A RANDOM WALK PROBLEM

By J. GILLIS

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1. We consider a random walk on a two-dimensional rectangular lattice in which steps are strictly between nearest neighbour points. The conditions of the walk are that the walker must, at each step, turn either to the right or to the left of his previous step with respective probabilities $\frac{1}{2}(1+\alpha)$, $\frac{1}{2}(1-\alpha)$, $(-1 \leqslant \alpha \leqslant 1)$. To fix the ideas it is assumed that he starts from the origin and the probability of each of the four possible starting directions is $\frac{1}{4}$. If A_r denotes the probability of return to the origin after r steps we shall show that

$$A_{4n} = \alpha^{2n}(1-\alpha^2)^{-1}\{[P_n(\beta)]^2 - \alpha^2[P_{n-1}(\beta)]^2\}, \tag{1}$$

where $\beta = \frac{1}{2}(\alpha+\alpha^{-1})$ and P_n is the nth Legendre polynomial. It is clear that A_r is zero for $r \not\equiv 0 \pmod 4$.

2. We denote the two coordinate axes by subscripts and let $B(m_1, m_2; n)$ be the probability that the walker finds himself, as a result of the nth step, at the point (m_1, m_2). Moreover, let $C_1(m_1, m_2; n)$ denote the probability that the nth step is from (m_1-1, m_2) to (m_1, m_2), $C_2(m_1, m_2; n)$ that it is from $(m+1, m_2)$ to (m_1, m_2); similarly, $C_3(m_1, m_2; n)$ and $C_4(m_1, m_2; n)$ are the respective probabilities that the nth step is from $(m_1, m_2 \mp 1)$ to (m_1, m_2). The following relations are self-evident

$$B(m_1, m_2; n) = \sum_{h=1}^{4} C_h(m_1, m_2; n) \tag{2}$$

and
$$\left.\begin{aligned}
C_1(m_1, m_2; n) &= \tfrac{1}{2}(1+\alpha)\, C_3(m_1-1, m_2; n-1) + \tfrac{1}{2}(1-\alpha)\, C_4(m_1-1, m_2; n-1),\\
C_2(m_1, m_2; n) &= \tfrac{1}{2}(1-\alpha)\, C_3(m_1+1, m_2; n-1) + \tfrac{1}{2}(1+\alpha)\, C_4(m_1+1, m_2; n-1),\\
C_3(m_1, m_2; n) &= \tfrac{1}{2}(1-\alpha)\, C_1(m_1, m_2-1; n-1) + \tfrac{1}{2}(1+\alpha)\, C_2(m_1, m_2-1; n-1),\\
C_4(m_1, m_2; n) &= \tfrac{1}{2}(1+\alpha)\, C_1(m_1, m_2+1; n-1) + \tfrac{1}{2}(1-\alpha)\, C_2(m_1, m_2+1; n-1).
\end{aligned}\right\} \tag{3}$$

We now define the respective generating functions

$$G_k(\theta_1, \theta_2; z) = \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \sum_{n=0}^{\infty} C_k(m_1, m_2; n) \exp\{i(m_1\theta_1 + m_2\theta_2)\} z^n \quad (k = 1, 2, 3, 4), \tag{4}$$

$$H(\theta_1, \theta_2; z) = \sum_{h=1}^{4} G_h(\theta_1, \theta_2; z), \tag{5}$$

and
$$R(z) = \sum_{r=0}^{\infty} A_r z^r. \tag{6}$$

$R(z)$ is the part of H independent of θ_1, θ_2, and so

$$R(z) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} H(\theta_1, \theta_2; z)\, d\theta_1\, d\theta_2. \tag{7}$$

It remains to calculate H and the integral (7).

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3. If we multiply both sides of each of the equations in (3) by the appropriate $\exp[i(m_1\theta_1+m_2\theta_2)]\,z^n$ and sum we get

$$G_1-\tfrac{1}{4}=\tfrac{1}{2}(1+\alpha)\,z\,e^{i\theta_1}G_3+\tfrac{1}{2}(1-\alpha)\,z\,e^{i\theta_1}G_4 \tag{8}$$

and three similar equations. The four equations can be solved and yield

$$\begin{aligned}4[1-(1-\alpha^2)z^2\cos\theta_1\cos\theta_2-\alpha^2z^4]\,G_1 &= 1+z\,e^{i\theta_1}+z^2[\tfrac{1}{4}(1+\alpha)^2\exp\{i(\theta_1+\theta_2)\}-\tfrac{1}{4}(1-\alpha^2)\,e^{-i\theta_1}+\tfrac{1}{4}(1-\alpha)^2\exp\{i(\theta_1-\theta_2)\}]\\ &\quad+\tfrac{1}{2}\alpha z^3[(1+\alpha)\,e^{i\theta_2}-(1-\alpha)\,e^{-i\theta_2}].\end{aligned} \tag{9}$$

It is not necessary to solve directly for G_2, G_3, G_4 since it is clear from (3) that G_2 may be derived from G_1 by changing the signs of α and θ_1; while G_3, G_4 may be obtained from G_2, G_1, respectively, by interchanging θ_1, θ_2. Performing these operations and combining the results with (9) we get

$$\begin{aligned}H &= \sum_{k=1}^{4} G_k\\ &= [1-(1-\alpha^2)z^2\cos\theta_1\cos\theta_2-\alpha^2z^4]^{-1}\\ &\quad\times\{1+\tfrac{1}{2}z(\cos\theta_1+\cos\theta_2)+\alpha^2z^2\cos\theta_1\cos\theta_2+\tfrac{1}{2}\alpha^2z^3(\cos\theta_1+\cos\theta_2)\}.\end{aligned} \tag{10}$$

The generating function $R(z)$ is now obtainable from (7) and (10). However, we notice that it must be an even function of z (since $A_{2r+1}\equiv 0$) and so the contribution to the integral in (7) from the odd part of H will be zero. Hence

$$\begin{aligned}R(z) &= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}[1+\alpha^2z^2\cos\theta_1\cos\theta_2][1-(1-\alpha^2)z^2\cos\theta_1\cos\theta_2-\alpha^2z^4]^{-1}\,d\theta_1\,d\theta_2\\
&= \frac{1}{4\pi^2}(1-\alpha^2)^{-1}\int_0^{2\pi}\int_0^{2\pi}\{-\alpha^2+(1-\alpha^4z^4)[1-(1-\alpha^2)z^2\cos\theta_1\cos\theta_2-\alpha^2z^4]^{-1}\}\,d\theta_1\,d\theta_2\\
&= -\alpha^2(1-\alpha^2)^{-1}+\frac{1}{2\pi}(1-\alpha^4z^4)(1-\alpha^2)^{-1}\int_0^{2\pi}\{(1-\alpha^2z^4)^2-z^4(1-\alpha^2)^2\cos^2\theta_1\}^{-\frac{1}{2}}\,d\theta_1\\
&= -\alpha^2(1-\alpha^2)^{-1}+\frac{1}{2\pi}(1-\alpha^4z^4)(1-\alpha^2)^{-1}\int_0^{2\pi}\{1-2\alpha^2z^4[\sin^2\theta_1+\tfrac{1}{2}(\alpha^2+\alpha^{-2})\cos^2\theta_1]\\
&\qquad+\alpha^4z^8\}^{-\frac{1}{2}}\,d\theta_1\\
&= -\alpha^2(1-\alpha^2)^{-1}+\frac{1}{2\pi}(1-\alpha^4z^4)(1-\alpha^2)^{-1}\sum_{n=0}^{\infty}\alpha^{2n}z^{4n}\int_0^{2\pi}P_n[\sin^2\theta_1\\
&\qquad+\tfrac{1}{2}(\alpha^2+\alpha^{-2})\cos^2\theta_1]\,d\theta_1,\\
&\quad\text{where } P_n \text{ is the } n\text{th Legendre polynomial}\\
&= -\alpha^2(1-\alpha^2)^{-1}+\frac{1}{2\pi}(1-\alpha^4z^4)(1-\alpha^2)^{-1}\sum_{n=0}^{\infty}\alpha^{2n}z^{4n}\int_0^{2\pi}P_n[1+2\nu^2\cos^2\theta_1]\,d\theta_1,\\
&\quad\text{where } \nu=\tfrac{1}{2}(\alpha-\alpha^{-1}),\\
&= -\alpha^2(1-\alpha^2)^{-1}+(1-\alpha^4z^4)(1-\alpha^2)^{-1}\sum_{n=0}^{\infty}\alpha^{2n}z^{4n}[P_n\sqrt{(1+\nu^2)}]^2,\\
&\quad\text{by a formula of Nicholson (2)}\\
&= -\alpha^2(1-\alpha^2)^{-1}+(1-\alpha^4z^4)(1-\alpha^2)^{-1}\sum_{n=0}^{\infty}\alpha^{2n}z^{4n}\{P_n(\beta)\}^2,\quad\text{where}\quad\beta=\tfrac{1}{2}(\alpha+\alpha^{-1}).\end{aligned} \tag{11}$$

The expression (1) for A_{4n} follows immediately.

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4. We now wish to estimate (1) asymptotically. Since it is an even function of α, we may restrict ourselves to the case $0 \leqslant \alpha \leqslant 1$, and so write $\alpha = e^{-\xi}$, where ξ is real and positive. By Darboux's formula (cf. Hobson (1), 307)

$$P_n(\cosh \xi) \sim \frac{e^{(n+\frac{1}{2})\xi}\Gamma(n+\frac{1}{2})}{n!(2\pi \sinh \xi)^{\frac{1}{2}}} = 2^{-2n}\binom{2n}{n}\alpha^{-n}(1-\alpha^2)^{-\frac{1}{2}}. \tag{12}$$

Using (12), for both n and $n-1$, in (1) gives us immediately

$$A_{4n} \sim \frac{1+\alpha^2}{1-\alpha^2}\,2^{-4n}\binom{2n}{n}^2 \sim \frac{1}{\pi n}\,\frac{1+\alpha^2}{1-\alpha^2}. \tag{13}$$

This breaks down, of course, when $\alpha = \pm 1$, but the degenerate nature of the walk is then obvious. It follows, incidentally, from (13) that $\sum_{n=0}^{\infty} A_{4n}$ diverges, so that the walk is recurrent for all values of α such that $|\alpha| < 1$.

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DEPARTMENT OF APPLIED MATHEMATICS
THE WEIZMANN INSTITUTE OF SCIENCE
REHOVOTH, ISRAEL

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