots < x_1$,

$$g(x_1, \dots, x_n) = g(x_1 - y, \dots, x_{n-1} - y, x_n - y). \tag{3.9}$$

Boundary conditions. We derive the boundary conditions that define $f(x_1, \dots, x_{n-1}, 0)$. Observe that given $\boldsymbol{X}(t) = \boldsymbol{x}$, where $\boldsymbol{x} \in \mathcal{A}$ is any state with $x_n = 0$, the time elapsed since the last (before time t) jump of the trajectory is equal to x_{n-1}, when by the definition $X_1(t - x_{n-1}-) > x_1 - x_{n-1}$; see Figure 2. We assume here that $t > x_{n-1}$.

Note that given $X_1(t - x_{n-1}-) = x > x_1 - x_{n-1}$, we have $\tau(x) = x_{n-1}$. Again using (3.5) and assuming that the process $\boldsymbol{X}$ is stationary, we first obtain

$$\begin{aligned}
&\mathbb{P}\{X_1(t) \in [x_1, x_1 + h), \dots, X_{n-1}(t) \in [x_{n-1}, x_{n-1} + h),\ X_n(t) \in [0, h)\} \\
&\quad = \mathbb{P}\{X_1(t - x_{n-1}) \ge x_1 - x_{n-1}, X_2(t - x_{n-1}) \\
&\qquad \in [x_1 - x_{n-1}, x_1 - x_{n-1} + h), \dots, X_n(t - x_{n-1}) \\
&\qquad \in [0, h),\ \tau(X_1(t - x_{n-1})) \in [x_{n-1}, x_{n-1} + h)\}(1 + o(h)).
\end{aligned}$$

Then dividing both sides by h^n and passing to the limit as $h \to 0$, we obtain

$$f(x_1, \dots, x_{n-1}, 0) = \int_{x_1 - x_{n-1}}^{\infty} f_{\tau(x)}(x_{n-1}) f(x, x_1 - x_{n-1}, \dots, x_{n-2} - x_{n-1}, 0)\,\mathrm{d}x. \tag{3.10}$$

Note that when $n = 2$, (3.10) has a particular closed form; namely,

$$f(x_1, 0) = \int_0^{\infty} f_{\tau(x)}(x_1) f(x, 0)\,\mathrm{d}x.$$

The uniqueness of the stationary distribution for $\boldsymbol{X}$ is established in the following proposition.

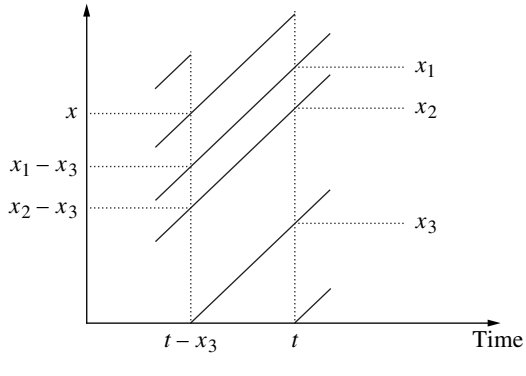


FIGURE 2: Boundary conditions for the $n = 4$ case.

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Proposition 3.1. *Let f be a continuous function on $\mathcal{A}$, such that (3.8) and (3.9) hold. Assume also that f satisfies the boundary condition (3.10). Then, we have*

$$g(x_1, \dots, x_{n-1}, 0) = C_n$$

for some constant $C_n > 0$ for all $x_1 > \cdots > x_{n-1} > 0$.

Proof. Assume that $n > 2$. (The case when $n = 2$ is simpler, and it can be treated straightforwardly. We can also allow $n = 2$ in the argument below with obvious simplifications.)

Using (1.1), (3.8), and (3.10), we derive

$$\begin{aligned}
&\exp\Big(-\int_0^{x_1} \Lambda(y)\,\mathrm{d}y\Big) g(x_1, \dots, x_{n-1}, 0) \\
&\quad = \int_{x_1 - x_{n-1}}^{\infty} \Lambda(x + x_{n-1}) \exp\Big(-\int_x^{x+x_{n-1}} \Lambda(y)\,\mathrm{d}y\Big) \\
&\qquad\qquad \times \exp\Big(-\int_0^{x} \Lambda(y)\,\mathrm{d}y\Big) g(x, x_1 - x_{n-1}, \dots, x_{n-2} - x_{n-1}, 0)\,\mathrm{d}x \\
&\quad = \int_{x_1}^{\infty} \Lambda(x) \exp\Big(-\int_0^{x} \Lambda(y)\,\mathrm{d}y\Big) g(x - x_{n-1}, x_1 - x_{n-1}, \dots, x_{n-2} - x_{n-1}, 0)\,\mathrm{d}x.
\end{aligned}$$

This together with (3.9) gives us

$$g(x_1, \dots, x_{n-1}, 0) = \int_{x_1}^{\infty} \Lambda(x) \exp\Big(-\int_{x_1}^{x} \Lambda(y)\,\mathrm{d}y\Big) g(x, x_1, \dots, x_{n-2}, x_{n-1})\,\mathrm{d}x. \quad (3.11)$$

Consider now the case when $x_1 > L$, which, by assumption, yields $\Lambda(x) = \lambda$ for all $x \geq x_1$, and, therefore, (3.11) can be written as

$$g(x_1, \dots, x_{n-1}, 0) = \int_{x_1}^{\infty} \lambda \exp(-\lambda(x - x_1)) g(x, x_1, \dots, x_{n-2}, x_{n-1})\,\mathrm{d}x. \quad (3.12)$$

By (3.9) this yields, for all $0 < x_n < x_{n-1}$ and $x_1 > L$,

$$\begin{aligned}
g(x_1, \dots, x_{n-1}, 0) &= \int_{x_1}^{\infty} \lambda \exp(-\lambda(x - x_1)) g(x - x_n, x_1 - x_n, \dots, x_{n-1} - x_n)\,\mathrm{d}x \\
&= \int_{x_1 - x_n}^{\infty} \lambda \exp(-\lambda(x - (x_1 - x_n))) g(x, x_1 - x_n, \dots, x_{n-1} - x_n)\,\mathrm{d}x \\
&= g(x_1 - x_n, x_2 - x_n, \dots, x_{n-1} - x_n, 0),
\end{aligned}$$

where the last equality is again due to (3.12). Again taking into account property (3.9), we derive, for all $0 < x_n < x_{n-1}$ and $x_1 > L$,

$$g(x_1, \dots, x_{n-1}, 0) = g(x_1, \dots, x_{n-1}, x_n), \quad (3.13)$$

which yields

$$\frac{\partial}{\partial x_n} g(x_1, \dots, x_n) = 0.$$

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This combined with (3.11) and (3.13) gives us

$$\begin{aligned}
&\frac{\partial}{\partial x_{n-1}} g(x_1, \ldots, x_{n-1}, x_n) \\
&\quad = \frac{\partial}{\partial x_{n-1}} g(x_1, \ldots, x_{n-1}, 0) \\
&\quad = \frac{\partial}{\partial x_{n-1}} \int_{x_1}^{\infty} \Lambda(x) \exp\left(- \int_{x_1}^{x} \Lambda(y)\, \mathrm{d}y \right) g(x, x_1, \ldots, x_{n-2}, x_{n-1})\, \mathrm{d}x \\
&\quad = \int_{x_1}^{\infty} \Lambda(x) \exp\left(- \int_{x_1}^{x} \Lambda(y)\, \mathrm{d}y \right) \left(\frac{\partial}{\partial y} g(x, x_1, \ldots, x_{n-2}, y) \right) \mathrm{d}x \\
&\quad = 0,
\end{aligned}$$

where we could take the derivative under the integral due to the absolute convergence of the integral and the assumptions on function g. Consecutively, we derive

$$\frac{\partial}{\partial x_k} g(x_1, \ldots, x_{n-1}, 0) = 0 \quad \text{for all } k = n-2, \ldots, 2.$$

We arrive at the conclusion that, for some positive constant C_n,

$$g(x_1, \ldots, x_{n-1}, 0) = C_n \quad \text{if } x_1 > L. \tag{3.14}$$

When $x_1 \le L$, by (3.11) and (3.14), we have

$$\begin{aligned}
&g(x_1, \ldots, x_{n-1}, 0) \\
&\quad = \int_{x_1}^{\infty} \Lambda(x) \exp\left(- \int_{x_1}^{x} \Lambda(y)\, \mathrm{d}y \right) g(x, x_1, \ldots, x_{n-2}, x_{n-1})\, \mathrm{d}x \\
&\quad = \int_{x_1}^{L+x_{n-1}} \Lambda(x) \exp\left(- \int_{x_1}^{x} \Lambda(y)\, \mathrm{d}y \right) g(x - x_{n-1}, x_1 - x_{n-1}, \ldots, x_{n-2} - x_{n-1}, 0)\, \mathrm{d}x \\
&\qquad + \int_{L+x_{n-1}}^{\infty} \Lambda(x) \exp\left(- \int_{x_1}^{x} \Lambda(y)\, \mathrm{d}y \right) C_n\, \mathrm{d}x.
\end{aligned} \tag{3.15}$$

We solve this equation with help from the following result.

Proposition 3.2. *Let $C > 0$ be an arbitrarily fixed constant. On the class of continuous nonnegative functions on $\{(x_1, \ldots, x_{n-1}) \in \boldsymbol{R}^{n-1} : 0 < x_{n-1} < \cdots < x_1\}$ with the boundary condition*

$$g(x_1, \ldots, x_{n-1}, 0) = C, \qquad x_1 \ge L, \qquad 0 < x_{n-1} < \cdots < x_1, \tag{3.16}$$

define an operator $\mathcal{T}$ such that

$$\begin{aligned}
&\mathcal{T}[g](x_1, \ldots, x_{n-1}, 0) \\
&\quad = \int_{x_1}^{L+x_{n-1}} \Lambda(x) \exp\left(- \int_{x_1}^{x} \Lambda(y)\, \mathrm{d}y \right) g(x - x_{n-1}, x_1 - x_{n-1}, \ldots, x_{n-2} - x_{n-1}, 0)\, \mathrm{d}x \\
&\qquad + \int_{L+x_{n-1}}^{\infty} \Lambda(x) \exp\left(- \int_{x_1}^{x} \Lambda(y)\, \mathrm{d}y \right) C\, \mathrm{d}x.
\end{aligned}$$

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Then the operator equation

$$g(x_1, \dots, x_{n-1}, 0) = \mathcal{T}[g](x_1, \dots, x_{n-1}, 0) \tag{3.17}$$

has a unique solution, which is

$$g(x_1, \dots, x_{n-1}, 0) = C, \qquad 0 < x_{n-1} < \cdots < x_1. \tag{3.18}$$

Proof. The proof is based on the fact that the operator $\mathcal{T}$ is contracting on the described class of functions.

Assume that g and $\widetilde{g}$ are functions which satisfy both (3.17) and (3.16). Then

$$\begin{aligned}
\|g - \widetilde{g}\| :=& \max_{0 \le x_{n-1} < \cdots < x_1 \le L} |g(x_1, \dots, x_{n-1}, 0) - \widetilde{g}(x_1, \dots, x_{n-1}, 0)| \\
=& \|\mathcal{T}[g] - \mathcal{T}[\widetilde{g}]\| \\
\le& \|g - \widetilde{g}\| \max_{0 \le x_{n-1} < x_1 \le L} \left(\int_{x_1}^{L + x_{n-1}} \Lambda(x) \exp\left(- \int_{x_1}^{x} \Lambda(y)\, \mathrm{d}y \right) \mathrm{d}x \right) \\
=& \|g - \widetilde{g}\| \max_{0 \le x_{n-1} < x_1 \le L} \left(1 - \exp\left(- \int_{x_1}^{L + x_{n-1}} \Lambda(y)\, \mathrm{d}y \right) \right) \\
\le& \|g - \widetilde{g}\| \left(1 - \exp\left(- \int_{0}^{2L} \Lambda(y)\, \mathrm{d}y \right) \right),
\end{aligned}$$

where $1 - \exp(-\int_0^{2L} \Lambda(y)\, \mathrm{d}y) < 1$. This confirms that $\|g - \widetilde{g}\| = 0$, and, thus, the solution is unique. Statement (3.18) follows by continuity. □

The last proposition also enables us to derive, from (3.15), that $g(x_1, \dots, x_{n-1}, 0) = C_n$ for all $L > x_1 > \cdots > x_n$. This completes the proof of Proposition 3.1. □

The above proofs of Propositions 3.1 and 3.2 immediately yield statement of Theorem 1.1. □

Proof of Corollary 1.1. We shall show that any invariant function is constant, i.e. if a bounded Borel function $g(\boldsymbol{x})$ satisfies

$$g(\boldsymbol{x}) = \mathbb{E} g(\boldsymbol{X}(t)), \qquad \boldsymbol{x} \in \mathcal{A},\ t \ge 0, \tag{3.19}$$

where $\boldsymbol{X}(0) = \boldsymbol{x}$, then $g(\boldsymbol{x})$ is constant on $\mathcal{A}$. Then the statement of the corollary follows by the criteria for weak ergodicity; see [7, Theorem 20.11].

Assume that (3.19) holds. Then

$$g(\boldsymbol{x}) = \lim_{t \to \infty} \mathbb{E} g(\boldsymbol{X}(t)).$$

Theorem 1.1 and the assumptions on g allow us to pass to the limit under the expectation and to derive

$$g(\boldsymbol{x}) = \int_{\mathcal{A}} f(\boldsymbol{y}) g(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y},$$

which readily proves that $g(\boldsymbol{x})$ is a constant. □

Acknowledgements

The authors thank the editor and the anonymous referee for useful remarks and suggestions, and they thank Anton Wakolbinger for a helpful discussion.

https://doi.org/10.1017/jpr.2016.34 Published online by Cambridge University Press

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