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NON-HOMOGENEOUS BINARY CUBIC FORMS

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1. Let $f(x_1, x_2, \ldots, x_n)$ be a homogeneous form with real coefficients in n variables $x_1, x_2, \ldots, x_n$. Let $a_1, a_2, \ldots, a_n$ be n real numbers. Define $m_f(a_1, \ldots, a_n)$ to be the lower bound of $|f(x_1+a_1, \ldots, x_n+a_n)|$ for integers $x_1, \ldots, x_n$. Let m_f be the upper bound of $m_f(a_1, \ldots, a_n)$ for all choices of $a_1, \ldots, a_n$. For many forms f it is known that there exist estimates for m_f in terms of the invariants alone of f. On the other hand, it follows from a theorem of Macbeath* that no such estimates exist if the region

$$|f(x_1, \ldots, x_n)| \leqslant 1$$

has a finite volume. However, for such forms there may be simple estimates for m_f dependent on the coefficients of f; for example, Chalk has conjectured that:

If $f(x, y)$ is a reduced† binary cubic form with a negative discriminant, then for any real a, b there exist integers x, y such that

$$|f(x+a, y+b)| \leqslant \max\{|f(\tfrac{1}{2}, 0)|, |f(0, \tfrac{1}{2})|, |f(\tfrac{1}{2}, \tfrac{1}{2})|, |f(\tfrac{1}{2}, -\tfrac{1}{2})|\}.$$

In the author's dissertation‡, the following theorem was proved:

THEOREM. *Let $f(x, y) = ax^3+bx^2y+cxy^2+dy^3$ be a binary cubic form with real coefficients and positive discriminant $D = 18abcd+b^2c^2-4ac^3-4db^3-27a^2d^2$. Then, if f is reduced in the sense of Hermite§ and x_0, y_0 are any two real numbers, there exist integers x, y such that*

$$|f(x+x_0, y+y_0)| \leqslant \max\{|f(\tfrac{1}{2}, 0)|, |f(0, \tfrac{1}{2})|, |f(\tfrac{1}{2}, \tfrac{1}{2})|, |f(\tfrac{1}{2}, -\tfrac{1}{2})|\}. \qquad (1)$$

Further, the result is best possible in the sense that there exist forms f for which the equality sign is necessary in (1).

The object of this note is to give a summary of the proof.

That the result is best possible can be easily seen by taking

$$f = y(y^2-3x^2) \quad \text{or} \quad x^3+x^2y-2xy^2-y^3 \quad \text{and} \quad (x_0, y_0) = (\tfrac{1}{2}, \tfrac{1}{2}).$$

We sketch the proof of the main part below.

* See p. 627.

† In the sense of Davenport, *J. London Math. Soc.* 20 (1945), 139–47.

‡ Cambridge Ph.D., 1950.

§ See Davenport, *J. London Math. Soc.* 20 (1945), 14–22.

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2. As f is reduced and has positive discriminant, it can be easily verified that we can find a real transformation $x = \alpha x' + \beta y'$, $y = \gamma x' + \delta y'$ which transforms $f(x, y)$ into $\phi(x', y') = \epsilon y'(y'^2 - 3x'^2)$, where $\epsilon = \pm 1$.

Since the quadratic covariant of $y'(3x'^2 - y'^2)$ is $9(x'^2 + y'^2)$, and this is a reduced form in x, y, the points A and B in the x', y' plane which correspond to the points $x = 1, y = 0$ and $x = 0, y = 1$ must satisfy the conditions:

(i) $OA \leqslant OB$,

(ii) the angle AOB lies between 60° and 120° inclusive.

Without loss of generality we can suppose that the sense of rotation from OA to OB is anti-clockwise. Since the form $y'(y'^2 - 3x'^2)$ is unaltered (except in sign) if the x', y' plane is rotated through 60°, we can further suppose without loss of generality that

(iii) A lies in the angle $0 \leqslant y' \leqslant x'\sqrt{3}$.

Let A be (p, q) and B be (r, s).

Let $\mathscr{R}$ be a region $|F(x, y)| \leqslant c$ and P a point (x_1, y_1). Then we shall write $\mathscr{R} + P$ for the region $|F(x - x_1, y - y_1)| \leqslant c$. In this notation it is easily seen that our theorem will follow if we can prove

LEMMA. *Let $C = A + B$ in the vector notation. Then the parallelogram $OACB$ is covered by the regions $\mathscr{R}$, $\mathscr{R} + A$, $\mathscr{R} + B$ and $\mathscr{R} + C$, where $\mathscr{R}$ is defined by*

$$|\phi(x', y')| = |y'(y'^2 - 3x'^2)| \leqslant m,$$

where

$$m = \max\{|\phi(\tfrac{1}{2}p, \tfrac{1}{2}q)|, |\phi(\tfrac{1}{2}r, \tfrac{1}{2}s)|, |\phi(\tfrac{1}{2}(p+r), \tfrac{1}{2}(q+s))|, |\phi(\tfrac{1}{2}(p-r), \tfrac{1}{2}(q-s))|\}.$$

The choice of m means that $\mathscr{R}$ is obtained by magnifying the region

$$|y'(y'^2 - 3x'^2)| \leqslant 1$$

until it contains, or has on its boundary, all four of the points $\frac{1}{2}A$, $\frac{1}{2}B$, $\frac{1}{2}(A + B)$ and $\frac{1}{2}(A - B)$.

Let the positive direction of the x'-axis be denoted by Ol_1 and the lines making angles of 60, 120, 180, 240 and 300° with Ol_1 by Ol_2, Ol_3, Ol_4, Ol_5 and Ol_6 respectively. Our proof of the lemma depends on two simple facts which are obvious consequences of the symmetry of $\mathscr{R}$ and of the convexity towards the origin of its boundary in each of the six sectors into which the lines Ol_i divide the plane. These two facts are as follows:

(*a*) Let $PQRS$ be a parallelogram with PQ and PS parallel to Ol_i and Ol_{i+1} ($i = 1, 2, \ldots, 6, Ol_7 = Ol_1$) respectively. Define $\mathscr{R}'$ by $|\phi(x', y')| \leqslant |\phi(\alpha, \beta)|$, where α, β are the coordinates of the point $\frac{1}{2}(R - P)$. Then $PQRS$ is covered by $\mathscr{R}' + P$ and $\mathscr{R}' + R$.

(*b*) Let PQR be an equilateral triangle with its sides parallel to Ol_1, Ol_2, Ol_3. Let $\mathscr{R}'$ be the region $|\phi(x', y')| \leqslant \gamma^3$, where γ is the distance of P from QR. Then PQR is covered by $\mathscr{R}' + P$.

The conditions (i) and (iii) imply that A lies between Ol_1 and Ol_2, while B is either between Ol_2 and Ol_3 or between Ol_3 and Ol_4. When B lies between Ol_2 and Ol_3, we denote the point of intersection of Ol_2 with the line through B parallel to Ol_6 by F and that of Ol_2 with the line through A parallel to Ol_4 by G (see Fig. 1). In our proof of the lemma we distinguish between the following four cases:

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(1) B lies between Ol_2 and Ol_3 and F lies above G.
(2) B lies between Ol_2 and Ol_3 and F lies below G.
(3) B lies between Ol_3 and Ol_4 and above A.
(4) B lies between Ol_3 and Ol_4 and below A.

As the proof in different cases varies only in details we give it for case (1) only.

As $OA \leqslant OB$ and B lies in the second sector, while A is in the first, it is easily seen that B is above A as shown in Fig. 1 above. Draw lines ADG and AE parallel to Ol_4 and Ol_3 respectively, and BFD and BE parallel to Ol_6 and Ol_1 respectively. The positions of the points D, E, F and G can be easily seen to be as shown in Fig. 1. Then by (a) we at once see that $ADBE$ is covered by $\mathscr{R}+A$ and $\mathscr{R}+B$, OAG by $\mathscr{R}$ and $\mathscr{R}+A$, and OFB by $\mathscr{R}$ and $\mathscr{R}+B$. Because of symmetry of $OACB$ about its centre, the lemma will be completely proved if we can show that the triangle DFG is covered by $\mathscr{R}$, $\mathscr{R}+A$, $\mathscr{R}+B$ and $\mathscr{R}+C$.

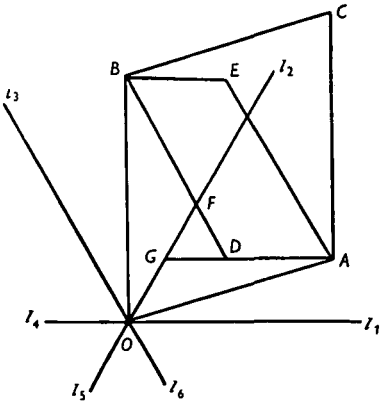


Fig. 1

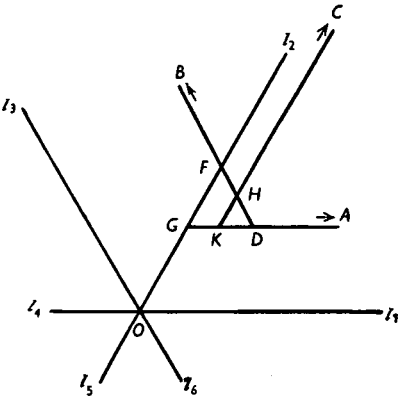


Fig. 2

By (b) it is easily seen that DFG is covered by each of the regions

$$|\phi(x', y')| \leqslant \tfrac{1}{8}(s+r\sqrt{3})^3 \quad \text{and} \quad \{|\phi(x', y')| \leqslant \tfrac{1}{8}(p\sqrt{3}-q)^3\}+A,$$

since $\frac{1}{2}(s+r\sqrt{3})$ is the distance from O to the line DF and $\frac{1}{2}(p\sqrt{3}-q)$ is the distance from A to the line FG (i.e. Ol_2). Also, if $\frac{1}{2}(s-r\sqrt{3}) < \frac{1}{2}(p\sqrt{3}-q)$, the line through C parallel to Ol_5 is nearer than Ol_2 to A and we have one of the following two possibilities:

Either the triangle DFG completely lies between Ol_2 and the line through C parallel to Ol_5 and is consequently covered by $\mathscr{R}$ and $\mathscr{R}+C$;

Or the line through C parallel to Ol_5 cuts the lines DF and DG at the points H and K, thus dividing the triangle DFG into the triangle DHK and the trapezium $FGKH$. Of these $FGKH$ is covered by $\mathscr{R}$ and $\mathscr{R}+C$, while DKH is covered by

$$\{|\phi(x', y')| \leqslant \tfrac{1}{8}(s-r\sqrt{3})^3\}+A,$$

since $\frac{1}{2}(s-r\sqrt{3})$ is the distance from A to HK.

Thus we see that the triangle DFG is covered by $\mathscr{R}$, $\mathscr{R}+C$ and one of $\mathscr{R}'$ and $\mathscr{R}'+A$, where $\mathscr{R}'$ is given by

$$|\phi(x', y')| \leqslant \min\{\tfrac{1}{8}(s+r\sqrt{3})^3, \tfrac{1}{8}(s-r\sqrt{3})^3, \tfrac{1}{8}(p\sqrt{3}-q)^3\}.$$

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Consequently we have only to show that

$$\min\{\tfrac{1}{8}(s+r\sqrt{3})^3, \tfrac{1}{8}(s-r\sqrt{3})^3, \tfrac{1}{8}(p\sqrt{3}-q)^3\} \leqslant m. \tag{2}$$

As $0 \leqslant |r|\sqrt{3} \leqslant s$ on the hypothesis of case (1), the inequality (2) follows from I and II below.

(I) Suppose that $0 \leqslant r\sqrt{3} \leqslant s$. Then

$$\begin{aligned} |\phi(\tfrac{1}{2}r, \tfrac{1}{2}s)| - \tfrac{1}{8}(s-r\sqrt{3})^3 &= \tfrac{1}{8}s(s^2-3r^2) - \tfrac{1}{8}(s-r\sqrt{3})^3 \\ &= \tfrac{3}{8}(s-r\sqrt{3})(s\sqrt{3}-r)r \\ &\geqslant 0. \end{aligned}$$

(II) Suppose that $0 \leqslant -r\sqrt{3} \leqslant s$. Then

$$\begin{aligned} |\phi(\tfrac{1}{2}r, \tfrac{1}{2}s)| - \tfrac{1}{8}(s+r\sqrt{3})^3 &= \tfrac{1}{8}s(s^2-3r^2) - \tfrac{1}{8}(s+r\sqrt{3})^3 \\ &= -\tfrac{3}{8}r(s+r\sqrt{3})(s\sqrt{3}+r) \\ &\geqslant 0. \end{aligned}$$

We may remark in conclusion that the proof also shows that the sign of equality in (i) is necessary at most when $(x_0, y_0) \equiv (\frac{1}{2}, 0), (0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}) \pmod 1$.

I am indebted to Prof. L. J. Mordell and Prof. H. Davenport for many useful suggestions.

Note added 1 *May* 1951. I have recently observed that the theorem of this note is true even when the form $f(x, y)$ is not restricted to be reduced. For the proof one applies a real linear transformation $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ which changes $f(x, y)$ into $\epsilon y'(y'^2 - 3x'^2)$ and at the same time transforms one of the pairs of points $\{(1, 0), (0, 1)\}$, $\{(0, 1), (1, 0)\}$, $\{(1, 0), (0, -1)\}$ or $\{(0, -1), (1, 0)\}$ into the pair $\{A\ (p, q), B\ (r, s)\}$ of the x', y' plane such that

(i) $OA \leqslant OB$,

(ii) the angle AOB is not greater than a right angle,

(iii) the rotation from OA to OB is anti-clockwise, and

(iv) A lies in the region $0 \leqslant y' \leqslant x'\sqrt{3}$.

After this transformation the proof of the theorem is unaltered except that one has to consider the additional case when both A and B lie between Ol_1 and Ol_2; but this case is easily disposed of, since $OACB$ is obviously covered by $\mathscr{R}$ and $\mathscr{R}+C$.

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