ON ITERATIVE PROCESSES AND FUNCTIONAL EQUATIONS

By C. DOMB and M. E. FISHER

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1. *Introduction: degenerate iterative processes.* A well-known class of iterative processes is specified by the equation

$$y_{n+1} = f(y_n) \quad (n = 0, 1, 2, \ldots). \tag{1}$$

Usually y_n is a number, but it may equally well be a vector (i.e. a group of numbers $y_n^{(1)}, y_n^{(2)}, \ldots, y_n^{(k)}$). Under suitable conditions y_n as specified by (1) tends to Y as $n \to \infty$, where Y is a root of the equation

$$f(y) - y = 0. \tag{2}$$

For a range of initial values y_0 the limit of the sequence (1) is independent of the initial term y_0. This means that errors made in a calculation based on (1) do not affect the finally computed value of the root Y.

Hartree (1) has shown that an important practical consideration is the order of the process (1). If we introduce η_n, the error at the nth stage, then for small enough η_n (1) may be written as

$$\eta_{n+1} \doteq a\eta_n^p \quad (\eta_n = y_n - Y). \tag{3}$$

(This equation is to be interpreted symbolically in the case where η is a two- or more-dimensional vector.) The index p defines the order of the process (1). It is generally desirable to transform (2) in such a manner that a second- or higher-order process is obtained.

In contrast to processes of type (1) there is a class of processes in which the limit of the sequence of iterations does depend on the initial terms. This can occur when y is a vector in two or more dimensions. For example, if $y_n = (u_n, v_n)$ and we have the process

$$\left.\begin{aligned} u_{n+1} &= (1-\rho)\,u_n + \rho v_n, \\ v_{n+1} &= \sigma u_n + (1-\sigma)\,v_n, \end{aligned}\right\} \tag{4}$$

where ρ and σ are constants, then u_n and v_n tend to a common limit $\dfrac{u_0\sigma + v_0\rho}{\sigma+\rho}$ determined by the initial value $y_0 = (u_0, v_0)$. The process converges if $|1-\rho-\sigma| < 1$. Further examples of this type of process are mentioned in Bromwich ((2), p. 23). An interesting one is the so-called *arithmetico-geometric* mean defined as the common limit l of the sequence

$$\left.\begin{aligned} u_{n+1} &= \tfrac{1}{2}(u_n + v_n), \\ v_{n+1} &= \sqrt{(u_n v_n)}. \end{aligned}\right\} \tag{5}$$

In fact l is given by an elliptic integral, viz.

$$\int_0^{\frac{1}{2}\pi} (u_0^2 \cos^2\theta + v_0^2 \sin^2\theta)^{-\frac{1}{2}}\, d\theta = \frac{\pi}{2l}.$$

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As mentioned by Bromwich the process (5) provides a rapid means of evaluating this elliptic integral. However, such processes, which we call degenerate iterative processes, will not have the pleasing error-correcting property mentioned above. None the less degenerate processes may be useful, especially if they can be made second order (as, in fact, is (5)).

In this note we investigate processes of this type and show that they correspond to the iterative solutions of a certain class of functional equations. Degenerate processes may thus be used to calculate numerical solutions of given functional equations (or to calculate functions satisfying known functional equations). We also study the extent to which the processes may be transformed to yield higher-order processes of greater utility. We illustrate our results by treating some functional equations satisfied by the hypergeometric function.

2. *Derivation of a functional equation from an iterative process.* Consider the iteration in the two variables u and v

$$u_{n+1} = \phi(u_n, v_n), \quad v_{n+1} = \psi(u_n, v_n); \tag{6}$$

this will be degenerate if the equations

$$\phi(u, v) - u = 0, \quad \psi(u, v) - v = 0 \tag{7}$$

are always satisfied by $u = v$. This is equivalent to the conditions

$$\text{(A)} \quad \phi(u, v) \quad \text{and} \quad \psi(u, v)$$

are homogeneous functions of the first order, and in particular,

$$\phi(u, v) = u\phi(1, v/u), \quad \psi(u, v) = u\psi(1, v/u);$$

and

$$\text{(B)} \quad \phi(1, 1) = \psi(1; 1) = 1. \tag{8}$$

If the process (6) is convergent for given u_0 and v_0, u_n and v_n will tend to a common limit $l = L(u_0, v_0)$. For this limit function we have

$$L(u_0, v_0) = L(u_1, v_1). \tag{9}$$

In virtue of condition (A) and (6) this may be written

$$u_0 L(1, y_0) = u_0 \phi(1, y_0) L\left\{1, \frac{\psi(1, y_0)}{\phi(1, y_0)}\right\}, \tag{10}$$

where $y = v/u$. But this is a functional equation of the general type

$$F(x) = \chi(x) F[\theta(x)], \tag{11}$$

in which $\chi(x)$ and $\theta(x)$ are known functions and $F(x)$ is to be determined. Simpler types of this equation were first considered by Abel ((3), p. 36) and Schröder (4).

Identifying corresponding terms in (10) and (11) we conclude that if the conditions

$$\chi(1) = 1, \quad \theta(1) = 1 \tag{12}$$

are fulfilled, then the process

$$u_{n+1} = u_n \chi(v_n/u_n), \quad v_{n+1} = u_n \chi(v_n/u_n)\, \theta(v_n/u_n), \tag{13}$$

with

$$u_0 = F(1) \quad \text{and} \quad y_0 = v_0/u_0 = x, \tag{14}$$

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will, if it converges, define a common limit for u_n and v_n equal to $F(x)$. Thus if a functional equation may be written in the standard form (11) in such a way that (12) holds, or if some required function satisfies (11), then the functional equation may be solved numerically or the required function evaluated by the iterative process (13).

3. *The convergence of the iterative process.* The convergence of the process (13) may be studied by introducing ξ_n and η_n by

$$u_{n+1} = \tfrac{1}{2}(u_n + v_n) + \xi_n, \quad v_{n+1} = \tfrac{1}{2}(u_n + v_n) + \eta_n. \tag{15}$$

Then, using (13) in the form (6),

$$\xi_{n+1} = \phi(u_{n+1}, v_{n+1}) - \tfrac{1}{2}(u_n + v_n) - \tfrac{1}{2}(\xi_n + \eta_n),$$
$$\eta_{n+1} = \psi(u_{n+1}, v_{n+1}) - \tfrac{1}{2}(u_n + v_n) - \tfrac{1}{2}(\xi_n + \eta_n).$$

Expanding ϕ and ψ about the point $u_{n+1} = v_{n+1} = \tfrac{1}{2}(u_n + v_n)$ as a Taylor series in ξ_n and η_n and using conditions (A) and (B) we obtain for small ξ_n and η_n

$$\begin{bmatrix} \xi_{n+1} \\ \eta_{n+1} \end{bmatrix} = \mathbf{M} \begin{bmatrix} \xi_n \\ \eta_n \end{bmatrix}. \tag{16}$$

The matrix $\mathbf{M}$ depends on the partial derivatives of $\phi(u, v)$ and $\psi(u, v)$, and hence, by (13), on the derivatives of $\chi(x)$ and $\theta(x)$. Equation (16) shows that the convergence of process (13) is determined by the latent roots of $\mathbf{M}$. These are 0 and $\theta'(1)$, so that a necessary condition for the convergence of the iterative solution (13) is

$$|\theta'(1)| \leqslant 1. \tag{17}$$

In practice the case of equality often does not give convergence, so that for most purposes it may be neglected. Except for this case a convergent process may always be found, for if $|\theta'(1)| > 1$ we may invert the functional equation (11) by writing $y = \theta(x)$ and $x = \theta_{-1}(y)$. Then it follows that $|\theta'_{-1}(1)| < 1$, so that the inverse process will converge.

The convergence of (13) will be better than first order if and only if $\theta'(1) = 0$. As examples we may examine the convergence of the two processes (4) and (5) mentioned in the Introduction. Comparing these equations with (13) we find that $\theta(x)$ is respectively

$$\frac{\sigma + (1-\sigma)x}{1-\rho+\rho x} \quad \text{and} \quad \frac{2\sqrt{x}}{1+x}.$$

In both cases $\theta(1) = 1$, but in the first case $\theta'(1) = 1-\rho-\sigma$ so that the process (4) is first order, whilst in the second case $\theta'(1) = 0$ and $\theta''(1) = -\frac{1}{4}$ which establishes the second-order nature of (5).

The question of the convergence of the iterative solution may also be approached directly from the functional equation (see Picard (5), Ch. IV). By repeated use of the equation (11) itself one obtains

$$F(x) = \prod_{l=0}^{n-1} \chi[\theta_l(x)]\, F[\theta_n(x)], \tag{18}$$

where

$$\theta_{n+1}(x) = \theta[\theta_n(x)] \quad \text{and} \quad \theta_0(x) = x. \tag{19}$$

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Then it is not difficult to see that

$$F(x) = F(1) \prod_{l=0}^{\infty} \chi[\theta_l(x)], \tag{20}$$

provided $\theta_n(x) \to 1$, which is assured if $|\theta'(1)| < 1$ (and x is within a certain region). This solution is equivalent to (13); it does, however, bring out the fact that convergence to the limit may be slow if $\chi'(1)$ is large, even though $|\theta'(1)|$ is small. This point will be taken up again later.

4. *Transformation of a functional equation when* $\chi(1) \neq 1$. We now inquire whether it is possible to relax the conditions (12) on the type of functional equation for which we may obtain solutions by (13). Considering first the case where $\theta(1) = 1$ but $\chi(1) \neq 1$, we may transform the functional equation (11) by dividing by an arbitrary function $k(x)$. The transformed equation is

$$G(x) = \chi_T(x)\, G[\theta(x)], \tag{21}$$

where

$$F(x) = k(x)\, G(x) \tag{22}$$

and

$$\chi_T(x) = \frac{k[\theta(x)]}{k(x)} \chi(x). \tag{23}$$

It should be noted that $\theta(x)$ is not altered by this transformation. The problem now is to choose $k(x)$ so that

$$\lim_{x \to 1} \frac{k[\theta(x)]\, \chi(x)}{k(x)} = 1. \tag{24}$$

(A limiting process must be involved, for, unless $k(1) = 0$ or ∞, (23) implies $\chi_T(1) = \chi(1)$.) If we write $x = 1 + \xi$ and suppose that for small enough ξ

$$\theta(x) = 1 + b\xi^{\nu} \quad \text{and} \quad \chi(x) = c\xi^{\mu}, \tag{25}$$

(24) becomes

$$k(x) = K(\xi) = c\xi^{\mu} K(b\xi^{\nu}). \tag{26}$$

Thus the required transformation function $K(\xi)$ is itself given by a functional equation! Evidently this functional equation cannot be solved by the classical method outlined in the last section. Indeed, since, in this case, $\chi(\xi) = c\xi^{\mu}$ and so in general $\chi(1) \neq 1$, the infinite product involved in (20) would diverge. An alternative general approach which enables a solution of (26) to be found, even though $\chi(1) \neq 1$ and $\theta(1) \neq 1$, is as follows. If we iterate the functional equation n times as in (18) with a particular fixed value of the variable x (or ξ), say α, the result may be viewed as determining $F[\theta_n(\alpha)]$ (or $K[\theta_n(\alpha)]$) in terms of $F(\alpha)$. We now write $\theta_n(\alpha) = z$. This equation is defined only for integral n, but, if it is possible to interpret it for *continuous* n, we may write $n = N(z)$, where $N(z)$ is a continuous function of z. Now, if we can also interpret the continued product in (18) for continuous n then, on elimination of n, our iterated equation determines $F(z)$ as a function of the continuous variable z. The details of the method may be seen more clearly in the working of case (a) below.

After n iterations with $\xi = \alpha$, where α is an arbitrary constant, we obtain from (26)

$$K\{b^{s(n)} \alpha^{\nu^n}\} = K(\alpha)\, c^{-n} b^{-\mu S(n)} \alpha^{-\mu s(n)}, \tag{27}$$

where

$$s(n) = \sum_{r=0}^{n-1} \nu^r \quad \text{and} \quad S(n) = \sum_{l=1}^{n-1} \sum_{r=0}^{l-1} \nu^r. \tag{28}$$

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There are various cases to consider:

(*a*) *First-order process.* $\nu = 1$ and we may suppose $b \neq 1$ since otherwise (26) would be a contradiction. Then we have $s(n) = n$ and $S(n) = \frac{1}{2}n(n-1)$, so that if we set $\alpha = 1$, (27) reduces to

$$K(b^n) = K(1)\, c^{-n} b^{-\frac{1}{2}\mu n(n-1)}. \tag{29}$$

As indicated above we now set $z = b^n$. This obviously has meaning for continuous n. In fact it can be written

$$n = \log_b z, \tag{30}$$

so that

$$K(z) = z^{-\log_b c - \frac{1}{2}\mu \log_b z/b} \quad (\nu = 1,\ b \neq 1), \tag{31}$$

which gives $K(z)$ as a function of continuous z. We have dropped the factor $K(1)$ since $K(z)$ is arbitrary to this extent. It is easily verified that (31) is indeed a solution of (26), so that if we put $z = x - 1$ we have a transformation function $k(x)$ which enables us to transform a functional equation with $\chi(1) \neq 1$ into one with $\chi_T(1) = 1$.

(*b*) *Higher-order process with* $b \neq 1$. Evaluating s and S for the general case $\nu \neq 1$ we proceed as above, obtaining

$$K(z) = c^{-n} b^{\mu(1+n\nu-n-\nu^n)(\nu-1)^{-2}} \quad (\nu \neq 1,\ b \neq 1), \tag{32}$$

where

$$n = \log_\nu \{1 + (\nu - 1)\log_b z\}. \tag{33}$$

(*c*) *Higher-order process with* $b = 1$. In this final case we take α so that $\log \alpha = 1$. With this choice and after dropping irrelevant factors, we obtain

$$K(z) = z^{\mu(1-\nu)^{-1}} (\log z)^{-\log_\nu c}. \tag{34}$$

In particular cases the formulae (31), (32) and (34) often simplify considerably. As an example consider the equation

$$F(\tfrac{1}{2}, \tfrac{1}{2}; 1; x) = (1 + x^{\frac{1}{2}})^{-1} F[\tfrac{1}{2}, \tfrac{1}{2}; 1; 4x^{\frac{1}{2}}(1 + x^{\frac{1}{2}})^{-2}] \tag{35}$$

for the hypergeometric function (see Erdélyi (6), Ch. II; this is actually a form of Landen's transformation of the complete elliptic integral). In this case $\theta(1) = 1$, $\theta'(1) = 0$ and $\theta''(1) = -\frac{1}{8}$, so that the corresponding iteration process for evaluating $F(\frac{1}{2}, \frac{1}{2}; 1; x)$ is second order. However, $\chi(1) = \frac{1}{2}$ so (35) must be transformed. Setting $b = -\frac{1}{16}$, $c = \frac{1}{2}$, $\nu = 2$ and $\mu = 0$ we obtain from (32) and (33)

$$K(z) = 1 + \frac{\log z}{\log(-\frac{1}{16})}. \tag{36}$$

Although simple, this expression is inconvenient from a practical viewpoint since it involves the logarithm of a negative quantity. This snag may be eliminated by multiplying by the constant $2\log(-\frac{1}{16})$ which gives

$$k(x) = \log[(x-1)^2/256] \tag{37}$$

as the transformation function for (35).

In a later section we consider transformations which may result in $\chi(x)$ behaving like

$$\chi(x) = (-\log \xi)^{-\mu} \tag{38}$$

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near $x = 1$. When $\mu \neq 0$ it is again necessary to consider transformations of $\chi(x)$ in order to make $\chi_T(1) = 1$. The desired $K(\xi)$ must satisfy

$$K(\xi) = (-\log \xi)^{-\mu} K(b\xi^{\nu}), \tag{39}$$

at least in the limit $\xi \to 0$. Using similar arguments to those above we find for the various cases:

(*a*) First-order process with $\nu = 1$, $b \neq 1$:

$$K(z) = \{\Gamma(\log_b z)\, z^{\log_b(-\log b)}\}^{\mu}. \tag{40}$$

(*b*) Higher-order process with $b = 1$, $\nu \neq 1$:

$$K(z) = (\log z)^{\frac{1}{2}\mu \log_{\nu}(-\log z/\nu)}. \tag{41}$$

(*c*) Higher-order process with $b \neq 1$, $\nu \neq 1$:

$$K(z) = \left(\frac{\log b}{1-\nu}\right)^{\mu n} \nu^{\frac{1}{2}\mu n(n-1)}, \tag{42}$$

where n is given in terms of z by (33). This last formula is only an asymptotic solution of (39) for small ξ.

5. *Transformation of a functional equation when* $\theta(1) \neq 1$. We conclude from the previous section that the condition $\chi(1) = 1$ is not a serious restriction. Unfortunately, the second condition, $\theta(1) = 1$, is rather more restrictive. To investigate this we observe that there are two methods of transforming $\theta(x)$. One of these consists of iterating the equation n times, as in (18). This gives

$$\theta_T(x) = \theta_n(x), \tag{43}$$

$$\chi_T(x) = \prod_{l=0}^{n-1} \chi[\theta_l(x)] \tag{44}$$

and

$$F_T(x) = F(x). \tag{45}$$

A more flexible method is to put $x = f(y)$ in the original functional equation, which gives

$$\theta_T(y) = f_{-1}\{\theta[f(y)]\}, \tag{46}$$

$$\chi_T(y) = \chi[f(y)], \tag{47}$$

$$F_T(y) = F[f(y)]. \tag{48}$$

In both these cases the transformation normally alters the behaviour of $\chi(x)$ at $x = 1$. However, if the $\theta_T(x)$ is of the required nature, $\chi_T(x)$ may be transformed to standard form without altering $\theta_T(x)$ as was mentioned previously.

If we impose the condition that $\theta_T(1) = 1$ we obtain from (43) and (46)

$$\theta_n[f(1)] = f(1).$$

This means that if $x = X$ is a root of the equation

$$\theta_n(x) - x = 0, \tag{49}$$

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then any function for which $f(1) = X$ will provide a suitable transformation. For convenience we may take

$$f(y) = y - 1 + X \quad \text{or} \quad f(y) = Xy \quad (X \neq 0). \tag{50}$$

For practical purposes we require a root of (49) to be real. Even without this requirement there may be no root of (49); for example, suppose $\theta(x) = x + \exp x$.

In such a case it follows that if $\theta(x)$ is a continuous function defined for all x then, since the graph of $\theta(x)$ cannot cross $y = x$, $|\theta(x)| \to \infty$ as $x \to +\infty$ or $-\infty$, or both; of course this may also happen when there is a root of (49). In either case the transformation

$$x = f(y) = A(y-1)^{-2} \tag{51}$$

gives

$$\theta_T(y) = 1 + \{A^{-1}\theta[A(y-1)^{-2}]\}^{-\frac{1}{2}}, \tag{52}$$

and evidently $\theta_T(1) = 1$. If $|\theta(x)| \to \infty$ whenever $|x| \to \infty$ (x real), then the simpler function

$$x = f(y) = (y-1)^{-1} \tag{53}$$

suffices to make $\theta_T(1) = 1$.

If $\theta(x)$ is discontinuous none of the foregoing transformations may be useful. Sometimes, however, they may be satisfactorily applied to a suitable iterate $\theta_n(x)$. An interesting example of this is afforded by the case when $|\theta(x)| \to \infty$ as $x \to B-$ or $B+$, and $\theta(x) \to B-$ or $B+$ respectively as $x \to +\infty$ or $-\infty$; $\theta(x) = B - (x-B)^{-3}$ would be such a function. When this is so $|\theta_2(x)| \to \infty$ as $x \to +\infty$ or $-\infty$ and the transformations (51) or (53) may be used; for the example quoted, $\theta_2(x) = B + (x-B)^6$.

Thus far we have shown how a tolerably large class of functional equations may be transformed into standard form with $\chi(1) = \theta(1) = 1$. Important exceptions are the equations of Γ-function type where, even though $\theta(x)$ is amenable to a transformation by (53), $\theta_T'(1) = b = 1$ and so $\chi_T(x)$ cannot in general be transformed. Even when transformation of $\chi_T(x)$ is not necessary the corresponding iterative process is not convergent.

6. *Transformation of the order of the iterative process.* We now take up the main question of interest from the viewpoint of computation. Can the order of the iteration process (13) for a functional equation in the standard form (11) and (12) be altered by suitable transformation?

A simple case when such a transformation is possible arises if $\theta(x) - x = 0$ has a root X for which $\theta'(X) = 0$; a transformation of type (50) immediately gives a second-order process. As an example suppose $\theta(x) = \frac{7}{3}x^2 - \frac{4}{3}x^4$; this represents a first-order convergent process since $\theta(1) = 1$ and $\theta'(1) = -\frac{2}{3}$. However, $\theta(0) = 0$ and $\theta'(0) = 0$, so that writing $x = y - 1$ we obtain $\theta_T(y) = 2 + \frac{1}{3}(2y - 17y^2 + 16y^3 - 4y^4)$, whence $\theta_T(1) = 1$, $\theta_T'(1) = 0$ and $\theta''(1) = \frac{14}{3}$. Thus the new process is second order.

A less trivial case arises when $|\theta| \to \infty$ as $x \to \pm\infty$ and a singular transformation of type (51) is possible. For suppose that

$$\theta(x) \sim kx^p \quad \text{with} \quad p > 0 \quad \text{as} \quad x \to \infty; \tag{54}$$

then using (52),

$$\theta_T(y) \simeq 1 + A^{\frac{1}{2}(1-p)}k^{-\frac{1}{2}}(y-1)^p \quad (y \to 1\pm), \tag{55}$$

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so that if $p > 1$ the resulting process is better than first order. (The actual order may be fractional.) If $p < 1$ the functional equation should be inverted before transformation. Merely for the purpose of illustrating this procedure we will outline the steps in the calculation of values of $F(1, 1; \frac{3}{2}; x)$ by a second-order iterative process. For this hypergeometric function we have the relation (6)

$$F(1, 1; \tfrac{3}{2}; x) = (1-2x)\,F[1, 1; \tfrac{3}{2}; 4x(1-x)]. \tag{56}$$

This equation could be brought into standard form by observing that $\theta(0) = 0$ or $\theta(\frac{3}{4}) = \frac{3}{4}$, but the transformations based on these points would give only first-order processes. If we set

$$x = (1-y)^{-1}, \quad y = 1-x^{-1}, \tag{57}$$

we obtain a second-order process. In fact

$$\theta_T(y) = 1 + (y-1)^2/4y = \frac{1}{y}\left(\frac{y+1}{2}\right)^2, \tag{58}$$

$$\chi_T(y) = \frac{y+1}{y-1} \tag{59}$$

and

$$F_T(y) = F[1, 1; \tfrac{3}{2}; (1-y)^{-1}]. \tag{60}$$

Now $\chi_T(y)$ has a singularity at $y = 1$ and so has to be transformed to standard form. Setting $\nu = 2$, $b = \frac{1}{4}$, $\mu = -1$ and $c = 2$ in (32) gives

$$k(y) = (y-1)\log\tfrac{1}{4}(y-1), \tag{61}$$

which leads to

$$\chi_{TT}(y) = \frac{y+1}{4y}\left\{2 - \frac{\log y}{\log\frac{1}{4}(y-1)}\right\}. \tag{62}$$

Thus the equation

$$G(y) = \chi_{TT}(y)\,G[\theta_T(y)], \quad G(1) = 1 \tag{63}$$

is in standard form and leads to a second-order process for $G(y)$. We have

$$F(1, 1; \tfrac{3}{2}; -x) = Ax^{-1}\log 4x\; G(1+x^{-1}). \tag{64}$$

To evaluate the normalization constant A we use the formulae for the asymptotic continuation of the hypergeometric function (5) which yield

$$F(1, 1; \tfrac{3}{2}; -z) \sim \tfrac{1}{2}z^{-1}\log 4z \quad \text{as} \quad -z \to \infty,$$

whence $A = \frac{1}{2}$. (The continuation formulae incidentally provide a first-order process for computing the hypergeometric function.)

As a numerical example we have set $y_0 = 2$ in the process (13) for (63). After three iterations u and v coincide in the first five decimal places; a further iteration gives $u = v = 0{\cdot}899\,125\,0 = G(2)$. (If sufficient figures had been carried u and v would have agreed in the first eleven places.) On setting $y_0 = 10$ five iterations bring agreement in the first eight places giving $G(10) = -0{\cdot}262\,552\,0$. By (64) these values give

$$F(1, 1; \tfrac{3}{2}; -1) = 0{\cdot}623\,226 \quad \text{and} \quad F(1, 1; \tfrac{3}{2}; -\tfrac{1}{9}) = 1{\cdot}382\,257.$$

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Unlike the hypergeometric function $F(\frac{1}{2}, \frac{1}{2}; 1; x)$ of (35) this hypergeometric function actually reduces to an elementary function, for

$$F(1, 1; \tfrac{3}{2}; x) = x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}} \sin^{-1} x^{\frac{1}{2}}. \tag{65}$$

This provides a direct check on the values calculated by the iterative process and is obviously a better method of evaluating the function when sufficiently accurate tables of $\sin x$ are available. Our iterative process may be viewed as a second-order process for $\sin^{-1} x$ and may be used when accurate tables of $\sin x$ are not available (provided accurate log tables are).

7. *A general transformation of the order.* The methods outlined in the previous section for obtaining second-order processes both depend on particular properties of $\theta(x)$. In fact, it is easy to show that with a non-singular transformation, not only is it impossible (in general) to alter the order of a first-order process, but it is even impossible to make $\theta'_T(1)$ different from $\theta'(1)$. For from (46) we have

$$\theta'_T(y) = f'_{-1}\{\theta[f(y)]\}\,\theta'[f(y)]f'(y), \tag{66}$$

and if $f(1) = X$, where $\theta(X) - X = 0$, then

$$\theta'_T(1) = f'_{-1}(X)\,\theta'(X)f'(1) = \theta'(X) \tag{67}$$

in general. (67) only breaks down if $f'_{-1}(X)$ or $f'(X)$ vanishes. Consider, for example, the transformation

$$x - 1 = (y-1)^{1/m}, \tag{68}$$

and suppose that

$$\theta(x) = 1 + b(x-1) + O\{(x-1)^2\} \quad (|\,b\,| < 1); \tag{69}$$

then

$$\theta_T(y) = 1 + b^m(y-1) + O\{(y-1)^2\}, \tag{70}$$

so that

$$\theta'_T(1) = [\theta'(1)]^m. \tag{71}$$

This transformation therefore does not alter the order of the process but merely improves the rate of convergence (if $m > 1$), essentially by compressing m steps of the iteration into one. This example suggests that the transformation

$$x = 1 + [\log A(y-1)]^{-1/m} \tag{72}$$

might be more powerful. Indeed, if $\theta(x)$ is given by (69), and (72) is used, we obtain

$$\theta_T(y) = 1 + A^{b^{-m}-1}(y-1)^{b^{-m}} \tag{73}$$

for small $|\,y-1\,|$. The order of the corresponding process is b^{-m} which, since $|\,b\,| < 1$, can be made as large as we please by choosing m large enough. (72) may result in $\chi_T(y)$ behaving as $[\log(y-1)]^{-\mu}$ at $y = 1$, but as shown in § 4 ((40), (41) and (42)) a transformation to standard form is still possible.

Evidently the transformation (72) always enables us to obtain a process of any order. This appears to mean that we can always find a second-order iterative process which yields the solution of the functional equation (12). That this is unfortunately not so may be seen when we notice that after applying a transformation of type (72) the behaviour of $\chi(x)$ for small ξ is, in general, given by

$$\chi(x) \simeq 1 + C[\log \xi]^{-l} \quad (l > 0). \tag{74}$$

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Suppose now we have an apparently higher-order process,

$$\theta(x) \simeq 1 + b\xi^{\nu} \quad (\nu > 1), \tag{75}$$

so that $$\theta_n(x) \simeq 1 + b^{s(n)}\xi^{\nu^n}, \quad s(n) = \frac{\nu^n - 1}{\nu - 1}.$$

Then for small ξ

$$\chi[\theta_n(x)] - 1 \simeq C(\log \xi)^{-1}\, \nu^{-ln}, \tag{76}$$

which indicates that $\chi[\theta_n(x)] \to 1$ only with first-order convergence (see (18)). In other words, although u_n and v_n in (13) may tend to each other at a rate which may be made as high as we please, we cannot alter the rate at which they approach their common limit $F(x)$. Thus the transformation (72), in general, yields processes which are only pseudo-second order and which are of little interest to the computer. There are, however, certain equations for which (72) leads to useful results. A particular case is the equation originally considered by Schröder (4) in which $\chi(x) = \chi_T(y) = c$.

It might also be remarked that if $\chi(x) = 1 + c\xi^{\mu} + O(\xi^{\mu+1})$ and $\mu > 1$, then u_n will tend to $F(x)$ more rapidly than will v_n although the order of the process is unaltered.

8. *Conclusions.* Having defined a degenerate iterative process in two variables we showed in § 2 how it is connected with a functional equation of type

$$F(x) = \chi(x)\, F[\theta(x)],$$

with $\chi(1) = 1$ and $\theta(1) = 1$. In § 3 we discussed the convergence of the iterative process for this equation and showed that it was largely determined by the function $\theta(x)$. If $\theta'(1) = 0$ we have a second- or higher-order process; otherwise the process is first order and we need $|\theta'(1)| < 1$ for convergence.

Functional equations in which $\chi(1) \neq 1$ but $\theta(1) = 1$ may be transformed to standard type by the transformations of § 4 whenever $\chi(x)$ is algebraic in x or in $\log(1-x)$ at $x = 1$. In deriving these transformations we presented a general method of solving functional equations which may be useful when the classical method is not available.

If $\theta(1) \neq 1$ transformation to standard form is more difficult. In § 5 we showed that it is always possible if there is a solution of $\theta_n(x) - x = 0$. If $|\theta(x)| \to \infty$ as $x \to \pm\infty$ a singular transformation is available. Γ-function type functional equations do not seem amenable to transformation.

The problem of transforming the order of the iterative process was discussed in § 6, where it was shown to be possible if $\theta(x)$ fulfilled certain relatively stringent conditions, in particular, if $\theta(x) \sim Ax^p$ as $x \to \pm\infty$, where $p > 0$, $p \neq 1$. In § 7 a general transformation to any order of process was presented. However, this turned out only to be of limited practical utility owing to the subsequent logarithmic behaviour of $\chi_T(x)$. No generally useful transformation to higher order appears to exist.

Apart from the elliptic integrals mentioned in §§ 1 and 4, few of the non-elementary functions of classical analysis seem to satisfy functional equations which enable them to be calculated by our methods. In certain cases, however, a second-order process may be useful even for an elementary function, as was mentioned in § 6 where $F(1, 1; \frac{3}{2}; x)$ was calculated.

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It should be noted that the theory of transformations of the functional equation (12) developed in this paper is applicable with trivial alteration to the equation

$$G(x) = G[\theta(x)] + \lambda(x), \tag{77}$$

analogous to the equation first considered by Abel. Since addition replaces multiplication in this approach it may be useful in some applications.

The general ideas of this note are easily extended to deal with degenerate processes in three and more variables and, correspondingly, with functional equations in many variables, of the general form

$$F(x_1, x_2, \ldots, x_m) = \chi(x_1, x_2, \ldots, x_m)\, F[\theta_1(x_1, x_2, \ldots, x_m)]. \tag{78}$$

Questions of convergence are now more difficult since the behaviour is determined by the latent roots of the matrix $\partial\theta_i/\partial x_j(1, 1, \ldots, 1)$.

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WHEATSTONE PHYSICS LABORATORY
KING'S COLLEGE, LONDON

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