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On arithmetically realizable classes

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1. *Introduction*

We fix a number field L and a finite group G, and write $\mathrm{Cl}(\mathbb{Z}_L[G])$ for the reduced Grothendieck group of the category of finitely generated projective $\mathbb{Z}_L[G]$-modules. We let R_G denote the ring of complex characters of G, with S_G the additive subgroup which is generated by the irreducible symplectic characters. We shall say that an element $\mathfrak{c} \in \mathrm{Cl}(\mathbb{Z}_L[G])$ is '(arithmetically) realizable' if there exists a tamely ramified Galois extension N/K of number fields with $L \subseteq K$ and an identification $\mathrm{Gal}(N/K) \xrightarrow{\sim} G$ via which $\mathfrak{c}$ is the class of some $\mathrm{Gal}(N/K)$-stable $\mathbb{Z}_N$-ideal. We let $\mathfrak{R}_L(G)$ denote the subgroup of $\mathrm{Cl}(\mathbb{Z}_L[G])$ which is generated by the realizable elements for varying N/K. Our interest in $\mathfrak{R}_L(G)$ arises from the fact that it is the largest subset of $\mathrm{Cl}(\mathbb{Z}_L[G])$ upon which the results of Chinburg and the author in [Bu,Ch] can be used to give an explicit module theoretic description of the action of the integral semi-group ring $\mathscr{A}_{L,G}$ of the Adams–Cassou–Noguès–Taylor operators $\{\Psi_{L,k} : k \in \mathbb{Z}, 2 \nmid k \text{ if } S_G \neq \{0\}\}$. Whilst the results of [Bu,Ch] can (at least partially) be understood 'geometrically' via the action of Bott cannibalistic elements on suitable Grothendieck groups (cf. [Ch,E,P,T], [Bu]), the underlying problem of finding an explicit module theoretic interpretation of the action of $\mathscr{A}_{L,G}$ on all elements of $\mathrm{Cl}(\mathbb{Z}_L[G])$ is of course essentially algebraic in nature. It is in this context that we were originally motivated to investigate $\mathfrak{R}_L(G)$. However, such realizability issues have in any case an intrinsic interest, and indeed certain aspects are already well understood. McCulloh has studied (and in the abelian case completely characterized) the subset $\mathfrak{R}_{L,0}(G)$ of $\mathrm{Cl}(\mathbb{Z}_L[G])$ which consists of the classes of $\mathbb{Z}_N$ as N/K varies over all tame extensions as above (cf. [M1], and for the non-abelian case [M2]). In addition, Wilson has shown that for any given tame extension N/K of group G the set of elements of $\mathrm{Cl}(\mathbb{Z}_K[G])$ which arise from all G-stable $\mathbb{Z}_N$-ideals is completely determined by the class of $\mathbb{Z}_N$ together with knowledge of the inertia subgroups of G and their standard representations over the respective residue fields of $\mathbb{Z}_K$ ([W]). These results however do not directly give explicit information concerning $\mathfrak{R}_L(G)$, and those explicit results which are known are only for very special classes of extensions (cf. for example [U]).

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We shall in this note deal only with coefficient fields L which are absolutely unramified at each prime divisor of $\#G$. In this case we shall (given the results of [M1]) explicitly describe $\mathfrak{R}_L(G)$ if G is abelian, and more generally describe strong restrictions on $\mathfrak{R}_L(G)$ as a subgroup of $\mathrm{Cl}(\mathbb{Z}_L[G])$. Since for any number field L the index of $\mathrm{res}^L_{\mathbb{Q}}\mathrm{Cl}(\mathbb{Z}_L[G])$ in $\mathrm{Cl}(\mathbb{Z}[G])$ is easily bounded explicitly and $\mathrm{res}^L_{\mathbb{Q}}\mathfrak{R}_L(G) \subseteq \mathfrak{R}_{\mathbb{Q}}(G)$ our result also sheds light on this issue for general coefficient fields L.

We state here a special case of the results proved in Section 2, and for this we need some notation. We call a tamely ramified extension N/K 'locally abelian' if the decomposition subgroup of each ramified prime is abelian. We let $\mathfrak{R}^a_L(G)$, resp. $\mathfrak{R}^a_{L,0}(G)$, denote the subgroup of $\mathfrak{R}_L(G)$, resp. $\mathfrak{R}_{L,0}(G)$, generated by elements which are realized by extensions N/K as above which are locally abelian. If G is abelian then of course $\mathfrak{R}^a_L(G) = \mathfrak{R}_L(G)$ and $\mathfrak{R}^a_{L,0}(G) = \mathfrak{R}_{L,0}(G)$. For any cyclic group C we let $\mathrm{Cl}(\mathbb{Z}_L[C])_\Sigma$ denote the subgroup of elements of $\mathrm{Cl}(\mathbb{Z}_L[C])$ which can be represented (in the ideal-theoretic Hom-description) by functions which are supported only above prime $\mathbb{Z}_L$-ideals which have absolute norm congruent to 1 modulo $\#C$. This is in general a proper subgroup of $\mathrm{Cl}(\mathbb{Z}_L[C])$. We let $\mathscr{C}$ denote the set of cyclic subgroups of G, and write $\mathrm{Cl}(\mathbb{Z}_L[G])_{\mathscr{C}}$, respectively $\mathrm{Cl}(\mathbb{Z}_L[G])_{\mathscr{C},\Sigma}$, for the subgroup $\sum_{C\in\mathscr{C}}\mathrm{ind}^G_C\mathrm{Cl}(\mathbb{Z}_L[C])$, respectively $\sum_{C\in\mathscr{C}}\mathrm{ind}^G_C\mathrm{Cl}(\mathbb{Z}_L[C])_\Sigma$, of $\mathrm{Cl}(\mathbb{Z}_L[G])$.

THEOREM 1·1. *Let L be a number field which is absolutely unramified at each prime divisor of $\#G$. Then one has $\mathfrak{R}^a_L(G) \subseteq \mathfrak{R}^a_{L,0}(G) + \mathrm{Cl}(\mathbb{Z}_L[G])_{\mathscr{C},\Sigma}$, with equality if G is abelian.*

Remarks 1·2. (i) Let $A(G)$ denote the Artin exponent of G. Since $\mathrm{Cl}(\mathbb{Z}_L[G])$ is a Frobenius module over the Frobenius functor $G_0(L[G])$ one has

$$A(G)\,\mathrm{Cl}(\mathbb{Z}_L[G]) \subset \mathrm{Cl}(\mathbb{Z}_L[G])_{\mathscr{C}}$$

(cf. [C,R], 38·14). In general however, the quotient $\mathrm{Cl}(\mathbb{Z}_L[G])/(\mathfrak{R}_{L,0}(G) + \mathrm{Cl}(\mathbb{Z}_L[G])_{\mathscr{C}})$ is large. For example, if p is a prime and G a non-cyclic group of order p^2 then $A(G) = p$, whilst if $L = \mathbb{Q}$ then $\mathfrak{R}_{\mathbb{Q},0}(G) = 0$ (by M. Taylor's theorem), $\#D(\mathbb{Z}[G])^-$ is $O(p^{p^2})$ (cf. [C,R], 50·50), and if p is regular then the index of $\mathrm{Cl}(\mathbb{Z}[G])_{\mathscr{C}}$ in $\mathrm{Cl}(\mathbb{Z}[G])$ is $\#D(\mathbb{Z}[G])$.

(ii) Since cyclotomic fields are genus extensions of $\mathbb{Q}$ it is not difficult to show that $\mathrm{Cl}(\mathbb{Z}[C]) = D(\mathbb{Z}[C] + \mathrm{Cl}(\mathbb{Z}[C])_\Sigma$ for any cyclic group C. I know of no group G for which $\mathrm{Cl}(\mathbb{Z}[G])_{\mathscr{C}} \neq \mathrm{Cl}(\mathbb{Z}[G])_{\mathscr{C},\Sigma}$, but this seems likely to be the case for most G. However, this is not so important when attempting to explicitly interpret the Adams operators of [C-N,T]. Indeed, Oliver has shown that $D(\mathbb{Z}[C])$ is a sum of generalized Swan groups attached to prime power order subgroups of C, and on these it is easy to explicitly describe the action of $\mathscr{A}_{\mathbb{Q},C}$ (cf. Appendix).

(iii) The main result of [Bu,Ch] implies that $\mathfrak{R}_L(G)$ is preserved under the action of $\mathscr{A}_{L,G}$, and this is a strong restriction on abstract subgroups of $\mathrm{Cl}(\mathbb{Z}_L[G])$. However, obtaining explicit information beyond the inclusion given in Theorem 1·1 is difficult for two reasons. Firstly, any restrictions on elements of $\mathrm{Cl}(\mathbb{Z}_L[G])$ arising from explicit realizations in extensions which are not locally abelian will depend on the precise conjugation action of each decomposition subgroup on its inertia subgroup (cf. §2). In addition, any existence results would require solutions to tame embedding

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problems of the type considered by Brinkhuis, and even now such a theory is satisfactory only for extensions with abelian kernel (cf. [Br], [P]).

(iv). There are results analogous to those of [M1] and also to Theorem 1·1 in the setting of tame abelian covers of smooth projective curves defined over finite fields (cf. [A,B]).

2. *On realizable classes*

We fix a finite group G and a number field L which is absolutely unramified at each prime divisor of $\#G$, and shall first analyse restrictions on elements of $\mathrm{Cl}(\mathbb{Z}_L[G])$ which arise from explicit realizations in the sense described above. In particular, we shall show that

$$\mathfrak{R}^a_L(G) \subseteq \mathfrak{R}^a_{L,0}(G) + \mathrm{Cl}(\mathbb{Z}_L[G])_{\mathscr{E},\Sigma}. \tag{2·1}$$

Our analysis relies upon the functional behaviour of the Hom-description of $\mathrm{Cl}(\mathbb{Z}_L[G])$ together with standard properties of tame local extensions.

Suppose then that $\mathfrak{c} \in \mathrm{Cl}(\mathbb{Z}_L[G])$ is realizable, corresponding to an extension N/K, an identification of $\mathrm{Gal}(N/K)$ with G (which we henceforth suppress) and a G-stable $\mathbb{Z}_N$-ideal $\mathfrak{U}$. Since obviously $(\mathbb{Z}_N) \in \mathfrak{R}_{L,0}(G)$ and $\mathfrak{c} = (\mathbb{Z}_N) + (\mathfrak{U}) - (\mathbb{Z}_N)$, we shall focus upon the class $\mathfrak{c}' := (\mathfrak{U}) - (\mathbb{Z}_N) \in \mathrm{Cl}(\mathbb{Z}_L[G])$.

To make full use of Hom-descriptions we must first introduce some notation. We choose a finite field extension E of N which is Galois over L, and such that all representations of all subgroups of G can be realized (up to conjugacy) over E. For each place $\mathfrak{r}$ of E, we choose an embedding $j_0 : E \hookrightarrow E_\mathfrak{r}$ which corresponds to $\mathfrak{r}$. We let $\mathfrak{q}$, resp. $\mathfrak{p}$, resp. $\wp$, denote the place of N, resp. K, resp. L, which is associated to the embedding $j_0 | N$, resp. $j_0 | K$, resp. $j_0 | L$. We identify the completion of E at any place above $\wp$ with $E_\mathfrak{r}$, and let J denote a set of embeddings $E \hookrightarrow E_\mathfrak{r}$ which are in bijective correspondence with the prime divisors of E above $\wp$ under the mapping $j \mapsto \mathfrak{r}_j$. We assume that if $\mathfrak{r}_j$ lies above $\mathfrak{q}$, resp. $\mathfrak{p}$, resp. $\wp$, then the restriction of j to N, resp. K, resp. L, coincides with that of j_0. We write r for the associated Ω_L-equivariant embedding of $(E \otimes_L L_\wp)^*$ into the group of ideles $\mathscr{I}(E)$ of E. We let $M_\mathfrak{q}$ denote the maximal unramified extension of $K_\mathfrak{p}$ in $N_\mathfrak{q}$, and write $G_\mathfrak{q}$, respectively $I_\mathfrak{q}$, for the Galois group of $N_\mathfrak{q}/K_\mathfrak{p}$, respectively $N_\mathfrak{q}/M_\mathfrak{q}$. Both $G_\mathfrak{q}$ and $I_\mathfrak{q}$ are identified with subgroups of G via the embedding j_0. We choose a $\mathbb{Z}_{K_\mathfrak{p}}[G_\mathfrak{q}]$-generator $a_{1,\mathfrak{q}}$, resp. $a_{2,\mathfrak{q}}$ for $\mathfrak{U}_\mathfrak{q}$, resp. $\mathbb{Z}_{N_\mathfrak{q}}$, and for each of $i = 1, 2$ let $b(q_i)_\mathfrak{p} \in N \otimes_K K_\mathfrak{p} \subseteq E \otimes_L L_\wp$ have $\mathfrak{r}_j$-component $a_{i,\mathfrak{q}}$, resp. 0, if $j = j_0$, resp. $j \neq j_0$. Then $b(a_1)_\mathfrak{p}$, resp. $b(a_2)_\mathfrak{p}$, is a $Z_{K_\mathfrak{p}}[G]$-generator of the semi-localization of $\mathfrak{U}$, resp. $\mathbb{Z}_N$, at $\mathfrak{p}$. For each place $\mathfrak{p} | \mathfrak{A}$ we let $g_\mathfrak{p} \in \mathrm{Hom}_{\Omega_L}(R_G, \mathscr{I}(E))$ denote the homomorphism which for each $\chi \in R_G$ has trivial archimedean components, and for each maximal $\mathbb{Z}_L$-ideal $\wp_1$ has $\wp_1$-component given by

$$g_\mathfrak{p}(\chi)_{\wp_1} = \begin{cases} \mathscr{N}_{K/L}((b(a_1)_\mathfrak{p} | \chi)^r / (b(a_2)_\mathfrak{p} | \chi)^r), & \text{if } \wp_1 = \wp; \\ 1, & \text{otherwise,} \end{cases} \tag{2·2}$$

where here $\mathscr{N}_{K/L}(- | \chi)$ is the norm-resolvent of ([F], chapter I). Writing $\mathfrak{c}_\mathfrak{p}$ for the element of $\mathrm{Cl}(\mathbb{Z}_L[G])$ which is represented by $g_\mathfrak{p}$ in the idelic Hom-description, standard theory implies that $\mathfrak{c}' = \sum_{\mathfrak{p} | \mathfrak{U}} \mathfrak{c}_\mathfrak{p}$ (cf. [F], chapter I). There is a natural conjugation action of $G_\mathfrak{q}/I_\mathfrak{q}$ on $I_\mathfrak{q}$, and we let $t_\mathfrak{p}$ denote the exponent of this action (which indeed only depends on $\mathfrak{p}$). We shall show that

$$t_\mathfrak{p}\, \mathfrak{c}_\mathfrak{p} \in \mathrm{ind}_G^{I_\mathfrak{q}} \mathrm{Cl}(\mathbb{Z}_L[I_\mathfrak{q}]) \tag{2·3}$$

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for each $\mathfrak{p} \mid \mathfrak{U}$. Since N/K is tame each $I_\mathfrak{q}$ is cyclic, and if $N_\mathfrak{q}/K_\mathfrak{p}$ is abelian then of course $t_\mathfrak{p} = 1$. In particular therefore, if N/K is locally abelian then (2·3) for each $\mathfrak{p} \mid \mathfrak{U}$ implies that $\mathfrak{c}' \in \mathrm{Cl}(\mathbb{Z}_L[G])_\mathscr{C}$, as is asserted in Theorem 1·1. We now fix places $\mathfrak{r}$, $\mathfrak{q}$, $\mathfrak{p}$, and $\wp$ as above with $\mathfrak{q} \mid \mathfrak{U}$, and write D for $G_\mathfrak{q}$, I for $I_\mathfrak{q}$, t for $t_\mathfrak{p}$, and M for the subfield N^I. If we let ind_G^I also denote the homomorphism

$$\mathrm{Hom}_{\Omega_L}(R_I, \mathscr{I}(E)) \to \mathrm{Hom}_{\Omega_L}(R_G, \mathscr{I}(E))$$

which is induced by the restriction map on characters $\mathrm{res}_I^G : R_G \to R_I$ then the following diagram commutes

$$\begin{array}{ccc} \mathrm{Hom}_{\Omega_L}(R_I, \mathscr{I}(E)) & \longrightarrow & \mathrm{Cl}(\mathbb{Z}_L[I]) \\ \Big\downarrow{\scriptstyle \mathrm{ind}_G^I} & & \Big\downarrow{\scriptstyle \mathrm{ind}_G^I} \\ \mathrm{Hom}_{\Omega_L}(R_G, \mathscr{I}(E)) & \longrightarrow & \mathrm{Cl}(\mathbb{Z}_L[G]) \end{array} \tag{2·4}$$

(cf. [F], theorem 12). Writing $\mathscr{U}\mathbb{Z}_L[G]$ for the group of unit ideles of $\mathbb{Z}_L[G]$ the group $\mathrm{Det}(\mathscr{U}\mathbb{Z}_L[G])$ lies in the denominator of the idelic Hom-description of $\mathrm{Cl}(\mathbb{Z}_L[G])$. Hence (2·3) will follow if we can define a homomorphism $g'_\mathfrak{p} \in \mathrm{Hom}_{\Omega_L}(R_I, \mathscr{I}(E))$ such that

$$(g_\mathfrak{p})^{t_\mathfrak{p}} \equiv g'_\mathfrak{p} \circ \mathrm{res}_I^G \text{ modulo } \mathrm{Det}(\mathscr{U}\mathbb{Z}_L[G]). \tag{2·5}$$

For each $\chi \in R_G$ we set $\chi_\mathfrak{p} := \mathrm{res}_D^G \chi \in R_D$, and for each $j \in J$ we then let $\chi_\mathfrak{p}^j$ denote the composite $\wp$-adic character $j \circ \chi_\mathfrak{p}$. Now the proof of ([F], theorem 19) (cf. also ([Bu,Ch], proposition 4·1)) implies that for each such character χ and embedding j there is an equality

$$\left(\mathscr{N}_{K/L} \frac{(b(a_1)_\mathfrak{p} \mid \chi)^r}{(b(a_2)_\mathfrak{p} \mid \chi)^r}\right)_{\tau_j} = \mathscr{N}_{K_\mathfrak{p}/L_\wp} \frac{(a_1 \mid \chi_\mathfrak{p}^j)}{(a_2 \mid \chi_\mathfrak{p}^j)}. \tag{2·6}$$

Next we use the proof of ([F], theorem 25(ii)) to note that for any free generator c_1, resp. c_2 of $\mathfrak{U}_\mathfrak{q}$, resp. $\mathbb{Z}_{N_\mathfrak{q}}$, over $\mathbb{Z}_{M_\mathfrak{q}}[I]$ there is a unit element λ_j of $\mathbb{Z}_{M_\mathfrak{q}}[D]$ (depending in fact on a_1, a_2, c_1, c_2, and j) which is such that for each $\chi \in R_G$ one has

$$\frac{(a_1 \mid \chi_\mathfrak{p}^j)}{(a_2 \mid \chi_\mathfrak{p}^j)} = \frac{(c_1 \mid \mathrm{res}_I^D \chi_\mathfrak{p}^j)}{(c_2 \mid \mathrm{res}_I^D \chi_\mathfrak{p}^j)} \mathrm{Det}(\lambda_j)(\chi_\mathfrak{p}^j). \tag{2·7}$$

To further analyse the right hand side of (2·7) we make use of the arithmetic of $N_\mathfrak{q}/K_\mathfrak{p}$. We let $v_\mathfrak{q}$ denote the standard valuation of $N_\mathfrak{q}$, and identify $\mathfrak{q}$ with the valuation ideal of $v_\mathfrak{q}$. We let $R_{D,\mathfrak{r}}$, respectively $R_{I,\mathfrak{r}}$, denote the ring of $E_\mathfrak{r}$-characters of D, respectively I, and write $I^\vee$ for the group of abelian $E_\mathfrak{r}$-characters of I. Since $N_\mathfrak{q}/M_\mathfrak{q}$ is totally tamely ramified the values of each element of $I^\vee$ lie in $M_\mathfrak{q}^*$, and one knows that the composition action of $D/I = \mathrm{Gal}(M_\mathfrak{q}/K_\mathfrak{p})$ on $I^\vee$ coincides with its conjugation action on I (that is, in each case any given element of D/I acts by raising elements to the same power). It follows that, for each character $\phi \in I^\vee$, the cardinality $t_D(\phi)$ of the D/I-orbit of ϕ is a divisor of t. Furthermore, if for each $\phi \in I^\vee$ we let e_ϕ denote the corresponding idempotent of $\mathbb{Z}_{M_\mathfrak{q}}[I]$ then there is a canonical generator $\chi_{N_\mathfrak{q}/M_\mathfrak{q}}$ of $I^\vee$ which is such that for each $x \in N_\mathfrak{q}$ and $\phi \in I^\vee$ for which $xe_\phi \neq 0$ one has $v_\mathfrak{q}(xe_\phi) \geqslant v_\mathfrak{q}(x)$, with equality if and only if $\phi = \chi_{N_\mathfrak{q}/M_\mathfrak{q}}^{v_\mathfrak{q}(x)}$. (For each of these results see for example ([Be], §1). Note also that the character $\chi_{N\mathfrak{q}/M_\mathfrak{q}}$ coincides with the character φ of ([Ch,E,P,T], Proposition 2·8) as defined for the extension $N_\mathfrak{q}/M_\mathfrak{q}$.)

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LEMMA 2·8. *For each choice of $\mathbb{Z}_{M_\mathfrak{q}}[I]$-generator c_1 of $\mathfrak{U}_\mathfrak{q}$ there is a $\mathbb{Z}_{M_\mathfrak{q}}[I]$-generator $\hat{c}_1$ of $\mathbb{Z}_{N_\mathfrak{q}}$ for which the function $h_\mathfrak{p}(c_1, \hat{c}_1)$ defined on $I^\vee$ by $h_\mathfrak{p}(c_1, \hat{c}_1)(\phi) := (c_1 \mid \phi)/(\hat{c}_1 \mid \phi)$ takes values in $K_\mathfrak{p}^*$.*

Proof. For any integer a one has $\mathfrak{q}^a = \bigoplus_{\phi \in I^\vee} \mathfrak{q}^a e_\phi$ where each $\mathfrak{q}^a e_\phi$ is a rank one $\mathbb{Z}_{M_\mathfrak{q}}$-lattice. Given the above described property of $\chi_{N_\mathfrak{q}/M_\mathfrak{q}}$ it follows that $\mathfrak{q}^a e_\phi = \mathfrak{q}^{a(\phi)} e_\phi$ where here $a(\phi)$ is the least integer satisfying both $a(\phi) \geqslant a$ and $\phi = \chi_{N_\mathfrak{q}/M_\mathfrak{q}}^{a(\phi)}$. Now for any integers a and b one has for each character $\phi \in I^\vee$ a congruence $a(\phi) \equiv b(\phi)$ modulo $\#I$, and hence there exists an element $x(a, b, \phi) \in K_\mathfrak{p}^*$ such that $\mathfrak{q}^b e_\phi = x(a, b, \phi)\, \mathfrak{q}^a e_\phi$. In particular therefore, having chosen a $\mathbb{Z}_{M_\mathfrak{q}}[I]$-generator c_1 of $\mathfrak{U}_\mathfrak{q}$, the element $\hat{c}_1 := \sum_{\phi \in I^\vee} x(v_\mathfrak{q}(\mathfrak{U}), 0, \phi)\, c_1 e_\phi$ is a $\mathbb{Z}_{M_\mathfrak{q}}[I]$-generator of $\mathbb{Z}_{N_\mathfrak{q}}$. Furthermore, for each $\phi \in I^\vee$ one has $h_\mathfrak{p}(c_1, \hat{c}_1)(\phi) = c_1 e_\phi / x(v_\mathfrak{q}(\mathfrak{U}), 0, \phi)\, c_1 e_\phi = x(v_\mathfrak{q}(\mathfrak{U}), 0, \phi)^{-1} \in K_\mathfrak{p}^*$. ∎

For each character $\phi \in I^\vee$ we let $T_D \phi \in R_{I,\mathfrak{r}}$ denote the sum over the D/I-orbit of ϕ. With $\hat{c}_1$ chosen as in (2·8) we define $\tilde{h}_\mathfrak{p}(c_1, \hat{c}_1)$ on $I^\vee$ by setting

$$\tilde{h}_\mathfrak{p}(c_1, \hat{c}_1)(\phi) = h_\mathfrak{p}(c_1, \hat{c}_1)^{t/t_D(\phi)}(T_D \phi) \in K_\mathfrak{p}^*.$$

Then $\tilde{h}_\mathfrak{p}(c_1, \hat{c}_1) \in \mathrm{Hom}_{\Omega_{K_\mathfrak{p}}}(R_{I,\mathfrak{r}}, E_\mathfrak{r}^*)$ and (by Frobenius reciprocity) the homomorphisms $\tilde{h}_\mathfrak{p}(c_1, \hat{c}_1)$ and $h_\mathfrak{p}(c_1, \hat{c}_1)^t$ agree on $\mathrm{res}_I^D R_{D,\mathfrak{r}}$. Note that in general $h_\mathfrak{p}(c_1, \hat{c}_1)^t$ is the smallest power of $h_\mathfrak{p}(c_1, \hat{c}_1)$ whose restriction to $\mathrm{res}_I^D R_{D,\mathfrak{r}}$ is the restriction of an element of $\mathrm{Hom}_{\Omega_{K_\mathfrak{p}}}(R_{I,\mathfrak{r}}, E_\mathfrak{r}^*)$.

Example 2·9. Let $N_\mathfrak{q}/K_\mathfrak{p}$ be any non-abelian Galois extension. Then, if c_1 is any $\mathbb{Z}_{M_\mathfrak{q}}[I]$-generator of $\mathfrak{q}^2$ there is no choice of $\mathbb{Z}_{M_\mathfrak{q}}[I]$-generator c_2 of $\mathbb{Z}_{N_\mathfrak{q}}$ for which the function $h_\mathfrak{p}(c_1, c_2)$ is $\Omega_{K_\mathfrak{p}}$-equivariant on $R_{I,\mathfrak{r}}$. It is of course true that with $\hat{c}_1$ chosen as in (2·8) the restriction of $h_\mathfrak{p}(c_1, \hat{c}_1)$ to $\mathrm{res}_I^D R_{D,\mathfrak{r}}$ is $\Omega_{K_\mathfrak{p}}$-equivariant, and so one's main interest is in finding an $\Omega_{K_\mathfrak{p}}$-equivariant homomorphism which completes the triangle

$$\begin{array}{ccc} \mathrm{res}_I^D R_{D,\mathfrak{r}} & \xrightarrow{\subset} & R_{I,\mathfrak{r}} \\ {\scriptstyle h_\mathfrak{p}(c_1, \hat{c}_1)}\downarrow & \swarrow & \\ E_\mathfrak{r}^* & & \end{array}$$

However, even if D is dihedral of order 6, there is in general no $\Omega_{K_\mathfrak{p}}$-equivariant completion of this triangle.

We now fix a choice of c_1 and $\hat{c}_1$ as in (2·8), write $\tilde{h}_\mathfrak{p}$ for $\tilde{h}_\mathfrak{p}(c_1, \hat{c}_1)$, and let $\hat{\lambda}_j$ denote the unit λ_j of $\mathbb{Z}_{M_\mathfrak{q}}[D]$ which satisfies (2·7) with respect to these choices. We let $F_\mathfrak{r}$ denote the maximal unramified extension of $L_\mathfrak{p}$ in $E_\mathfrak{r}$. Since $M_\mathfrak{q}/K_\mathfrak{p}$ is unramified, one has $\mathcal{N}_{K_\mathfrak{p}/L_\mathfrak{p}}(\mathrm{Det}(\hat{\lambda}_j)^t) = \mathrm{Det}(\mu_j)$ for some unit $\mu_j \in \mathbb{Z}_{F_\mathfrak{r}}[D]^*$ (cf. [F], theorem 13). For each $\chi \in R_G$ and $j \in J$ the equation (2·7) thus implies that

$$\left(\mathcal{N}_{K_\mathfrak{p}/L_\mathfrak{p}} \frac{(a_1 \mid \chi_\mathfrak{p}^j)}{(a_2 \mid \chi_\mathfrak{p}^j)}\right)^t = (\mathcal{N}_{K_\mathfrak{p}/L_\mathfrak{p}} \tilde{h}_\mathfrak{p}(\mathrm{res}_I^D \chi_\mathfrak{p}^j))\, \mathrm{Det}(\mu_j)(\chi_\mathfrak{p}^j). \tag{2·10}$$

To pass from this to a global equality we define a function $g_\mathfrak{p}' \in \mathrm{Hom}(R_I, (E \otimes_L L_\mathfrak{p})^*)$ by specifying its components at each $\mathfrak{r}_j$: for each $\phi \in R_I$ and $j \in J$ we set

$$(g_\mathfrak{p}'(\phi)^\tau)_{\mathfrak{r}_j} = \mathcal{N}_{K_\mathfrak{p}/L_\mathfrak{p}} \tilde{h}_\mathfrak{p}(\phi^j). \tag{2·11}$$

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Since $\tilde{h}_\mathfrak{p}$ is $\Omega_{K_\mathfrak{p}}$-equivariant it follows that $\mathcal{N}_{K_\mathfrak{P}/L_\mathfrak{p}}\tilde{h}_\mathfrak{p}$ is $\Omega_{L_\mathfrak{p}}$-equivariant and hence one has

$$g'_\mathfrak{p} \in \mathrm{Hom}_{\Omega_L}(R_I, (E \otimes_L L_\mathfrak{p})^*). \tag{2·12}$$

Next, we let $\mu_\mathfrak{p} \in \prod_{j\in J} \mathbb{Z}_{F_{\mathfrak{r}_j}}[D]^* \subseteq \prod_{j\in J} \mathbb{Z}_{F_{\mathfrak{r}_j}}[G]^*$ denote the unit specified by

$$(\mu_\mathfrak{p})_{\mathfrak{r}_j} = \mu_j \in \mathbb{Z}_{F_{\mathfrak{r}_j}}[D]^* \tag{2·13}$$

for each $j \in J$. For each $\chi \in R_G$ and $j \in J$ this satisfies

$$(\mathrm{Det}(\mu_\mathfrak{p})(\chi))_{\mathfrak{r}_j} = \mathrm{Det}(\mu_j)(\chi^j) = \mathrm{Det}(\mu_j)(\chi^j_\mathfrak{p}). \tag{2·14}$$

Now from (2·2), (2·6), (2·10), (2·11) and (2·14) it follows that for each $\chi \in R_G$ one has

$$g_\mathfrak{p}(\chi)^t = g'_\mathfrak{p}(\mathrm{res}^G_I \chi)^r \mathrm{Det}(\mu_\mathfrak{p})(\chi). \tag{2·15}$$

Given (2·12) the congruence (2·5) will thus follow if we can show that

$$\mathrm{Det}(\mu_\mathfrak{p}) \in \mathrm{Det}(\mathbb{Z}_{L_\mathfrak{p}}[G]^*). \tag{2·16}$$

Note firstly that since both $g_\mathfrak{p}$ and $g'^r_\mathfrak{p}$ are Ω_L-equivariant the equation (2·15) implies that $\mathrm{Det}(\mu_\mathfrak{p})$ is Ω_L-equivariant. In particular therefore, if $\mathfrak{p} \nmid \#G$ then the inclusion (2·16) is obvious. On the other hand, if $\mathfrak{p} \mid \#G$ then by our assumption on L the completion $L_\mathfrak{p}$ is an absolutely unramified field, and so one can deduce (2·16) by applying Taylor's fixed point theorem for the group determinant ([T], chapter 8, theorem 1·4) just as in the proof of ([F], theorem 10).

To conclude the proof of (2·1) we shall now show that if in the above analysis $N_\mathfrak{q}/K_\mathfrak{p}$ is abelian then the class $\mathfrak{c}_2 \in \mathrm{Cl}(\mathbb{Z}_L[I])$ which is represented by the function $g'_\mathfrak{p}$ of (2·12) actually belongs to $\mathrm{Cl}(\mathbb{Z}_L[I])_\Sigma$. To do this (and also what follows afterwards) it is convenient to switch to an ideal-theoretic Hom-description, and so we shall first recall a few details.

Let us fix a cyclic subgroup C of G. We set $k := \#C$, and fix a primitive root of unity $\zeta \in \mathbb{Q}^c$ of order k. We write $\mathscr{D}_C$ for the set of Ω_L-conjugacy classes of $\mathbb{Q}^c$-valued abelian characters of C – since $L/\mathbb{Q}$ is unramified at each prime divisor of $\#G$ this set coincides with the set of $\Omega_\mathbb{Q}$-conjugacy classes of characters. It follows that if for each $D \in \mathscr{D}_C$ we let $|D|$ denote the (common) order of the kernel of each $\chi \in D$ then the relation $\succ$ defined on $\mathscr{D}_C$ by $D_1 \succ D_2$ if $|D_1| > |D_2|$ is a linear ordering. For each $\mathbb{Q}^c$-character χ of G we let L_χ denote the field extension of L which is generated by the values of χ, and write $\mathbb{Z}_\chi$ for the integral closure of $\mathbb{Z}$ in L_χ. The field L_χ depends only upon the conjugacy class D to which χ belongs, and so we define the symbols L_D and $\mathbb{Z}_D$ in the obvious way. For each $D \in \mathscr{D}_C$ we let $\mathrm{Cl}_k(D)$ denote the ray class field modulo k of L_D, and write $\mathrm{Cl}_{k,\Sigma}(D)$ for the subgroup of $\mathrm{Cl}_k(D)$ which is generated by the classes of ideals which have trivial Frobenius on $L(\zeta)$. The Tchebotarev density theorem implies that $\mathrm{Cl}_{k,\Sigma}(D)$ coincides with the subgroup of $\mathrm{Cl}_k(D)$ which is generated by the classes of prime $\mathbb{Z}_D$-ideals which lie above prime $\mathbb{Z}_L$-ideals having absolute norm congruent to 1 modulo k. We fix once and for all a character $\chi_D \in D$ for each $D \in \mathscr{D}_C$, and we define a homomorphism

$$\pi_{L,C} : \bigoplus_{D\in\mathscr{D}_C} \mathrm{Cl}_k(D) \to \mathrm{Cl}(\mathbb{Z}_L[C])$$

in the following way: for each $\mathbb{Z}_D$-ideal $\mathfrak{a}_D$ which is coprime to k we let a_D denote its

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class in $\mathrm{Cl}_k(D)$ and define $\pi_{L,C}((a_D)_{D\in\mathscr{D}_C})$ to be the function which on a character χ of C takes the value a_D^ω where here $\chi = \chi_D^\omega$ for some $D\in\mathscr{D}_C$ and $\omega\in\Omega_L$. Such a function represents an element of $\mathrm{Cl}(\mathbb{Z}_L[C])$ via the Hom-description of ([F], notes to chapter I, [4]). The map $\pi_{L,C}$ is well defined and surjective, and maps $\bigoplus_{D\in\mathscr{D}_C}\mathrm{Cl}_{k,\Sigma}(D)$ onto $\mathrm{Cl}(\mathbb{Z}_L[C])_\Sigma$.

Returning to the proof of (2·1) we now set $C := I_{\mathfrak{q}}$, so that k is the inertial degree of $\mathfrak{p}$ in N/K. Since N/K is tame one has $\mathfrak{p}\nmid k$ and so for each character χ of C the $\mathbb{Z}_\chi$-ideal $(g'_{\mathfrak{p}}(\chi))$ generated by $g'_{\mathfrak{p}}(\chi)$ is coprime to k. Hence $\mathfrak{c}_2 = \pi_{L,C}\hat{\mathfrak{c}}_2$ where here

$$\hat{\mathfrak{c}}_2 := \sum_{D\in\mathscr{D}_C}(g'_{\mathfrak{p}}(\chi_D))\in\bigoplus_{D\in\mathscr{D}_C}\mathrm{Cl}_k(D).$$

For each maximal $\mathbb{Z}_\chi$-ideal $\mathfrak{l}$ we let $v_{\mathfrak{l}}(\chi)$ denote the valuation of $(g'_{\mathfrak{p}}(\chi))$ at $\mathfrak{l}$. Thus $v_{\mathfrak{l}}(\chi) = 0$ unless $\mathfrak{l}\mid\wp$. If now $N_{\mathfrak{q}}/K_{\mathfrak{p}}$ is abelian then the absolute norm of $\mathfrak{p}$ is congruent 1 modulo k. From (2·11) it follows that for each character χ of C and maximal $\mathbb{Z}_\chi$-ideal $\mathfrak{l}$ the absolute norm of $\mathfrak{l}^{v_{\mathfrak{l}}(\chi)}$ is congruent 1 modulo k. The Frobenius of each ideal $(g'_{\mathfrak{p}}(\chi))$ is therefore trivial on $L(\zeta)$, and hence the element $\hat{\mathfrak{c}}_2$ actually belongs to $\bigoplus_{D\in\mathscr{D}_C}\mathrm{Cl}_{k,\Sigma}(D)$. This in turn implies that $\mathfrak{c}_2\in\mathrm{Cl}(\mathbb{Z}_L[C])_\Sigma$ as was required to complete the proof of (2·1).

To complete the proof of Theorem 1·1 we henceforth assume that G is abelian and shall prove that $\mathfrak{R}_{L,0}(G)+\mathrm{Cl}(\mathbb{Z}_L[G])_{\mathscr{E},\Sigma}\subset\mathfrak{R}_L(G)$. To do this, it is obviously sufficient to fix an arbitrary cyclic subgroup C of G and prove that $\mathrm{ind}_G^C(\mathrm{Cl}(\mathbb{Z}_L[C])_\Sigma)\subset\mathfrak{R}_L(G)$. Moreover, since $\mathfrak{R}_L(G)$ consists of arbitrary finite linear combinations of realizable elements this last inclusion will follow if we can show that for each $D_0\in\mathscr{D}_C$, and each class $x_{D_0}\in\mathrm{Cl}_{k,\Sigma}(D_0)$ one has

$$\mathrm{ind}_G^C(\pi_{L,C}((c_D)_{D\in\mathscr{D}_C}))\in\mathfrak{R}_L(G)$$

for an element $(c_D)_{D\in\mathscr{D}_C}\in\bigoplus_{D\in\mathscr{D}_C}\mathrm{Cl}_k(D)$ which satisfies $c_{D_0} = x_{D_0}$ and $c_D = 0$ for each $D\succ D_0$.

We choose a maximal cyclic subgroup C' of G which contains C, and a subgroup H of G for which $G = H\times C'$. We fix a tame extension F_1 of L which is disjoint from the ray class field modulo k of $L(\zeta)$ and has group $\mathrm{Gal}(F_1/L)$ isomorphic to H. We fix an isomorphism of $\mathrm{Gal}(F_1/L)$ with H, an element $D_0\in\mathscr{D}_C$ and a class $x_{D_0}\in\mathrm{Cl}_{k,\Sigma}(D_0)$. Using the Tchebotarev density theorem we may choose a prime $\mathbb{Z}_{D_0}$-ideal $\mathfrak{l}_0$ which is coprime to $\#G$, has absolute norm congruent to 1 modulo k, splits completely in F_1/L, and has class $x_{D_0}\in\mathrm{Cl}_k(D_0)$. Letting l_0 denote the prime $\mathbb{Z}_L$-ideal below $\mathfrak{l}_0$ there exists a tame cyclic extension F of L of degree $\#C'$ which is disjoint from F_1/L, and in which l_0 has both inertial and decomposition subgroups of order k (this follows easily from the Grunwald–Wang theorem (cf. [A,T], chapter 10), or if one takes $\mathfrak{l}_0$ of absolute degree 1 by simple use of cyclotomic fields). We let N denote the compositum of F_1 and F. We next specify an identification of $\mathrm{Gal}(F/L)$ with C', and hence of $\mathrm{Gal}(N/L) = \mathrm{Gal}(F_1/L)\times\mathrm{Gal}(F/L)$ with G. For this we let F_0 denote the subfield of F which corresponds to the kernel of any element of D_0, with F_2 the maximal extension of L in F in which l_0 is unramified. Thus $F_2\subseteq F_0$ and $\mathrm{Gal}(F/F_2)\cong C$. We write L_0 for the completion of L at l_0, fix an embedding $j_0: L^c\hookrightarrow L_0^c$ which corresponds on L_{D_0} to $\mathfrak{l}_0$, and write M_0 for the completion of F_0 at the place corresponding to j_0. The completion of F_2 at the place corresponding to j_0 is L_0 and so we can identify $\mathrm{Gal}(F_0/F_2)$ and $\mathrm{Gal}(M_0/L_0)$ via the embedding j_0. We thus regard χ_{D_0} as a character

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of $\mathrm{Gal}(M_0/L_0)$, and fix an identification of $\mathrm{Gal}(F/L)$ with C' such that $j_0 \circ \chi_{D_0}$ is the canonical generator χ_{M_0/L_0} of the group of abelian l_0-adic characters of $\mathrm{Gal}(M_0/L_0)$.

For each intermediate field N' of N/L we let $l_0(N')$ denote the radical of l_0 in $\mathbb{Z}_{N'}$. If Γ is any subgroup of G then for each irreducible $\mathbb{Q}^c$-character χ of Γ we let Γ_χ denote its kernel, and e_χ the corresponding idempotents of both $L_\chi[\Gamma]$ and $L_\chi[\Gamma/\Gamma_\chi]$. For any pair of integers m, n we define a function $f_{\Gamma,m,n}$ on irreducible $\mathbb{Q}^c$-characters of Γ by the module index

$$f_{\Gamma,m,n}(\chi) = [(l_0(N)^m \otimes \mathbb{Z}_\chi)e_\chi : (l_0(N)^n \otimes \mathbb{Z}_\chi)e_\chi]_{\mathbb{Z}_\chi}$$

(all tensor products here and in what follows are over $\mathbb{Z}_L$). We similarly define a function $f^*_{m,n}$ on characters of C by

$$f^*_{m,n}(\chi) = [(l_0(F)^m \otimes \mathbb{Z}_\chi)e_\chi : (l_0(F)^n \otimes \mathbb{Z}_\chi)e_\chi]_{\mathbb{Z}_\chi}.$$

Now for each pair of integers m, n the class

$$c(m,n) := (l_0(N)^m) - (l_0(N)^n) \in \mathrm{Cl}(\mathbb{Z}_L[G])$$

is represented in the ideal theoretic Hom-description by the function $f_{G,m,n}$. Note that under restriction from N to F the inertia subgroup I of l_0 in G identifies with C. Furthermore since l_0 splits completely in F_2F_1/L it is easy to show that for each character χ of G there are equalities

$$f_{G,m,n}(\chi) = f_{I,m,n}(\mathrm{res}^G_I \chi) \quad \text{and} \quad f_{I,m,n}(\mathrm{res}^G_I \chi) = f^*_{m,n}(\mathrm{res}^G_C \chi).$$

It follows that $c(m,n) = \mathrm{ind}^C_G \hat{c}(m,n)$ with $\hat{c}(m,n) \in \mathrm{Cl}(\mathbb{Z}_L[C])$ represented by the function $f^*_{m,n}$. One now obtains the required classes by choice of integers m and n according to the following lemma.

LEMMA 2·17. *One has* $\hat{c}(1, |D_0|+1) = \pi_{L,C}((c_D)_{D \in \mathscr{D}_C}) \in \mathrm{Cl}(\mathbb{Z}_L[C])$ *with* $c_{D_0} = x_{D_0}$ *and* $c_D = 0$ *if* $D \succ D_0$.

Proof. We set $d_0 = |D_0|$. For each character χ of C we set $d_\chi := \# C_\chi$ and write F_χ for F^{C_χ}. If $d_\chi > d_0$ then

$$\begin{aligned} f^*_{1,d_0+1}(\chi) &= [(l_0(F) \otimes \mathbb{Z}_\chi)e_\chi : (l_0(F)^{d_0+1} \otimes \mathbb{Z}_\chi)e_\chi]_{\mathbb{Z}_\chi} \\ &= [(l_0(F_\chi) \otimes \mathbb{Z}_\chi)e_\chi : (l_0(F_\chi) \otimes \mathbb{Z}_\chi)e_\chi]_{\mathbb{Z}_\chi} = \mathbb{Z}_\chi. \end{aligned}$$

If on the other hand $\chi \in D_0$ then one has

$$f^*_{1,d_0+1}(\chi) = [(l_0(F_0) \otimes \mathbb{Z}_\chi)e_\chi : (l_0(F_0)^2 \otimes \mathbb{Z}_\chi)e_\chi]_{\mathbb{Z}_\chi}.$$

To describe this index more explicitly we identify the completion of F_0 at any place above l_0 with M_0, and let $\mathfrak{m}$ denote the valuation ideal of M_0. If now $j : L_{D_0} \hookrightarrow L_0$ is any embedding then the l_0-adic closure of $f^*_{1,d_0+1}(\chi)^j$ is equal to $[\mathfrak{m}e_{j\circ\chi} : \mathfrak{m}^2 e_{j\circ\chi}]_{\mathbb{Z}_{L_0}}$. In addition, for each abelian l_0-adic character ϕ of $\mathrm{Gal}(M_0/L_0)$ one has

$$\mathfrak{m}^2 e_\phi = \begin{cases} l_0 \mathfrak{m} e_\phi, & \text{if } \phi = \chi_{M_0/L_0} \\ \mathfrak{m} e_\phi, & \text{otherwise} \end{cases}$$

(cf. the proof of Lemma 1·8). From our chosen identification of $\mathrm{Gal}(F/L)$ with C' it follows that $f^*_{1,d_0+1}(\chi_{D_0}) = l_0$. But since l_0 was chosen to have class $x_{D_0} \in \mathrm{Cl}_k(D_0)$ the result is now clear. ∎

This completes the proof of Theorem 1·1.

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Appendix. Adams operators on generalized Swan subgroups

Throughout this appendix G is abelian. For each subgroup H of G we set $h := \#H$, and write $\mathbb{Z}_{(h)}$ for the finite ring $\mathbb{Z}/h\mathbb{Z}$. We set $\sigma_H := \sum_{\gamma \in H} \gamma \in \mathbb{Z}[H]$, and let $\mathscr{A}_{H,G}$ denote the quotient ring $\mathbb{Z}[G]/(\sigma_H)$. For each such H there is a natural fibre diagram

$$\begin{array}{ccc} \mathbb{Z}[G] & \longrightarrow & \mathbb{Z}[G/H] \\ \downarrow & & \downarrow \\ \mathscr{A}_{H,G} & \longrightarrow & \mathbb{Z}_{(h)}[G/H] \end{array} \tag{A 1}$$

associated to which there is a Mayer–Vietoris sequence (cf. [C,R], 49·39)

$$K_1(\mathbb{Z}_{(h)}[G/H]) \xrightarrow{\delta_H} D(\mathbb{Z}[G]) \longrightarrow D(\mathscr{A}_{H,G}) \oplus D(\mathbb{Z}[G/H]) \longrightarrow 0.$$

The (generalized) Swan subgroup $T_H(G)$ is defined to be the image of the boundary homomorphism δ_H. With $\mathscr{P}(G)$ denoting the set of subgroups of G which have prime power order, Oliver has shown that

$$\sum_{H \in \mathscr{P}(G)} T_H(G) = D(\mathbb{Z}[G]) \tag{A 2}$$

(cf. [O], Theorem 2). Let us fix a subgroup H for which h is a power of a prime l. Of course, if $h = 1$ then $T_H(G) = 0$ and so we may assume that $h > 1$. For each $w \in K_1(\mathbb{Z}_{(h)}[G/H]) = (\mathbb{Z}_{(h)}[G/H])^*$ there exists an element $u \in \mathbb{Z}[G] \cap \mathbb{Z}_l[G]^*$ which has image in $\mathbb{Z}_{(h)}[G/H]$ under the diagram (A 1) equal to w. We let $S_H(u)$ denote the $\mathbb{Z}[G]$-ideal which is generated by u and σ_H – this is a locally-free $\mathbb{Z}[G]$-lattice which has class $\delta_H(w) \in \mathrm{Cl}(\mathbb{Z}[G])$. In the case $h = 1$ we shall for convenience set $S_H(u) := 0$ for each $u \in \mathbb{Z}[G]$.

For any prime p we write π_p for the ring endomorphism of $\mathbb{Z}[G]$ which is induced by raising elements of G to the power p. Via (A 2) we can now explicitly describe the action of the Adams operators defined in [C-N,T] on the elements of $D(\mathbb{Z}[G])$ (cf. Remark 1·2(ii)).

LEMMA A3. *Let H and u be as above. Then for any prime p one has $\pi_p u \in \mathbb{Z}[G^p] \cap \mathbb{Z}_l[G^p]^*$ and*

$$\Psi_{\mathbb{Q},p}(S_H(u)) = \mathrm{ind}_G^{G^p}(S_{H^p}(\pi_p u)) \in \mathrm{Cl}(\mathbb{Z}[G]).$$

Proof. Since $u \in \mathbb{Z}[G] \cap \mathbb{Z}_l[G]^*$ and π_p is a ring endomorphism it is clear that $\pi_p u \in \mathbb{Z}[G^p] \cap \mathbb{Z}_l[G^p]^*$. To prove the rest we need a little more notation. For a finite abelian group Γ we write $\langle , \rangle_\Gamma$ for the standard inner product on the complex representation ring R_Γ. We write $\Gamma^\vee$ for the set of irreducible complex characters of Γ, and let ϵ_Γ denote the trivial character of Γ. We extend each character $\chi \in \Gamma^\vee$ to a homomorphism $\tilde{\chi} \in \mathrm{Hom}(\mathbb{Z}[\Gamma], \mathbb{Q}^c)$.

Now, by adapting an argument of Ullom (cf. [F], example 2, p. 25) one can show that $(S_H(u)) \in \mathrm{Cl}(\mathbb{Z}[G])$ is represented in the Hom-description by the homomorphism $f \in \mathrm{Hom}_{\Omega_\mathbb{Q}}(R_G, \mathscr{I}(\mathbb{Q}^c))$ which has components at each $\chi \in G^\vee$ given by

$$f(\chi)_q = \begin{cases} 1, & \text{if } q \neq l \\ (\tilde{\chi}(u))^{\langle \epsilon_H, \mathrm{res}_H^G \chi \rangle_H}, & \text{if } q = l. \end{cases}$$

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One checks similarly that the class of $S_{H^p}(\pi_p u)$ in $\mathrm{Cl}(\mathbb{Z}[G^p])$ is represented by the homomorphism $g \in \mathrm{Hom}_{\Omega_Q}(R_{G^p}, \mathscr{J}(\mathbb{Q}^c))$ which is defined componentwise for each $\phi \in (G^p)^\vee$ by

$$g(\phi)_q = \begin{cases} 1, & \text{if } q \neq l \\ (\tilde{\phi}(\pi_p u))^{\langle \epsilon_{H^p}, \mathrm{res}^{G^p}_{H^p} \phi \rangle_{H^p}}, & \text{if } q = l. \end{cases}$$

We write ψ_p for the the p-th Adams operator on R_G. Given the commutativity of the square (2·4) together with the definition of $\Psi_{\mathbb{Q},p}$ (cf. [C-N,T]) and the above interpretation of f and g as representing homomorphisms, the result of the lemma follows immediately if we can show that $f \circ \psi_p = g \circ \mathrm{res}^G_{G^p}$ on R_G. But this is now easily checked. Indeed, writing $u = \sum_{g \in G} c_g g \in \mathbb{Z}[G]$ then for each character $\chi \in G^\vee$ one has

$$\widetilde{(\mathrm{res}^G_{G^p} \chi)}(\pi_p u) = \sum_{g \in G} c_g \chi(g^p) = \sum_{g \in G} c_p (\psi_p \circ \chi)(g) = \widetilde{(\psi_p \circ \chi)}(u).$$

In addition, for each $\chi \in G^\vee$ one has

$$\langle \epsilon_{H^p}, \mathrm{res}^G_{H^p} \chi \rangle_{H^p} \neq 0 \Leftrightarrow \chi(H^p) = 1 \Leftrightarrow \psi_p \circ \chi(H) = 1 \Leftrightarrow \langle \epsilon_H, \mathrm{res}^G_H \psi_p \circ \chi \rangle_H \neq 0$$

and so

$$\langle \epsilon_{H^p}, \mathrm{res}^{G^p}_{H^p} (\mathrm{res}^G_{G^p} \chi) \rangle_{H^p} = \langle \epsilon_H, \mathrm{res}^G_H \psi_p \circ \chi \rangle_H. \quad \blacksquare$$

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