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# Notes on general topology

## III. A non-metric image of an ordered compactum

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It has been known for some time that the product of a non-metrizable Hausdorff space and any (non-trivial) Hausdorff space cannot be the continuous image of an ordered continuum. (For a survey of this and related problems, see Mardešić and Papić ((1)).) Further, it has been shown by Treybig ((2)) (and independently by the present author) that the product of two Hausdorff spaces cannot even be the continuous image of an ordered compactum unless both the spaces are metrizable or one is finite. It is therefore of some interest to give a simple example of a space $X$ which is the continuous image of an ordered compactum $K$ and contains the product of a non-metrizable space and an infinite discrete space, imbedded in such a way as to form a sequence of homeomorphic subsets with a connected (non-trivial) topological limit.

We first define the ordered compactum $K$, whose elements are pairs of real numbers $(\xi, \eta)$ ordered 'lexicographically'; that is $(\xi, \eta) < (\xi', \eta')$ iff $\xi < \xi'$ or $\xi = \xi'$ and $\eta < \eta'$.

Let $J_0$ be the Cantor set of real numbers in $[0, 1]$, and for each $m$ let

$$I_{mn} \quad (n = 1, 2, \ldots, 2^{m-1})$$

be those complementary intervals of $J_0$ which are of length $3^{-m}$, arranged so that if $\xi \in I_{mn}$, $\xi' \in I_{mn'}$ $(n < n')$, then $\xi < \xi'$. Let $J_{mn}$ be the middle third of $I_{mn}$, taken closed. Now let $K_0 = \{(\xi, 0), \xi \in J_0\}$, $K_{mn} = \{(\xi, n); \xi \in J_{mn},\ 0 \leqslant \eta \leqslant 1/m\}$, and let $K$ be the union of $K_0$ and all $K_{mn}$. It is clear that every subset of $K$ has exact upper and lower bounds (relative to $K$), so that $K$, with its order topology, is an ordered compactum.

The space $X$ has as its elements triplets of real numbers $(x, y, z)$ and is the union of sets $X_n$, $n = 0, 1, 2, \ldots$, defined thus:

$$\text{for} \quad n = 0; \quad 0 \leqslant x \leqslant 1, \quad y = 0, \quad z = 0;$$

$$\text{for} \quad n \geqslant 1; \quad 0 \leqslant x \leqslant 1, \quad 0 \leqslant y \leqslant 1/n, \quad z = 1/n.$$

We define the topology by giving the neighbourhoods of each point. The neighbourhoods of a point $a$ in $X_n$, $n \geqslant 1$, are those sets which contain a subset of $X_n$ which is a neighbourhood of $a$ in the lexicographic order-topology of $X_n$. The neighbourhoods of $(x_0, 0, 0)$ are those sets which contain, for some $\epsilon > 0$, all points $(x, y, z)$ of $X$ satisfying $|x - x_0| < \epsilon$, $y < \epsilon$, $z < \epsilon$. It is easily seen that these neighbourhood systems satisfy the conditions to define a Hausdorff topology.

We now define the mapping $f$ of $K$ onto $X$. For a point $(\xi, 0)$ of $J_0$ we write $\xi$ in the form $\sum_{n=1}^{\infty} a_r 3^{-r}$, where each $a_r$ is 0 or 2, and define $f(\xi, 0) = [g(\xi), 0, 0]$ where $g(\xi) = \sum_{n=1}^{\infty} \frac{1}{2} a_r 2^{-r}$.



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It is well known that $g$ is a monotone map of $J_0$ onto $[0, 1]$, which is continuous (in the metric topologies of $J_0$ and $[0, 1]$, which coincide with their order topologies).

For a point $(\xi, \eta)$ of $K_{mn}$ we define $f(\xi, \eta) = [h(\xi), \eta, 1/m]$, where $h(\xi)$ is a linear map of $J_{mn}$ on to $[(2n-2)\,2^{-m}, (2n)\,2^{-m}]$. It is obvious that $f$ is onto $X$, and is trivially continuous at each point of each $K_{mn}$, and continuous over $K_0$ at each point of $K_0$. As the $K_{mn}$ are open and closed in $K$, it remains only to consider the limit of $f(\xi, \eta)$ as $(\xi, \eta)$ tends to a point in $K_0$, say $(\xi_0, 0)$, through points not in $K_0$.

Given any integer $m_0$, there exists $\epsilon > 0$ such that $J_{mn} \cap (\xi_0 - \epsilon, \xi_0 + \epsilon) = \emptyset$ for $m < m_0$; thus $|\xi - \xi_0| < \epsilon$ and $f(\xi, \eta) = (x, y, z)$ imply $y$ and $z \leqslant 1/m_0$. Now it is easily seen that any $J_{mn}$ lies immediately between two points $a_{mn}$, $b_{mn}$ say of $J_0$ (the end-points of $I_{mn}$) such that $g(a_{mn}) = g(b_{mn}) = (2n-1)\,2^{-m}$; thus for $\xi$ in $J_{mn}$,

$$|h(\xi) - g(a_{mn})| \leqslant 2^{-m} \leqslant 2^{-m_0}$$

if $|\xi - \xi_0| < \epsilon$. It easily follows that the function which coincides with $g$ on $J_0$ and with $h$ on each $J_{mn}$ is continuous at $\xi_0$ (in the ordinary sense), and hence that $f(\xi, \eta)$ is continuous (with our topologies) at $(\xi_0, 0)$.

It is easy to join the end-points of $X_0$ and each $X_n$ by an ordinary curve and thus obtain a connected set which is the continuous image of an ordered compactum but, not being locally connected, not the image of an ordered continuum. (It is of course obvious that the union of a non-locally connected metric continuum and a non-metric ordered continuum, with one point in common, has this property, but our example is 'essentially' non-metric.) The problem of characterizing such sets appears to be difficult. Failing a complete characterization, it would be interesting to know (*a*) whether there exists such a set which is locally connected, and (*b*) whether there exists such a set which contains a sequence (or more generally a directed net) of disjoint connected closed sets with a non-metrizable topological limit. (It is easy to see that the lexicographic square, an ordered continuum, contains a sequence of finite sets whose topological limit is a non-metrizable ordered compactum, in fact the set of all points $(\xi, 0)$ and $(\xi, 1)$, $0 \leqslant \xi \leqslant 1$. The same space contains a sequence of disjoint non-metrizable subspaces (intervals) whose topological limit is a single point.) In connexion with this last problem we remark that, by the same method as is used for a product-space, it is easy to prove the following theorem: if $X$ is a Hausdorff space containing a sequence $\{X_n\}$ of disjoint closed sets and a non-enumerable system of disjoint open sets, each of which meets the lower topological limit of the sequence $\{X_n\}$, then $X$ is not the continuous image of an ordered compactum. I have shown further that the image of an ordered compactum cannot contain a sequence of disjoint closed sets with a non-metrizable, *locally connected* topological limit.

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