BORN'S ELECTRODYNAMICS IN COMPLEX FORM

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Introduction

Born's electrodynamics, together with its quantization, has been developed in several papers by Born and Infeld†. It is a general concept, containing many different systems of electrodynamics, of which Maxwell's theory is one particular case. Each system is characterized by a Lagrangian (or Hamiltonian) function of two invariants F and G (or P and Q). These invariants are built up by means of two antisymmetrical tensors, which are the "field quantities".

It is the purely classical aspect of the theory with which the present paper is concerned. Thus we do not deal with the field quantities as functions of the world coordinates, or the field equations or the variation principle. The separation of these two parts of Born's electrodynamics is not only possible but seems also to be convenient, since the formalism developed here turns out to suit the former part well, but at the present stage cannot be made to fit the latter. Moreover, the connection between Lagrangian, Hamiltonian and field quantities holds unaltered in the quantum theory, so that our restriction to the former part of electrodynamics leaves the door open to an eventual application of the present considerations to quantum electrodynamics.

In the first three sections we try to develop all relations between the field quantities and invariants in as general and as symmetrical a form as possible. Some new quantities and relations are obtained in this way. In the later part we introduce a complex formalism, which enables us to state and to prove all these relations in a much simpler and shorter form. The result is that, with one restriction which requires only unessential modifications of the existing field theories, each field theory can be described by one analytic function, the "complex Lagrangian" (or "complex Hamiltonian"), of one complex variable. Moreover, this complex function has to satisfy certain general conditions. Thus the study of Born's electrodynamics is reduced to a large extent to the study of a certain class of analytic functions.

A complex formalism has previously been developed by Schrödinger‡ and

† M. Born, *Proc. Roy. Soc.* A, 143 (1934), 410, quoted as I; M. Born and L. Infeld, *Proc. Roy. Soc.* A, 144 (1934), 425, quoted as II; M. Born and L. Infeld, *Proc. Roy. Soc.* A, 147 (1934), 522, quoted as III; M. Born and L. Infeld, *Proc. Roy. Soc.* A, 150 (1935), 141, quoted as IV; L. Infeld, *Proc. Camb. Phil. Soc.* 32 (1936), 127, quoted as V.

‡ E. Schrödinger, *Proc. Roy. Soc.* A, 150 (1935), 465.

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has been applied to the study of the "new field theory", which is a particular system of Born's electrodynamics. It is different from ours in that Schrödinger combines corresponding components of different tensors to form complex quantities, whereas we combine different components of the same tensor.

1. ALGEBRAIC TENSOR RELATIONS

We recall that Born's electrodynamics deals with two antisymmetrical tensors f_{kl} and p^{kl}, or, expressed in dual form, f^{*kl} and p^{*}_{kl}, which describe the electromagnetic field, and with a symmetrical tensor g_{kl}, which describes the metric. By means of the g_{kl} the pulling up and down of suffixes is defined, so that we can also form the tensors f^{kl}, p_{kl}, f^{*}_{kl} and p^{*kl}. We recall furthermore the space-vector notation, defined by

$$\left.\begin{aligned} (f_{23}, f_{31}, f_{12}) &= \mathbf{B}, & (f_{14}, f_{24}, f_{34}) &= \mathbf{E}, \\ (p_{23}, p_{31}, p_{12}) &= \mathbf{H}, & (p_{14}, p_{24}, p_{34}) &= \mathbf{D}. \end{aligned}\right\} \tag{1·1}$$

There exist the following six invariants:

$$\tfrac{1}{4} f_{kl} f^{kl} = -\tfrac{1}{4} f^{*kl} f^{*}_{kl} = F, \quad \tfrac{1}{4} f_{kl} f^{*kl} = \tfrac{1}{4} f^{*}_{kl} f^{kl} = G, \tag{1·2}$$

$$\tfrac{1}{4} p^{*}_{kl} p^{*}_{kl} = -\tfrac{1}{4} p^{kl} p_{kl} = P, \quad \tfrac{1}{4} p^{*}_{kl} p^{kl} = \tfrac{1}{4} p_{kl} p^{*kl} = Q, \tag{1·3}$$

$$\tfrac{1}{4} f_{kl} p^{kl} = -\tfrac{1}{4} f^{*kl} p^{*}_{kl} = R, \quad \tfrac{1}{4} f_{kl} p^{*kl} = \tfrac{1}{4} f^{*kl} p_{kl} = S. \tag{1·4}$$

The invariants F, G; P, Q have been defined previously†.

In space-vector form and in a Lorentz frame of reference these invariants read

$$F = \tfrac{1}{2}(\mathbf{B}^2 - \mathbf{E}^2), \qquad G = (\mathbf{B}.\mathbf{E}), \tag{1·2a}$$

$$P = \tfrac{1}{2}(\mathbf{D}^2 - \mathbf{H}^2), \qquad Q = (\mathbf{D}.\mathbf{H}), \tag{1·3a}$$

$$R = \tfrac{1}{2}((\mathbf{B}.\mathbf{H}) - (\mathbf{D}.\mathbf{E})), \qquad S = \tfrac{1}{2}((\mathbf{B}.\mathbf{D}) + (\mathbf{E}.\mathbf{H})). \tag{1·4a}$$

Between these tensors and invariants the following relations hold:

$$\tfrac{1}{2}(f_{kr} f^{lr} - f^{*}_{kr} f^{*lr}) = F\delta^{l}_{k}, \quad f_{kr} f^{*lr} = f^{*}_{kr} f^{lr} = G\delta^{l}_{k}, \tag{1·5}$$

$$\tfrac{1}{2}(p^{*}_{kr} p^{*lr} - p_{kr} p^{lr}) = P\delta^{l}_{k}, \quad p^{*}_{kr} p^{lr} = p_{kr} p^{*lr} = Q\delta^{l}_{k}, \tag{1·6}$$

$$\tfrac{1}{2}(f_{kr} p^{lr} - p^{*}_{kr} f^{*lr}) = R\delta^{l}_{k}, \quad \tfrac{1}{2}(f_{kr} p^{*lr} + p_{kr} f^{*lr}) = S\delta^{l}_{k}. \tag{1·7}$$

They can all be proved by straightforward calculation in a Lorentz coordinate system‡. Their general validity then follows from their tensor character.

From (1·7) we deduce the equations

$$\tfrac{1}{2}(f_{kr} p^{lr} + p^{*}_{kr} f^{*lr}) = f_{kr} p^{lr} - R\delta^{l}_{k} = p^{*}_{kr} f^{*lr} + R\delta^{l}_{k}, \tag{1·8}$$

$$\tfrac{1}{2}(f_{kr} p^{*lr} - p_{kr} f^{*lr}) = f_{kr} p^{*lr} - S\delta^{l}_{k} = -p_{kr} f^{*lr} + S\delta^{l}_{k}. \tag{1·9}$$

† II, pp. 431, 435. Our F and P differ from the previous ones by a factor $\frac{1}{2}$. This turns out to be more convenient, because it avoids unnecessary numerical factors.

‡ The second equation of (1·5) and the first of (1·7) are given in II as (2·25) and (4·17).

Note added in proof. Equations (1·2)–(1·4a) have recently been given by B. S. Madhava Rao, cf. *Proc. Ind. Acad. Science*, IV, 5 (1936), pp. 578–9, and applied to the "new field theory".

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The non-diagonal terms of (1·8) represent, apart from the sign, the energy-momentum tensor T_k^l. Now it will be observed that there is a symmetry between (1·8) and (1·9), and the non-diagonal terms of (1·9) thus represent a kind of counterpart of the energy-momentum tensor.

We write down both these tensors in space-vector notation, using a Lorentz coordinate system and leaving out the diagonal terms. We get

$$\begin{bmatrix} & B_yH_x+E_xD_y & B_zH_x+E_xD_z & B_yD_z-B_zD_y \\ B_xH_y+E_yD_x & & B_zH_y+E_yD_z & B_zD_x-B_xD_z \\ B_xH_z+E_zD_x & B_yH_z+E_zD_y & & B_xD_y-B_yD_x \\ E_yH_z-E_zH_y & E_zH_x-E_xH_z & E_xH_y-E_yH_x & \end{bmatrix}, \tag{1·8a}$$

$$\begin{bmatrix} & B_yD_x-E_xH_y & B_zD_x-E_xH_z & -B_yH_z+B_zH_y \\ B_xD_y-E_yH_x & & B_zD_y-E_yH_z & -B_zH_x+B_xH_z \\ B_xD_z-E_zH_x & B_yD_z-E_zH_y & & -B_xH_y+B_yH_x \\ E_yD_z-E_zD_y & E_zD_x-E_xD_z & E_xD_y-E_yD_x & \end{bmatrix}. \tag{1·9a}$$

The tensor (1·9a) has so far not occurred in Born's electrodynamics. Its physical significance will be seen at the end of § 7.

2. The symmetry condition of the energy-momentum tensor

It has been shown† that, since the tensors f_{kl} and p^{kl} are connected with each other by means of an invariant Lagrangian or Hamiltonian function, the contravariant energy-momentum tensor T^{kl} is the derivative of this function with respect to the metric tensor g_{kl}, namely

$$T^{kl}=\frac{\partial L}{\partial g_{kl}}=\frac{\partial H}{\partial g_{kl}}. \tag{2·1}$$

Consequently the tensor T^{kl} is symmetrical, and from (1·8) we have the relations

$$g^{kj}f_{jr}p^{rl}=g^{lj}f_{jr}p^{rk}, \quad g^{kj}p^*_{jr}f^{*rl}=g^{lj}p^*_{jr}f^{*rk}, \tag{2·2}$$

the two sets being equivalent to one another. In space-vector form and a Lorentz frame these relations read‡

$$(\mathbf{B}\times\mathbf{H})=(\mathbf{E}\times\mathbf{D}), \qquad (\mathbf{B}\times\mathbf{D})=(\mathbf{H}\times\mathbf{E}). \tag{2·3}$$

Now it will be seen, by translating the mixed tensors (1·8a) and (1·9a) into contravariant form, that the symmetry conditions for the tensor (1·9a) are precisely the same as those for (1·8a). It therefore follows from (1·9) that these conditions may also be written

$$g^{kj}f_{jr}p^{*rl}=g^{lj}f_{jr}p^{*rk}, \quad g^{kj}p_{jr}f^{*rl}=g^{lj}p_{jr}f^{*rk}. \tag{2·4}$$

† Cf. H. Weyl, *Raum, Zeit, Materie*, 5th ed. (Berlin, 1923), pp. 232–5.

‡ Compare IV, p. 159.

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Again the two sets of (2·4) are equivalent to one another, and they are also equivalent to (2·2).

The sets of equations (2·2) and (2·4) constitute four different but equivalent expressions of the same symmetry condition. There are four more expressions of this condition which we shall need later. They read

$$p_{kr}f^{rl}=f_{kr}p^{rl}, \quad f^{*}_{kr}p^{*rl}=p^{*}_{kr}f^{*rl}, \tag{2·5}$$

$$p^{*}_{kr}f^{rl}=f_{kr}p^{*rl}, \quad f^{*}_{kr}p^{rl}=p_{kr}f^{*rl}. \tag{2·6}$$

For the proof of the equivalence it is most convenient to use matrix notation. If we denote the tensors f_{kl}, f^{*kl}, p^{kl}, p^{*}_{kl}, g_{kl} by the matrices $\mathbf{f}$, $\mathbf{f^*}$, $\mathbf{p}$, $\mathbf{p^*}$, $\mathbf{g}$ respectively, equations (2·2) read

$$\mathbf{g^{-1}fp=pfg^{-1}}, \quad \mathbf{g^{-1}p^*f^*=f^*p^*g^{-1}}, \tag{2·2a}$$

and equations (2·5) read

$$\mathbf{(gpg)(g^{-1}fg^{-1})=fp}, \quad \mathbf{(gf^*g)(g^{-1}p^*g^{-1})=p^*f^*}. \tag{2·5a}$$

The equivalence of (2·2a) and (2·5a) becomes evident if we remove the brackets in (2·5a). The proof for (2·4) and (2·6) is quite similar.

With the help of (2·5) and (2·6) the equations (1·7) and the left-hand sides of (1·8) and (1·9) may be written in the form

$$\left.\begin{aligned}\tfrac{1}{2}(p_{kr}f^{lr}-p^{*}_{kr}f^{*lr})&=\tfrac{1}{2}(f_{kr}p^{lr}-f^{*}_{kr}p^{*lr})=R\delta^{l}_{k},\\ \tfrac{1}{2}(p^{*}_{kr}f^{lr}+p_{kr}f^{*lr})&=\tfrac{1}{2}(f_{kr}p^{*lr}+f^{*}_{kr}p^{lr})=S\delta^{l}_{k},\end{aligned}\right\} \tag{2·7}$$

$$\tfrac{1}{2}(p_{kr}f^{lr}+p^{*}_{kr}f^{*lr})=\tfrac{1}{2}(f_{kr}p^{lr}+f^{*}_{kr}p^{*lr}), \tag{2·8}$$

$$\tfrac{1}{2}(p^{*}_{kr}f^{lr}-p_{kr}f^{*lr})=\tfrac{1}{2}(f_{kr}p^{*lr}-f^{*}_{kr}p^{lr}). \tag{2·9}$$

The important feature in the formulation of (2·7), (2·8) and (2·9) is that in each expression one tensor, say f_{kl}, and its dual occur either only in contravariant or only in covariant form, but never in both forms together. This fact, which could only be established owing to the symmetry condition, is important in our later considerations.

Another consequence of the symmetry condition is that the six invariants F, P, G, Q, R, S are not independent, but that there exist two relations between them which read

$$FP+GQ=S^2-R^2, \tag{2·10}$$

$$FQ-PG=2RS. \tag{2·11}$$

A proof of these relations can be given, but we postpone it, as it would be rather laborious at the present stage of our considerations. In § 4 we shall find a very simple proof by a complex method.

All these considerations are quite independent of the particular choice of the invariant Lagrangian or Hamiltonian function. Our formalism, however, suggests an assumption which implies a restriction of a general nature on the Lagrangian and Hamiltonian. This is shown in the following section.

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3. A RESTRICTION ON THE FIELD THEORY

The energy-momentum tensor T^l_k can be expressed in the alternative forms†

$$T^l_k = L\delta^l_k - f_{kr}p^{lr} = H\delta^l_k - p^*_{kr}f^{*lr}, \tag{3·1}$$

where the Lagrangian L and the Hamiltonian H are functions of f_{kl} and of p^*_{kl} respectively.

The tensors f_{kl} and p^{kl} are connected with one another by the relations

$$\frac{\partial L}{\partial f_{kl}} = p^{kl}, \quad \frac{\partial H}{\partial p^*_{kl}} = f^{*kl}, \tag{3·2}$$

which may alternatively be written

$$\frac{\partial L}{\partial f^*_{kl}} = -p^{*kl}, \quad \frac{\partial H}{\partial p_{kl}} = -f^{kl}. \tag{3·3}$$

L and H are connected by the relation

$$L - H = 2R. \tag{3·4}$$

The non-diagonal elements of T^l_k are given by (1·8).

We have seen in § 1 that there is a tensor whose non-diagonal elements, given by (1·9), form a counterpart to the energy-momentum tensor (3·1). We now wish to extend this analogy to the diagonal elements. We therefore consider two functions $M(f_{kl})$ and $N(p_{kl})$, connected by the relation

$$M - N = 2S, \tag{3·5}$$

and form with their help the tensor

$$V^l_k = M\delta^l_k - f_{kr}p^{*lr} = N\delta^l_k + p_{kr}f^{*lr}. \tag{3·6}$$

Its non-diagonal elements are in fact given by (1·9). The consistency of (3·5) and (3·6) also follows from (1·9). It will be seen that (3·6) and (3·5) correspond to (3·1) and (3·4) respectively, just as (1·9) corresponds to (1·8).

Now in correspondence with (3·2) we assume the relations

$$\frac{\partial M}{\partial f_{kl}} = p^{*kl}, \quad \frac{\partial N}{\partial p_{kl}} = -f^{*kl}, \tag{3·7}$$

which may alternatively, corresponding to (3·3), be written

$$\frac{\partial M}{\partial f^*_{kl}} = p^{kl}, \quad \frac{\partial N}{\partial p^*_{kl}} = -f^{kl}. \tag{3·8}$$

This assumption implicitly imposes a restriction on the field theory. For, from (3·2), (3·3) and (3·7), (3·8) it follows that L and M are connected by the relations

$$\frac{\partial L}{\partial f_{kl}} = \frac{\partial M}{\partial f^*_{kl}}, \quad \frac{\partial L}{\partial f^*_{kl}} = -\frac{\partial M}{\partial f_{kl}}, \tag{3·9}$$

† Cf. II, pp. 434, 436.

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and H and N by the relations

$$\frac{\partial H}{\partial p^*_{kl}} = \frac{\partial N}{\partial(-p_{kl})}, \quad \frac{\partial H}{\partial(-p_{kl})} = -\frac{\partial N}{\partial p^*_{kl}}. \tag{3·10}$$

The equations (3·9) are the Cauchy-Riemann equations expressing the condition that L and M are the real and imaginary parts of a complex function $L+iM$ of the complex variables $f_{kl}+if_{kl}$. In the same way the equations (3·10) express the condition that H and N are the real and imaginary parts of a complex function $H+iN$ of the complex variables $p^*_{kl}-ip_{kl}$†.

Consequently, each of the four functions L, M, H, N satisfies Laplace's equation, namely,

$$\frac{\partial^2 L}{\partial f_{kl}^2} + \frac{\partial^2 L}{\partial f^{*2}_{kl}} = 0. \tag{3·11}$$

This is the restriction mentioned. We investigate it later in terms of invariants.

We have thus been led to introduce the formalism of complex function theory, and in the following sections we build it up from the beginning.

Finally we mention that, on account of (1·8), (1·9) and (2·8), (2·9), the tensors T^l_k and V^l_k may be written in the form

$$T^l_k = T\delta^l_k - \tfrac{1}{2}(f_{kr}p^{lr} + f^*_{kr}p^{*lr}) = T\delta^l_k - \tfrac{1}{2}(p_{kr}f^{lr} + p^*_{kr}f^{*lr}), \tag{3·12}$$

$$V^l_k = V\delta^l_k - \tfrac{1}{2}(f_{kr}p^{*lr} - f^*_{kr}p^{lr}) = V\delta^l_k - \tfrac{1}{2}(p^*_{kr}f^{lr} - p_{kr}f^{*lr}), \tag{3·13}$$

where we have put

$$\tfrac{1}{2}(L+H) = T‡, \tag{3·14}$$

$$\tfrac{1}{2}(M+N) = V. \tag{3·15}$$

These relations are symmetrical between Lagrangian and Hamiltonian and adapted for later use.

Since M and N are invariants, we have the relations

$$V^{kl} = \frac{\partial M}{\partial g_{kl}} = \frac{\partial N}{\partial g_{kl}}, \tag{3·16}$$

similar to (2·1).

The physical significance of M, N, V and V^l_k will be seen at the end of § 7.

4. THE COMPLEX FORMALISM IN SPACE-VECTOR FORM

We introduce the complex formalism first in space-vector form, because some of its advantages can be made clear without involving complicated manipulations with suffixes, which are unavoidably connected with the tensor notation. In this section we must, however, restrict ourselves to Lorentz coordinate systems.

† The difference in the sign could have been avoided if, in order to make the symmetry between f_{kl} and p^*_{kl} complete, we had written $p^{**}_{kl} = -p_{kl}$ instead of p_{kl}.

‡ This T differs from the one defined by Infeld by the factor $\frac{1}{2}$. Compare V, p. 129.

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The principal idea is not new†. We form the complex vectors

$$\mathbf{B} \pm i\mathbf{E}, \qquad \mathbf{D} \pm i\mathbf{H}. \tag{4·1}$$

Forming their squares, we obtain, on account of (1·2a) and (1·3a),

$$(\mathbf{B} \pm i\mathbf{E})^2 = 2(F \pm iG), \quad (\mathbf{D} \mp i\mathbf{H})^2 = 2(P \mp iQ); \tag{4·2}$$

and, forming their scalar product, we get, on account of (1·4a),

$$(\mathbf{B} \pm i\mathbf{E})(\mathbf{D} \mp i\mathbf{H}) = 2(S \mp iR). \tag{4·3}$$

Since on the right-hand side of (4·2) and (4·3) there appear only world invariants and in fact all the world invariants, we can state

THEOREM 1. *The group of transformations of an antisymmetrical tensor of the second rank in a four-dimensional real space, induced by the Lorentz group, is isomorphic with the group of orthogonal transformations in a three-dimensional complex space.*

From the equations (2·3), which express the symmetry condition in space-vector form, it follows that the vector product of the two vectors in (4·1) vanishes:

$$(\mathbf{B} \pm i\mathbf{E}) \times (\mathbf{D} \mp i\mathbf{H}) = 0. \tag{4·4}$$

Hence the product of the squares of these two vectors is equal to the square of their scalar product, so that we have

$$(F \pm iG)(P \mp iQ) = (S \mp iR)^2. \tag{4·5}$$

If we split up this relation into its real and imaginary parts, we obtain the equations (2·10) and (2·11), which are hereby proved.

We shall meet again with all relations of this section when we develop the complex formalism in tensor notation.

5. THE COMPLEX FORMALISM IN TENSOR FORM. GENERAL THEORY

We introduce the complex tensors ϕ_{kl} and ψ_{kl} by the transformation‡

$$\left.\begin{aligned} \phi_{kl} &= \frac{1}{\sqrt{2}}(f_{kl} + if^*_{kl}), & \psi_{kl} &= \frac{1}{\sqrt{2}}(p^*_{kl} - ip_{kl}), \\ \bar{\phi}_{kl} &= \frac{1}{\sqrt{2}}(f_{kl} - if^*_{kl}), & \bar{\psi}_{kl} &= \frac{1}{\sqrt{2}}(p^*_{kl} + ip_{kl}). \end{aligned}\right\} \tag{5·1}$$

Here the bar denotes the conjugate complex quantity. The inverse transformation reads

$$\left.\begin{aligned} f_{kl} &= \frac{1}{\sqrt{2}}(\phi_{kl} + \bar{\phi}_{kl}), & p^*_{kl} &= \frac{1}{\sqrt{2}}(\psi_{kl} + \bar{\psi}_{kl}), \\ f^*_{kl} &= -\frac{i}{\sqrt{2}}(\phi_{kl} - \bar{\phi}_{kl}), & p_{kl} &= \frac{i}{\sqrt{2}}(\psi_{kl} - \bar{\psi}_{kl}). \end{aligned}\right\} \tag{5·2}$$

† Cf. F. Klein, *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert*, 2 (Berlin, 1927), p. 82.

‡ This transformation has a close connection with the formalism of spinors and semi-vectors. Cf. e.g. A. Einstein and W. Mayer, *Berl. Sitz. Ber.* (1932), pp. 526–6; B. S. Madhava Rao, *Proc. Ind. Acad. Science* IV, 4 (1936), pp. 437–9; E. T. Whittaker, *Proc. Roy. Soc.* 158 (1937), pp. 38–41.

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The tensors ϕ_{kl}, ψ_{kl} and their conjugate complex tensors transform like any other antisymmetrical tensor of the second rank, but only three of their components are independent. The other three are determined by the relations

$$\phi^*_{kl} = -i\phi_{kl}, \quad \psi^*_{kl} = -i\psi_{kl}, \tag{5·3}$$

and the conjugate complex relations. These follow directly from (5·1). To each relation which we obtain there corresponds a conjugate complex relation. We omit the latter when it is obvious.

If we choose in (5·1) the components (23), (31) and (12) as independent and use a Lorentz frame, we obtain, apart from the factor $1/\sqrt{2}$, the same quantities as in the previous section, but the present considerations are generally invariant, not only Lorentz invariant.

We now define complex invariants ϕ, ψ and ρ by

$$\tfrac{1}{4}\phi_{kl}\phi^{kl} = \phi, \tag{5·4}$$

$$\tfrac{1}{4}\psi_{kl}\psi^{kl} = \psi, \tag{5·5}$$

$$\tfrac{1}{4}\phi_{kl}\psi^{kl} = -i\rho. \tag{5·6}$$

From (1·2), (1·3), (1·4) together with (5·1), it follows that

$$\phi = F + iG, \tag{5·7}$$

$$\psi = P - iQ, \tag{5·8}$$

$$\rho = R + iS. \tag{5·9}$$

We now wish to express the relations (1·5)–(1·7) in complex form. From (1·5) and (1·6) we get

$$\phi_{kr}\phi^{lr} = \phi\delta^l_k, \tag{5·10}$$

$$\psi_{kr}\psi^{lr} = \psi\delta^l_k. \tag{5·11}$$

We cannot express (1·7), as it stands, in a similar form, but we can so express (2·7), which is equivalent to (1·7) on account of the symmetry condition of § 2, and get

$$\psi_{kr}\phi^{lr} = \phi_{kr}\psi^{lr} = -i\rho\delta^l_k. \tag{5·12}$$

Similarly, we cannot combine (1·8) and (1·9), as they stand, into one complex expression, but we can do this with the equivalent expressions (2·8) and (2·9) and obtain

$$\bar{\psi}_{kr}\phi^{lr} = \phi_{kr}\bar{\psi}^{lr}. \tag{5·13}$$

Thus for the formation of (5·12) and (5·13) the symmetry condition of § 2 is essential. Moreover, the non-diagonal parts of (5·12) and (5·13) are the expression of this symmetry condition in our complex formalism.

From (5·10)–(5·12) we can derive a connection between the invariants ϕ, ψ, ρ. We form

$$\phi_{kr}(\psi^{jr}\psi_{js})\phi^{ls} = (\phi_{kr}\psi^{jr})(\psi_{js}\phi^{ls}).$$

The left-hand side of this identity becomes

$$\phi_{kr}\phi^{ls}\psi\delta^r_s = \phi\psi\delta^l_k,$$

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and the right-hand side becomes

$$(-i\rho\delta^l_k)(-i\rho\delta^l_j) = -\rho^2\delta^l_k.$$

Thus we have† $$\phi\psi = -\rho^2. \tag{5·14}$$

This is precisely the relation (4·5).

We now substitute (5·2) into the Lagrangian L and Hamiltonian H. Then we have from (5·2)

$$\frac{\partial L}{\partial \phi_{kl}} = \frac{1}{\sqrt{2}}\left(\frac{\partial L}{\partial f_{kl}} - i\frac{\partial L}{\partial f^*_{kl}}\right), \quad \frac{\partial H}{\partial \psi_{kl}} = \frac{1}{\sqrt{2}}\left(\frac{\partial H}{\partial p^*_{kl}} + i\frac{\partial H}{\partial p_{kl}}\right). \tag{5·15}$$

Owing to the invariance, L and H are functions of $\phi, \bar{\phi}$ and $\psi, \bar{\psi}$ respectively, and we obtain from (5·4) and (5·5)

$$\frac{\partial L}{\partial \phi_{kl}} = 2\frac{\partial L}{\partial \phi}\phi^{kl}, \quad \frac{\partial H}{\partial \psi_{kl}} = 2\frac{\partial H}{\partial \psi}\psi^{kl}. \tag{5·16}$$

Substituting (3·2) and (3·3) into (5·15) and comparing with (5·16), we obtain

$$2\frac{\partial L}{\partial \phi}\phi^{kl} = \frac{1}{\sqrt{2}}(p^{kl} + ip^{*kl}), \quad 2\frac{\partial H}{\partial \psi}\psi^{kl} = \frac{1}{\sqrt{2}}(f^{*kl} - if^{kl}),$$

and from (5·1) $$2\frac{\partial L}{\partial \phi}\phi^{kl} = i\psi^{kl}, \quad 2\frac{\partial H}{\partial \psi}\psi^{kl} = -i\phi^{kl}. \tag{5·17}$$

Multiplying by ϕ_{kl} and ψ_{kl} respectively we get

$$2\frac{\partial L}{\partial \phi}\phi = \rho, \quad 2\frac{\partial H}{\partial \psi}\psi = -\rho, \tag{5·18}$$

and similarly $$2\frac{\partial L}{\partial \phi}\rho = -\psi, \quad 2\frac{\partial H}{\partial \psi}\rho = \phi. \tag{5·19}$$

Eliminating ρ from (5·18) and (5·19), we obtain finally

$$4\left(\frac{\partial L}{\partial \phi}\right)^2\phi = -\psi, \quad 4\left(\frac{\partial H}{\partial \psi}\right)^2\psi = -\phi. \tag{5·20}$$

From this and the conjugate complex relation, ψ and $\bar{\psi}$ can easily be calculated when $L(\phi, \bar{\phi})$ is given, and ϕ and $\bar{\phi}$ can be calculated when $H(\psi, \bar{\psi})$ is given. The corresponding relations in real notation are substantially more complicated.

6. Restriction by Laplace's equation and reduction to complex function theory

We now assume the existence of the functions M and N, which have been introduced in § 3, and consequently restrict L and H by Laplace's equation (3·11). It reads, after the transformation (5·1),

$$\frac{\partial^2 L}{\partial \phi_{kl}\partial \bar{\phi}_{kl}} = 0, \quad \frac{\partial^2 H}{\partial \psi_{kl}\partial \bar{\psi}_k} = 0. \tag{6·1}$$

† Compare V, p. 130 (3·6). There the imaginary parts of ϕ, ψ, ρ are zero.

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By (5·16), this is equivalent to

$$\frac{\partial^2 L}{\partial\phi\partial\bar{\phi}}=0, \quad \frac{\partial^2 H}{\partial\psi\partial\bar{\psi}}=0. \tag{6·2}$$

And from (5·7) and (5·8) we obtain as a *necessary and sufficient restriction for invariant Lagrangian and Hamiltonian functions* the following relations in real notation:

$$\frac{\partial^2 L}{\partial F^2}+\frac{\partial^2 L}{\partial G^2}=0, \quad \frac{\partial^2 H}{\partial P^2}+\frac{\partial^2 H}{\partial Q^2}=0, \tag{6·3}$$

that is, Laplace's equation with respect to the invariants F, G and P, Q respectively. This is the expression of (3·11) in terms of invariants.

From (3·7) and (3·8) we obtain for M and N, by an argument similar to that which led to (5·17),

$$2\frac{\partial M}{\partial\phi}\phi^{kl}=\psi^{kl}, \quad 2\frac{\partial N}{\partial\psi}\psi^{kl}=-\phi^{kl}. \tag{6·4}$$

Hence

$$2\frac{\partial M}{\partial\phi}\phi=-i\rho, \quad 2\frac{\partial N}{\partial\psi}\psi=i\rho. \tag{6·5}$$

Here M and N are supposed to be invariants, like L and H. Comparing (5·18) and (6·5) and the conjugate complex relations, we get

$$\frac{\partial L}{\partial\phi}=i\frac{\partial M}{\partial\phi}, \quad \frac{\partial H}{\partial\psi}=i\frac{\partial N}{\partial\psi}, \tag{6·6}$$

$$\frac{\partial L}{\partial\bar{\phi}}=-i\frac{\partial M}{\partial\bar{\phi}}, \quad \frac{\partial H}{\partial\bar{\psi}}=-i\frac{\partial N}{\partial\bar{\psi}}. \tag{6·7}$$

These are the Cauchy-Riemann equations expressing the condition that the complex quantities

$$\Lambda=L+iM, \quad \Xi=H+iN \tag{6·8}$$

are analytic functions of the variables ϕ and ψ respectively. These functions will be called the "complex Lagrangian" and the "complex Hamiltonian" respectively. Their derivatives will be denoted by a dash.

From (5·17) and (6·4) it follows that

$$\Lambda'\phi^{kl}=i\psi^{kl}, \quad \Xi'\psi^{kl}=-i\phi^{kl}. \tag{6·9}$$

Hence we obtain

$$\Lambda'\phi=\rho, \quad \Xi'\psi=-\rho, \tag{6·10}$$

$$\Lambda'\rho=-\psi, \quad \Xi'\rho=\phi, \tag{6·11}$$

and from these relations, on eliminating ρ, we get

$$\Lambda'^2\phi=-\psi, \quad \Xi'^2\psi=-\phi. \tag{6·12}$$

The equations (3·4) and (3·5) can be combined into the equation

$$\Lambda-\Xi=2\rho, \tag{6·13}$$

from which we obtain, by (6·10),

$$\Xi=\Lambda-2\Lambda'\phi, \quad \Lambda=\Xi-2\Xi'\psi. \tag{6·14}$$

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Herewith the field theory is reduced to the study of two analytic functions, Λ and Ξ, of one variable each, which are connected with each other by a kind of Legendre transformation.

We now form the complex tensor

$$\Sigma_k^l = T_k^l + iV_k^l \tag{6·15}$$

and obtain from (3·12) and (3·13)

$$\Sigma_k^l = \Sigma\delta_k^l - i\bar{\phi}_{kr}\psi^{lr} = \Sigma\delta_k^l - i\psi_{kr}\bar{\phi}^{lr}, \tag{6·16}$$

where we have put, in agreement with (3·14) and (3·15),

$$\Sigma = T + iV. \tag{6·17}$$

It follows from (3·14) and (3·15) that

$$\Sigma = \tfrac{1}{2}(\Lambda + \Xi). \tag{6·18}$$

This Σ is the complex form of Infeld's action function†.

Finally, from (2·1) and (3·16), we obtain

$$\Sigma^{kl} = \frac{\partial\Lambda}{\partial g_{kl}} = \frac{\partial\Xi}{\partial g_{kl}}. \tag{6·19}$$

The tensor Σ_k^l or Σ^{kl} will be called "the complex energy-momentum tensor".

It should be noted that the g_{kl} form *real* parameters in the complex functions $\Lambda(\phi)$ and $\Xi(\psi)$.

7. Deduction of the condition of self-conjugacy

So far we have not taken account in our complex formalism of the fact that the Lagrangian and Hamiltonian functions of the real formalism contain the invariants G and Q respectively as squares only‡:

$$L = L(F, G^2), \quad H = H(P, Q^2). \tag{7·1}$$

This feature means§ in our complex formalism that the analytic functions Λ and Ξ are *self-conjugate*, that is, that they satisfy the relations

$$\overline{\Lambda(\phi)} = \Lambda(\bar{\phi}), \quad \overline{\Xi(\psi)} = \Xi(\bar{\psi}), \tag{7·2}$$

provided they are one-valued in the region under consideration‖. Then it follows from (6·18) that Σ is also self-conjugate.

† Cf. V, p. 129.

‡ This fact, though it has never been stated explicitly, arises from the method of formation of the invariants. Cf. II, pp. 430–1.

§ Let $w(z) = u(x, y) + iv(x, y)$ be such that $u(x, y) = u(x, -y)$. Then, if

$$\Delta = v(x, y) + v(x, -y),$$

the Cauchy-Riemann equations show that

$$\partial\Delta/\partial x = \partial\Delta/\partial y = 0, \quad \text{and so} \quad v(x, y) = -v(x, -y),$$

neglecting a trivial constant. Hence $\overline{w(z)} = w(\bar{z})$.

‖ An example of a many-valued function will be discussed in § 8.

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Any finite expression or any power series, by which these functions are represented, has therefore real coefficients.

From (7·2) it follows in particular that Λ is real for real ϕ, that is, when $G=0$, and that Ξ is real for real ψ, that is, when $Q=0$†. Moreover, if (7·2) holds for an analytic function, it also holds for its derivative, and therefore it follows from (6·12) that G and Q always vanish simultaneously. Thus we have

THEOREM 2. *The real axes of the ϕ-plane and the ψ-plane correspond to each other and to those of the Λ-plane, the Ξ-plane and the Σ-plane.*

We can therefore refer to the "real axis" as such, without mentioning the particular complex variable under consideration.

We find, furthermore, from (6·12) that on the real axis the corresponding values of F and P have opposite signs. If $\Lambda'(\phi)$ and $\Xi'(\psi)$ do not vanish at $\phi=0$ and $\psi=0$ respectively, the origins of the ϕ-plane and ψ-plane correspond.

If ϕ and ψ remain at the origin or on the real axis throughout a particular electromagnetic field, the physical meaning of origin and real axis is evident; the origin of the ϕ- and ψ-plane corresponds to plane waves ($F=G=P=Q=0$), and the rest of the real axis corresponds to fields of purely static character, that is, fields which can be transformed into pure static fields, electrostatic for $F<0$, $P>0$, magnetostatic for $F>0$, $P<0$.

All these facts are independent of the particular field theory under consideration. But in some cases, in the "new field theory" for instance, a complex variable, ϕ say, is restricted to a limited region of the real axis, as we shall see in § 8.

From the preceding considerations, together with equation (6·19), it follows that the tensor Σ_k^l is self-conjugate in the sense that every component of it assumes conjugate complex values for conjugate complex values of the invariant ϕ (and ψ)‡. In particular we obtain

THEOREM 3. *The complex energy-momentum tensor is real on the real axis.*

Theorems 2 and 3 throw some light on the physical nature of M, N, V and V_k^l; these quantities appear only when G (and therefore Q) does not vanish.

† This condition is already sufficient for self-conjugacy. Cf. Titchmarsh, *Theory of functions* (Oxford, 1932), p. 155.

‡ The weaker property that Σ_k^l assumes conjugate complex values for conjugate complex values of the tensor ϕ_{kl} (or ψ_{kl}) follows also from the equations (6·9) and (6·16). Our proposition is much stronger and its validity is due to the more far-reaching relation (6·19). On the other hand it will be seen from (6·9) that ψ_{kl} does not assume conjugate complex values for conjugate complex values of ϕ_{kl}, but that

$$\overline{\psi_{kl}(\phi_{rs})} = -\psi_{kl}(\bar{\phi}_{rs}),$$

and vice versa.

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8. Discussion of special cases and Maxwell's limiting condition

We have established the result that to every field theory there corresponds a "self-conjugate analytic function" $\Lambda(\phi)$ (and $\Xi(\psi)$), provided the Lagrangian (and Hamiltonian) of the field theory satisfies Laplace's equation with respect to F and G (or P and Q). We shall now discuss some examples.

(*a*) *Maxwell's theory*

$$L = F, \quad H = P. \tag{8·1}$$

Laplace's equation (6·3) is satisfied. For M and N we obtain from the Cauchy-Riemann equations the relations

$$M = G, \quad N = -Q. \tag{8·2}$$

Hence the analytic functions of Maxwell's theory are

$$\Lambda = \phi, \quad \Xi = \psi. \tag{8·3}$$

(*b*) *The new field theory*†

Two Lagrangians have been considered in the new field theory, namely,

$$L = (1+2F)^{\frac{1}{2}} - 1 \quad \text{and} \quad L = (1+2F-G^2)^{\frac{1}{2}} - 1. \tag{8·4}$$

Neither of them satisfies Laplace's equation. Generally, no Lagrangian which depends on F only (like the first one of (8·4)) satisfies Laplace's equation, except Maxwell's Lagrangian (more precisely $L = cF + d$). Thus the new field theory has to be modified in order to fit into our scheme. The required modification is, however, such that it does not alter any important feature of the theory. For the new field theory has been discussed so far in the static case only, where G is zero, that is on the real axis. In this case the two Lagrangians (8·4) have the same form. To find the general case, we use the principle of "analytic continuation" and obtain

$$\Lambda = (1+2\phi)^{\frac{1}{2}} - 1. \tag{8·5}$$

This is, from our point of view, the proper complex Lagrangian of the new field theory. The proper real Lagrangian L can be calculated from (8·5) by splitting up Λ into its real and imaginary parts. The power series for L about the origin has the form

$$L = F - \tfrac{1}{2}F^2 + \tfrac{1}{2}G^2 + \tfrac{1}{2}F^3 - \tfrac{3}{2}FG^2 - \ldots, \tag{8·5a}$$

while the power series of the two Lagrangians (8·4) have the forms

$$\left.\begin{aligned} &L = F - \tfrac{1}{2}F^2 \qquad\quad + \tfrac{1}{2}F^3 \qquad\quad - \ldots \\ \text{and} \quad &L = F - \tfrac{1}{2}F^2 - \tfrac{1}{2}G^2 + \tfrac{1}{2}F^3 + \tfrac{1}{2}FG^2 - \ldots. \end{aligned}\right\} \tag{8·4a}$$

We see that the series differ already in the second powers, provided G does not vanish.

† I, II. The absolute field is taken as unity.

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An electric singularity of the new field theory corresponds to the point $\phi = -\frac{1}{2}$ in the ϕ-plane. This is the branch point of the Riemann surface for $\Lambda(\phi)$. In the interval $-\infty < F < -\frac{1}{2}$ on the real axis Λ becomes imaginary, and the condition (7·2) breaks down. Values of ϕ in this interval are physically inadmissible.

From (6·12) we obtain the relation

$$\phi + 2\phi\psi + \psi = 0, \tag{8·6}$$

and the complex Hamiltonian turns out to be†

$$\Xi = (1 + 2\psi)^{\frac{1}{2}} - 1. \tag{8·7}$$

The branch point $-\frac{1}{2}$ of the ϕ-plane corresponds to the point ∞ of the ψ-plane.

(c) *Infeld's new action function*‡

Instead of a Lagrangian L or a Hamiltonian H, Infeld uses as starting point the T-function which he assumes to depend on F, P, R. The corresponding procedure by our complex method consists in considering a Σ-function which depends on ϕ, ψ, ρ. With these changes, Infeld's considerations can be taken over completely, and for his new action function we obtain

$$\Sigma = \log\left(-\frac{\phi}{\psi}\right) - \frac{\phi + \psi}{\rho}. \tag{8·8}$$

This action function leads to the relation

$$\rho^2 = 2\rho + \psi - \phi. \tag{8·9}$$

On the real axis these formulae go over into the corresponding formulae of Infeld§. Using (5·14), we obtain from (8·9) an algebraic relation between ϕ and ψ only, namely

$$\phi^2\psi^2 - 2\phi^2\psi + 2\phi\psi^2 + \phi^2 + 2\phi\psi + \psi^2 = 0. \tag{8·10}$$

There are several branches on the Riemann surface of the function $\phi(\psi)$ corresponding to the different solutions obtained by Infeld in the static case.

The equations (6·12) allow us to determine Λ and Ξ from the relation (8·10). We see that Λ and Ξ become Abelian integrals, while Σ is a more elementary function. This fact explains the advantage of the T- (or Σ-) representation for Infeld's action function.

The general conclusion to be drawn from the discussion of these examples is the following:

Every self-conjugate analytic function of a complex variable characterizes an electrodynamic field theory of the type proposed by Born. If a field theory is given by a real analytic expression for the static case, where it involves F or P only and not

† Compare I, p. 427.
‡ V, pp. 129–30, 134–5.
§ Viz. V, p. 134, formulae (7·1), (7·2).

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G or Q, the general case can be obtained immediately by substituting the complex quantity ϕ or ψ for the real quantity F or P respectively and then separating the real and imaginary parts.

A field theory may be characterized in various ways: by means of a Lagrangian Λ, a Hamiltonian Ξ, a $\sum$-function or by a relation between ϕ and ψ or ϕ, ψ and ρ. From any one form of characterization all the others can be deduced, as has been explained in the preceding examples.

For physical reasons we have to require from any field theory that it goes over into Maxwell's theory for small values of the invariants, that is, near the origin. From this condition it follows firstly that $\Lambda(\phi)$ and $\Xi(\psi)$ are regular at the origin and satisfy the relations

$$\left.\begin{aligned} \Lambda(0) &= 0, \quad \Xi(0) = 0, \\ \Lambda'(0) &= 1, \quad \Xi'(0) = 1. \end{aligned}\right\} \tag{8·11}$$

Secondly it follows that, as long as ϕ is finite, ψ vanishes if and only if ϕ vanishes, and vice versa. We therefore obtain from (6·12) the conditions

$$\Lambda'(\phi) \neq 0, \quad \Xi'(\psi) \neq 0, \tag{8·12}$$

which are valid throughout the finite and regular part of the ϕ- and the ψ-plane. The new field theory and Infeld's theory satisfy these conditions.

The condition (8·12) restricts to a great extent the class of functions which are admissible for the description of a field theory. A further restriction arises from the condition of "finite self-energy", which rules out Maxwell's theory. This condition has so far been discussed only in the special case of an electrostatic field with central symmetry. A more general formulation of this condition seems to be required if it is to fit into our present considerations. Up to now, however, we have not been able to obtain such a general formulation. Therefore, the suitability of any new electrodynamics can at present be judged only by considering the special case of an electrostatic solution with central symmetry, as has been done in the previous papers of Born and Infeld.

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