J. Plasma Physics (1998), *vol.* 60, *part* 2, *pp.* 299–329. Printed in the United Kingdom
© 1998 Cambridge University Press

Nonlinear particle dynamics in a broadband turbulence wave spectrum

BRIGITTE R. RAGOT

Max-Planck-Institut für Radioastronomie, Postfach 2024, D-53010 Bonn, Germany

(Received 24 September 1997 and in revised form 3 December 1997)

In the statistical quasilinear theory of weak plasma turbulence, charged particles moving in electrostatic fluctuations diffuse in velocity, i.e. the velocity variance $\langle \Delta v^2(t) \rangle$ increases linearly with time t, for times long compared with the autocorrelation time τ_{ac} of the field, which may be estimated as the reciprocal of the spectral width of the fluctuations. Recent test-particle simulations have revealed a new regime at very long timescales $t \gg \tau_{ac}$ where quasilinear theory breaks down, for intermediate field amplitudes. As this behaviour is not consistent with a diffusion on quasilinear timescales, the problem of the motion of particles in a broadband wave field, for the case of a slowly growing field, is considered here from a purely dynamical point of view, introducing no statistics on the field and no restriction on the amplitude of this field. By determining, on a given timescale, and in the frame of wave–particle interaction, the spectral width over which waves interact efficiently with a particle, a new timescale is found: the nonlinear time of wave–particle interaction $\tau_{NL} \propto (\text{spectral density of energy})^{-1/3} \gg \tau_{ac}$. This is the correlation time of the dynamics. For times shorter than τ_{NL}, the particles trajectories remain globally regular, and do not separate: they follow a quasifractal set of dimension 2. For times long compared with τ_{NL}, there appears a 'true' diffusive regime with mixing and decorrelation, due to nonlinear mixing in phase space and the localization of the wave–particle interaction. These theoretical results are confirmed by a numerical study of the velocity variance as a function of time. In particular, the particle dynamics really do become diffusive on timescales several orders of magnitude longer than that predicted by quasilinear theory (namely $\gg \tau_{NL} \gg \tau_{ac}$). Finally, deviations from the quasilinear value of the diffusion coefficient and wave growth rate, discussed in the literature, are explained.

1. Introduction

Quasilinear theory is a statistical theory aimed at describing the evolution of a weak turbulent field and a distribution of resonant particles in this field when wave–particle interactions dominate (see Vedenov *et al.* 1962; Drummond and Pines 1962; Sagdeev and Galeev 1969; Davidson 1972; Delcroix and Bers 1994). The fast, weak and warm beam–plasma instability, presented in all introductions to plasma physics (see e.g. Nicholson 1983; Delcroix and Bers 1994) constitutes the central paradigm of this problem. This statistical quasilinear theory considers a set of Vlasov plasmas, and studies the evolution in time of the average properties of such a statistical set of systems. However, it is clear that only the extreme difficulty of

https://doi.org/10.1017/S0022377898006795 Published online by Cambridge University Press

this problem of weak turbulence motivates the introduction of statistics, and that the initial, physically relevant problem remains the evolution of *one* field and *one* distribution of particles interacting with this field. We shall handle the evolution of a wave field and of particles interacting with this field from a purely dynamical point of view, and – except in Sec. 6 for checking coherence with the previous results by Cary *et al.* (1990, 1992), Pesme *et al.* (1993), Helander and Kjellberg (1994) and Ishihara *et al.* (1993), which first numerically invalidated the quasilinear theory – we shall not incorporate statistics in our theory.

Of course, we do not claim that we are able to solve this problem self-consistently from a dynamical viewpoint; this still remains a dream. However, when the field evolves slowly, i.e. when its growth time γ^{-1} is long compared with the typical nonlinear time τ_{NL} of wave–particle interaction, this problem reduces, on the timescale of a few τ_{NL}, to the dynamics of charged particles in a fixed field. Note that, since we want to reconsider the problem at its root, with a novel point of view, we cannot attribute any definite value to the timescale τ_{NL} at the beginning of this work. One of the main goals will thus consist in precisely defining this timescale. This will arise naturally in the course of our study.

The interaction of particles with a broadband wave spectrum has already been considered from a dynamical point of view (see Delcroix and Bers 1994). But the proposed theory of wave–particle interaction mainly relies on an assumption of rapid decorrelation, which is not justified. Indeed, the dynamics is supposed to obey a diffusive process attributed, without further demonstration, to a decorrelation mechanism that, in the absence of collisions, would be due to an intrinsic stochasticity in the nonlinear dynamics. Earlier interpretations attributed the diffusion to an extrinsic stochasticity, connected to the random wave phases. The fact that a diffusion can be an intrinsic property of a deterministic system was realized by Chirikov (1978) and Rechester and White (1980) via the study of the standard map of Chirikov–Taylor, which describes the dynamics of independent particles interacting with regularly phased waves. This already constitutes a substantial progress in the understanding of the phenomena. However, nonlinearity does not necessarily imply stochasticity, and it remains unclear how the process of diffusion comes about in the phase space, and on which timescale. We shall endeavour to clarify these points.

To that end, we shall determine the spectral width of the field that really acts on the particles on a given timescale, and, in the approximation of a fixed field, solve the nonlinear dynamics with a given precision on this timescale. The only conditions of validity for our theory will be $\gamma\tau_{NL} \ll 1$, as mentioned above, and a limitation on the level of the field energy for the particles not to be trapped in the field. To a large extent, the present work is based on the results of Ragot and Shklyar (1995) (which deals with the nonlinear growth rate), but requires a more precise handling of the nonlinear dynamics on all the timescales. The particle trajectories will be studied, in a nonlinear and nonperturbative way, individually and relative to each other. This will allow us to state the timescale for which mixing appears in the phase space. In fact, we shall prove that the correlation time of the dynamics is the nonlinear time τ_{NL} of wave–particle interaction, which scales as a power of the spectral energy density, $W_\omega^{-1/3}$. Furthermore, we shall establish the existence of two diffusion regimes of fundamentally different natures. The first results not from any memory loss or phase mixing, but from the stochasticity of

https://doi.org/10.1017/S0022377898006795 Published online by Cambridge University Press

the field and the self-similarity of the dynamics on a range of timescales ($< \tau_{NL}$), and will be called the 'quasi-fractal' diffusion regime, whereas the second (the long-time diffusion regime, on times $\gg \tau_{NL}$), more classical, results from a mixing and a decorrelation in the phase space, due to the nonlinear dynamics of the particles and the localization of the wave–particle interaction.

In Sec. 2, we shall briefly introduce the beam–plasma instability, and present the state of the art. Section 3 will address the field representation, and Sec. 4 the nonlinear dynamics as outlined above. In Sec. 5, we shall compare our predictions with numerical simulations of the velocity variance as a function of time, and discuss more thoroughly the long-time diffusion regime. Finally, in Sec. 6, we shall compare our theory and its results with previous work, such as the resonance broadening theory of Dupree (1966), and test-particle simulations by Cary *et al.* (1990, 1992), Pesme *et al.* (1993), Helander and Kjellberg (1994) and Ishihara *et al.* (1993). In particular, we shall prove that our theory is consistent with these recent numerical results that invalidated the quasilinear theory.

2. The beam–plasma instability: a paradigm of the wave–particle interaction

A warm electron beam in a noncollisional plasma excites Langmuir modes, via the Landau effect, in the velocity range where the slope of the electron distribution function is positive. For a sufficiently weak beam density, the wave propagation characteristics of the plasma are not substantially affected by the beam. The dispersion relation for these Langmuir modes can thus be deduced from the Hermitian dielectric function that describes the linear dielectric constituted by the nonresonant plasma bulk. When the beam is sufficiently fast, the dispersion relation takes the usual form

$$\omega = \omega_{pe}(1 + \tfrac{3}{2}k^2\lambda_{De}^2), \tag{2.1}$$

where ω_{pe} and λ_{De} are respectively the plasma frequency and the electronic Debye length. Indeed, this relation is known to be valid in the limit of a wavelength λ long compared with the Debye length λ_{De}, i.e. for phase velocities fast compared with the thermal velocity of the plasma bulk.

2.1. Quasilinear approach to beam–plasma interaction

The evolution of the Langmuir field and of the resonant particle distribution function is usually described by quasilinear theory (Vedenov *et al.* 1962; Drummond and Pines 1962; Sagdeev and Galeev 1969; Davidson 1972; Nicholson 1983; Delcroix and Bers 1994). This theory assumes a diffusion of the particles in velocity, through the spectrum of the wave field, and an exponential growth of the modes with the Landau growth rate (as in the monochromatic case)

$$\gamma_L = \frac{2\pi^2 e^2 \omega_0}{m k_0^2} \frac{\partial f}{\partial v}\left(\frac{\omega_0}{k_0}, t\right), \tag{2.2}$$

determined by the averaged distribution function deduced from the diffusion equation

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v}\left(D\frac{\partial f}{\partial v}\right). \tag{2.3}$$

Quasilinear theory justifies the diffusion of the charged particles in the broad-

https://doi.org/10.1017/S0022377898006795 Published online by Cambridge University Press

band wave spectrum by a decorrelation of the dynamics for times longer than the autocorrelation time of the field seen by the particles, $\tau_{ac} \propto$ (complete spectral width)$^{-1}$, and proposes a linear derivation of the diffusion coefficient, which neglects the statistical modifications of the particle orbits in the field. So, the particles are supposed to interact resonantly with the wave packet during a time of the order of

$$\tau_{ac} \equiv \frac{2\pi}{k_0 \Delta v_\varphi}, \tag{2.4}$$

where Δv_φ is the total width of the wave spectrum in phase velocity, and k_0 is the typical wavenumber. This resonant interaction of particles with randomly phased waves produces a variation of the average quadratic velocity of the particles that would be given by $\langle (\Delta v)^2 \rangle = (e/m)^2 \langle E^2 \rangle \tau_{ac}^2$, as in the linear theory. This yields the diffusion coefficient

$$D_{QL} = \frac{\langle (\Delta v)^2 \rangle}{2\tau_{ac}} = \pi \left(\frac{e}{m} \right)^2 \frac{\langle E^2 \rangle}{k_0 \Delta v_\varphi}. \tag{2.5}$$

2.2. Recent criticisms of quasilinear theory

The validity of the above diffusion coefficient has been discussed ever since its derivation in the 1960s (and more recently by Rechester *et al.* 1979*a*, *b*; Adam *et al.* 1979; Laval and Pesme 1984; Cary *et al.* 1990, 1992; Ishihara *et al.* 1993; Liang and Diamond 1993; Pesme *et al.* 1993; Helander and Kjellberg 1994). In particular, test-particle simulations presented by Cary *et al.* (1990, 1992), Pesme *et al.* (1993), Helander and Kjellberg (1994) and Ishihara *et al.* (1993) reveal that the slope of the variance $\langle (\Delta v)^2 \rangle$ as a function of time starts with the quasilinear value, but may significantly differ from quasilinearity (by a factor > 2) after times much longer than τ_{ac}, for intermediate values of the spectral density of energy, and would remain so, following Ishihara *et al.* (1993), up to a time of the order of few tens of τ_{NL}, when the slope would again become quasilinear.

Furthermore, Pesme *et al.* (1993) deal with the case when the phases of the waves (of the discrete field representation) are correlated over a finite spectral width. It appears that the convergence of the diffusion coefficient to its quasilinear value was not found when increasing the overlap parameter of the waves, which makes the authors conclude that the statistical properties of the field, and especially 'non-Gaussianity' of the field statistics, may imply nonquasilinear behaviour, even in the limit of infinite overlap parameter.

In fact, it is not the numerical deviation from quasilinear values that is most meaningful, but the timescales and the underlying reason for this nonquasilinear behaviour of the statistical dynamics. Cary *et al.* (1990, 1992) relate the time for the change of slope to the discretization time $2\pi/(k_0 \delta v_\varphi)$, where δv_φ is the distance, in phase velocity, between two successive harmonics of the discrete spectrum used for the numerical simulations. But the simulations of Pesme *et al.* (1993) invalidate this assertion, since the time for the change of slope will not significantly increase when δv_φ decreases, if the spectral correlation length of the waves' phases is fixed, as we checked.

We shall consider, in Secs 3 and 4, uncorrelated phases and much larger spectral densities than Cary *et al.* (1990, 1992), Pesme *et al.* (1993) and Helander and Kjellberg (1994) do, so that both regimes will give a quasilinear value for the diffusion coefficient. But the two slopes observed by Cary *et al.* (1990, 1992), Pesme *et al.*

https://doi.org/10.1017/S0022377898006795 Published online by Cambridge University Press

(1993) and Helander and Kjellberg (1994) correspond to the two regimes found in this paper. In Sec. 5, we shall prove that our understanding of the nonlinear dynamics (presented in the next sections), together with our analytical solution for the velocity variation (derived by Ragot and Shklyar 1995), explain how the slope of the velocity variance $\langle \Delta v^2(t) \rangle$ as a function of time may change, for some spectral density of energy, after times already considered as very long compared with the time of the diffusion by quasilinear theory. We shall also analytically find the value of the supraquasilinear diffusion coefficient obtained by Cary *et al.* (1990, 1992), Pesme *et al.* (1993) and Helander and Kjellberg (1994), for an overlap parameter of 18, which corresponds to the maximum deviation in their paper. Moreover, we shall show that, even in the case of a finite spectral correlation length of the waves' phases studied by Pesme *et al.* (1993), the diffusion coefficient tends to its quasilinear value (up to a few percent) for a high spectral energy density.

These nonquasilinear diffusion regimes were not the only striking results that threw shadows on quasilinear theory. The simple evolution of the wave spectrum, predicted by quasilinear theory, has also been called into question by experiments in a travelling-wave tube in San Diego (see Tsunoda *et al.* 1987, 1991), and numerical simulations by Cary *et al.* (1992) and Doxas (1994 private communication) based on the self-consistent Hamiltonian derived by Mynick and Kaufman (1978) from the equations of motion introduced by O'Neil *et al.* (1974). Both experiments and simulations showed evidence of strong correlations between the modes, which is not really surprising, since the assumed linear independence of the modes in a regime of high resonance overlap is theoretically difficult to justify. Indeed, the growth rate is determined by the distribution of the particles in the resonant region, where linear theory breaks down since the velocity modulations due to different harmonics are no longer linearly independent. The expression for the growth rate in the case of a wide spectrum thus required a specific derivation, especially aimed at its determination, without hypothesis of linear independence of the modes. This was achieved by Ragot and Shklyar (1995), and improved by Ragot (1995), where the nonlinear dynamics of resonant particles in the broadband field was first studied, in a regime of sufficiently slow growth of the wave field ($\gamma \tau_{NL} \ll 1$), in order to deduce the modulation at one given frequency of the charge density, and, from this modulation, the growth rate.

This study of the particle dynamics allowed the establishment of an evolution equation for a short-time Fourier harmonic of the field. The global correction to the linear Landau growth rate was found to be less than 10%, which is in agreement with the measurements in the San Diego travelling-wave tube. The growth rate was found by Cary *et al.* (1992) to be greater than 1.5 times that predicted by quasilinear theory, which was attributed to self-consistent effects and spontaneous discretization of the spectrum due to mode-coupling effects. However, we believe that this enhancement is most probably due to the convexity of the resonant distribution function. Indeed, the derivation of the Landau growth rate includes an expansion of $f(v)$ to the first order, and, in the case when the enhancement is found, Fig. 5 in the paper by Cary *et al.* (1992) (as well as in the paper by Doxas and Cary, 1997) clearly shows that $f''(v)\Delta v^2$ is of the same order as $f'(v)\Delta v$. The theory presented by Ragot and Shklyar (1995) and, Ragot (1995), as well as the quasilinear one, suppose that the convexity of the resonant distribution function may be neglected.

The integro-differential equation due to Ragot and Shklyar (1995) and Ragot (1995) also implies that a harmonic with relatively small amplitude can have a

https://doi.org/10.1017/S0022377898006795 Published online by Cambridge University Press

much faster and more complex behaviour, as has been observed in the experiments in San Diego, as well as in the numerical simulations based on the self-consistent Hamiltonian.

For the derivation of the growth rate of the ω_0 harmonic by Ragot and Shklyar (1995) and Ragot (1995), the fact that only the ω_0-oscillating part had to be extracted from the solution of the particle velocity variation allowed us to simplify the study of the nonlinear dynamics. Here, since the goal is to precisely understand the nonlinear dynamics of the particles in a broadband wave spectrum, and how diffusion may appear in a completely deterministic system, we need to handle the problem of the nonlinear dynamics and of the determination of the velocity variation in a more rigorous way. This will be our purpose in the next two sections.

3. Field and discretization

So, we study the dynamics of charged particles (with a charge $-e$ and a mass m) in one-dimensional plasma turbulence, for the case when the evolution of the field is sufficiently slow for it to be considered as fixed on the timescales on which we want to study the dynamics, i.e. for $\gamma\tau_{NL} \ll 1$. We might consider general one-dimensional slowly growing plasma turbulence. However, though our theory may as well be applied, for instance, to whistler modes in parallel propagation as demonstrated by Ragot (1995), we shall reduce our present dissertation to the case of 1D Langmuir turbulence, for the sake of simplicity.

We shall assume that all the conditions described above (in Sec. 2) are fulfilled, so that the dispersion relation of our modes can be written in the form (2.1). This will permit us to make precise the conditions of validity of our theory, whenever necessary.

$\mathscr{E}(x,t)$ will denote the electrostatic field, composed of many high-frequency modes propagating in one direction, with characteristic amplitude A, frequencies spaced on the average by $\overline{\delta\omega}$, and independent phases ϕ_n. With the real field $\mathscr{E}(x,t)$, we shall associate a complex analytical field $\tilde{E}(x,t)$, by addition to the real field of an imaginary part equal to its Hilbert transform, so that only positive frequencies will appear in the spectrum of $\tilde{E}(x,t)$:

$$\mathscr{E}(x,t) = \mathrm{Re}\{\tilde{E}(x,t)\} = \mathrm{Re}\left\{\sum_n A_n \exp[i\,(k_n x - \omega_n t + \phi_n)]\right\}. \tag{3.1}$$

k_n and ω_n are connected by the nonlinear dispersion relation (2.1), and A_n are the amplitudes of the different modes of the field. We shall denote by $\delta\omega$ the quantity $\overline{\delta\omega}\,|1 - v_\varphi/v_g|$, where v_φ and v_g are the phase and group velocities. $\delta\omega = k_0\delta v_\varphi$ (to first order), where δv_φ is the average distance between two successive harmonics measured in phase velocity. The Hamiltonian of the dynamics reads

$$H(x,v,t) = \frac{v^2}{2} + \frac{e}{m}\sum_n \frac{A_n}{k_n}\sin(k_n x - \omega_n t + \phi_n)\,. \tag{3.2}$$

The expression (3.1) is readily accepted as a general representation of Langmuir turbulence in a plasma. However, the dispersion relation for Langmuir waves yields, in an unbounded medium, a continuum of solution modes. The writing of the field in the form of a discrete sum thus constitutes only an imperfect representation of the field. We shall first make precise the conditions under which it can be considered

https://doi.org/10.1017/S0022377898006795 Published online by Cambridge University Press

as correct. Strictly speaking,

$$\tilde{E}(x,t) = \int d\omega\, \mathscr{A}(\omega) e^{i[k(\omega)x-\omega t]}\,, \tag{3.3}$$

where $\mathscr{A}(\omega)$ is a complex amplitude. Let $\{\omega_n\}$ be a series of frequencies of the $\tilde{E}$ spectrum, such that

$$\langle \omega_{n+1} - \omega_n \rangle_n = \overline{\delta\omega} = \frac{\delta\omega}{1 - v_\varphi/v_g}\,, \tag{3.4}$$

and $\{I_n\}$ an $\mathbb{R}$-partition, such that each interval I_n is centred on the frequency ω_n. $\tilde{E}(x,t)$ may also be written as

$$\begin{aligned}\tilde{E}(x,t) &= \sum_n \int_{I_n} d\omega\, \mathscr{A}(\omega) e^{i[k(\omega)x-\omega t]} \\ &= \sum_n e^{i(k_n x-\omega_n t)} \int_{I_n} d\omega\, \mathscr{A}(\omega) e^{i\{[k(\omega)-k_n]x-(\omega-\omega_n)t\}}.\end{aligned} \tag{3.5}$$

As the functions $(x,t) \to e^{i(k_n x-\omega_n t)}$ are independent, the representation (3.1) is valid if the arguments of the exponentials are equal:

$$A_n e^{i\phi_n} = \int_{I_n} d\omega\, \mathscr{A}(\omega)\, e^{i\{[k(\omega)-k_n]x-(\omega-\omega_n)t\}}.$$

Hence $\int_{I_n} d\omega\, \mathscr{A}(\omega)\, e^{i\{[k(\omega)-k_n]x-(\omega-\omega_n)t\}}$ should not depend on time or on the position. This is true if

$$\overline{\delta\omega}\,\Delta t \ll \pi \quad \text{and} \quad \overline{\delta k}\,\Delta x \ll \pi,$$

i.e.

$$\Delta t \ll \frac{\pi}{\delta\omega}\left|1 - \frac{v_\varphi}{v_g}\right| \sim \frac{\pi}{\delta\omega}\frac{1}{3k^2\lambda_{De}^2}$$

and

$$\Delta x \ll \frac{\pi}{\delta\omega} v_g \left|1 - \frac{v_\varphi}{v_g}\right| \sim \frac{\pi}{\delta\omega}\frac{\omega_p}{k}, \tag{3.6}$$

since

$$\overline{\delta k} \approx \left.\frac{\partial k}{\partial\omega}\right|_{\omega_0} \overline{\delta\omega} \approx \frac{\overline{\delta\omega}}{v_g}.$$

The condition

$$\Delta t \ll \frac{\pi}{\delta\omega}\frac{1}{3k^2\lambda_{De}^2}$$

is not very restrictive, since $k^2\lambda_{De}^2 \ll 1$. For the solar wind, for instance, $3k^2\lambda_{De}^2$ is typically 10^{-2} ($\lambda_{De} \approx 30$ m and $k \approx 2 \times 10^{-3}\,\mathrm{m}^{-1}$). But when one follows a given particle on a time interval Δt, the condition on Δx implies that $v\Delta t$ should remain small compared with $(\pi/\delta\omega)\omega_p/k$, i.e. for a resonant particle, $\Delta t \ll \pi/\delta\omega$, which is more restrictive. This condition is barely fulfilled in the simulations by Cary *et al.* (1990, 1992), Helander and Kjellberg (1994) giving a supraquasilinear diffusion coefficient. We shall not discuss this point in detail in this section. However, we shall keep this condition in mind for further considerations, and, since it may be the key point for deciding about the relevance of observed nonquasilinear regimes, we shall come back to it in Sec. 6.

https://doi.org/10.1017/S0022377898006795 Published online by Cambridge University Press

4. Nonlinear dynamics, decorrelation time and diffusion regimes

The work presented hereinafter is based on the idea that to describe the dynamics of a resonant particle correctly on a given timescale, it is not necessary to take into account all the modes of the spectrum, provided that the particle velocity remains sufficiently localized on this timescale. Indeed, fast oscillations should be averaged out. In a first part, the main goal will thus consist in proving this assertion and making it precise, using the solution derived by Ragot and Shklyar (1995) for the velocity of a particle interacting with a wave packet of width $M\delta\omega$, on its characteristic nonlinear time $\tau_{(M)}$ of amplitude variation.

4.1. Nonlinear dynamics in a wave packet composed of M modes

Let us consider the reduced field $\tilde{E}_M(\xi, t)$, composed of the M nearest modes around one given harmonic $(\omega_0, k(\omega_0))$.

4.1.1. Field representation of a wave packet. As in Ragot and Shklyar (1995), in the frame of this harmonic moving at phase speed ω_0/k_0, we can put the reduced field – in the variables $\xi = k_0 x - \omega_0 t$, $u = k_0 v - \omega_0$, with $k_0 = k(\omega_0)$ – in the form

$$\tilde{E}_M(\xi, t) = \sum_{M\delta\omega} A_n \exp\left\{ i \left[\xi + \left(\frac{v_{\varphi 0}}{v_{g0}} - 1 \right) (\omega_n - \omega_0) t + \phi_n + \frac{k_n - k_0}{k_0} \xi \right] \right\}, \tag{4.1}$$

where $v_{\varphi 0}$ and v_{g0} are the phase and group velocities of the harmonic $(\omega_0, k(\omega_0))$. The expression (4.1) results from a first-order expansion of the exponential arguments in (3.1). $M\delta k \ll k_0$ is required now. For this, we define M_k as the largest integer such that $M_k \delta k$ can still be considered small compared with k_0; and we choose $M \leqslant M_k$. As we shall see later, the nonlinear time τ_{NL} of wave–particle interaction corresponds to a variation of the order of 1 in $\mid \xi \mid$. This means that, for shorter timescales, $M\delta k \ll k_0$ allows us to neglect the last term in the exponent of (4.1), which already contains arbitrary phases ϕ_n. With this simplification, $\exp(i\xi)$ can be taken out of the sum:

$$\tilde{E}_M(\xi, t) = \exp(i\xi) \sum_{M\delta\omega} A_n \exp\left\{ i \left[\left(\frac{v_{\varphi 0}}{v_{g0}} - 1 \right) (\omega_n - \omega_0) t + \phi_n \right] \right\}. \tag{4.2}$$

Expressing the sum, which now depends only on time, as a function of the instantaneous amplitude $\mathscr{E}_M(t)$ and phase $\varphi(t)$, we have

$$\tilde{E}_M(\xi, t) = \mathscr{E}_M(t) \exp[i\xi + i\varphi(t)]. \tag{4.3}$$

We have performed extensive numerical simulations of the wave packet (4.2) to study the behavior of $\mathscr{E}_M(t)$ and $\varphi(t)$, choosing the modes as described at the beginning of Sec. 3. Figure 1 shows a typical – randomly chosen – example of the time variations of $\mathscr{E}_M(t)$ and $\varphi(t)$: the instantaneous amplitude $\mathscr{E}_M(t)$ follows a series of bursts (see also Fig. 4), and inside each of these bursts, the instantaneous phase $\varphi(t)$ varies slowly and almost linearly. A projection of the field (4.2) on optimized Gabor wavelets (cf. Ruskai *et al.* 1992), as done by Ragot and Shklyar (1995) and Ragot (1995), corroborates this slow variation of the phase $\varphi(t)$ around the maxima of amplitude.

Each upper part of the amplitude bursts can be easily fitted by a Gaussian. As the wave–particle interaction is most important when the amplitude of the field is large (i.e. at the top of the bursts), we shall approximate each burst of amplitude by a Gaussian function with, for each burst, slightly different parameters, whose

https://doi.org/10.1017/S0022377898006795 Published online by Cambridge University Press

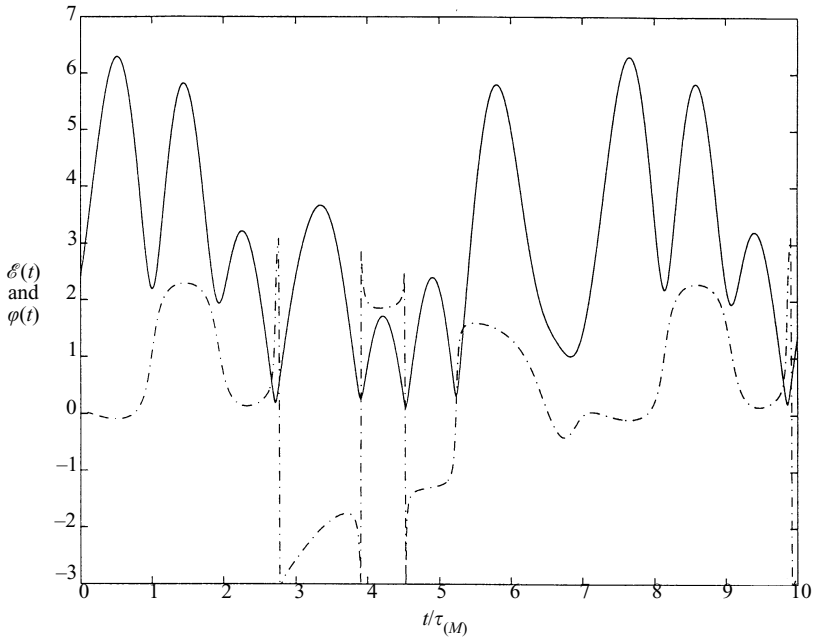


Figure 1. Variations of the instantaneous amplitude $\mathscr{E}(t)$ (——) and the instantaneous phase $\varphi(t)$ (- - -) as functions of the nondimensional time $t/\tau_{(M)}$, for $M = 11$.

statistical properties we shall study below, in particular as functions of the number of modes retained in the reduced spectrum. This approximation of Gaussian bursts of the instantaneous amplitude $\mathscr{E}_M(t)$ is justified by the fact that $\mathscr{E}_M(t)$ only departs from a Gaussian function at low values where the velocity of the particles is no longer significantly modified by the field $\tilde{E}_M$. In addition to the low value of the field amplitude, we argue that in the time intervals where the deviation of $\mathscr{E}_M(t)$ from Gaussian behaviour becomes important, the phase $\varphi(t)$ of the field varies strongly, which further reduces the efficiency of the wave–particle interaction.

Still referring to Ragot and Shklyar (1995), we shall thus describe the field as a series of Gaussian bursts of the instantaneous amplitude $\mathscr{E}_M(t)$, with a slowly varying instantaneous phase $\varphi(t)$ inside each burst, and write:

$$\mathscr{E}_M(t) = \mathscr{E}_p \exp[-\tfrac{1}{2}\mu_p(t-t_p)^2], \qquad \varphi(t) = \varphi_p + \dot{\varphi}_p(t-t_p), \tag{4.4}$$

where $\mathscr{E}_p$, φ_p and $\dot{\varphi}_p$ denote the amplitude, the phase and its derivative at the amplitude maxima, which happen at times t_p. The phase φ undergoes fast nonlinear variations at the transition of the amplitude $\mathscr{E}_M(t)$ through each of its minima, so that the φ_ps may be considered as independent – in fact, most important is that their average value is zero. (Note again that these fast variations of the phase φ are essential to validate the model of Gaussian bursts for our problem of wave–particle interaction.)

The particular form (4.4) adopted for $\mathscr{E}_M$ and φ initially arises from computer simulations of the wave field (see Fig. 1) – similarly for the numerical coefficients appearing thereafter. The quantities $\mathscr{E}_p$, φ_p and $\dot{\varphi}_p$ vary from one burst to the next, and we have numerically studied their statistical properties as functions of M. In the case when the amplitudes A_n and the phases ϕ_n are randomly distributed, and

https://doi.org/10.1017/S0022377898006795 Published online by Cambridge University Press

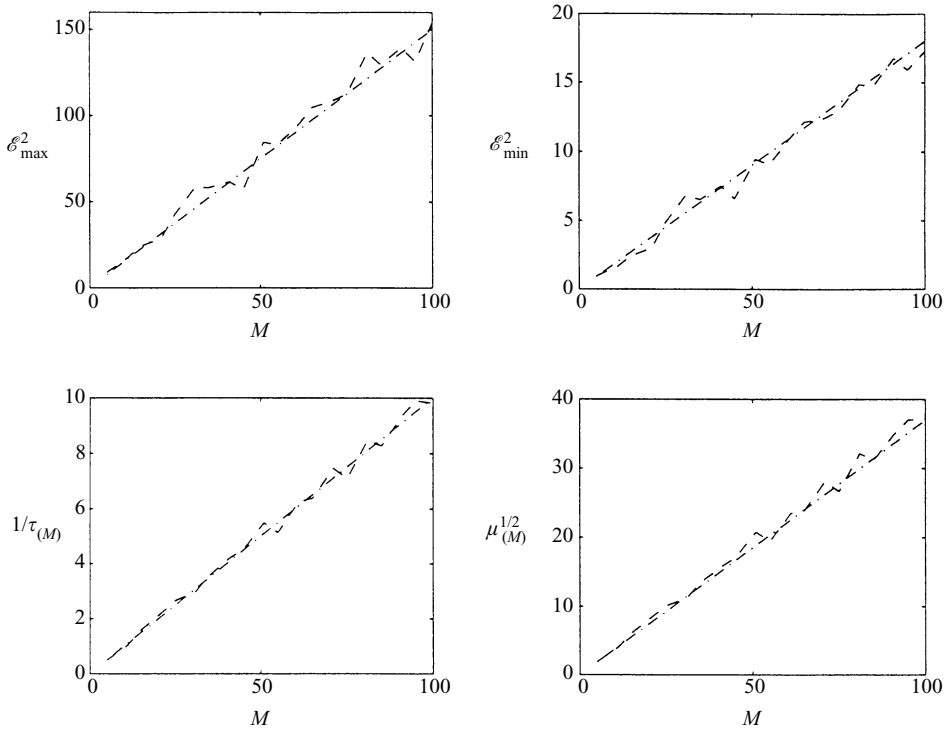


Figure 2. Variations of $\mathscr{E}^2_{\max}$, $\mathscr{E}^2_{\min}$, $1/\tau_{(M)}$ and $\mu^{1/2}_{(M)}$ with respect to the number of harmonics M kept in the spectrum. The powers of $\mathscr{E}_{\max}$, $\mathscr{E}_{\min}$, $\tau_{(M)}$ and $\mu_{(M)}$ are chosen so that the variations as functions of the number of harmonics M are linear, and the scaling laws can be easily checked: A and $\delta\omega$ are equal to 1.

for frequency intervals as described above, we find that the time-average values of $\mathscr{E}_p$ and μ_p are (see Fig. 2)

$$\mathscr{E} \sim 1.2M^{1/2}A, \qquad \mu \sim 0.14(\Delta\omega)^2 \sim 0.14(M\delta\omega)^2. \tag{4.5}$$

(By time-average, we mean that we average the values of $\mathscr{E}_p$ and μ_p over many successive bursts of amplitude.) These scalings, which are valid for uncorrelated phases and M large enough, say $M \geqslant M_{QL} \approx 10$, will be used for further estimates of the velocity variance and 'diffusion coefficient', since we assume here a sufficiently high spectral density of energy of the turbulence.

Another important feature of the reduced field $\tilde{E}_M$ is its characteristic nonlinear time $\tau_{(M)}$ of amplitude variation. This corresponds to the mean time interval between two successive amplitude minima of the wave packet, and scales as

$$\tau_{(M)} \sim \frac{10}{M\delta\omega} \tag{4.6}$$

(see the third quadrant of Fig. 2). Let us first consider the dynamics on this timescale, that is in one burst of the reduced field amplitude. On putting

$$\zeta = \xi + \varphi(t) + \pi, \qquad w = \mu_p^{-1/2}(u + \dot{\varphi}_p), \qquad \tilde{t} = \mu_p^{1/2}(t - t_p), \tag{4.7}$$

https://doi.org/10.1017/S0022377898006795 Published online by Cambridge University Press

the equations of motion can be written as (omitting the tilde on $\tilde{t}$)

$$\frac{d\zeta}{dt} = w, \qquad \frac{dw}{dt} = \beta_M \exp(-\tfrac{1}{2}t^2)\cos\zeta, \tag{4.8}$$

where the value of the parameter $\beta_M = ek_0\mathscr{E}_p/m\mu_p$ depends on the number of harmonics taken into account to the power $-\frac{3}{2}$, since $\mathscr{E}_p$ scales as $1.2M^{1/2}A$ and μ_p as $0.14(M\delta\omega)^2$.

4.1.2. Nonlinear velocity variation in one burst of instantaneous amplitude. In order to study the diffusivity of the particles and to be able to estimate a diffusion coefficient, if one exists, we need to know the velocity variation. We shall try to find, as a first step, the velocity variation $\Delta_M w(t)$ inside one burst of instantaneous amplitude, i.e. the difference between the current nondimensional velocity $w(t)$ and the corresponding velocity w_0 at the beginning of the burst that we are considering.

To that end, we put

$$z = -\tfrac{1}{2}t^2 + i\zeta, \qquad p = -t + i\frac{d\zeta}{dt}, \tag{4.9}$$

so that the equations of motion become

$$\frac{dz}{dt} = p, \qquad \frac{dp}{dt} = i\beta_M e^z - \alpha, \tag{4.10}$$

where

$$\alpha = 1 - \beta_M \exp(-\tfrac{1}{2}t^2)\sin\zeta. \tag{4.11}$$

With this set of equations there is associated a complex energy

$$\varepsilon = \tfrac{1}{2}p^2 + \alpha(t)z - i\beta_M e^z. \tag{4.12}$$

Then we put

$$\mathscr{P} = p - i\beta_M\lambda\int_{-\infty}^{z} \frac{e^\theta d\theta}{[2(\varepsilon - \alpha\theta + i\beta_M e^\theta)]^{1/2}}, \tag{4.13}$$

with $\lambda = -1$ (respectively $+1$) for $t < 0$ (respectively $t > 0$), and where we take the branch of the square root with a positive real part. This quantity obeys

$$\frac{d\mathscr{P}}{dt} = -\alpha, \tag{4.14}$$

neglecting the terms of second order in β_M with small coefficient. Let us define the function h by

$$\theta = y + ih(y), \qquad \text{with } h(y) = \tilde{\beta}e^\theta. \tag{4.15}$$

We find

$$h(y) = \sum_{n=1}^{\infty} a_n\tilde{\beta}^n e^{ny}, \qquad a_n = i^{n-1}\frac{n^{n-1}}{n!}, \tag{4.16}$$

which allows us to integrate and write $\mathscr{P}$ in the form

$$\mathscr{P} = p + i\alpha^{1/2}\sum_{n=1}^{\infty}\left(\frac{n\pi}{2}\right)^{1/2} a_n\left(\frac{\beta_M}{\alpha}\right)^n \exp\left(\frac{n\varepsilon}{\alpha}\right)\left[\Phi\left(\left(\frac{n}{2\alpha}\right)^{1/2}p\right) + \lambda\right], \tag{4.17}$$

where Φ stands for the error function

$$\Phi(x) = \frac{1}{\pi^{1/2}}\int_0^{x^2}\frac{e^{-t}}{t^{1/2}}\,dt. \tag{4.18}$$

https://doi.org/10.1017/S0022377898006795 Published online by Cambridge University Press

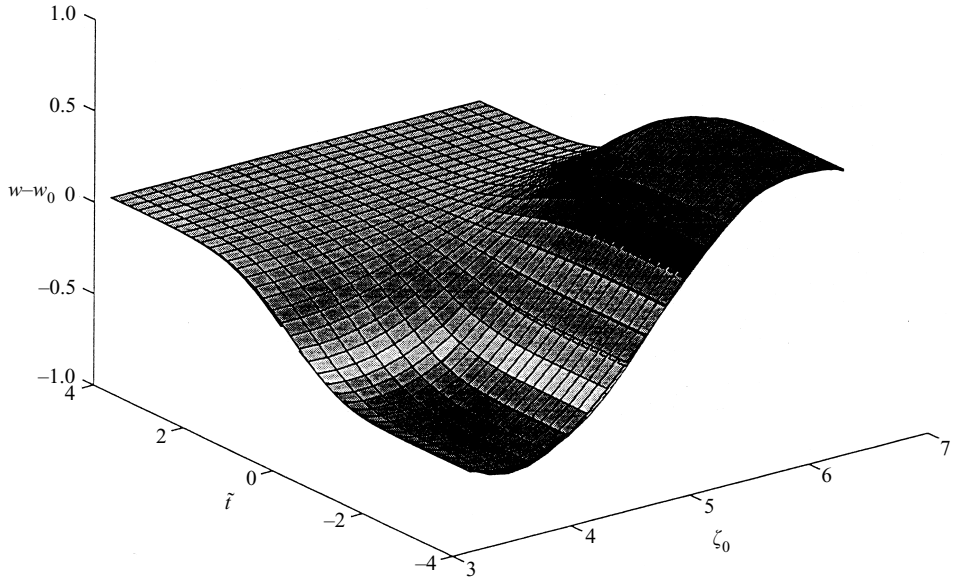


Figure 3. $w - w_0$ as a function of ζ_0 and $\tilde{t}$, for $\beta = 0.25$ and $w_0 = 0$: comparison of numerical and analytical results. $\zeta_0 \in [\pi, 2\pi]$ and $\tilde{t} \in [-4, 4]$.

Then, taking account of the fact that α is real, we obtain the velocity variation in implicit form:

$$\begin{aligned}\Delta_M w(t) &= w_0 - w(t)\\ &= \frac{\alpha^{1/2}}{2}\sum_{n\geqslant 1}\left(\frac{n\pi}{2}\right)^{1/2} a_n\left(\frac{\beta_M}{\alpha}\right)^n \exp\left(\frac{n\varepsilon}{\alpha}\right)\left[\Phi\left(\left(\frac{n}{2\alpha}\right)^{1/2} p\right) + \lambda\right]\\ &\quad -\nu\alpha_m^{1/2}\sum_{n\geqslant 1}\left(\frac{n\pi}{2}\right)^{1/2} a_n\left(\frac{\beta_M}{\alpha_m}\right)^n \exp\left(\frac{n\varepsilon_m}{\alpha_m}\right) + \text{c.c.},\end{aligned} \tag{4.19}$$

with $\nu = 1$ (respectively 0) if $t > 0$ (respectively $t < 0$), and where the index m indicates the values at the amplitude maximum, corresponding to $t = 0$.

This analytical integration is presented in greater detail by Ragot and Shklyar (1995) and Ragot (1995). The series converges for sufficiently small β_M, that is, approximately, $\beta_M < 1/e$. We shall check further that this condition is fulfilled for the values of M we are interested in.

Figure 3 compares the values of $w_0 - w$ obtained by numerical integration with those deduced from (4.19), for a range of initial phases $\zeta_0 \in [\pi, 2\pi]$, an initial velocity $w_0 = 0$, and times $\tilde{t}$ between -4 and 4. This figure shows that w is a regular function of time in one burst of the reduced field amplitude, and the sensitivity to initial conditions is moderate. Moreover, the integration that leads to the formula (4.19) is reliable. Equation (4.19) also allows us to estimate the first order of the velocity variation in one burst of the instantaneous amplitude as

$$\Delta_M w \sim (2\pi)^{1/2}\beta_M \exp(-\tfrac{1}{2}w_m^2)\cos\zeta_m. \tag{4.20}$$

It can be checked from (4.19) that the correction to this first-order term is small. It will be neglected in further considerations, and in particular in Sec. 6, when we shall look for the corrections to quasilinearity obtained by Cary *et al.* (1990, 1992),

https://doi.org/10.1017/S0022377898006795 Published online by Cambridge University Press

Pesme *et al.* (1993) and Helander and Kjellberg (1994). Indeed, these corrections are corrections of order 1, which either are already contained in the term (4.20) or cannot be explained by the present theory.

4.1.3. Limitation on M and definition of τ_{NL}. Now, there is an obvious limitation on the values that we may reasonably consider for M. Indeed, we need to take into account at least sufficiently many modes for the velocity variation of the resonant particles to be covered by the width, in phase velocity, of the spectrum we keep for describing the dynamics. We shall denote by N the smallest value for which this condition is satisfied for all the particle trajectories that, during the time $\tau_{(N)}$, cross the line $v = \omega_0/k_0$. $\tau_{(N)}$ defines the nonlinear time τ_{NL} of wave–particle interaction, which scales as the spectral density of energy ($W_\omega = A^2/\delta\omega$) to the power $-\frac{1}{3}$, since

$$\tau_{NL} \sim \frac{10}{N\delta\omega} \tag{4.21}$$

and (4.20) yields, as a first estimate,

$$N \sim \left[\frac{15ek_0A}{m(\delta\omega)^2}\right]^{2/3}. \tag{4.22}$$

(To obtain the factor 15, we proceed numerically with (4.19), looking for the smallest value of M such that the largest $k_0\Delta_M v$ equals $\frac{1}{2}M\delta\omega$, i.e. $\approx \frac{1}{2}N\delta\omega$, according to the definition of N. This is not an exact way of computing it, but should already give a good estimate.)

Note that for Langmuir waves, the condition $N\delta k \ll k_0$ reduces to $\Delta_{(N)}v \ll v$. And, for a fast beam, the nonlinear velocity variation of resonant particles is obviously negligible with respect to their velocity in the fixed frame. Hence $N \ll M_k$.

Another limitation on M is connected to the total width of the spectrum. We consider a broadband wave spectrum, but of finite width, which imposes the upper limit for the spectral density of energy W_ω. Indeed, using the definition (2.4) of the autocorrelation time τ_{ac} of the field, and writing this condition for N, which is the smallest value of M, we find the inequality

$$0.23\left(\frac{ek_0}{m}\right)^2 W_\omega < \tau_{ac}^{-3}.$$

The conditions for our present theory to have some range of applicability can thus be summarized as

$$\gamma^3 \ll 0.23\left(\frac{ek_0}{m}\right)^2 W_\omega < \tau_{ac}^{-3}. \tag{4.23}$$

4.2. Quasifractal dynamics in the whole spectrum, and correlation time

As we know the velocity variation induced by a wave packet composed of M modes on the timescale $\tau_{(M)}$, we can estimate the error generated by replacing the whole spectrum by a reduced one on this timescale. With this aim, we compare the smallest velocity variation, induced on a ω_0-resonant particle, by a wave packet that contains M modes centred around (ω_0, k_0), with the largest variation induced on this particle by a wave packet of the same width, but shifted in frequency by $aM\delta\omega$. To determine the largest possible error, we consider a particle with velocity $w = \frac{1}{2}$ at the amplitude maxima of both wave packets (which is unlikely to occur, especially for $M > N$, but will give a wide estimate). Using (4.20), we find that the

https://doi.org/10.1017/S0022377898006795 Published online by Cambridge University Press

external wave packet has a relative influence on its velocity variation of the order of $\exp[-\frac{3}{2}(a-\frac{1}{2})^2+\frac{3}{8}]$ compared with the central one, i.e. approximately 4.9% for $a=2$. As for the harmonics further outside this external wave packet, their influence is completely negligible (for $a=2$) on the timescale $\tau_{(M)}$, since the decrease of this influence is a Gaussian function of the distance from the harmonic ω_0. From this, we conclude that a wave packet of $3M$ harmonics around ω_0 is sufficient to determine the velocity variation of a resonant particle, on the timescale $\tau_{(M)}$.

The trajectory of a resonant particle is regular – nonlinear but regular – when we omit the perturbation of negligible average due to modes outside the wave packet of $3M$ modes, on the timescale $\tau_{(M)}$. Since $\tau_{(3M)}=\frac{1}{3}\tau_{(M)}$ (see (4.6)), it is typically composed of three regular jumps in velocity. However, to calculate the velocity after a time of order $\tau_{(3M)}$, we have to take into account three times more modes, so that each of the regular jumps should in fact be viewed as composed of three regular jumps of smaller amplitude, and so on. Hence we come to a conclusion about the self-similarity of the solution (for the velocity) at all the timescales shorter than τ_{NL}, down to $\tau_m=\max(\tau_{ac},10/(M_k\delta\omega))$ (where $M_k\delta k$ is the largest wavenumber width that can still be considered as small compared with k_0). Note that for Langmuir waves produced by a beam of temperature T and speed v_b, we have $\tau_m=\tau_{ac}$ if $T^{1/2}\ll v_b$.

Figure 4 illustrates our understanding of the dynamics on times shorter than τ_{NL}. The instantaneous amplitudes $\mathscr{E}_0(t)$ of reduced fields, composed respectively of 8, 24 and 72 modes, are plotted in (*a*). (We keep the central modes unchanged, when increasing the width of the spectra.) Figure 4(*b*) shows the resonant velocity u computed in these three reduced fields, as a function of time. Its slope changes significantly at each new amplitude burst of the field in which it is computed. Of course, only the velocity variation after three jumps can be considered as quite rigorously estimated (in the whole spectrum). Moreover, the third jump does not exactly finish at the end of a jump obtained on a three-times longer timescale, since the minima of the reduced field amplitudes do not precisely occur at the same moments. But on a few of the longest jumps, it becomes clear that the more perturbed curve just follows the smooth one.

To summarize, even if the trajectory is composed of nonlinear jumps in velocity at all the timescales between τ_{NL} and τ_m, this trajectory will appear as regular on the timescale τ_{NL}, as well as all intermediate timescales, by an averaging effect. Finally, if the spectral width of the wave were infinite, and the equations of motion could be put in the form (4.8) whatever the spectral width (which is not true, because the condition $M\delta k\ll k_0$ would break down), the trajectory would be fractal with Hausdorff–Besicovitch dimension 2. This dimension $\mathscr{D}$ is defined as $\lim_{s\to 0}\log[\mathscr{N}(s)]/\log(1/s)$, where $\mathscr{N}(s)$ is the number of cubes of size s required for covering one given part of the trajectory (cf. Lichtenberg and Liebermann 1992; Bergé *et al.* 1992). And, in our case,

$$\mathscr{D}=\lim_{M\to+\infty}\frac{\log(3M/N)}{\log[\delta\omega/(k_0\Delta_{(3M)}v)]}=2. \tag{4.24}$$

Of course, in a physical case, it is not possible to let the scale tend to zero as we would need in order to estimate this dimension. Nevertheless, there is a scale invariance, from τ_{NL} down to τ_m, and the decorrelation time of the dynamics is not τ_{ac}, but at least τ_{NL}.

https://doi.org/10.1017/S0022377898006795 Published online by Cambridge University Press

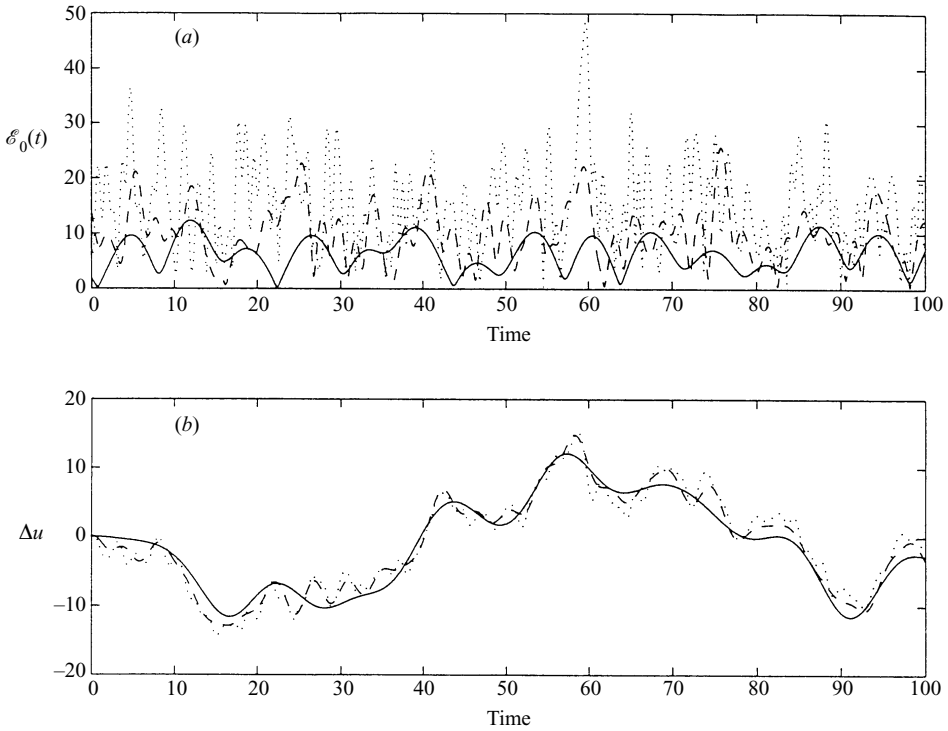


Figure 4. (*a*) Amplitudes $\mathscr{E}_0$ of the fields composed respectively of 8, 24 and 72 modes, as functions of time. The slowest variation corresponds to the 8-mode case, while the sharp high peaks are generated by 72 modes. Of course, the central (8 and then 24) modes are kept unchanged. (*b*) Variation of the velocity u as a function of time, in the wave packets described above. On the timescale of a few amplitude bursts of a given wave packet, the effect of external modes is averaged out and the trajectory looks regular. The total duration is much shorter than τ_{NL} in this example, so that trajectories with slightly different initial conditions do not separate on this timescale. They are not represented for the clarity of the figure.

4.3. Diffusion regimes and time transition from regular to chaotic motion

A forthright consequence of this scale invariance is what we shall call the 'quasi-fractal diffusion' regime, on timescales between τ_m and τ_{NL} – under the condition that the scalings giving β_M are valid for all the values of M considered, which is not exact when N and thus M can approach values of the order of a few ($< M_{QL} \approx 10$). Indeed, (4.20) gives in dimensional units (see Sec. 4.1.1, (4.5), (4.7) and (4.8))

$$\Delta_{(3M)}v \approx \frac{(\mu_{(3M)})^{1/2}}{k_0}\Delta_{3M}w \approx \frac{3M\delta\omega(0.14)^{1/2}}{k_0}(2\pi)^{1/2}\beta_{3M}\cos\zeta_m, \tag{4.25}$$

$$\frac{\overline{(\Delta_{(3M)}v)^2}}{2\tau_{(3M)}} \approx \frac{\pi}{2}\left(\frac{e}{m}\right)^2\frac{A^2}{\delta\omega} = D_{QL} \tag{4.26}$$

for $M > N$, which allows us to speak about a 'diffusion coefficient'. $\overline{a}$ signifies that we average over the particle phases (so that $\overline{\cos^2\zeta_m}$ gives $\frac{1}{2}$), and on a time of the order of a few $\tau_{(M)}$.

When speaking here about a 'diffusion coefficient', we do not claim that $\langle\Delta v^2(t)\rangle$ (averaged over the initial conditions) has a constant slope over a time interval

https://doi.org/10.1017/S0022377898006795 Published online by Cambridge University Press

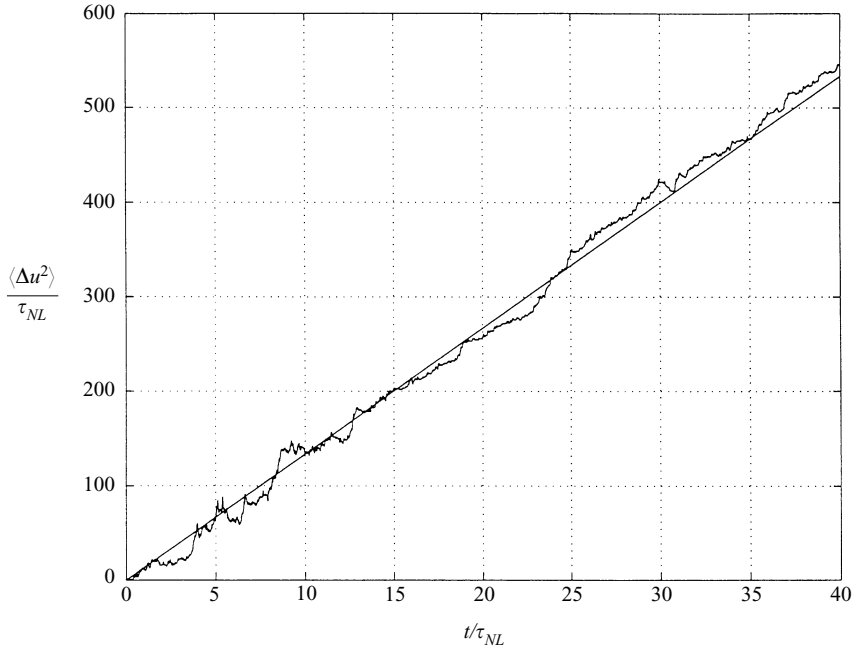


Figure 5. Velocity variance $\langle \Delta u^2 \rangle / \tau_{NL}$ as a function of the nondimensional time t/τ_{NL}, for a set of 800 particles with initial velocity $u = 0$ and random initial phases ξ between 0 and 2π, for $\delta\omega = 1$ and $A = 2$. The straight line has a quasilinear slope. It fits the curve for times shorter than τ_{NL}, corresponding to the 'quasifractal diffusion' regime. In the transient regime, that is for times shorter than many τ_{NL} ($\approx 10\tau_{NL}$), the irregularity of the curve demonstrates the nondiffusivity of the dynamics (tested for up to 10^4 particles). And, for longer times, when the phase mixing starts to be efficient, the curve asymptotes the straight line, which confirms the long-time diffusion regime.

(fluctuations may be very large), but that $\langle \Delta v^2(t) \rangle$ takes the quasilinear value on discrete times between τ_m and τ_{NL}, which gives the impression of a quasilinear slope when observed on a longer timescale (see Fig. 5).

Moreover, the dynamics is clearly nondiffusive, in its usual sense, since we can really estimate the position of a particle on a time of the order of τ_{NL}, which implies that there is no 'practical' memory loss; nor is there phase mixing.

But we may also predict that this regime is followed, after a transient period of several τ_{NL}, by a second, more classical diffusion regime. Indeed, for longer times, the fast jump of the instantaneous phase of the efficient field, associated with a strong nonlinear phase and velocity variation, as well as some sensitivity to initial conditions, permits us to forecast a good mixing of trajectories.

We present in Fig. 5 the variance $\langle \Delta u^2 \rangle$ as a function of time. Its slope starts with the quasilinear value (when examined on longer timescales). Then it varies strongly and we need to wait at least for $10\tau_{NL}$ to again observe a fixed slope. We propose below the underlying explanation, and defer to Sec. 5 more detailed comments on this curve.

On short timescales, many harmonics have to be taken into account in the dynamics. Thus, by normalization, $w \ll 1$, since μ is large, u is bounded (since it stays inside a spectral band of width $N\delta\omega$ around ω_0), and $\dot{\varphi}$ is small compared with μ. If w is small, the phase ζ will undergo a very small variation on a burst of amplitude

https://doi.org/10.1017/S0022377898006795 Published online by Cambridge University Press

(cf. (4.8)). And the variation of this phase $\zeta = \xi + \varphi + \pi$ happens essentially at the transition of the considered field amplitude through each of its minima, where the phase φ undergoes strong variations. This implies that all the particles will be submitted to the same significant variations of their phases, at the same times, so that initially nearby trajectories will follow each other on many bursts. Moreover, the jumps in velocity space are independent, though of the same mean amplitude ($\propto \exp(-\frac{1}{2}w_m^2) \approx 1$).

Furthermore, for an individual particle, since w is small, and thus varies very little, since the average of φ falls to zero on a timescale long compared with the one on which the dynamics is solved, and since the variation of the phase ζ is principally due to the variations of φ, we can predict that the trajectory of this particle will return near its initial position in the phase space (ζ, w). Since the sensitivity to initial conditions is moderate, on the average, the motion will follow a regular curve. And indeed, if we solve the dynamics on a timescale long compared with the previous one, and if we take into account only those harmonics that have an effect on this timescale, the motion in one amplitude burst of this new timescale will appear regular, corresponding to the average motion of the previous scale. From this, we can say that the dynamics remains correlated, at least on times of order τ_{NL}.

This reasoning is transposable up to the timescale τ_{NL}, or slightly shorter, but not beyond, since the time that would be necessary to find a phase near the initial one would be so long that the velocity would have significantly changed meanwhile, and thus the probability of again finding the initial phase ζ and velocity w would be extremely low.

So, the difference between short and long timescales consists in the fact that the variation of ζ is, in one case, essentially due to the variation of the field phase at each transition through the amplitude minima, and in the other case, in equivalent proportion, to the variation of ξ itself. This is essential since, in the first case, the particles of the system undergo almost the same variation of their phase, which limits the mixing of these trajectories in the phase space, while in the second case, the variation of ζ depends on the initial phase of each particle, and thus differs for each one. There is thus nonlinear decorrelation of the ensemble motion on the long timescale. And the decorrelation time of the dynamics really is of the order of a few τ_{NL}.

We consequently have an initial regime of 'ordered diffusion', on timescales shorter than τ_{NL}, whose characteristics differ radically from a classical diffusion regime, since there is neither phase mixing nor memory loss, and which is connected with the stochasticity of the field, followed after a relatively long, transient period of phase mixing by a second diffusion regime, on timescales long compared with τ_{NL}, more classical, where the nonlinearity of the particle dynamics plays a crucial role.

5. Diffusion regimes: numerical test and further discussion

In the previous section, we have established the correlation time τ_{NL} of the particle dynamics in a broadband wave spectrum. We have deduced from wave–particle interaction analysis, computer modelling and analytical derivation, the dynamical transition between a regular, quasifractal behaviour of the particle trajectories, without separation of initially close trajectories, and a chaotic behaviour of these trajectories characterized by a nonlinear mixing in the phase space.

https://doi.org/10.1017/S0022377898006795 Published online by Cambridge University Press

5.1. Quasifractal diffusion regime

From the first of these results, we can predict that the velocity variance $\langle \Delta u^2 \rangle_\xi$, as a function of time, although locally irregular, should globally (when examined on a timescale of a few τ_{NL}) increase linearly with the quasilinear slope up to the timescale $\min(\tau_{NL}, \tau_{(M_{QL})})$, where M_{QL} defines the minimal number M of harmonics in the efficient wave packet for which the quasilinear scalings (4.4) are valid ($M_{QL} \approx 10$). Indeed, when we perform numerical test-particle simulations, we check that for any realization of the wave field (4.2) (such as described in Sec. 3), the increase of $\langle \Delta u^2(t) \rangle_\xi$ looks linear, with a quasilinear slope at least up to the time $\min(\tau_{NL}, \tau_{(M_{QL})})$. When the upper time is connected to the breakdown of the quasilinear scalings (i.e. $\tau_{(M_{QL})}$), it is not modified by a change of the wave energy (as long as $\tau_{NL} > \tau_{(M_{QL})}$).

When $\tau_{NL} < \tau_{(M_{QL})}$, the quasilinear slope extends up to slightly longer times (between τ_{NL} and $1.5\tau_{NL}$). This is not in contradiction with the discussion in Sec. 4. In fact, we estimated the time when neighbouring trajectories start to separate in the case when their velocity variation is the largest. But in the simulation, we average over the initial particle phases, and not all trajectories will precisely start to separate from neighbouring ones at that time. Those particles for which the velocity variation on τ_{NL} is significantly smaller than the width, in phase velocity, of the efficient wave packet, will still be under the influence of the same wave packet on longer times. And the effect of the 'fastest' harmonics of this wave packet will be progressively averaged out on these longer times, in a similar way as the effect of external harmonics has already fallen to 'zero' on the timescale τ_{NL}. Our reasoning of Sec. 4 can thus be extended for these particles to narrower wave packets, and the corresponding timescales. This can be applied to a non-negligible proportion of the particles (see Fig. 3). Hence the slightly longer time of validity of the 'quasifractal diffusion regime'.

5.2. Long-time diffusion regime

We shall now turn our attention to the long-time diffusion regime, connected with the nonlinear phase mixing in Sec. 4. For this, we will study the spreading of a set of particles on this long timescale, and give complementary arguments in favour of this long-time diffusion regime.

Figure 5 shows the velocity variance of the particles as a function of time, for a set of initial conditions such that $u(t = 0) = 0$ and $\xi(t = 0) \in [\pi; 2\pi]$. What strikes us at once is the irregularity of the curve, which persists no matter how high the number of particles in the initial conditions (tested for up to 10^4 particles). This confirms the non diffusivity of the particle dynamics on times shorter than many nonlinear times of resonant interaction (and longer than τ_{NL}), corresponding to the transient regime forecast in Sec. 4.

This irregularity progressively lessens for times increasing up to values large compared with τ_{NL}. Now, even if the curve extended with the same irregularity (as at the beginning), we could speak about diffusion on a very long timescale, since the slope of $\langle \Delta u^2 \rangle$ as a function of t would tend asymptotically to a fixed value, under the condition that no barriers such as KAM tori were encountered by the particles, i.e. that the wave spectrum were sufficiently wide.

Hence, from the separation of the particle trajectories and the nonlinear mixing of these trajectories in the phase space, which starts on the timescale τ_{NL} of nonlinear wave–particle interaction, and requires a few τ_{NL} to be efficient, we are tempted

https://doi.org/10.1017/S0022377898006795 Published online by Cambridge University Press

to predict 'the long-time diffusion regime', characterized by the linear behaviour of the velocity variance $\langle \Delta u^2 \rangle$ as a function of time (see Fig. 5).

There is a belief that the mixing of trajectories implies a diffusion. While this might be a valid argument, the many unanswered questions concerning this point do not allow us to affirm that the nonlinear phase mixing alone explains the observed long-time diffusion. So, we shall give strong complementary arguments in favor of this long-time diffusion regime.

As we have seen in Sec. 4, the difference between short- ($< \frac{1}{3}\tau_{NL}$) and long-timescale regimes is the result of a transition from a one-dimensional to a two-dimensional problem in the (ζ, w) coordinates. Whereas the normalized velocity w can be set equal to zero for short timescales, it becomes significant for longer times, which makes the nonlinearity of the motion equations play a crucial role and induce the nonlinear phase mixing. In Sec. 4, our efficient wave packets were always centred around the same harmonic (ω_0, k_0), and we stopped our reasoning at the timescale τ_{NL}, when some particle velocities become too large to be further determined by the same reduced wave packet. Now, it is clear that on longer timescales, when the particles spread out in velocity, the wave–particle interaction zones corresponding to the different particle positions in the phase space, identical for $t \ll \tau_{NL}$ (since all the particles have the same velocity), will progressively dissociate as the particles go away from their starting point at a distance larger than $N\delta\omega/k_0$. The particles subjected to the strongest accelerations or decelerations will be the first to learn the initial zone, and will rapidly be subjected, in their average motion, to the influence of harmonics 'outside' the initial zone, with phases independent of those of the harmonics acting in the initial period. As time goes by, the number of particles escaping their first resonance zone increases, since the particle distribution in velocity is a smoothing Gaussian around $u = 0$ if one admits a diffusion, which we shall not do but which, we believe, gives a correct first sight of reality. Of course, the particle dynamics can no longer be described as in Sec. 4 (that is, by the interaction with one given wave packet of width $3M\delta\omega$ on the timescale $\tau_{(M)}$). But we understand that for times very long compared with τ_{NL}, the dynamics of a particles set with initial velocity $u = 0$ becomes diffusive, since, in addition to the arguments concerning the nonlinear phase mixing that starts after $\frac{1}{3}\tau_{NL}$, these particles will be submitted to groups of harmonics with independent phases, and thus to noncorrelated forces.

This is a direct consequence of the localization of the wave–particle interaction that we demonstrated in Sec. 4. It may (as well as the nonlinear phase mixing) explain why the irregularity of the curve giving the variance of u as a function of time progressively lessens for increasing times.

Of course, a particle that is successively submitted to different independent wave packets can return in its initial wave–particle interaction zone; and we absolutely do not say here that the particle will have forgotten its stay in this initial zone. On the contrary, a study based on the standard map of Chirikov–Taylor and its relatives, presented by Ragot (1995), states that, however long the particle may have a diffusive-like dynamics, it will remember the phases of the waves that have interacted with it at any time of its history. (The measured spreading rate is a characteristic of the whole spectrum interacting with the particles.) But when a particle arrives in an interaction zone that it has already visited in the past, its new phase is decorrelated from the one it had when leaving this zone, in the sense

https://doi.org/10.1017/S0022377898006795 Published online by Cambridge University Press

that it is the result of forces that are completely independent of the force produced by this particular interaction zone.

Consequently, beyond the strict nonlinear phase mixing demonstrated in Sec. 4, we also invoke the localization of the wave–particle interaction, and the decorrelation it induces on timescales $\gg \tau_{NL}$, to explain the long-time diffusion regime corroborated by the curve on Fig. 5, and which we may call the true diffusion regime, in opposition to the first quasifractal one.

Now, we need to remark that the diffusion coefficient D_0, obtained when following a set of given particles with initial velocity v for a long time, is not precisely the $D(v)$ that we should like to introduce in a diffusion equation, for describing the evolution of the distribution function. It characterizes the spreading of a set of particles over a very long time (many times τ_{NL}), and thus in a very wide domain of the phase space (of very broad expanse compared with $\Delta_M\omega/k_0$), which does not allow us to attribute to D_0 the quality of a function of the velocity v any longer, unless the phase space is uniform, in which case $D(v)$ is a constant, and many interesting possibilities are excluded.

On the other hand, the precision in velocity of the diffusion coefficient $D(v)$ is intrinsically limited by $1/k_0\tau_{NL}$, that is the mean velocity width on which the particle trajectories are regular. Moreover, $D(v)$ has to describe the transport of all the particles inside this range of velocities, whatever the time they stay inside. The coefficient D_0, which does not take into account the contribution to the transport coefficient of a large number of particles whose velocity, even if initially very different, can approach v during the measurement period, is not the right candidate for a diffusion equation.

However, a definite conclusion of this study is that the diffusion coefficient $D(v)$ of the wave–particle interaction (with a broadband spectrum) is only meaningful for describing the particle transport on a time long compared with the time of resonant interaction τ_{NL}, in a domain of the phase space of width of order $\Delta v = 1/k_0\tau_{NL}$ in velocity.

In the next section, we shall show that there is no contradiction with the aforementioned papers, which proved the existence of nonquasilinear regimes, but did not notice the nondiffusivity on such long timescales.

6. Correlation with previous work

The system that we are dealing with is perfectly deterministic, since it is described by Hamiltonian equations. The average that we perform for studying the velocity variance on long timescales concerns the initial conditions of the particles, and especially their spatial position, whereas quasilinear theory uses statistics on the field's realizations.

The simulations that lead to Fig. 5 look similar to those performed by Cary *et al.* (1990, 1992), Pesme *et al.* (1993) and Helander and Kjellberg (1994); however, there is a crucial difference. The simulations by these other authors aim at testing the validity of quasilinear theory, and thus naturally deal with the problem of wave–particle interaction from a statistical point of view. Statistics are introduced by a sampling of the field: the phases ϕ_n of the field (3.1) are randomly chosen, and changed for each new initial condition, that is, for each particle, which amounts not only to averaging over initial conditions, but also to letting particles evolve with the same initial velocity and random initial phases (which does not play any essential

https://doi.org/10.1017/S0022377898006795 Published online by Cambridge University Press

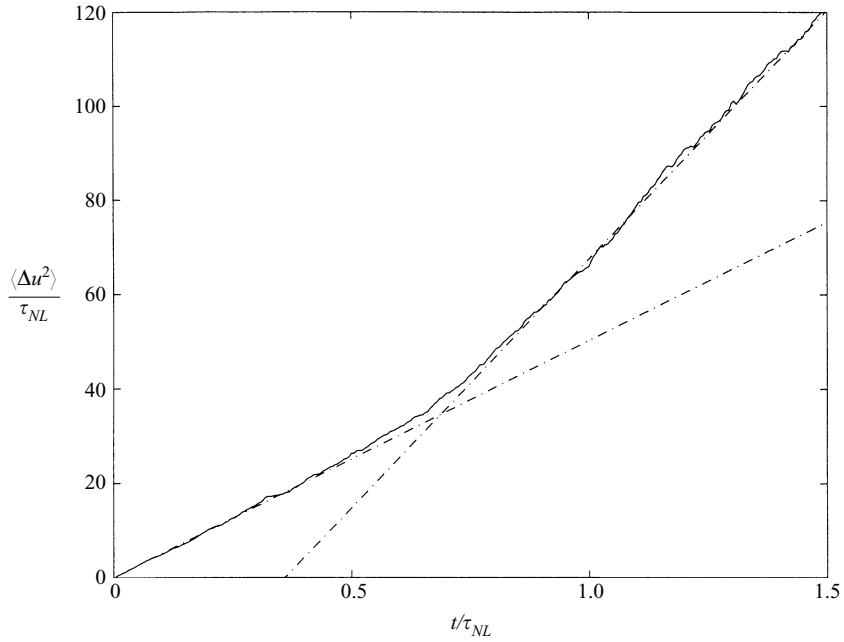


Figure 6. Velocity variance $\langle \Delta u^2 \rangle / \tau_{NL}$ for 800 particles, as a function of the nondimensional time t/τ_{NL}, when the phases of the field are randomly changed for each new initial condition taken at $u = 0$ and with random initial phases between 0 and 2π. For this simulation, $\delta\omega$ was chosen equal to 1, and A equal to 0.45, so that $4\pi^2 A/(\delta\omega)^2 \approx 18$, which corresponds to the overlap parameter giving the maximum for D/D_{QL}. The slopes of the straight lines are respectively D_{QL} and $2.1 D_{QL}$.

role in that special case) in physically distinct systems, producing at the same times completely uncorrelated forces. The particles evolve in different realizations of the field in accordance with the spirit of quasilinear theory. This is why the velocity variance, presented as a function of time in Fig. 3 of the paper by Cary *et al.* (1992) (and also here in Fig. 6), looks so regular on times longer than τ_{NL} (and shown up to $13\tau_{NL}$ by Cary *et al.* 1992). As the slope is found to be more than twice as large as the quasilinear one for relatively small overlap parameters, and also (or primarily) because this enhancement appears for times much longer than τ_{ac} (actually, of the order of $\tau_{(M_{QL})}$), after a quasilinear regime, this invalidates quasilinear theory for these overlap parameters. But these simulations are not intended to prove the diffusivity of particles interacting with a given wave field. In fact, we know from Sec. 4 that the mixing of trajectories cannot be efficient before a few τ_{NL}. And since we cannot appeal to any memory loss to justify a diffusion on these timescales, it is clear that the long-time diffusion cannot occur before at least few τ_{NL}, as confirmed by Fig. 5 here. The observed regularity of the curves by Cary *et al.* (1992) and in Fig. 6 is not connected to the 'diffusive dynamics' of the particles, but to the field statistics.

6.1. Resonance broadening, Lyapunov and nonlinear wave–particle interaction times

One important remark is in order now, concerning the nonlinear time τ_{NL} of wave–particle interaction and its relation to the time of resonance broadening introduced by Dupree (1966). The 'trapping time' or 'resonance broadening' time

https://doi.org/10.1017/S0022377898006795 Published online by Cambridge University Press

$\tau_{RB} = (\frac{1}{3}k_0^2 D)^{-1/3}$ is interpreted as the time required for a particle's position to diffuse over a wavelength k_0^{-1}, and thus the time for the particle's position to become randomized with respect to wave phases due to its random acceleration.

Now, if we replace D by its quasilinear expression in the definition of τ_{RB}, we realize that τ_{RB} scales as τ_{NL}, and even that the factor of proportionality is very close to 1. So, even if it would be senseless to connect the definition of τ_{NL} to any diffusion coefficient, since τ_{NL} is proved to be the timescale on which the particles' trajectories remain regular and do not separate, the definitions of these two quantitatively equivalent times should contain common features.

The resonance broadening theory of Dupree is a statistical theory, based on the supposition of a diffusion *ab initio* (at least for times $> \tau_{ac}$ in the case $\tau_{RB} \gg \tau_{ac}$). This theory, which we believe provides an incorrect picture of wave–particle interaction because of its underlying hypothesis of *ab initio* diffusion, does not encounter any internal contradiction and, as we have just seen, even predicts an essential timescale of the problem. This is due to the facts that there is actually a 'quasifractal' diffusion of the particles' trajectories on timescales shorter than τ_{NL}, with a 'diffusion coefficient' estimated to be of the order of the quasilinear one – i.e. the resulting average velocity variation on τ_{NL} is practically the same as the one that would result from an *ab initio* quasilinear diffusion – and that, as we can easily check, our time τ_{NL} satisfies a similar property, namely a difference $\Delta_{(N)}v$ between particles' velocities implies, after a time τ_{NL}, a spatial shift of the order of k_0^{-1}, i.e. a typical separation of a wavelength.

This timescale also appeared in the numerical work by Rechester *et al.* (1979*a*, *b*). The Lyapunov time of the dynamics of particles in a broadband wave spectrum, which measures the time of exponential separation of trajectories in the system, was found to be of the order of a fraction of τ_{NL}. It is difficult to guess from the papers by Rechester *et al.* (1979*a*, *b*) whether the simulations were performed in one given realization of the field, or in a statistical system, with the waves' phases changing from one particle to another. What we can rely on, hopefully, is the fact that the error in τ_{NL} cannot exceed a few percent, and can in no case give a factor of much more than unity. So, we think that Rechester *et al.* just gave the scaling law for the Lyapunov time, or that they dealt with a statistical system, in which case the value obtained would reflect the value of the average time in which the particle velocities undergo a significant nonlinear variation due to the resonant interaction with the wave, and would perfectly corroborate what we find here.

The nonlinearity taken into account by Dupree (1966), in the resonance broadening theory, did not provide for any deviation from quasilinearity in the case $\tau_{NL} \gg \tau_{ac}$. We still think, however, that deviations from quasilinearity may occur, which are connected to nonlinearity (as is most probably the case in Cary *et al.* 1990, 1992; Pesme *et al.* 1993; Helander and Kjellberg 1994), only that this nonlinearity should not be incorporated in the theory in a perturbative way. We shall now endeavour to check that our nonlinear and nonperturbative treatment of the dynamics in Sec. 4 gives the deviations from quasilinearity demonstrated by Cary *et al.* (1990, 1992), Pesme *et al.* (1993), Helander and Kjellberg (1994) and Ishihara *et al.* (1993), and precisely explains the behaviour of $\langle \Delta u^2 \rangle$ as a function of the time.

6.2. Supraquasilinear regime

The simulations by Cary *et al.* (1990, 1992), Pesme *et al.* (1993) and Helander and Kjellberg (1994) are in agreement with what we find. Indeed, the fact that two

https://doi.org/10.1017/S0022377898006795 Published online by Cambridge University Press

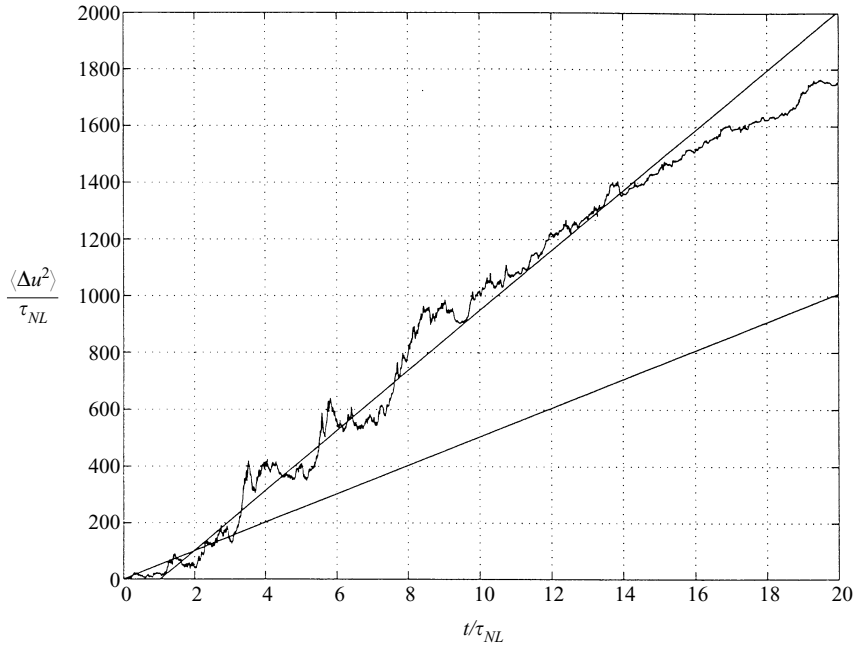


Figure 7. Velocity variance $\langle \Delta u^2 \rangle / \tau_{NL}$ for 800 particles, as a function of the nondimensional time t/τ_{NL}, in one realization of the field. Initial conditions and field parameters are chosen as in Fig. 6. The slopes of the straight lines are also D_{QL} and $2.1 D_{QL}$. For times longer than $10 \times \pi/\delta\omega$ (i.e., in this particular case, approximately $10\tau_{NL}$), the slope decreases again to the quasilinear value. The quasifractal regime (up to $\tau_{(M_{QL})} \approx \frac{1}{3}\tau_{NL}$) is hardly distinguishable because of the scales.

slopes are observed by Cary *et al.* (1992) for the velocity variance $\langle \Delta u^2 \rangle$ as a function of time is consistent with the existence of two 'diffusion' regimes of different natures demonstrated in Secs 4 and 5.

The effect that gives rise to the enhanced diffusion coefficient measured by Cary *et al.* (1990, 1992), Pesme *et al.* (1993) and Helander and Kjellberg (1994), and obtained here in a similar way (see Fig. 6), certainly also influences the transport in a single realization of the field. This influence is quite difficult to observe directly on the variance $\langle \Delta u^2 \rangle$ for times that are physically relevant with respect to the discrete description of the field. Indeed, the long-time diffusion regime is not reached, and the curve is extremely irregular, corresponding to the transient regime. But we can check in Fig. 7 that the average slope is of the order of twice that predicted by quasilinear theory. Further (and even already for shorter times), since $\pi/\delta\omega$ corresponds in that particular case to approximately τ_{NL}, the field representation cannot be considered as correct for describing a 1D Langmuir turbulence (cf. Sec. 3). However, if the field really is discrete, it is worth noticing that the slope again becomes quasilinear, which is consistent with the fact that quasilinear theory gives a time-asymptotic estimate of the diffusion coefficient.

Now, since we are able to estimate Δu analytically in one realization of the field, we should easily obtain its average value in the case of a sampling of the field.

We need first to remark that for the values of the overlap parameter giving a deviation from quasilinearity in the papers by Cary *et al.* (1990, 1992), Pesme *et al.* (1993) and Helander and Kjellberg (1994), the number N of effectively acting

https://doi.org/10.1017/S0022377898006795 Published online by Cambridge University Press

harmonics is of the order of 3 to 5, and that the theory developed here is not really suitable, since it supposes in fact that $N \gg 1$. More precisely, we have written the condition $N \geqslant M_{QL} \approx 10$ for the quasilinear scalings (4.4) to be valid. We shall now consider the limit $N = 3$ (cf. (4.22)), corresponding to the overlap parameter that gives, approximately, the maximum deviation from quasilinearity, in order to check that our solution (4.19), derived by Ragot and Shklyar (1995), contains this supraquasilinear effect.

In the simulations of Cary *et al.* (1990, 1992), Pesme *et al.* (1993) and Helander and Kjellberg (1994), the spectrum was chosen to have a uniform quasilinear diffusion coefficient and a uniform overlap parameter, which means that the choice of amplitudes and frequency shifts allows us to write the reduced field, composed of three harmonics, in the form

$$\begin{aligned}\tilde{E}(\xi, t) &= A[1 + e^{i(\delta\omega t + \phi_1)} + e^{-i(\delta\omega t - \phi_{-1})}]e^{i\xi} \\ &= A[1 + 2\cos(\delta\omega t + \alpha_1)e^{i\alpha_1'}]e^{i\xi}, \end{aligned} \tag{6.1}$$

where

$$\alpha_1 = \tfrac{1}{2}(\phi_1 - \phi_{-1}), \qquad \alpha_1' = \tfrac{1}{2}(\phi_1 + \phi_{-1})$$

(if we put the phase of the central harmonic into ξ). We can write it again as a function of its instantaneous amplitude and phase, at a fixed point in the frame of the central harmonic, as

$$\tilde{E}(\xi, t) = \mathscr{E}(t)e^{i[\xi + \varphi(t)]}, \tag{6.2a}$$

where

$$\mathscr{E}(t) = A^2[1 + 4\cos^2(\delta\omega t + \alpha_1) + 4\cos(\delta\omega t + \alpha_1)\cos\alpha_1'], \tag{6.2b}$$

$$\tan\varphi(t) = \frac{2\cos(\delta\omega t + \alpha_1)\sin\alpha_1'}{1 + 2\cos(\delta\omega t + \alpha_1)\cos\alpha_1'}\,. \tag{6.2c}$$

A study of variations of $\mathscr{E}(t)$ shows that the maxima are reached for $\delta\omega t + \alpha_1 = 0$ and π, and are respectively equal to $A(5 + 4\cos\alpha_1')^{1/2}$ and $A(5 - 4\cos\alpha_1')^{1/2}$. Moreover, the function $\mathscr{E}(t)$ goes through its minima, of value one, for $\delta\omega t + \alpha_1 = X_+$ and $2\pi + X_-$, with $X_\pm = \pm\arccos[-\frac{1}{2}(\cos\alpha_1')]$.

We now have to estimate the 'diffusion coefficient' (in the conditions under which it is observed, that is, with a sampling of the field). For this, we shall use the solution (4.19) of the nonlinear equations of motion and the particular characteristics of the amplitude bursts in the case $N = 3$. The first order of the analytical solution (4.19) of the equations of motion yields

$$\frac{\Delta v^2}{2\tau_{NL}} \sim \frac{\pi}{2}\left(\frac{e}{m}\right)^2 \frac{\mathscr{E}^2}{\mu\tau_{NL}} \exp\left(-\frac{\dot{\varphi}^2}{\mu}\right). \tag{6.3}$$

We can check, in the general case, where the scalings are†

$$\begin{aligned}\frac{1}{\mu\tau_{NL}} &\approx \frac{7}{(N\delta\omega)^2}\frac{N\delta\omega}{10} \approx \frac{7}{10N\delta\omega}, \\ \mathscr{E}^2 &\approx \tfrac{3}{2}NA^2, \end{aligned}$$

† We recall that the numerical coefficients are computed estimates (see Sec. 4.1).

https://doi.org/10.1017/S0022377898006795 Published online by Cambridge University Press

that the diffusion coefficient can be estimated in the first approximation as

$$\frac{\Delta v^2}{2\tau_{NL}} \approx \frac{\pi}{2}\left(\frac{e}{m}\right)^2 \frac{A^2}{\delta\omega}\frac{21}{20}\exp\left(-\frac{\dot{\varphi}^2}{\mu}\right) \approx \frac{\pi}{2}\left(\frac{e}{m}\right)^2 \frac{A^2}{\delta\omega}\,, \tag{6.4}$$

i.e. the quasilinear diffusion coefficient up to few percent. And this is the coefficient for the first 'diffusion regime', on timescales shorter than $\tau_{(M_{QL})}$, where more than M_{QL} harmonics have to be taken into account.

But in the limit case $N = 3$, these scalings are modified on timescales $> \tau_{(M_{QL})}$. More precisely, on the resolution timescale τ_{NL}, we have two types of burst, with respective amplitudes

$$A(5 + 4\cos\alpha_1')^{1/2} \quad \text{and} \quad A(5 - 4\cos\alpha_1')^{1/2},$$

respective characteristic variation rates

$$\frac{4(1 + 0.5\cos\alpha_1')\delta\omega^2}{(5 + 4\cos\alpha_1')} \quad \text{and} \quad \frac{4(1 - 0.5\cos\alpha_1')\delta\omega^2}{(5 - 4\cos\alpha_1')}$$

and respective lengths

$$\frac{2}{\delta\omega}\,\mathrm{Arccos}\,[-\tfrac{1}{2}(\cos\alpha_1')] \quad \text{and} \quad \frac{2\pi}{\delta\omega} - \frac{2}{\delta\omega}\,\mathrm{Arccos}\,[-\tfrac{1}{2}(\cos\alpha_1')].$$

Since for each particle new phases are randomly chosen for the field, the measured slope for the velocity variance corresponds to an average of $\mathscr{E}/\mu\tau$ as soon as the nonlinear variations of the particle velocities are significant (i.e. for times shorter than or of the order of τ_{NL}), and not only after a few τ_{NL}. As in this special case, one realization of the field is periodic ($\omega_{n+1} - \omega_n = \text{const}$) and can be represented by the two amplitude bursts with parameters as described above, the observed slope for $\langle\Delta v^2\rangle/(2\tau D_{QL})$ is

$$\begin{aligned}
\frac{\delta\omega}{A^2}\left\langle\frac{\mathscr{E}^2}{\mu\tau}\right\rangle_{\alpha_i} &= \frac{\delta\omega}{A^2}\frac{\delta\omega}{2\pi}\left\langle\frac{\mathscr{E}_1^2}{\mu_1} + \frac{\mathscr{E}_2^2}{\mu_2}\right\rangle_{\alpha_i} \\
&= \frac{1}{2\pi}\left\langle\frac{(5 + 4\cos\alpha_1')^2}{4 + 2\cos\alpha_1'} + \frac{(5 - 4\cos\alpha_1')^2}{4 - 2\cos\alpha_1'}\right\rangle_{\alpha_1'} \\
&= \frac{1}{\pi}\left\langle\frac{25 - 4\cos^2\alpha_1'}{4 - \cos^2\alpha_1'}\right\rangle_{\alpha_1'} \approx 2.1\,.
\end{aligned} \tag{6.5}$$

The value found in numerical simulations lies between 2 and 2.2 in the papers by Cary *et al.* (1990, 1992) and Helander and Kjellberg (1994), and at about 2.1 here in Fig. 6 for $N = 3$.

As already mentioned, for times longer than τ_{NL}, the particles can go out of their initial resonance zone. And the reasoning above holds only for times of the order of τ_{NL} such that most of the particles are still inside the initial resonance zone. Consequently, if this permits us to find the right value for the slope of $\langle\Delta v^2\rangle/2D_{QL}$ as function of the time, given by the numerical simulations, it certainly does not allow us to argue that for times $t \gg \tau_{NL}$, the slope should keep the same value. In fact, simulations by Ishihara *et al.* (1993) revealed that the slope again becomes quasilinear for very long times. This result is confirmed here in Fig. 7.

A derivation presented by Ragot (1995) also led to a similar conclusion. There, the diffusion coefficient is expressed as a function of the growth rate in the form of

https://doi.org/10.1017/S0022377898006795 Published online by Cambridge University Press

an integral relation, where the wavenumber interval over which the usual quasilinear formula should be averaged clearly appears. If the correlations of the modes (see Ragot and Shklyar 1995; Ragot 1995; and Sec. 2.2) average to zero over this interval, this integral relation shows that in the limit of very long timescales ($\gg \pi/\delta\omega$), the diffusion coefficient again takes its quasilinear value. And this is not surprising, since the quasilinear diffusion coefficient, as initially derived, is a time-asymptotic estimate, and thus should be time-asymptotically correct.

Furthermore, for times very much shorter than τ_{NL}, the field cannot be reduced to three harmonics, since the effect of other harmonics is not averaged out yet. And for these short times, the value of the slope has to be 1, since the scalings taking into account more harmonics ($\geqslant M_{QL} \approx 10$) give the quasilinear value for $\langle \Delta v^2 \rangle / 2\tau$ (cf. (6.4)). We are thus satisfied to see that the curve starts with the slope 1. Then, by continuity, the slope has to take all values between 1 and 2.1.

As we have already noticed in Sec. 2, the discrete description of Langmuir turbulence is not valid for times of order $\pi/\delta\omega$, when one follows a given set of particles on the whole duration of the measure. Now, the supraquasilinear regime is observed for times of that order. This does not mean that a discrete description is not reasonable for the level of turbulence considered in the papers by Cary *et al.* (1990, 1992), Pesme *et al.* (1993), Ishihara *et al.* (1993) and Helander and Kjellberg (1994), but that the field should be represented by more modes, in order that the description remain valid for longer times. Then the overlap parameter (in $A/(\delta\omega)^2$) is much higher for a given spectral density of energy, W_ω (in $A^2/\delta\omega$), and the long-time diffusion coefficient may again be quasilinear, but the phases may also become correlated, in which case nonquasilinear behaviours extend to higher overlap parameters, as demonstrated by Pesme *et al.* (1993).

6.3. Nonquasilinear regime in the case of correlated phases

It is usually taken that, as the size L of the medium tends to $+\infty$, the amplitude $\mathscr{A}$ of the harmonics, written as a function of $\mathbf{k}$, need not become a continuous function of $\mathbf{k}$ (cf. Dupree 1966; Cary *et al.* 1990, 1992) – and for this reason that it would be more convenient to use a Fourier series than a Fourier integral. We do not agree with this point of view that the most realistic description of turbulence would correspond to an overlap parameter ($\propto A/(\delta\omega)^2$) tending to $+\infty$ (understood with random harmonic phases), which implies a zero spectral correlation length of the field phase.

The problem lies first in the definition of an overlap parameter for a nonperiodic field. Knowing its spectral density of energy and the total width of its spectrum is obviously not enough – hence the temptation to take the limit of an infinite overlap parameter with a fixed energy level. However, as we shall see in the Appendix, the spectral correlation length of the field phase has a finite value, due at least to the collisions that cannot be completely neglected on very long timescales, but perhaps also to nonlinear spectral structuring of the wave (which is beyond the scope of this paper) – hence the interest in the problem with correlated phases.

The case of phases correlated on a finite-frequency length has been studied numerically by Pesme *et al.* (1992). When the phases are linearly correlated on 10 successive modes (by dividing the previous frequency intervals), the computed diffusion coefficient remains of the order of $\frac{1}{2}D_{QL}$ for relatively high overlap parameters (tested up to $25 \times 10^{3/2}$ by Pesme *et al.* 1993). However, for smaller $\delta\omega$ (divided by 10 in this special case), one needs, in order to describe turbulence of the same

https://doi.org/10.1017/S0022377898006795 Published online by Cambridge University Press

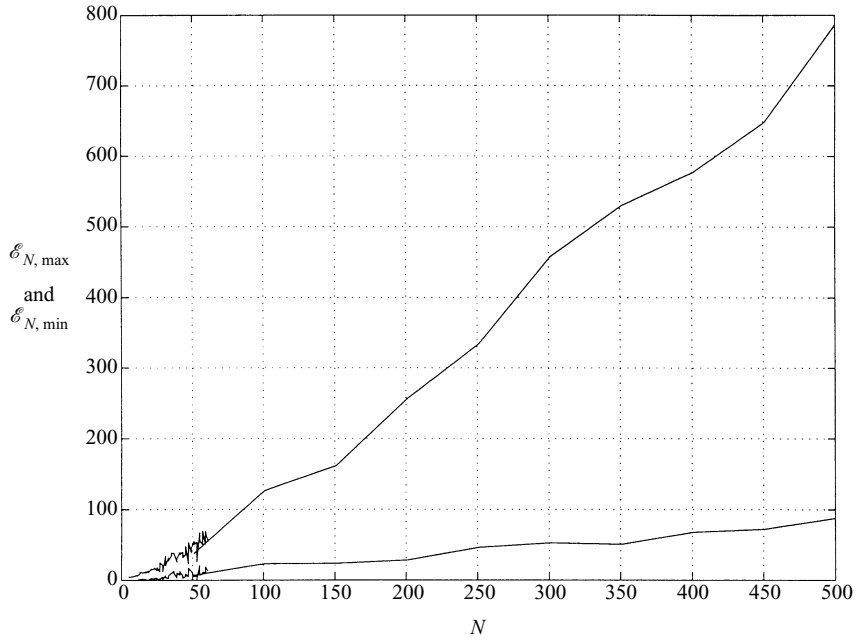


Figure 8. Average values of the maxima and minima of the (N) efficient field instantaneous amplitude as functions of N. For $N \gg 50$, these variation laws yield the quasilinear value for the diffusion coefficient. For $N < 50$, the scaling laws are modified, and produce a 2 times slower diffusion coefficient. Hence the subquasilinear diffusion observed by Pesme *et al.* (1993).

level of energy, to consider overlap parameters much higher ($\times 10^{3/2}$) than in the case without phase correlation. Thus the convergence to quasilinearity observed in the case of uncorrelated phases is not yet reached in the paper by Pesme *et al.* (1993) for the considered values of the overlap parameter, since overlap parameters need to be larger than 100 for the deviation from quasilinearity to become insignificant in the case without phase correlation.

The deviation from quasilinearity comes from scalings that are different from those obtained with random phases (and sufficiently high amplitudes). Figure 8 shows the average values for the maxima and minima of the efficient field instantaneous amplitude as functions of N. The regime observed by Pesme *et al.* (1993) corresponds to the beginnings of the curves. For higher energies, the generic scalings (4.5) giving the quasilinear value for the diffusion coefficient are found again.

7. Conclusions

In order to understand the nonquasilinear behaviours observed numerically by Cary *et al.* (1990, 1992), Pesme *et al.* (1993), Ishihara *et al.* (1993) and Helander and Kjellberg (1994), but primarily how diffusion may appear in a completely deterministic system, we studied the nonlinear dynamics of resonant particles in a broadband wave spectrum, in a nonlinear and nonperturbative way. This study proves the localization of the wave–particle interaction, and precisely determines, on each timescale, the efficient spectral width of the field. It establishes the tran-

https://doi.org/10.1017/S0022377898006795 Published online by Cambridge University Press

sition from a regular quasifractal behaviour of the particle dynamics to a chaotic behaviour with mixing and spreading, which justifies a diffusion in velocity on the long timescale.

From this study of the nonlinear dynamics, we found the timescale τ_{NL} on which the nonlinear phenomena are important. This varies as the spectral density of energy to the power $-\frac{1}{3}$, and is the longest timescale on which the particle dynamics are globally regular, and the trajectories do not mix. If it appears that τ_{NL} is in fact of the order of the time of resonance broadening $\tau_{RB} \equiv (\frac{1}{3}k_0^2 D_{QL})^{-1/3}$ introduced by Dupree (1966), we cannot assimilate them, owing to the completely different approaches that have led to their introduction. The theory of Dupree is a statistical theory, which supposes an *ab initio* diffusion, whereas in our dynamical (*nonstatistical*) theory, we show that the particle dynamics remain correlated on the timescale τ_{NL}, which prevents us from connecting the definition of τ_{NL} to any diffusion coefficient.

Our theory reproduces the deviations from quasilinearity of the long-time diffusion coefficient at low spectral density of energy. These deviations, at a given low energy, are drastically influenced by the discret representation of the field (which determines the value of the overlap parameter), and especially the correlation of its harmonic phases. Actually, an instructive piece of work would consist in the systematic study of the behaviour of D/D_{QL} at fixed spectral density of energy as a function of the spectral correlation length of the phases, or in the direct measurement of this length for typical spectra. Indeed, without this essential data, we cannot conclude definitively about the physical relevance of the deviations from quasilinearity obtained for the diffusion coefficient.

Nevertheless, we have proved that these deviations are due to the nonlinearity of the dynamics, and appear at the timescale $\tau_{(M_{QL})}$, where the generic quasilinear scalings break down. They should remain on a timescale of the order of few τ_{NL}. But for *very long* timescales ($\gg \pi/\delta\omega$), the quasilinear diffusion coefficient, which is a time-asymptotic estimate, should be found again (as demonstrated numerically here in Fig. 7 and by Ishihara *et al.* 1993).

To summarize, the correlation time of the particle dynamics in a broadband high-frequency wave spectrum is long compared with the autocorrelation time τ_{ac} of the field seen by the particles: it is of the order of τ_{NL}. This permits one to understand how the slope of the velocity variance as a function of time may change, for some overlap parameter, after times already considered as very long compared with the time of the 'diffusion' by quasilinear theory. The reason is that on time scales between τ_{ac} and τ_{NL}, the particle trajectories are globally regular, and do not separate. The linear behaviour of $\langle \Delta v^2(t) \rangle$ as a function of time when statistics are performed on the fields is merely an effect of these statistics, which make the particles evolve in uncorrelated fields already after τ_{ac}. Without field statistics, the increase of $\overline{\Delta v^2(t)}$ only appears linear for these timescales when it is observed on a longer timescale (initial quasilinear slope). This arises from the self-similarity of the dynamics – particles follow a 'quasifractal' set of Hausdorff dimension 2. This regime of 'ordered' quasifractal diffusion is connected to the *extrinsic* stochasticity of the field, which acts as a thermal bath. Beyond τ_{NL}, our approach shows how mixing comes about in the phase space, and decorrelation occurs, owing to the localization of the wave–particle interaction. It thus justifies the diffusion as it is traditionally understood, but for times long compared with τ_{NL} ($\sim 10\times$), which in turn is very long compared with τ_{ac} (typically a few hundreds of τ_{ac} in

https://doi.org/10.1017/S0022377898006795 Published online by Cambridge University Press

the solar wind). This long-time diffusion is due both to the *intrinsic* stochasticity of the dynamics, which implies the nonlinear mixing in the phase space, and to *induced extrinsic* stochasticity, resulting from the localization of the wave–particle interaction.

Acknowledgements

This work started in collaboration with D. R. Shklyar (see Ragot and Shklyar 1995), whom I wish to thank here. It was performed while I was Elève Normalienne, and thus was supported by the French Ministère de la Recherche et de l'Enseignement Supérieur. I am very grateful to P. Duffy, J. Kirk and R. Schlickeiser for helping to improve the readability of the manuscript.

Appendix. Choice of the discretization parameters

The spectral density of energy and the total width of the spectrum constitute the essential parameters that permit us to describe the wave field. However, in order to represent the field in a discrete form, we are led to introduce two parameters instead of the spectral density of energy, W_ω, namely, the average amplitude A of the new representation harmonics and $\overline{\delta\omega}$, the average distance in frequency between nearby harmonics. These two parameters are connected by the relation $W_\omega = A^2/\delta\omega$. But this is not enough to determine them uniquely.

Of course, the discrete representation is only a numerical auxiliary. Note that the particular values of A and $\delta\omega$ do not appear in any dimensional parameter of the problem. Nevertheless, the value of N depends strongly on A and $\delta\omega$, and it is important to check that, for typical experimental values, it is actually large compared with 1, which constitutes the essential condition for the reasoning given in this paper.

We shall thus think now about the reasonable value we may attribute to $\delta\omega$ for the field to be well described on the one hand, and the harmonics to have uncorrelated phases on the other hand. With regard to the first point, it is clear that the narrower the harmonics, the better the description (see Sec. 2). We still have to search for the lower limit for $\delta\omega$, and check whether the harmonics with independent phases can be narrow enough for N to be large compared with 1.

Let us consider a particular harmonic (ω_0, k_0), and place ourselves in the frame moving at speed ω_0/k_0. For other harmonics of the discrete spectrum to be reasonably considered as independent, the electron density modulations due to these harmonics must not have time to be modified by the existence of the first one. The interaction between harmonics that we are talking about here is a purely microscopic interaction, where only collisions between particles play a role. This interaction is optimal for nearby harmonics, since their relative evolution is the slowest.

In order that collisions do not have time to be effective, the density variation due to the narrower harmonic should be, locally, faster than the collisional relaxation. 'Locally' here, means, in a fixed point in the frame of the 'main' harmonic (as opposed to its neighbour, for which we endeavour to know at which minimum distance in frequency we shall be allowed to consider it as independent from the first one, or rather, below which distance we surely cannot consider it as independent). If we denote by τ_c the collisional relaxation time, this condition can be written as

$$\tau_c \delta\omega \gg 2\pi.$$

https://doi.org/10.1017/S0022377898006795 Published online by Cambridge University Press

In a plasma, the deflection of a charged particle essentially results from multiple small-angle collisions, rather than from large-angle collisions that are fewer (cf. Nicholson 1983). Indeed, the relaxation due to small-angle collisions occurs on a timescale of the order of $2\log\Lambda$ times shorter than the average time between two collisions with a large deflection, where Λ is the large plasma parameter, equal to $n_0\lambda_{De}^3$, and the typical value for $2\log\Lambda$ lies between 10 and 100. Hence the collisional relaxation time

$$\tau_c = \frac{m^2 v_0^3}{8\pi n_b e^4 \log\Lambda} ,$$

where v_0 is the average relative speed of the resonant particles that collide (and is thus connected to the beam temperature) and n_b is the beam density.

To determine precisely the order of magnitude of $\delta\omega$, and of the harmonic numbers N and $\mathscr{N}$ that we may expect to be able to take for describing the particle dynamics and the complete spectrum of the Langmuir wave field, we shall estimate the limits imposed by $\tau_c\delta\omega \gg 2\pi$ on typical parameter values for the solar wind.

The collisional relaxation frequency reads

$$\nu_c = \frac{8\pi n_b e^4 \ln\Lambda}{m^2 \Delta v_b^3} .$$

The beam density will be taken to be of the order of $1/100$ of the ambient plasma density, say of order $10^{-1}\,\mathrm{cm}^{-3}$, and the beam thermal velocity to be around $5\times 10^8\,\mathrm{cm\,s}^{-1}$. In CGS units, we also have $m\approx 9.1\times10^{-28}\,\mathrm{g}$ and $e\approx 4.8\times10^{-10}\,\mathrm{esu}$. Hence $\nu_c\approx 1.3\times10^{-9}\ln\Lambda\,\mathrm{Hz}\approx 4.6\times10^{-8}\,\mathrm{Hz}\approx 5\times10^{-8}\,\mathrm{Hz}$, $\ln\Lambda\sim\ln(n_f\lambda_{De_f}^3)\approx \ln(27\times10^{14})\approx 35$ and $\delta\omega\gg 3\times10^{-7}\,\mathrm{Hz}$. If we take $\delta\omega$ of order $10^{-5}\,\mathrm{Hz}$, $\mathscr{N} = D\omega/\delta\omega\approx k_0\Delta v_f/\delta\omega\approx 2\times10^{-5}\times5\times10^8/10^{-5}\approx 10^9$. Moreover, $E\approx\frac{3}{2}\mathscr{N}^{1/2}A\approx 3\times10^{-5}\,\mathrm{V\,m}^{-1}\approx 10^{-9}$ statvolt cm^{-1}. Thus $A\approx 2\times10^{-14}\,\mathrm{statvolt\,cm}^{-1}$. And

$$N\approx\left[\frac{15ek_0A}{m(\delta\omega)^2}\right]^{2/3}$$
$$\approx\left[\frac{15\times4.8\times10^{-10}\times2\times10^{-5}\times2.10^{-14}}{9.1\times10^{-28}\times(10^{-5})^2}\right]^{2/3}\approx 10^7 ,$$

which would give us a very wide margin for the choice of $\delta\omega$. Note that $\tau_{NL}>150\tau_{ac}$ in this example.

For the aurora, since the density is 10^5 times higher, we should have typically found $\delta\omega\gg 10^{-1}-10\,\mathrm{Hz}$ and $N\approx 10^3-10^5$.

References

Adam, J. C., Laval, G. and Pesme, D. 1979 *Phys. Rev. Lett.* **43**, 1671.
Bergé, P., Pomeau, Y. and Vidal, Ch. 1992 *L'ordre dans le chaos*. Hermann, Paris.
Cary, J. R., Doxas, I., Escande, D. F. and Verga, A. D. 1992 *Phys. Fluids* **B4**, 2062.
Cary, J. R., Escande, D. F. and Verga, A. D. 1990 *Phys. Rev. Lett.* **65**, 3132.
Chirikov, B. V. 1979 *Phys. Rep.* **52**, 265.
Davidson, R. C. 1972 *Methods in Nonlinear Plasma Theory*. Academic Press, New York.
Delcroix, J.-L. and Bers, A. 1994 *Physique des Plasmas*. InterEditions/CNRS Editions, Paris.
Doxas, I. and Cary J. R. 1997 *Phys. Plasmas* **4**, 2508.
Drummond, W. E. and Pines, D. 1962 *Nucl. Fusion Suppl.* **3**, 1049.
Dupree, T. H. 1966 *Phys. Fluids* **B9**, 1773.

https://doi.org/10.1017/S0022377898006795 Published online by Cambridge University Press

Helander, P. and Kjellberg, L. 1994 *Phys. Plasmas* **1**, 210.
Ishihara, O., Xia, H. and Watanabe, S. 1993 *Phys. Fluids* **B5**, 2786.
Laval, G. and Pesme, D. 1984 *Phys. Rev. Lett.* **53**, 270.
Liang, Y.-M. and Diamond, P. H. 1993 *Phys. Fluids* **B5**, 4333.
Lichtenberg, A. J. and Lieberman, M. A. 1992 *Regular and Chaotic Dynamics*. Springer-Verlag, New York.
Mynick, H. E. and Kaufman, A. N. 1978 *Phys. Fluids* **B21**, 653.
Nicholson, D. R. 1983 *Introduction to Plasma Theory*. Wiley, New York.
O'Neil, T. M., Winfrey, J. H. and Malmberg, J. H. 1974 *Phys. Fluids* **B14**, 1204.
Pesme, D., Mora, P., Verga, A. and Laval, G. 1993 *Spatio-Temporal Analysis for Resolving Plasma Turbulence*, (ed. R. Schmidt). Aussois.
Ragot, B. R. 1995 Dynamique non linéaire de l'interaction onde–particule dans le cas de spectres larges: transport et instabilité. Thesis, Université de Provence.
Ragot, B. R. and Shklyar, D. R. 1995 *Phys. Plasmas* **2**, 63.
Rechester, A. B. and White, R. B. 1980 *Phys. Rev. Lett.* **44**, 1586.
Rechester, A. B, Rosenbluth, M. N. and White, R. B. 1979*a* *Phys. Rev. Lett.* **42**, 1247.
Rechester, A. B., White, R. B. and Rosenbluth, M. N. 1979*b* *Intrinsic Stochasticity in Plasmas*, p. 239. Les Editions de Physique, Orsay.
Ruskai, M. B., Beylkin, G., Coifman, R., Daubechies, I., Mallat, S., Meyer, Y. and Raphael, L. 1992 *Wavelets and Their Applications*. Jones and Bartlett, Boston.
Sagdeev, R. Z. and Galeev, A. A. 1969 *Nonlinear Plasma Theory*. Benjamin, New York.
Tsunoda, S. I., Doveil, F. and Malmberg, J. H. 1987 *Phys. Rev. Lett.* **58**, 1112.
Tsunoda, S. I., Doveil, F. and Malmberg, J. H. 1991 *Phys. Fluids* **B3**, 2747.
Vedenov, A. A, Velikhov, E. P. and Sagdeev, R. Z. 1962 *Nucl. Fusion Suppl.* **2**, 465.

https://doi.org/10.1017/S0022377898006795 Published online by Cambridge University Press