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RESEARCH ARTICLE

Counting rational points close to p-adic integers and applications in Diophantine approximation

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Abstract

We find upper and lower bounds on the number of rational points with bounded denominators that are contained in a rectangular neighbourhood of some n-dimensional p-adic integer. To find the upper bound, we use lattice point counting techniques on p-adic approximation lattices, and for the corresponding lower-bound statement, a classical pigeonhole principle-style argument is used. We apply this counting result to prove a statement in the setting of weighted simultaneous p-adic Diophantine approximation on coordinate hyperplanes. For the lower-bound Hausdorff dimension result, we construct a local ubiquitous system of rectangles and then apply the recent Mass Transference Principle result of Wang and Wu (Math. Ann., 2021).

1. Introduction

The study of rational points on, or close to, curves or manifolds has a wide variety of applications in many areas of mathematics. In the setting of $\mathbb{R}^n$, there have been many results of this type; in particular, see [7, 32, 47, 48, 50] in the case of curves and [4, 13, 30, 31] for results on manifolds. The equivalent problem in the p-adic setting is far less understood. In [2, 3], a bound on the number of rational points that lie on the p-adic Veronese curve was found, but as yet, no results for general manifolds or curves are known. In this article, we provide an upper and lower bound on the number of rational points within a rectangular neighbourhood of a p-adic integer. This allows us to prove a result in the setting of p-adic Diophantine approximation on manifolds, which has been vigorously studied in the past few decades [3, 21, 39, 43].

Throughout, our setting is as follows. Fix a prime number $p \in \mathbb{N}$ and let $|.|_p$ denote the p-adic norm. Define $\mathbb{Q}_p$ to be the set of p-adic numbers and denote by $\mathbb{Z}_p := \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$ the ring of p-adic integers. For an introduction to p-adic numbers, see [26]. We will use the bold notation $\boldsymbol{x} \in \mathbb{Z}_p^n$ to denote a n-dimensional p-adic integer. Let $\Psi = (\psi_1, \dots, \psi_n)$ be an n-tuple of monotonic decreasing functions of the form $\psi_i : \mathbb{N} \to \mathbb{R}_+$, with $\psi_i(q) \to 0$ as $q \to \infty$ for each $1 \leq i \leq n$. Often we will consider Ψ to be of the form $\psi_i(q) = q^{-\tau_i}$ $(i = 1, \dots, n)$ for some vector $\boldsymbol{\tau} = (\tau_1, \dots, \tau_n) \in \mathbb{R}^n_{>0}$. For fixed $\boldsymbol{x} \in \mathbb{Z}_p^n$, fixed $N \in \mathbb{N}$ and function Ψ, we study the cardinality of the set

$$\mathcal{Q}_n(\boldsymbol{x}, \Psi, N) := \left\{ (q_0, q_1, \dots, q_n) \in \mathbb{Z}^{n+1} : \begin{array}{c} 0 < q_0 \leq N, \\ \max_{1 \leq i \leq n} |q_i| \leq N, \end{array} \ |q_0 x_i - q_i|_p < \psi_i(N), \ 1 \leq i \leq n \right\}.$$

That is, the set of rational points $(\frac{q_1}{q_0}, \dots, \frac{q_n}{q_0}) \in \mathbb{Q}^n$ with denominator and numerator bounded by N that lie in a p-adic rectangle with sidelengths $\frac{\psi_i(N)}{|q_0|_p}$ in the ith coordinate axis and centre $\boldsymbol{x} \in \mathbb{Z}_p^n$. If the functions ψ_i are of the form $\psi_i(q) = q^{-\tau_i}$ for some vector $\boldsymbol{\tau} = (\tau_1, \dots, \tau_n) \in \mathbb{R}^n_{>0}$, we will use the notation $\mathcal{Q}_n(\boldsymbol{x}, \boldsymbol{\tau}, N)$. Throughout, we write $a \ll b$ to mean there exists some constant $c > 0$ such that $a \leq cb$. If $a \ll b$ and $a \gg b$, we write $a \asymp b$. In order to obtain good bounds on the cardinality of $\mathcal{Q}_n(\boldsymbol{x}, \Psi, N)$, we use the

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Diophantine exponent $\tau(\boldsymbol{x})$ defined as

$$\tau(\boldsymbol{x}) := \sup\left\{\sum_{i=1}^{n}\tau_i : \max_{1\le i\le n}\left\{|q_0x_i - q_i|_p^{\frac{1}{\tau_i}}\right\} < \left(\max_{0\le i\le n}|q_i|\right)^{-1}, \quad \text{for i.m. } (q_0,\dots,q_n)\in\mathbb{Z}^{n+1}\right\},$$

where 'i.m.' denotes 'infinitely many'. Generally, the Diophantine exponent $\tau(\boldsymbol{x})$ tells us how well $\boldsymbol{x}$ can be approximated by integer rational points. It was proven by Mahler (see for example [49]) that for all $x\in\mathbb{Z}_p$, $\tau(x)\ge 2$ and that $\tau(\boldsymbol{x}) = n+1$ for almost all $\boldsymbol{x}\in\mathbb{Z}_p^n$ [36]. By almost all here, we mean with respect to the n-dimensional p-adic Haar measure μ_n on $\mathbb{Q}_p^n$ normalised by $\mu_n(\mathbb{Z}_p^n) = 1$.

We have the following result on the cardinality of $\mathcal{Q}_n(\boldsymbol{x},\tau,N)$.

Theorem 1.1. *Let $\boldsymbol{x}\in\mathbb{Z}_p^n$ and suppose that $\tau(\boldsymbol{x}) = n+1$. Let $\Psi = (\psi_1,\dots,\psi_n)$ be an n-tuple of functions $\psi_i:\mathbb{N}\to(0,\infty)$ with each*

$$q^{-1-\frac{1}{n}+\varepsilon} < \psi_i(q) < q^{-1}, \qquad \text{for all } q\in\mathbb{N} \text{ and } 1\le i\le n, \tag{1}$$

for some $\varepsilon>0$. Then there exists $N_0\in\mathbb{N}$ such that for all $N\ge N_0$,

$$\#\mathcal{Q}_n(\boldsymbol{x},\Psi,N) \le C_1N^{n+1}\prod_{i=1}^{n}\psi_i(N),$$

where

$$C_1 = \max\left\{3(6\sqrt{n})^n, \frac{(n+2)!\pi^{n/2}\sqrt{n}^{n+1}}{\Gamma\left(\frac{n}{2}+1\right)}\right\}.$$

This type of result has been proven in the real case; see [8, Lemma 6.1]. If we want to strengthen the statement to all $\mathbb{Z}_p^n$, we need to consider the Diophantine exponent of our point of interest. For this statement, we restrict it to the case when the approximation functions are power functions and $n=1$.

Lemma 1.2. *Let $x\in\mathbb{Z}_p$ with Diophantine exponent $\tau(x)$ and let $\psi(q) = q^{-\tau}$ for some $\tau\in\mathbb{R}_+$ with $\max\{1,\tau(x)-1\} < \tau < \tau(x)$. Then for any $\epsilon>0$, there exists a sufficiently large $N_0\in\mathbb{N}$ such that for all $N\ge N_0$*

$$\#\mathcal{Q}_1(x,\tau,N) \le N^{\tau(x)-\tau+\epsilon}.$$

In order to deduce the upper bound appearing in Lemma 1.2, we need to know the value of $\tau(x)$, which may not be immediate. By the aforementioned result that $\tau(x) = 2$ for almost all $x\in\mathbb{Z}_p$, if $1<\tau<2$ then for any $\epsilon>0$ there exists $N_0\in\mathbb{N}$ such that for all $N\ge N_0$

$$\#\mathcal{Q}_1(x,\tau,N) \le N^{2-\tau+\epsilon}.$$

A natural question to ask is about the sharpness of the bounds presented in Theorem 1.1 and Lemma 1.2. We have the following result.

Theorem 1.3. *Let $\boldsymbol{\tau} = (\tau_1,\dots,\tau_n)\in\mathbb{R}_{>0}^n$ and suppose that*

$$1 < \tau_i < 1 + \frac{1}{n}$$

for each $1\le i\le n$ and that $\tau(\boldsymbol{x}) = n+1$. Then there exists $N_0\in\mathbb{N}$ such that for all $N\ge N_0$ we have

$$\#\mathcal{Q}_n(\boldsymbol{x},\boldsymbol{\tau},N) \asymp N^{n+1-\sum_{i=1}^{n}\tau_i}.$$

Furthermore, the lower bound holds for all $\boldsymbol{x}\in\mathbb{Z}_p^n$.

As with Theorem 1.1, the lower bound statement has been proven in $\mathbb{R}^n$, see [45, Lemma 3].

The proof of Lemma 1.2 and Theorem 1.3 use elementary techniques. The proof of Theorem 1.1 is more substantial and uses p-adic approximation lattices and lattice counting techniques. Prior to the proofs of these results, we give an application in Diophantine approximation.

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2. p-adic Diophantine approximation

The study of Diophantine approximation is an intensively studied area of research in number theory. For a beautiful introduction to the topic, see the notes [12]. In this article, we consider p-adic Diophantine approximation. Define the set of p-adic weighted simultaneously approximable points as follows. For an n-tuple of approximation functions $\Psi = (\psi_1, \dots, \psi_n)$ and $q_0 \in \mathbb{N}$ let

$$\mathfrak{A}_{q_0}(\Psi) = \bigcup_{\substack{|q_i| \le q_0,\ gcd(q_i, q_0)=1 \\ 1 \le i \le n}} \left\{ \boldsymbol{x} \in \mathbb{Z}_p^n : \left| x_i - \frac{q_i}{q_0} \right|_p < \psi_i(q_0) \right\},$$

where $\boldsymbol{x} = (x_1, \dots, x_n)$. Define the set of weighted Ψ-approximable p-adic points as

$$\mathfrak{W}_n(\Psi) := \limsup_{q_0 \to \infty} \mathfrak{A}_{q_0}(\Psi).$$

That is, a point $\boldsymbol{x}$ is weighted Ψ-approximable if there exists infinitely many hyperrectangles, with centre $\frac{p}{q} \in \mathbb{Q}^n$ and sidelengths depending on Ψ and q, that contain $\boldsymbol{x}$. If the approximation functions are the same in each coordinate axis, i.e., $\Psi = (\psi, \dots, \psi)$, then we will use the notation $\mathfrak{W}_n(\Psi) = \mathfrak{W}_n(\psi)$, which is often referred to as the set of simultaneously ψ-approximable points. If each $\psi_i(q) = q^{-\tau_i}$ for some $\boldsymbol{\tau} = (\tau_1, \dots, \tau_n) \in \mathbb{R}^n_{>0}$ use the notation $\mathfrak{W}_n(\Psi) = \mathfrak{W}_n(\boldsymbol{\tau})$.

Dirichlet's Theorem in the p-adic setting implies that for $\Psi = (pq^{-\tau_1}, \dots, pq^{-\tau_n})$ the set $\mathfrak{W}_n(\boldsymbol{\tau}) = \mathbb{Z}_p^n$ provided $\sum_{i=1}^n \tau_i \le n+1$, see for example [42] for the $n = 1$ version, or [11, Lemma 3.3] (Lemma 3.1 below) from which it can be deduced. Metric results on the set $\mathfrak{W}_n(\Psi)$ analogous to those of Khintchine [37], Besicovitch [17] and Jarnik [35] from the classical setting have been proven.

The Khintchine-type result in the one dimensional p-adic setting was proven in [36], and since then results in the p-adic dual setting [41], and the weighted simultaneous setting [11] have been proven. See also [27] for the analogous Duffin-Schaeffer type theorem in the p-adics and [40] for the recent resolution.

For sets of zero Haar measure, we use Hausdorff measure, denoted $\mathcal{H}^f$ for dimension function f, and Hausdorff dimension, denoted $\dim_H$, to provide more accurate notions of size. The definitions of Hausdorff measure and dimension readily follow in p-adic space. See [25] for the definition of Hausdorff measure and dimension, and [11] for the relevant definitions in the p-adic setting.

Hausdorff measure and dimension results on the set $\mathfrak{W}_n(\Psi)$ are also known. For the p-adic analogue of the Jarnik-Besicovitch Theorem see [1], and see [6, Theorem 16] for the p-adic dual and simultaneous approximation setting. In the p-adic weighted simultaneous setting, analogous to [46] in the real setting, we have the following.

Theorem 2.1 *([11]). Let* $\sum_{i=1}^n \tau_i > n+1$. *Then*

$$\dim_H \mathfrak{W}_n(\boldsymbol{\tau}) = \min_{1 \le i \le n} \left\{ \frac{n+1+\sum_{j=i}^n (\tau_i - \tau_j)}{\tau_i} \right\}.$$

2.1. p-adic approximation on manifolds

In recent decades, Diophantine approximation on curves and manifolds has been intensively studied, see for example the fundamental works [4, 38] in the real setting, and for an in-depth discussion on the topic see [16]. There has also been many results in the analogous p-adic setting. Kleinbock and Tomanov proved the extremality (i.e., almost all points w.r.t. the induced Haar measure on the manifold have $\tau(\boldsymbol{x}) = n+1$) of p-adic manifolds, provided some non-degeneracy conditions are satisfied; see [39] for more details. There are a variety of results for p-adic dual approximation (linear forms) on manifolds, see for example [5, 9, 20, 23, 43]; however, results in the simultaneous case are lacking.

For the Hausdorff dimension, there are results on simultaneously approximable points over the p-adic Veronese curve, $\mathcal{C}_f = \{(x, x^2, \dots, x^n) : x \in \mathbb{Z}_p\}$, provided the approximation functions have sufficiently

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large Diophantine exponents, see [3, 21]. In [11], a lower bound for the Hausdorff dimension was found for general n-dimensional manifolds satisfying a strong differentiability condition (see [11] for more details). A key reason the upper bound could not be obtained was a lack of results on the distribution of rational points close to p-adic curves.

In this article, we consider approximation on p-adic hyperplanes. Note that in much of the previous work the manifold under consideration is usually "sufficiently curved" (formally, the requirement is often that the manifold is *non-degenerate;* see [38] for details), and so hyperplanes are not usually considered. However, there are still several results for Diophantine approximation on hyperplanes; see [10, 44, 45] for results in the real setting.

Consider the following setup. Fix $1 \leq m \leq n-1$ and set $d = n - m$. For any vector of p-adic integers $\boldsymbol{\alpha} \in \mathbb{Z}_p^m$ define the coordinate hyperplane

$$\Pi_{\boldsymbol{\alpha}} := \{(x_1, \dots, x_d, \boldsymbol{\alpha}) : (x_1, \dots, x_d) \in \mathbb{Z}_p^d\} \subset \mathbb{Z}_p^n, \tag{2}$$

We are interested in the set of weighted Ψ-approximable points on coordinate hyperplanes, i.e., the set $\mathfrak{W}_n(\boldsymbol{\tau}) \cap \Pi_{\boldsymbol{\alpha}}$. Note that trivially $\dim_H \mathfrak{W}_n(\boldsymbol{\tau}) \cap \Pi_{\boldsymbol{\alpha}} \leq d$, with equality when $\sum_{i=1}^n \tau_i \leq n+1$. Theorems 1.1–1.3 of this paper provide us with a good understanding of the behaviour of rational points close to coordinate hyperplanes, allowing us to prove the following result.

Theorem 2.2. *Let $\Pi_{\boldsymbol{\alpha}}$ be a coordinate hyperplane of $\mathbb{Z}_p^n$ of the form (2), let $\boldsymbol{\alpha} \in \mathbb{Z}_p^m$ satisfy $\tau(\boldsymbol{\alpha}) = m+1$. Let $\boldsymbol{\tau} = (\tau_1, \dots, \tau_n) \in \mathbb{R}_+^n$ be a weight vector with the properties that*

$$m + \frac{m}{n} \leq \sum_{i=1}^{m} \tau_{d+i} < m+1, \quad \sum_{i=1}^{n} \tau_i > n+1, \quad \tau_i > 1,$$

for all $1 \leq i \leq n$. Then

$$\dim_H \mathfrak{W}_n(\boldsymbol{\tau}) \cap \Pi_{\boldsymbol{\alpha}} = \min_{1 \leq i \leq d} \left\{ \frac{n+1-\sum_{i=1}^{m} \tau_{d+i} + \sum_{\substack{\tau_j \leq \tau_i \\ 1 \leq j \leq d}} (\tau_i - \tau_j)}{\tau_i} \right\} = s.$$

Furthermore $\mathcal{H}^s(\mathfrak{W}_n(\boldsymbol{\tau}) \cap \Pi_{\boldsymbol{\alpha}}) = \infty$.

Remark 2.3. *In the special case of simultaneous approximation, we have that*

$$\dim_H \mathfrak{W}_n(\tau) \cap \Pi_{\boldsymbol{\alpha}} = \frac{n+1}{\tau} - m \quad \text{provided} \quad 1 + \frac{1}{n} < \tau < 1 + \frac{1}{m}.$$

This gives us the expected dimension of $\dim_H \mathfrak{W}_n(\tau)$ less the codimension of the hyperplane $\Pi_{\boldsymbol{\alpha}}$.

Remark 2.4. *The upper bound constraints on $(\tau_{d+1}, \dots, \tau_n)$ ensure that we can apply Theorem 1.1, whereas the lower bound constraints ensure we can obtain a ubiquity hypothesis; see § 4.1 below. The condition that $\sum_{i=1}^n \tau_i > n+1$ is merely present to ensure that we do not include the trivial case when $\mathfrak{W}_n(\boldsymbol{\tau}) = \mathbb{Z}_p^n$, in which case $\dim_H \mathfrak{W}_n(\boldsymbol{\tau}) \cap \Pi_{\boldsymbol{\alpha}} = d$.*

Remark 2.5. *We note that coordinate hyperplanes, the main focus of this paper, satisfies the differentiability condition introduced in [11] and so part of the lower bound in Theorem 2.2 has been proven in [11]. To highlight the key difference between the lower bound obtained here and the lower bound achieved in [11], note that the approximation exponents on the independent variable do not need to be greater than the approximation exponents on $\boldsymbol{\alpha}$. The reason being in this case the 'dependent variable' is fixed and thus independent of the first d coordinates.*

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For general approximation functions $\Psi = (\psi_1, \ldots, \psi_n)$, recall the order of infinity of ψ_i to be

$$v_i = \lim_{q\to\infty} \frac{-\log(\psi_i(q))}{\log q}, \qquad i = 1, \ldots, n. \tag{3}$$

Providing the order of infinity of ψ_i exists and is positive and finite for each $1 \le i \le n$, then define $\Psi^* = (v_1, \ldots, v_n)$.

Corollary 2.6. *Let $\Psi = (\psi_1, \ldots, \psi_n)$ be an n-tuple of approximation functions with the order of infinity of ψ_i being positive finite in each coordinate. If Ψ^* satisfy the same conditions as in Theorem 2.2, then for all $\boldsymbol{\alpha} \in \mathbb{Z}_p^m$ with $\tau(\boldsymbol{\alpha}) = m + 1$,*

$$\dim_H \mathfrak{W}_n(\Psi) \cap \Pi_{\boldsymbol{\alpha}} = \dim_H \mathfrak{W}_n(\Psi^*) \cap \Pi_{\boldsymbol{\alpha}}.$$

The corollary easily follows from the observation that (3) implies there exists a sufficiently large $q \in \mathbb{N}$ such that

$$q^{-v_i-\epsilon_i} \le \psi_i(q) \le q^{-v_i+\epsilon_i}$$

for all $1 \le i \le n$ and $\epsilon = (\epsilon_1, \ldots, \epsilon_n) > 0$ with $\epsilon_i \to 0$ as $q \to \infty$. Thus $\mathfrak{W}_n(\Psi^* + \epsilon) \subseteq \mathfrak{W}_n(\Psi) \subseteq \mathfrak{W}_n(\Psi^* - \epsilon)$, and so letting $\epsilon \to 0$ we obtain the desired result. While Corollary 2.6 provides a result for general Ψ satisfying (3), note there are many functions where such an order of infinity does not exist.

The following section provides auxiliary results needed to prove Theorem 2.2. In particular, the framework for the Mass Transference Principle from rectangles to rectangles [51] is provided.

3. Auxiliary results for Theorem 2.2

The following known results are required to prove Theorem 2.2. We begin with the p-adic version of Minkowski's theorem for systems of linear forms. The proof is an application of the pigeonhole principle and can be found in [11].

Lemma 3.1 ([11, Lemma 3.3]). *Let each $L_i(\boldsymbol{x}) : \mathbb{Z}_p^n \to \mathbb{Z}_p$, with $i = 1, \ldots, n$, be a linear form with p-adic integer coefficients. Let $\sum_{i=1}^n \tau_i = n + 1$ for $\tau_i \in \mathbb{R}_+$ and $\sum_{i=1}^n \sigma_i = n$ for $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathbb{R}^n$. Then there exists $H_\sigma > 1$ such that for all integers $H_0, H_1, \ldots, H_n$ with $T^{n+1} = (H_0 + 1)(H_1 + 1)\ldots(H_n + 1) \ge H_\sigma$ there exists a non-zero rational integer vector $\boldsymbol{x} = (x_0, x_1, \ldots, x_n) \in \mathbb{Z}^n$ with*

$$|x_i| \le H_i \quad 0 \le i \le n,$$

satisfying the system of inequalities

$$|L_i(\boldsymbol{x})|_p < p^{\sigma_i} T^{-\tau_i} \quad \text{for } i = 1, \ldots, n.$$

We also require the following lemma, which tells us that the measure of a lim sup set of balls remains unaltered when the radii are multiplied by some constant. The Euclidean version of this result is well known; see for example [6, 14]. We need the following ultrametric version.

Lemma 3.2 ([11, Lemma 4.1]). *Let (X, d) be a separable ultrametric space and μ be a Borel regular measure on X. Let $(B_i)_{i\in\mathbb{N}}$ be a sequence of balls in X with radii $r_i \to 0$ as $i \to \infty$. Let $(U_i)_{i\in\mathbb{N}}$ be a sequence of μ-measurable sets such that $U_i \subset B_i$ for all i. Assume that for some $c > 0$*

$$|U_i| \ge c|B_i| \qquad \text{for all } i.$$

Then the limsup sets

$$\limsup_{i\to\infty} U_i := \bigcap_{j=1}^{\infty} \bigcup_{i\ge j} U_i \qquad \text{and} \qquad \limsup_{i\to\infty} B_i := \bigcap_{j=1}^{\infty} \bigcup_{i\ge j} B_i$$

have the same μ-measure.

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3.1. The weighted mass transference principle

Another key result is the following Mass Transference Principle type theorem. In order to state the result of Wang and Wu [51], we need the notion of local ubiquity for rectangles, a variation of the notion of ubiquitous systems introduced by Beresnevich, Dickinson, and Velani [6].

Fix an integer $n \geq 1$, and for each $1 \leq i \leq n$ let $(X_i, |.|_i, m_i)$ be a bounded locally compact metric space with m_i a δ_i-Ahlfors regular measure. Consider the product space $(X, |.|, m)$, defined by

$$X = \prod_{i=1}^{n} X_i, \quad m = \prod_{i=1}^{n} m_i, \quad |.| = \max_{1 \leq i \leq n} |.|_i \, .$$

For any $x \in X$ and $r \in \mathbb{R}_+$ define the open ball

$$B(x, r) = \left\{ y \in X : \max_{1 \leq i \leq n} |x_i - y_i|_i < r \right\} = \prod_{i=1}^{n} B_i(x_i, r),$$

where B_i denotes the ball associated with the i^{th} metric space. Let J be an infinite countable index set and $\beta : J \to \mathbb{R}_+$ a positive function. Let l_k, u_k be two sequences in $\mathbb{R}_+$ such that $u_k \geq l_k$ with $l_k \to \infty$ as $k \to \infty$. Define

$$J_k = \{\alpha \in J : l_k \leq \beta_\alpha \leq u_k\} \, .$$

Let $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function with $\rho(\beta_\alpha) \to 0$ as $\beta_\alpha \to \infty$. For each $1 \leq i \leq n$, let $\{R_{\alpha,i}\}_{\alpha \in J}$ be a sequence of subsets in X_i. As with the standard setting of ubiquitous systems define the resonant sets

$$\left\{ R_\alpha = \prod_{i=1}^{n} R_{\alpha,i} \right\}_{\alpha \in J} .$$

For $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{R}_+^n$ denote the set of hyperrectangles

$$\Delta(R_\alpha, \rho(r)^{\boldsymbol{a}}) = \prod_{i=1}^{n} \Delta(R_{\alpha,i}, \rho(r)^{a_i}),$$

where for some set A and $b \in \mathbb{R}_+$ let $\Delta(A, b) = \bigcup_{a \in A} B(a, b)$.

Definition 3.3 (local ubiquitous system of rectangles). *Call $(\{R_\alpha\}_{\alpha \in J}, \beta)$ a local ubiquitous system of rectangles with respect to $(\rho, \boldsymbol{a})$ if there exists a constant $c > 0$ such that for any ball $B \subset X$,*

$$\limsup_{k \to \infty} m\left(B \cap \bigcup_{\alpha \in J_k} \Delta(R_\alpha, \rho(u_k)^{\boldsymbol{a}}) \right) \geq cm(B).$$

For $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{R}_+^n$ define

$$W(\mathbf{t}) = \limsup_{\alpha \in J} \Delta\left(R_\alpha, \rho(\beta_\alpha)^{\boldsymbol{a}+\mathbf{t}}\right) .$$

Given the above notations and definitions, we can state the Mass Transference Principle from rectangles to rectangles (MTPRR) of [51].

Theorem 3.4 (Mass Transference Principle from rectangles to rectangles). *Under the settings above assume that $(\{R_\alpha\}_{\alpha \in J}, \beta)$ satisfies the local ubiquity for systems of rectangles condition with respect to $(\rho, \boldsymbol{a})$. Then*

$$\dim_H W(\mathbf{t}) \geq \min_{A_i \in A} \left\{ \sum_{j \in K_1} \delta_j + \sum_{j \in K_2} \delta_j + \frac{\sum_{j \in K_3} a_j \delta_j - \sum_{j \in K_2} t_j \delta_j}{A_i} \right\} = s,$$

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where $A = \{a_i, a_i + t_i, 1 \le i \le n\}$ *and* K_1, K_2, K_3 *are a partition of* $\{1, \dots, n\}$ *defined as*

$$K_1 = \{j : a_j \ge A_i\}, \quad K_2 = \{j : a_j + t_j \le A_i\} \backslash K_1, \quad K_3 = \{1, \dots n\} \backslash (K_1 \cup K_2).$$

Furthermore $\mathcal{H}^s(B \cap W(\boldsymbol{\tau})) = \mathcal{H}^s(B)$.

Hence, provided we can find a lim sup set of hyperrectangles that satisfy the local ubiquity property for rectangles, then we have a lower bound on the Hausdorff dimension of the corresponding lim sup set of shrunken hyperrectangles.

4. Proof of Theorem 2.2

We split the proof into the upper and lower bounds and solve each case separately. In both cases we will use the following simplified set. Let π be the projection $\pi : \mathbb{Z}_p^n \to \mathbb{Z}_p^{n-m}$, defined by

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_d).$$

By a well-known result of fractal geometry (see [25, Proposition 3.3]), as π is a bi-Lipschitz mapping over $\mathfrak{W}_n(\boldsymbol{\tau}) \cap \Pi_{\boldsymbol{\alpha}}$, we have that

$$\dim_H \mathfrak{W}_n(\boldsymbol{\tau}) \cap \Pi_{\boldsymbol{\alpha}} = \dim_H \pi(\mathfrak{W}_n(\boldsymbol{\tau}) \cap \Pi_{\boldsymbol{\alpha}}). \tag{4}$$

Let $\boldsymbol{\tau}_m = (\tau_{d+1}, \dots, \tau_n)$ denote the m-tuple of approximation exponents of $\boldsymbol{\alpha}$, and similarly, let $\boldsymbol{\tau}_d = (\tau_1, \dots, \tau_d)$ denote the d-tuple of approximation exponents of the independent variables of $\Pi_{\boldsymbol{\alpha}}$. Consider the set of integers

$$\mathcal{Q}(\boldsymbol{\alpha}, \boldsymbol{\tau}_m) := \left\{ q_0 \in \mathbb{N} : \left| \alpha_i - \frac{q_{d+i}}{q_0} \right|_p < q_0^{-\tau_{d+i}}, \text{ for some } \begin{array}{c} |q_{d+i}| \le q_0, \\ gcd(q_{d+i}, q_0) = 1, \end{array} 1 \le i \le m \right\},$$

and the union of sets

$$\mathfrak{A}^*_{q_0}(\boldsymbol{\tau}_d) = \bigcup_{\substack{|q_i| \le q_0, \ gcd(q_i, q_0) = 1 \\ 1 \le i \le d}} \left\{ \boldsymbol{x} \in \mathbb{Z}_p^d : \left| x_i - \frac{q_i}{q_0} \right|_p < q_0^{-\tau_i} \right\}.$$

Then, we have that

$$\pi\ (\mathfrak{W}_n(\boldsymbol{\tau}) \cap \Pi_{\boldsymbol{\alpha}}) = \limsup_{q_0 \in \mathcal{Q}(\boldsymbol{\alpha}, \boldsymbol{\tau}_m)} \mathfrak{A}^*_{q_0}(\boldsymbol{\tau}_d),$$

and so, by (4), our problem is reduced to finding upper and lower bounds for $\dim_H \limsup_{q_0 \in \mathcal{Q}(\boldsymbol{\alpha}, \boldsymbol{\tau}_m)} \mathcal{A}^*_{q_0}(\boldsymbol{\tau}_d)$.

4.1. *Upper bound*

For the upper bound we take the standard cover of hyperrectangles used in the construction of $\mathfrak{A}^*_q(\boldsymbol{\tau}_d)$. By hyperrectangles in the p-adic setting we mean Cartesian products of p-adic balls in each coordinate axis. Concisely,

$$\left\{ \boldsymbol{x} \in \mathbb{Z}_p^d : \left| x_i - \frac{q_i}{q_0} \right|_p < q_0^{-\tau_i} \right\} = \prod_{i=1}^{d} B\left(\frac{q_i}{q_0}, q_0^{-\tau_i} \right), \tag{5}$$

where $B(x, r)$ denotes a ball in $\mathbb{Z}_p$ induced by the p-adic norm. Observe that, unlike the Euclidean case, the centre of the ball is not uniquely defined. In particular, if $|x - y|_p < r$, then by the strong triangle inequality $B(x, r) = B(y, r)$. It is critical in the upper bound that we can estimate the size of $\mathcal{Q}(\boldsymbol{\alpha}, \boldsymbol{\tau}_m)$. By a standard geometrical argument, note that each hyperrectangle of the form (5) in the construction of $\mathfrak{A}^*_q(\boldsymbol{\tau}_d)$ can be covered by a finite collection of balls $\mathfrak{B}_q(\tau_i)$ of radius $q^{-\tau_i}$ for $1 \le i \le d$. Without loss of generality, we can assume that $\tau_1 \ge \cdots \ge \tau_d$, since if not, then take some bi-Lipschitz mapping to reorder

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the coordinate axes so this is the case. Fix some $1 \leq i \leq d$. For each coordinate $1 \leq j \leq d$, consider the ball $B(\frac{q_j}{q_0}, p^{-\ell_j})$, where $\ell_j \in \mathbb{N}$ is uniquely chosen so that

$$B\left(\frac{q_j}{q_0}, p^{-\ell_j}\right) \supseteq B\left(\frac{q_j}{q_0}, q_0^{-\tau_j}\right) \supset B\left(\frac{q_i}{q_0}, p^{-\ell_j-1}\right) .$$

Each ball $B(\frac{q_j}{q_0}, p^{-\ell_j})$ can be covered by $p^{\max\{0,\ell_i-\ell_j\}}$ balls $\mathcal{G}_j$ of radii $p^{-\ell_i}$. Note that

$$p^{\max\{0,\ell_i-\ell_j\}} \leq \max\left\{1, p^2\frac{q^{-\tau_j}}{q^{-\tau_i}}\right\} .$$

Thus, by considering the Cartesian product of all balls in each set $\mathcal{G}_j$, we have a covering by balls of radius $p^{-\ell_i}$ of the hyperrectangle of type (5), and the cardinality of the cover is at most

$$\ll \prod_{j=i}^{d} \frac{q^{-\tau_j}}{q^{-\tau_i}} = q^{\sum_{j=i}^{d}(\tau_i-\tau_j)}.$$

Next, we need to know how many hyperrectangles are in the set $\mathfrak{A}_q^*(\boldsymbol{\tau}_d)$. Working this out per layer is hard, but using our main result, Theorem 1.1, for each $k \in \mathbb{N}$, we can estimate the number of hyperrectangles in the set

$$C(\boldsymbol{\alpha}, \boldsymbol{\tau}_m, k) := \left\{q_0 \in \mathcal{Q}(\boldsymbol{\alpha}, \boldsymbol{\tau}_m) : 2^k \leq q_0 < 2^{k+1}\right\} .$$

To do this, observe the following. Let

$$\mathcal{Q}_m^{'}(\boldsymbol{\alpha}, \boldsymbol{\tau}_m, N) := \{q_0 \in \mathbb{N} : (q_0, \dots, q_m) \in \mathcal{Q}_m(\boldsymbol{\alpha}, \Psi_m, N) \text{ and } gcd(q_i, q_0) = 1 \; \forall \; 1 \leq i \leq m\} ,$$

where Ψ_m has functions of the form $q \mapsto (2^{-1}q)^{-\tau_i}$ in each of the m coordinates. Since $\tau_i > 1$ for each $1 \leq i \leq m$ we have that each $q_0 \in \mathcal{Q}_m^{'}(\boldsymbol{\alpha}, \boldsymbol{\tau}_m, N)$ has unique associated $(q_0, q_1, \dots, q_m) \in \mathcal{Q}_m(\boldsymbol{\alpha}, \Psi_m, N)$. So, with the additional coprimality condition, we have that

$$\#\mathcal{Q}_m^{'}(\boldsymbol{\alpha}, \boldsymbol{\tau}_m, N) \leq \#\mathcal{Q}_m(\boldsymbol{\alpha}, \Psi_m, N) . \tag{6}$$

Furthermore, by the coprimality condition, note that

$$|q_0\alpha_i - q_i|_p < (2^{-1}N)^{-\tau_i}, \quad \text{and} \quad \left|\alpha_i - \frac{q_i}{q_0}\right|_p < (2^{-1}N)^{-\tau_i}$$

are equivalent since $p \nmid q_0$. To see that $p \nmid q_0$ suppose that $q_0 = p^k q_0'$ for some $k \in \mathbb{N}$. Since $gcd(q_i, q_0) = 1$ we clearly have $p \nmid q_i$. As $\alpha_i \in \mathbb{Z}_p$ this leads us to conclude, by the strong triangle inequality, that

$$|q_0\alpha_i - q_i|_p = \max\{|\alpha_i|_p p^{-k}, 1\} = 1 .$$

This cannot be true for N sufficiently large, and so $p \nmid q_0$. Thus,

$$\mathcal{Q}_m^{'}(\boldsymbol{\alpha}, \boldsymbol{\tau}_m, N) = \left\{ q_0 \in \mathbb{N} : \begin{array}{c} 0 < q_0 \leq N , \\ \max_{1\leq i\leq m} |q_i| \leq N , \\ gcd(q_i, q_0) = 1 , \end{array} \left|\alpha_i - \frac{q_i}{q_0}\right|_p < (2^{-1}N)^{-\tau_i} \quad 1 \leq i \leq m \right\} .$$

Consider any $q_0 \in C(\boldsymbol{\alpha}, \boldsymbol{\tau}_m, k)$. By definition of $\mathcal{Q}(\boldsymbol{\alpha}, \boldsymbol{\tau}_m)$ there exists $(q_{d+1}, \dots, q_{d+m}) \in \mathbb{Z}^m$ with $gcd(q_{d+i}, q_0) = 1$ and $\max_i |q_{d+i}| \leq q_0$. Since $2^k \leq q_0 < 2^{k+1}$ we have $\max_i |q_{d+i}| \leq 2^{k+1}$ and

$$\left|\alpha_i - \frac{q_{d+i}}{q_0}\right|_p < q_0^{-\tau_{d+i}} \leq (2^k)^{-\tau_{d+i}} .$$

Thus $q_0 \in \mathcal{Q}_m^{'}(\boldsymbol{\alpha}, \Psi_m, 2^{k+1})$, and so

$$\#C(\boldsymbol{\alpha}, \boldsymbol{\tau}_m, k) \leq \#\mathcal{Q}_m(\boldsymbol{\alpha}, \Psi, 2^{k+1})$$

by (6). Each τ_i-approximation function is monotonic decreasing and so

$$\limsup_{k\to\infty} \bigcup_{q_0 \in C(\boldsymbol{\alpha}, \boldsymbol{\tau}_m, k)} \bigcup_{\substack{|q_i| \leq q_0, \\ 1\leq i\leq d}} \left\{\boldsymbol{x} \in \mathbb{Z}_p^d : \left|x_i - \frac{q_i}{q_0}\right|_p < (2^k)^{-\tau_i}\right\} ,$$

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is a cover of our set $\pi\ (\mathfrak{W}_n(\boldsymbol{\tau}) \cap \Pi_{\boldsymbol{\alpha}})$. Hence, for any $k_0 \in \mathbb{N}$ sufficiently large, we have that

$$\begin{aligned}\mathcal{H}^s\left(\pi\ (\mathfrak{W}_n(\boldsymbol{\tau}) \cap \Pi_{\boldsymbol{\alpha}})\right) &\leq \sum_{k=k_0}^{\infty} \sum_{q \in C(\boldsymbol{\alpha}, \boldsymbol{\tau}_m, k)} q^d \#\mathfrak{B}_{2^k}(\tau_i).((2^k)^{-\tau_i})^s, \\ &\overset{\text{Theorem 1.1}}{\ll} \sum_{k=k_0}^{\infty} 2^{(k+1)(m+1-\sum_{i=1}^{m}\tau_{d+i})}(2^{k+1})^d (2^k)^{\sum_{j=i}^{d}(\tau_i-\tau_j)}(2^k)^{-\tau_i s}, \\ &\ll \sum_{k=k_0}^{\infty} 2^{k(n+1-\sum_{i=1}^{m}\tau_{d+i}+\sum_{j=i}^{d}(\tau_i-\tau_j)-\tau_i s)},\end{aligned}$$

The above sum converges when

$$s \geq \frac{n+1-\sum_{i=1}^{m}\tau_{d+i}+\sum_{j=i}^{d}(\tau_i-\tau_j)}{\tau_i} + \epsilon,$$

for any $\epsilon > 0$. This is true for each $1 \leq i \leq d$, and as ϵ is arbitrary we have that

$$s \geq \min_{1\leq i\leq d}\left\{\frac{n+1-\sum_{i=1}^{m}\tau_{d+i}+\sum_{j=i}^{d}(\tau_i-\tau_j)}{\tau_i}\right\},$$

completing the upper bound result.

4.2. Lower bound

In order to use the weighted mass Transference Principle (Theorem 3.4) we need to construct a ubiquitous system of rectangles. Let

$$J = \mathcal{Q}(\boldsymbol{\alpha}, \boldsymbol{\tau}_m),\ R_{q,i} = \left\{\frac{q_i}{q} \in \mathbb{Q} : \begin{matrix}|q_i| \leq q, \\ gcd(q_i, q) = 1\end{matrix}\right\}, \quad R_q = \prod_{i=1}^{d} R_{q,i},$$
$$\beta(q) = q, \qquad \rho(q) = Cq^{-1}, \qquad l_k = M^k,\ u_k = M^{k+1},$$

with $M \in \mathbb{N}$ and $C > 0$ constant, both to be determined later. We prove the following.

Proposition 4.1. *Let R_q, ρ, and J_k be defined as above. Let $\tilde{v} = (v_1, \ldots, v_d) \in \mathbb{R}^d_{>0}$ with each $v_i > 1$ and*

$$\sum_{i=1}^{d} v_i = n+1-\sum_{i=1}^{m}\tau_{d+i} = n+1-\tilde{\tau},$$

for

$$m + \frac{m}{n} < \sum_{i=1}^{m}\tau_{d+i} < m+1, \tag{7}$$

and each $\tau_i > 1$. Then for any ball $B = B(x, r) \subset \mathbb{Z}_p^d$, with centre $x \in \mathbb{Z}_p^d$ and radius $0 < r < r_0$ for some $r_0 \in \mathbb{R}_+$, there exists sufficiently large $k, M \in \mathbb{N}$ and a constant $c > 0$ such that

$$\mu_d\left(B \cap \bigcup_{q\in J_k} \Delta(R_{\boldsymbol{q}}, \rho(u_k)^{\tilde{v}})\right) \geq c\mu_d(B).$$

The proof follows the same style of similar results in $\mathbb{R}^n$. See [12, Theorem 1.3] for the one dimensional real case, or [11, Proposition 5.1] for the n-dimensional p-adic case.

Proof. Fix some ball $B = B(\boldsymbol{y}, r)$ for some $\boldsymbol{y} \in \mathbb{Z}_p^n$ and $r \in \{p^i : i \in \mathbb{N} \cup \{0\}\}$. We will assume that k is sufficiently large so that $M^k r \geq 1$. For any $y = (y_1, \ldots, y_d) \in (\mathbb{Z}_p \backslash \mathbb{Q})^d$, consider the system of inequalities

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$$\begin{cases} |q_0\alpha_i - q_{d+i}|_p < (M^{k+\frac{1}{n+1}})^{-\tau_{d+i}}, & 1 \le i \le m, \\ |q_0 y_i - q_i|_p < p^{\frac{n}{d}}(M^{k+\frac{1}{n+1}})^{-v_i}, & 1 \le i \le d, \\ \max_{1\le i\le n}|q_i| \le M^k, \\ |q_0| \le M^{k+1}. \end{cases} \tag{8}$$

By the condition on $\tilde{v}$, we have, by Lemma 3.1, that there exists a non-zero integer solution $(q_0, \dots, q_n) \in \mathbb{Z}^{n+1}$ to (8) for any $\boldsymbol{y} \in B$. We can assume without loss of generality that $q_0 \ge 0$ and, furthermore, that $q_0 \ne 0$ since each $v_i, \tau_{d+i} > 1$ for all $1 \le i \le d$ and $1 \le j \le m$. As we wish to have a statement for rectangles, rather than linear forms, we need to divide through by $|q_0|_p$. To ensure we do not divide by a value too small, we remove the set of q_0 that contain large powers of p, that is, all q_0 such that $|q_0|_p \le p^{-\lambda_0}$ for some fixed $\lambda_0 \in \mathbb{N}$. Since $v_i, \tau_{d+j} > 1$ for each $1 \le i \le d$ and $1 \le j \le m$, (8) combined with $0 < q_0 \le M^{k+1}$ implies that $|q_i|_p \le |q_0|_p$ for each $1 \le i \le n$, provided that k is sufficiently large.

For fixed solution $(q_0, \dots, q_n)$ to (8) let λ be the integer such that $|q_0|_p = p^{-\lambda}$. Write $q_0' = q_0 p^{-\lambda}$ and $q_i' = q_i p^{-\lambda}$. Observe that $q_0', q_i' \in \mathbb{Z}$, that

$$(q_0', q_{d+1}', \dots, q_n') \in \mathcal{Q}_m\left(\boldsymbol{\alpha}, \boldsymbol{\tau}, \frac{M^{k+1}}{p^\lambda}\right), \quad 0 < q_0' \le p^{-\lambda}M^{k+1}, \quad |q_i'| \le p^{-\lambda}M^k, \tag{9}$$

for each $1 \le i \le n$, and that

$$\left|y_i - \frac{q_i'}{q_0'}\right|_p = p^\lambda |q_0 y_i - q_i|_p < p^{\lambda+\frac{n}{d}}\left(M^{k+\frac{1}{n+1}}\right)^{-v_i}, \qquad i = 1, \dots, d\,. \tag{10}$$

We have similar for the inequalities on $\boldsymbol{\alpha}$. We want to count the number of $\frac{\boldsymbol{q}}{q_0} = \left(\frac{q_1}{q_0}, \dots, \frac{q_d}{q_0}\right) \in R_{q_0}$ such that

$$B \cap \Delta\left(\frac{\boldsymbol{q}}{q_0}, \rho(M^{k+1})^{\tilde{v}}\right) = B(\boldsymbol{y}, r) \cap \prod_{i=1}^{d} B\left(\frac{q_i}{q_0}, \rho(M^{k+1})^{v_i}\right) \ne \emptyset\,. \tag{11}$$

By using our assumptions that $r \ge M^{-k}$ and each $v_i > 1$, we have that $r > \rho(M^{k+1})^{v_i}$ for each $i = 1, \dots, d$. So, by the strong triangle inequality, we need

$$\left|y_i - \frac{q_i}{q_0}\right|_p < r, \quad i = 1, \dots, d \tag{12}$$

in order for $\frac{\boldsymbol{q}}{q_0}$ to satisfy (11). For q_0 fixed and each $|q_i| \le q_0$ by congruence classes, we have that there are at most $(2q_0 r + 1)^d < 3^d M^{kd} r^d$ suitable values of $\boldsymbol{q}$ solving (12). Hence, we have that

$$\begin{aligned}
&\mu_d\left(B \cap \bigcup_{\lambda \ge \lambda_0} \bigcup_{\substack{|q_i'| \le \frac{M^k}{p^\lambda} \\ 0 < q_0' \le \frac{M^{k+1}}{p^\lambda} :\, q_0' \in \mathcal{Q}_m'\left(\boldsymbol{\alpha}, \boldsymbol{\tau}, \frac{M^{k+1}}{p^\lambda}\right)}} \bigcup_{\left(\frac{q_1'}{q_0'}, \dots, \frac{q_d'}{q_0'}\right) \in B} \prod_{i=1}^{d} B\left(\frac{q_i'}{q_0'}, p^{\lambda+\frac{n}{d}}\left(M^{k+\frac{1}{n+1}}\right)^{-v_i}\right)\right) \\
&\le \sum_{\lambda \ge \lambda_0} \#\mathcal{Q}_m'\left(\boldsymbol{\alpha}, \boldsymbol{\tau}, \frac{M^{k+1}}{p^\lambda}\right)\left(3\frac{M^k}{p^\lambda} r\right)^d p^{d\lambda+n} M^{-k(n+1-\tilde{\tau})-1+\frac{\tilde{\tau}}{n+1}}, \\
&\overset{Theorem 1.1}{\le} \sum_{\lambda \ge \lambda_0} C_1\left(\frac{M^{k+1}}{p^\lambda}\right)^{m+1-\tilde{\tau}} 3^d \left(\frac{M^k}{p^\lambda}\right)^d p^{d\lambda+n} M^{-\left(k(n+1-\tilde{\tau})+1-\frac{\tilde{\tau}}{n+1}\right)} \mu_d(B) \\
&= \sum_{\lambda \ge \lambda_0} \mu_d(B) C_1 3^d p^{n-\lambda(m+1-\tilde{\tau})} M^{m-\tilde{\tau}+\frac{\tilde{\tau}}{n+1}}, \\
&\le C_1 3^d M^{m-\tilde{\tau}+\frac{\tilde{\tau}}{n+1}} \frac{p^{n+(1-\lambda_0)(m+1-\tilde{\tau})}}{p^{m+1-\tilde{\tau}} - 1} \mu_d(B).
\end{aligned}$$

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Take λ_0 sufficiently large, say

$$p^{\lambda_0} > \left(2\frac{C_1 3^d p^{n+m+1-\tilde{\tau}} M^{m-\tilde{\tau}+\frac{\tilde{\tau}}{n+1}}}{p^{m+1-\tilde{\tau}}-1}\right)^{\frac{1}{m+1-\tilde{\tau}}}.$$

Observe that since $\tilde{\tau} > m + \frac{m}{n}$ the fact that λ_0 is dependent on M is irrelevant since the value decreases as M increases; hence, we could replace $M^{m-\tilde{\tau}+\frac{\tilde{\tau}}{n+1}}$ by 1. Then we have that

$$\mu_d\left(B \cap \bigcup_{\lambda\geq\lambda_0} \bigcup_{\substack{|q_i|\leq \frac{M^k}{p^\lambda} \\ 0<q_0'\leq \frac{M^{k+1}}{p^\lambda}:\, q_0'\in\mathcal{Q}_m'\left(\boldsymbol{\alpha},\boldsymbol{\tau},\frac{M^{k+1}}{p^\lambda}\right)}} \bigcup_{\left(\frac{q_1'}{q_0'},\ldots,\frac{q_d'}{q_0'}\right)\in B} \prod_{i=1}^{d} B\left(\frac{q_i'}{q_0'}, p^{\lambda+\frac{n}{d}}\left(M^{k+\frac{1}{n+1}}\right)^{-\nu_i}\right)\right) \leq \frac{1}{2}\mu_d(B).$$

and so

$$\begin{aligned}
&\mu_d\left(B \cap \bigcup_{\substack{|q_i|\leq M^k \\ 0<q_0'\leq M^{k+1}:\, q_0'\in\mathcal{Q}_m'\left(\boldsymbol{\alpha},\boldsymbol{\tau},M^{k+1}\right),\ |q_0|_p\geq p^{-\lambda_0}}} \prod_{i=1}^{n} B\left(\frac{q_i}{q_0}, p^{\lambda_0+\frac{n}{d}}\left(M^{k+\frac{1}{n+1}}\right)^{-\alpha_i}\right)\right) \\
&\geq \mu_d\left(B \cap \bigcup_{\substack{|q_i|\leq M^k \\ 0<q_0'\leq M^{k+1}:\, q_0'\in\mathcal{Q}_m'\left(\boldsymbol{\alpha},\boldsymbol{\tau},M^{k+1}\right)}} \prod_{i=1}^{n} B\left(\frac{q_i}{q_0}, p^{\lambda+\frac{n}{d}}\left(M^{k+\frac{1}{n+1}}\right)^{-\alpha_i}\right)\right) \\
&\quad - \mu_d\left(B \cap \bigcup_{\substack{|q_i|\leq M^k \\ 0<q_0'\leq M^{k+1}:\, q_0'\in\mathcal{Q}_m'\left(\boldsymbol{\alpha},\boldsymbol{\tau},M^{k+1}\right),\ |q_0|_p<p^{-\lambda_0}}} \prod_{i=1}^{n} B\left(\frac{q_i}{q_0}, p^{\lambda+\frac{n}{d}}\left(M^{k+\frac{1}{n+1}}\right)^{-\alpha_i}\right)\right) \\
&\geq \frac{1}{2}\mu_d(B).
\end{aligned}$$

Similarly to the above we can deduce that

$$\begin{aligned}
&\mu_d\left(B \cap \bigcup_{\substack{|q_i|\leq M^k \\ M^k<q_0\leq M^{k+1}:\, q_0\in\mathcal{Q}_m'\left(\boldsymbol{\alpha},\boldsymbol{\tau},M^{k+1}\right),\ |q_0|_p\geq p^{-\lambda_0}}} \prod_{i=1}^{n} B\left(\frac{q_i}{q_0}, p^{\lambda_0+\frac{n}{d}}\left(M^{k+\frac{1}{n+1}}\right)^{-\alpha_i}\right)\right) \\
&\geq \mu_d\left(B \cap \bigcup_{\substack{|q_i|\leq M^k \\ 0<q_0<M^{k+1}:\, q_0\in\mathcal{Q}_m'\left(\boldsymbol{\alpha},\boldsymbol{\tau},M^{k+1}\right),\ |q_0|_p\geq p^{-\lambda_0}}} \prod_{i=1}^{n} B\left(\frac{q_i}{q_0}, p^{\lambda_0+\frac{n}{d}}\left(M^{k+\frac{1}{n+1}}\right)^{-\alpha_i}\right)\right) \qquad (13) \\
&\quad - \mu_d\left(B \cap \bigcup_{\substack{|q_i|\leq M^k \\ 0<q_0\leq M^k:\, q_0\in\mathcal{Q}_m'\left(\boldsymbol{\alpha},\boldsymbol{\tau},M^k\right),\ |q_0|_p\geq p^{-\lambda_0}}} \prod_{i=1}^{n} B\left(\frac{q_i}{q_0}, p^{\lambda_0+\frac{n}{d}}\left(M^{k+\frac{1}{n+1}}\right)^{-\alpha_i}\right)\right).
\end{aligned}$$

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Note that we are justified in taking the smaller set $\mathcal{Q}_m'(\boldsymbol{\alpha}, \boldsymbol{\tau}, M^k)$ rather than $\mathcal{Q}_m(\boldsymbol{\alpha}, \boldsymbol{\tau}, M^{k+1})$ in the last row of the (13) since

$$\left\{0 < q_0 \leq M^k : q_0 \in \mathcal{Q}_m^{'}(\boldsymbol{\alpha}, \boldsymbol{\tau}, M^k)\right\} \supseteq \left\{0 < q_0 \leq M^k : q_0 \in \mathcal{Q}_m^{'}(\boldsymbol{\alpha}, \boldsymbol{\tau}, M^{k+1})\right\}.$$

Calculating the measure of the 3rd row of (13) gives us that

$$\begin{aligned}
&\mu_d\left(B \cap \bigcup_{\substack{|q_i| \leq M^k \\ 0<q_0 \leq M^k:\ q_0 \in \mathcal{Q}_m'\left(\boldsymbol{\alpha}, \boldsymbol{\tau}, M^k\right),\ |q_0|_p \geq p^{-\lambda_0}}} \prod_{i=1}^{n} B\left(\frac{q_i}{q_0}, p^{\lambda_0+1}\left(M^{k+\frac{1}{n+1}}\right)^{-\alpha_i}\right)\right) \\
&\leq \#\mathcal{Q}_m'(\boldsymbol{\alpha}, \boldsymbol{\tau}, M^k)\left(3M^k r\right)^d p^{d\lambda_0+n} M^{-\left(k(n+1-\tilde{\tau})+1-\frac{\tilde{\tau}}{n+1}\right)} \\
&\overset{\text{Theorem 1.1}}{\leq} C_1 M^{k(m+1-\tilde{\tau})} 3^d M^{kd} p^{d\lambda_0+n} M^{-\left(k(n+1-\tilde{\tau})+1-\frac{\tilde{\tau}}{n+1}\right)} \mu_d(B) \\
&\leq 3^d p^{d\lambda_0+n} M^{-(1-\frac{\tilde{\tau}}{n+1})} \mu_d(B).
\end{aligned}$$

Taking

$$M \geq (c_1 3^d p^{d\lambda_0+n})^{\frac{n+1}{n+1-\tilde{\tau}}}$$

for some constant $c_1 < \frac{1}{2}$ and applying this to (13) gives us that

$$\mu_d\left(B \cap \bigcup_{\substack{|q_i| \leq M^k \\ M^k < q_0 \leq M^{k+1}: \\ q_0 \in \mathcal{Q}_m'\left(\boldsymbol{\alpha}, \boldsymbol{\tau}, M^{k+1}\right),\ |q_0|_p \geq p^{-\lambda_0}}} \prod_{i=1}^{n} B\left(\frac{q_i}{q_0}, p^{\lambda_0+\frac{n}{d}}\left(M^{k+\frac{1}{n+1}}\right)^{-\alpha_i}\right)\right) \geq \left(\frac{1}{2} - c_1\right)\mu_d(B).$$

To complete the proof take the constant C associated with the function ρ to be

$$C = \max_{1 \leq i \leq n} p^{\frac{\lambda_0+\frac{n}{d}}{\alpha_i}} M^{1-\frac{1}{n+1}}. \qquad \square$$

Given Proposition 4.1, we have that (R_q, β) is a local ubiquitous system of rectangles with respect to $(\rho, \tilde{\nu})$, provided $\sum_{i=1}^{d} \nu_i = n + 1 - \sum_{i=1}^{m} \tau_{d+i}$ and (7) is satisfied. Given $\boldsymbol{\tau}_d = (\tau_1, \ldots, \tau_d) \in \mathbb{R}^d_{>0}$ assume without loss of generality that $\tau_1 > \tau_2 > \ldots > \tau_d$ and define each ν_{d-i} recursively by

$$\nu_{d-i} = \min\left\{\tau_{d-i}, \frac{n + 1 - \sum_{i=1}^{m} \tau_{d+i} - \sum_{j=d-i+1}^{d} \nu_j}{d-i}\right\}.$$

By the condition on $\boldsymbol{\tau}_d$ of Theorem 2.2, there exists a $k \in \{1, \ldots, d\}$ such that

$$\nu_l = \frac{n + 1 - \sum_{i=1}^{m} \tau_{d+i} - \sum_{j=d-k+1}^{d} \nu_j}{d-k},$$

for all $1 \leq l \leq d-k$. Clearly each $\nu_i \leq \tau_i$ for $1 \leq i \leq d$, and so the associated vector $\mathbf{t} = (t_1, \ldots t_{n-1}) \in \mathbb{R}^{n-1}_{\geq 0}$ is well defined by

$$t_i = \tau_i - \nu_i, \quad 1 \leq i \leq d.$$

Consider the set

$$A = \{\nu_1, \ldots, \nu_d, \tau_1, \ldots, \tau_d\}.$$

For each $A_i \in A$ observe the following:

i) $A_i \in \{\nu_1, \ldots, \nu_d\}$: Then we have the sets

$$K_1 = \{1, \ldots, \max\{i, d-k\}\},\ K_2 = \{\max\{i+1, d-k+1\}, \ldots, d\},\ K_3 = \emptyset.$$

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By Theorem 3.4 we have that

$$\begin{aligned}&\dim_H \mathfrak{W}_n(\boldsymbol{\tau}) \cap \Pi_{\boldsymbol{\alpha}}\\ &\geq \min_{A_i}\left\{\frac{\max\{i,d-k\}v_i + (d-\max\{i+1,d-k+1\}+1)v_i - \sum_{j=\max\{i+1,d-k+1\}}^{d} t_j}{v_i}\right\},\\ &= \min_{A_i}\left\{\frac{dv_i - \sum_{j=\max\{i+1,d-k+1\}}^{d} t_j}{v_i}\right\}.\end{aligned}$$

Since $t_j = 0$ for $d-k+1 \leq j \leq d$ the above equation gives a dimension result of $d = n-m$, the maximal dimension of $\mathfrak{W}_n(\boldsymbol{\tau}) \cap \Pi_{\boldsymbol{\alpha}}$.

ii) $A_i \in \{\tau_1, \ldots, \tau_d\}$: Since $\tau_i = v_i$ for $d-k+1 \leq i \leq d$ the above argument covers such case, so we only need to consider τ_i for $1 \leq i \leq d-k$. For such τ_i we have the sets

$$K_1 = \emptyset,\ K_2 = \{i, \ldots, d\},\ K_3 = \{1, \ldots, i-1\}.$$

Applying Theorem 3.4 we have

$$\begin{aligned}\dim_H \mathfrak{W}_n(\boldsymbol{\tau}) \cap \Pi_{\boldsymbol{\alpha}} &\geq \min_{A_i}\left\{\frac{(d-i+1)\tau_i + \sum_{j=1}^{i-1} v_j - \sum_{j=i}^{d} t_j}{\tau_i}\right\},\\ &= \min_{A_i}\left\{\frac{(d-i+1)\tau_i + (d-k)\left(\frac{n+1-\sum_{i=1}^{m}\tau_{d+i} - \sum_{j=d-k+1}^{d} v_j}{d-k}\right) - \sum_{j=i}^{d-k}\tau_j}{\tau_i}\right\},\\ &= \min_{A_i}\left\{\frac{n+1-\sum_{i=1}^{m}\tau_{d+i} + \sum_{j=i}^{d}(\tau_i - \tau_j)}{\tau_i}\right\}.\end{aligned}$$

Combining i) and ii) we have that

$$\dim_H \mathfrak{W}_n(\boldsymbol{\tau}) \cap \Pi_{\boldsymbol{\alpha}} \geq \min_{1\leq i\leq d}\left\{\frac{n+1-\sum_{i=1}^{m}\tau_{d+i} + \sum_{j=i}^{d}(\tau_i - \tau_j)}{\tau_i}\right\},$$

completing the proof of the lower bound, and thus completing the proof of Theorem 2.2.

5. Proof of the counting results

Recall, we are providing bounds on the set

$$\mathcal{Q}_n(\boldsymbol{x}, \Psi, N) := \left\{(q_0, \ldots, q_n) \in \mathbb{Z}^{n+1} : \begin{array}{c} 0 < q_0 \leq N, \\ \max_{1\leq i\leq n}|q_i| \leq N, \end{array} \ |q_0 x_i - q_i|_p < \psi_i(N),\ 1 \leq i \leq n\right\}.$$

We begin with the two easier statements, namely the lower bound statement of Theorem 1.3 and Lemma 1.2. The style of proof used in the lower bound of Theorem 1.3 follows a similar method to the proof in the real case analogue, see [45, Lemma 3].

Proof of Theorem 1.3: Fix $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{Z}_p^n$ and take $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{N}^n$ to be the integers such that

$$p^{-t_i} \leq N^{-\tau_i} < p^{-t_i+1}, \quad 1 \leq i \leq n.$$

Denote by $P = \prod_{i=1}^{n} p^{t_i}$. Consider a set of open disjoint rectangles $\{R_i\}_{i=1}^{P}$, each with some centre point $k_i = (k_{i,1}, \ldots, k_{i,n}) \in \mathbb{Z}^n$ and sidelights p^{-t_i}. Choose the set of centre points $\{k_i\}$ such that $\mathbb{Z}_p^n \subseteq \bigcup_{i=1}^{P} R_i$. Consider the $(N+1)^{n+1}$ set of points of the form

$$(q_0\boldsymbol{x} - \boldsymbol{q}) = (q_0x_1 - q_1, \ldots, q_0x_n - q_n) \in \mathbb{Z}_p^n,$$

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with $q_i \in [0, N]$ for each $0 \leq i \leq n$. By the pigeon-hole principle there exists at least one rectangle, say R_j, containing at least

$$\frac{(N+1)^{n+1}}{P} > \frac{1}{p^n} N^{n+1-\sum_{i=1}^{n} \tau_i}$$

points. As $\sum_{i=1}^{n} \tau_i < n+1$ we can choose N sufficiently large so that $p^{-n} N^{n+1-\sum_{i=1^n} \tau_i} > 2$. Order the points $(q_0, \ldots, q_n)$, corresponding to the points $q_0 \boldsymbol{x} - \boldsymbol{q}$ contained in R_j, by the absolute value of the q_0 component. If the q_0 components are equal then order by q_1 and so on. Suppose that the vector $(m_0, \ldots, m_n)$ is the smallest by our ordering. Then for all other vectors $(r_0, \ldots, r_n)$ contained in R_j we have that

$$|k_{j,i} - (m_0 x_i - m_i) - (k_{j,i} - (r_0 x_i - r_i)|_p < p^{-t_i},$$
$$|(r_0 - m_0)x_i - (r_i - m_i)|_p < p^{-t_i} \leq N^{-\tau_i}.$$

Hence, the vectors $(r_0 - m_0, \ldots, r_n - m_n) \in \mathbb{Z}^{n+1}$ solve the inequality of $\mathcal{Q}_n(\boldsymbol{x}, \boldsymbol{\tau}, N)$. Furthermore, $(r_i - m_i) \in [-N, N]$, and by the ordering stated above $r_0 - m_0 \in [0, N]$. To exclude the case where $r_0 - m_0 = 0$ observe that each $\tau_i > 1$ and so we would have that

$$N^{-1} \leq |r_i - m_i|_p < p^{-t_i} < N^{-1}$$

for $1 \leq i \leq n$, a contradiction. The above argument yields $p^{-n} N^{n+1-\sum_{i=1^n} \tau_i} - 1$ such points. Taking N sufficiently large we obtain our lower bound counting result, completing the proof modulo Theorem 1.1. □

Lemma 1.2 also has a relatively simple proof. The following method of assuming a contradiction and then using the pigeonhole principle to prove otherwise is a well know technique used in a variety of texts [15, 49].

Proof of Lemma 1.2: We use a proof by contradiction. Suppose that

$$\#\mathcal{Q}_1(x, \tau, N) > 2N^{\tau(x)-\tau+\epsilon}. \tag{14}$$

We use the following notations. Let $X \in \mathbb{N}$ be an integer such that

$$|x - X|_p < p^{-M},$$

for some suitably large $M \in \mathbb{N}$, in particular we may take

$$X = \sum_{i=0}^{M} x_i p^i,$$

where $(x_i)_{i \in \mathbb{N}}$ is the p-adic expansion of x. . Define V_N^+ and V_N^- to be the sets

$$V_N^+ := \{(q, q_1) \in \mathbb{N} \times \mathbb{Z} : 0 < q \leq N,\ 0 \leq q_1 \leq N,\ \},$$
$$V_N^- := \{(q, q_1) \in \mathbb{N} \times \mathbb{Z} : 0 < q \leq N,\ -N \leq q_1 \leq 0,\ \}.$$

Let $t \in \mathbb{N}$ be the integer such that

$$p^{-t} \leq N^{-\tau} < p^{-t+1},$$

and similarly $k \in \mathbb{N}$ be the integer such that

$$p^{-k} \leq N^{-(\tau(x)+\epsilon)} < p^{-k+1}.$$

Note that as $\tau(x) > \tau$, we have that $k \geq t$, and so $p^{k-t} \in \mathbb{N}$. Further, observe that

$$p^{k-t} < p N^{\tau(x)-\tau+\epsilon}. \tag{15}$$

Lastly, by the definition of $\tau(x)$, we have that there exists only finitely many $Q \in \mathbb{N}$ such that

$$|qx - q_1|_p < Q^{-(\tau(x)+\epsilon)},$$

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for $0 < q, |q_1| \leq Q$. Hence, we may choose a sufficiently large N_0 such that for all $N > N_0$ for any pair $0 < q, |q_1| \leq N$,

$$|qx - q_1|_p \geq N^{-(\tau(x)+\epsilon)}, \tag{16}$$

for all $\epsilon > 0$. Consider the set of points in $\mathcal{Q}_1(x, \tau, N)$. Note that $(q, q_1) \in \mathcal{Q}_1(x, \tau, N)$ if and only if $(q, q_1) \in V_N^+ \cup V_N^-$, and

$$qX - q_1 \equiv 0 \mod p^t. \tag{17}$$

Thus, for all $(q, q_1) \in \mathcal{Q}_1(x, \tau, N)$ we have that

$$qX - q_1 = \lambda p^t,$$

for some $\lambda \in \mathbb{Z}$. Split the set of points in $\mathcal{Q}_1(x, \tau, N)$ into two disjoint sets, the set of pairs in V_N^+, and the set of pairs in V_N^-. As there are greater than $2N^{\tau(x)-\tau+\epsilon}$ pairs, at least one of the sets has greater than $N^{\tau(x)-\tau+\epsilon}$ pairs. Without loss of generality assume such set of points belong in V_N^+. Considering the range of values of λp^t for t fixed and λ varying we observe there are p^{k-t} possible values of λp^t modulo p^k. By (14) and (15) we have, by the pigeon-hole principle, that there exists at least two pairs, say (a, a_1) and (b, b_1), such that

$$(a - b)X - (a_1 - b_1) \equiv 0 \mod p^k.$$

This implies

$$|(a - b)x - (a_1 - b_1)|_p \leq p^{-k} \leq N^{-(\tau(x)+\epsilon)},$$

with $(a - b, a_1 - b_1) \in V_N^+ \cup V_N^-$, as $0 < a - b \leq N$ by our choice of ordering of a, b, and $|a_1 - b_1| \leq N$ by the fact that the pairs $(a, a_1), (b, b_1) \in V_N^+$. However, such result contradicts (16) which follows from the definition of $\tau(x)$, thus (14) must be false. □

5.1. p-adic approximation lattices

Prior to the proof of Theorem 1.1, we recall some basic definitions and results from the geometry of numbers that will be needed. Define a lattice Λ as a discrete additive subgroup of $\mathbb{R}^n$. If $\Lambda \subseteq \mathbb{Z}^n$ the n Λ is an integer lattice. A set of linearly independent vectors $\mathbf{b}_1, \ldots, \mathbf{b}_n$ that generate Λ as a $\mathbb{Z}$-module is called a basis of Λ. Let B be a $n \times n$ matrix with columns $\mathbf{b}_i$, then call B a basis matrix. Define the fundamental region as

$$\mathcal{F}(B) := \left\{ \sum_{i=1}^{n} a_i \mathbf{b}_i : a_i \in \mathbb{R}, \ 0 \leq a_i < 1 \right\}.$$

If B is a basis matrix for Λ then $\mathcal{F}(B)$ contains no lattice points other than the origin (see Chapter 3, Lemma 6 of [22]).

The volume of the fundamental region can be found by taking the determinant of the basis matrix, that is $\mathrm{vol}(\mathcal{F}(B)) = |\det B|$. A basis matrix is not unique for each Λ, however for any lattice Λ the volume of the fundamental region is the same regardless of choice of basis matrix. For this reason we use the notation $\det \Lambda$ to denote the volume of the fundamental region. If $U \in \mathbb{Z}^{n \times n}$ is a unimodular matrix and B_1 is a basis matrix for Λ then $B_2 = B_1 U$ is also a basis matrix for Λ.

Let $B_n = B(0, 1)$ denote the n-dimensional Euclidean unit ball. For $c \in \mathbb{R}_+$ we use the notation $cB_n = B(0, c)$. Define the successive minima of a lattice $\Lambda \subset \mathbb{R}^n$ of rank n as the set of values

$$\lambda_i(\Lambda) := \min\{\lambda > 0 : \dim(\Lambda \cap \lambda B) \geq i\},$$

for $i = 1, \ldots, n$. By Minkowski's inequalities on the successive minima (see e.g. [29]) we have that

$$\mathrm{vol}(B_n) \prod_{i=1}^{n} \lambda_i(\Lambda) \leq 2^n \det \Lambda. \tag{18}$$

For a count on the number of lattice points within a convex body we have the follow theorem.

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Theorem 5.1 ([19]). *Let $\Lambda \subset \mathbb{R}^n$ be a lattice of rank n and let $V \subset \mathbb{R}^n$ be a convex body such about the origin that* $\operatorname{rank}(\Lambda \cap V) = n$. *Then*

$$\#(\Lambda \cap V) \leq n! \frac{\operatorname{vol}(V)}{\det \Lambda} + n.$$

We also have the following counting result due to Henk where the convex body of interest need not have full rank when intersected with the lattice.

Theorem 5.2 ([18, 28]). *Let $n \geq 2$, $B(0, K)$ a n-dimensional ball of radius $K > 0$ centred at the origin and Λ a n-dimensional lattice. Then*

$$\#(\Lambda \cap K) < 2^{n-1} \prod_{i=1}^{n} \left\lfloor \frac{2K}{\lambda_i(\Lambda)} + 1 \right\rfloor.$$

We remark that if $\operatorname{rank}(\Lambda \cap B(0, M)) < n$ then we must have at least that $\lambda_n(\Lambda) \geq M$. Thus the n^{th} value of the product in Theorem 5.2 would be bounded above by 3.

For the proof of Theorem 1.1 we use p-adic approximation lattices. Such lattices have been used in p-adic Diophantine approximation regularly, for example De Weger [24] used them to prove a variety of results in classical p-adic Diophantine approximation, including the p-adic analogue of Hurwitz Theorem in classical Diophantine approximation. See also [33, 34].

For a n-tuple of approximation functions $\Psi = (\psi_1, \dots, \psi_n)$, an integer $N \in \mathbb{N}$, and a fixed $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{Z}_p^n$ define the Ψ-approximation lattice $\Lambda_{N,\boldsymbol{x}}$ by

$$\Lambda_{N,\boldsymbol{x}} = \{(a_0, \dots, a_n) \in \mathbb{Z}^{n+1} : |a_0 x_i - a_i|_p \leq \psi_i(N), \ \ 1 \leq i \leq n\}.$$

To briefly justify the above claim note that the collection of points are discrete, and that any integer linear combination of such points is also contained within the set due to the strong triangle inequality. Observe that such claim is clearly false if we were to consider the real case analogous to the above setting. Observe that

$$\mathcal{Q}_n(\boldsymbol{x}, \Psi, N) \subseteq \Lambda_{N,\boldsymbol{x}} \cap B(0, \sqrt{n}N),$$

since the Euclidean ball $B(0, \sqrt{n}N)$ contains all integer points satisfying $\max_{0 \leq i \leq n} |q_i| \leq N$.

For any $\boldsymbol{x} \in \mathbb{Z}_p^n$ we may write each x_j as the p-adic expansion

$$x_j = \sum_{i=0}^{\infty} x_{j,i} p^i, \quad x_{j,i} \in \{0, 1, \dots, p-1\}.$$

Let $X_{j,N} \in \mathbb{Z}$ be the integer

$$X_{j,N} = \sum_{i=0}^{t_j} x_{j,i} p^i,$$

where each $t_j \in \mathbb{N}$ is the unique value associated with N satisfying

$$p^{-t_j} \leq \psi_j(N) < p^{-t_j+1}. \tag{19}$$

Lastly, for each $1 \leq j \leq n$ let $\psi_{j,N}^* = p^{t_j}$. Then the set of vectors

$$B = \left\{ \begin{pmatrix} 1 \\ X_{1,N} \\ \vdots \\ X_{n,N} \end{pmatrix}, \begin{pmatrix} 0 \\ \psi_{1,N}^* \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \psi_{n,N}^* \end{pmatrix} \right\}, \tag{20}$$

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form a basis for $\Lambda_{N,\boldsymbol{x}}$. To support this claim observe that

$$|x_i - X_{i,N}|_p = \left| \sum_{j=t_i+1}^{\infty} x_{i,j} p^j \right|_p < p^{-t_i} < \psi_i(N),$$

and

$$|0 \cdot x_i - \psi^*_{i,N}|_p = |p^{t_i}|_p = p^{-t_i} < \psi_i(N),$$

for $1 \leq i \leq n$. Thus, each vector in B is a point in $\Lambda_{N,\boldsymbol{x}}$; hence, the span of vectors B at least produces a sublattice of $\Lambda_{N,\boldsymbol{x}}$. To show B is a basis of $\Lambda_{N,\boldsymbol{x}}$ we consider the fundamental region $\mathcal{F}(B)$ and show that the only lattice point contained is $\mathbf{0}$. Suppose there exists a non-zero lattice point in $\mathcal{F}(B)$ of the form

$$\sum_{i=1}^{n+1} c_i b_i$$

where b_i are the vectors of B and each $0 \leq c_i < 1$. Since $\Lambda_{N,\boldsymbol{x}} \subseteq \mathbb{Z}^{n+1}$ and the first vector of B is the only vector with a non-zero entry in the first row we must have $c_1 = 0$. For the remaining vectors observe that each non-zero term is a prime power, so each c_i must be of the form p^{-r} for some $r \in \mathbb{N}$. However, by the construction of each $\psi^*_{i,N}$ we have that

$$|\psi^*_{i,N}|_p < \psi_i(N) \quad \text{and} \quad |p^{-1}\psi^*_{i,N}|_p \geq \psi_i(N),$$

and so the only possible value for each c_i is zero. Thus $\sum_{i=1}^{n+1} c_i b_i = \mathbf{0}$.

Given basis B we can deduce that

$$|\det \Lambda_{N,\boldsymbol{x}}| = \prod_{i=1}^{n} \psi^*_{i,N} \asymp \left(\prod_{i=1}^{n} \psi_i(N) \right)^{-1},$$

where the implied constants can be easily found using (19) to obtain

$$\left(\prod_{i=1}^{n} \psi_i(N) \right)^{-1} \leq |\det \Lambda_{N,\boldsymbol{x}}| \leq p^n \left(\prod_{i=1}^{n} \psi_i(N) \right)^{-1}. \tag{21}$$

In the simultaneous case, $\Psi = (\psi, \ldots, \psi)$, it was proven in [34] that

$$\lambda_1(\Lambda_{N,\boldsymbol{x}}) \ll \psi(N)^{-\frac{n}{n+1}}.$$

In order to prove Theorem 1.1 we find a lower bound on $\lambda_1(\Lambda_{N,\boldsymbol{x}})$ by considering $\boldsymbol{x} \in \mathbb{Z}_p$ satisfying certain Diophantine exponent properties. For completeness we also prove the upper bound.

Lemma 5.3. *Let $\Lambda_{N,\boldsymbol{x}}$ be defined above with $\tau(\boldsymbol{x}) = n+1$, and suppose that*

$$\prod_{i=1}^{n} \psi_i(N) < N^{-n}. \tag{22}$$

Then for any $\frac{1}{n+1} > \delta > 0$ the exists sufficiently large $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$,

$$\left(\frac{1}{\prod_{i=1}^{n} \psi_i(N)} \right)^{\frac{1}{n+1}-\delta} \leq \lambda_1(\Lambda_{N,\boldsymbol{x}}) \leq C_2 \left(\frac{1}{\prod_{i=1}^{n} \psi_i(N)} \right)^{\frac{1}{n+1}},$$

where

$$C_2 = 2 \left(\frac{\Gamma\left(\frac{n+1}{2}+1\right) p^n}{\pi^{\frac{n+1}{2}}} \right)^{\frac{1}{n+1}}.$$

Proof. We prove the upper bound case first. Such proof is a standard application of Minkowski's Theorem on successive minima and follows almost immediately by the above calculation of $|\det \Lambda_{M,\boldsymbol{x}}|$.

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Concisely, we have that

$$\lambda_1(\Lambda_{N\boldsymbol{x}})^{n+1}\mathrm{vol}(B(0,1)) \leq 2^{n+1}|\det \Lambda_{N\boldsymbol{x}}|.$$

Rearranging for $\lambda_1(\Lambda_{N\boldsymbol{x}})$, using (21), and recalling the volume of an $(n+1)$- dimensional Euclidean ball we obtain the upper bound on $\lambda_1(\Lambda_{N\boldsymbol{x}})$.

We now prove the lower bound. For any $\boldsymbol{x} \in \mathbb{Z}_p^n$ observe that

$$\begin{cases} \prod_{i=1}^n |q_0 x_i - q_i|_p < p^n N^{-(n+1)}, \\ \max_{0\leq i\leq n} |q_i| \leq N \end{cases}$$

has solutions for infinitely many N by Lemma 3.1. Since $\tau(\boldsymbol{x}) = n+1$ for any $\delta > 0$ there exists N_δ so that for all $N \geq N_\delta$ any rational integer vectors $(q_0, \ldots, q_n)$ with $\max_{0\leq i\leq n} |q_i| = N$ satisfies

$$\prod_{i=1}^n |q_0 x_i - q_i|_p \geq N^{-(n+1+\delta)}. \tag{23}$$

Let N_1 be sufficiently large such that for all $N > N_1$

$$\left(\prod_{i=1}^n \psi_i(N)\right)^{-\left(\frac{1}{n+1}-\delta\right)} \geq N_\delta.$$

Such N is possible since $\prod_{i=1}^n \psi_i(N) < N^{-n}$ and so the value on the left hand side tends to infinity as $N \to \infty$ (for any $\delta < \frac{1}{n+1}$). Henceforth fix $\frac{1}{n+1} > \delta > 0$ and assume $N \geq N_0 = \max\{N_\delta, N_1\}$.

Suppose that $(q_0, \ldots, q_n) \in \mathbb{Z}^{n+1}$ is a minimum length non-zero vector of $\Lambda_{N\boldsymbol{x}}$. Then $\lambda_1(\Lambda_{N\boldsymbol{x}}) \geq \max_{0\leq i\leq n} |q_i|$. Suppose that

$$\max_{0\leq i\leq n} |q_i| < \left(\prod_{i=1}^n \psi_i(N)\right)^{-\left(\frac{1}{n+1}-\delta\right)}. \tag{24}$$

We prove (24) to be false. Observe that

$$\prod_{i=1}^n |q_0 x_i - q_i|_p < \prod_{i=1}^n \psi_i(N),$$

since $(q_0, \ldots, q_n) \in \Lambda_{N\boldsymbol{x}}$, and by rearranging (24) we can conclude that

$$\prod_{i=1}^n |q_0 x_i - q_i|_p < \left(\max_{0\leq i\leq n} |q_i|\right)^{-\frac{n+1}{1-\delta(n+1)}}.$$

This contradicts (23) since

$$\frac{n+1}{1-\delta(n+1)} > n+1+\delta$$

for any $\frac{1}{n+1} > \delta > 0$. Hence, we must have that (24) is false, and so

$$\lambda_1(\Lambda_{N\boldsymbol{x}}) \geq \left(\prod_{i=1}^n \psi_i(N)\right)^{-\left(\frac{1}{n+1}-\delta\right)},$$

completing the proof. □

Given Lemma 5.3 we can proceed with the following.

Proof of Theorem 1.1: Consider the following two cases:

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i) $\operatorname{rank}(\Lambda_{N,\boldsymbol{x}} \cap B(0, \sqrt{n}N)) = n+1$: By Theorem 5.1 we have that

$$\begin{aligned}\#(\Lambda_{N,\boldsymbol{x}} \cap B(0, \sqrt{n}N)) &\leq (n+1)! \frac{\operatorname{vol}(B(0, \sqrt{n}N))}{\det \Lambda_{N,\boldsymbol{x}}} + n + 1, \\ &\leq \frac{(n+1)!\pi^{n/2}\sqrt{n}^{n+1}}{\Gamma\left(\frac{n}{2}+1\right)} N^{n+1} \Big(\prod_{i=1}^{n} \psi_{i,N}^*\Big)^{-1} + n + 1, \\ &\leq \frac{(n+2)!\pi^{n/2}\sqrt{n}^{n+1}}{\Gamma\left(\frac{n}{2}+1\right)} N^{n+1} \prod_{i=1}^{n} \psi_i(N).\end{aligned}$$

Note that the last inequality follows since $\prod_{i=1}^{n} \psi_i(N) > N^{-(n+1-\varepsilon n)}$ by (1). This proves Theorem 1.1 for the rank $n+1$ case.

ii) $\operatorname{rank}(\Lambda_{N,\boldsymbol{x}} \cap B(0, \sqrt{n}N)) < n+1$: Since $\operatorname{rank}(\Lambda_{N,\boldsymbol{x}} \cap B(0, \sqrt{n}N)) < n+1$, we must have $\lambda_{n+1}(\Lambda_{N,\boldsymbol{x}}) > \sqrt{n}N$. Hence, by Theorem 5.2, we have that

$$\#(\Lambda_{N,\boldsymbol{x}} \cap B(0, \sqrt{n}N)) < 2^n 3 \prod_{i=1}^{n} \left(\frac{2\sqrt{n}N}{\lambda_i(\Lambda_{N,\boldsymbol{x}})} + 1 \right).$$

Note that we have used here the remark following Theorem 5.2. Namely, we are only considering the above product over the first n-successive minima, rather than all $(n+1)$-successive minima. Each $\lambda_i(\Lambda_{N,\boldsymbol{x}})$ is bounded below by $\lambda_1(\Lambda_{N,\boldsymbol{x}})$ for $1 \leq i \leq n$. Condition (22) of Lemma 5.3 is satisfied by (1). Choose $\frac{1}{n+1} > \delta > 0$ such that $\delta < \varepsilon(n+1)^{-2}$. By Lemma 5.3 there exists $N_0 \in \mathbb{N}$ such that for all $N > N_0$

$$\begin{aligned}\lambda_1(\Lambda_{N,\boldsymbol{x}}) &\geq \left(\frac{1}{\prod_{i=1}^{n} \psi_i(N)} \right)^{\frac{1}{n+1} - \delta}, \\ &\geq \left(\left(\frac{1}{\prod_{i=1}^{n} \psi_i(N)} \right)^{n - \delta n(n+1)} \right)^{\frac{1}{n(n+1)}}, \\ &\geq \left(\left(\prod_{i=1}^{n} \psi_i(N) \right)^{\delta n(n+1)} \left(\frac{1}{\prod_{i=1}^{n} \psi_i(N)} \right)^{n} \right)^{\frac{1}{n(n+1)}}, \\ &\overset{(*)}{\geq} \left(\frac{1}{N^{n+1} \prod_{i=1}^{n} \psi_i(N)} \left(\frac{1}{\prod_{i=1}^{n} \psi_i(N)} \right)^{n} \right)^{\frac{1}{n(n+1)}}, \\ &\geq \left(\frac{1}{N \prod_{i=1}^{n} \psi_i(N)} \right)^{1/n}.\end{aligned}$$

We can deduce $(*)$ by observing that by our choice of δ we have

$$\delta < \frac{\varepsilon}{(n+1)^2} < \frac{\varepsilon}{n+1} \frac{\log N}{(n+1)\log N} \overset{(1)}{<} \frac{\varepsilon}{n+1} \frac{\log N}{\log\left(\prod_{i=1}^{n} \psi_i(N)\right)^{-1}}, \tag{25}$$

for any $N \geq N_0$. Thus

$$\left(\prod_{i=1}^{n} \psi_i(N) \right)^{\delta n(n+1)} \overset{(25)}{>} N^{-\varepsilon n} \overset{(1)}{\geq} \left(N^{n+1} \prod_{i=1}^{n} \psi_i(N) \right)^{-1}.$$

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combining the two statements above, and Theorem 5.2, we have that

$$\begin{aligned}\#(\Lambda_{N,\boldsymbol{x}} \cap B(0,\sqrt{n}N)) &< 2^n 3 \prod_{i=1}^{n} \left(\frac{2\sqrt{n}N}{\lambda_1(\Lambda_{N,\boldsymbol{x}})} + 1 \right), \\ &< 2^n 3 \left(2\sqrt{n}N^{1+1/n} \left(\prod_{i=1}^{n} \psi_i(N) \right)^{1/n} + 1 \right)^n, \\ &< 3(6\sqrt{n})^n N^{n+1} \prod_{i=1}^{n} \psi_i(N).\end{aligned}$$

Thus, in either case *i*) or *ii*) we have that

$$\#(\Lambda_{N,\boldsymbol{x}} \cap B(0,\sqrt{n}N)) \leq C_1 N^{n+1} \prod_{i=1}^{n} \psi_i(N),$$

with

$$C_1 = \max\left\{ 3(6\sqrt{n})^n, \frac{(n+2)!\pi^{n/2}\sqrt{n}^{n+1}}{\Gamma\left(\frac{n}{2}+1\right)} \right\}.$$

□

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