$f(x) = x^2$ the sequence is defined by the Root-Mean Square of the two preceding terms and $\lim_{n\to\infty} a_n = \sqrt{\frac{a_1^2 + 2a_2^2}{3}}$; if $f(x) = x^m$ the sequence is defined by the Power Mean of the two preceding terms and $\lim_{n\to\infty} a_n = \sqrt[m]{\frac{a_1^m + 2a_2^m}{3}}$; finally, if $f(x) = \frac{1}{x}$ the sequence is defined by the Harmonic Mean of the two preceding terms and $\lim_{n\to\infty} a_n = \frac{3}{\frac{1}{a_1} + \frac{2}{a_2}}$.

Reference

1. David Borwein, Jonathan M. Borwein, Brailey Sims, On the solution of linear mean recurrences, *Amer. Math. Month.* **121** (6) (2014) pp. 486-498.

doi:10.1017/mag.2016.21

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100.13 The Steinhaus 'four-colour' tiling

This Note extends no. 237 of Hugo Steinhaus' *Mathematical Snapshots* [1]. Each face of a regular tetrahedron receives a different coloured paint and the tetrahedron rolls about the plane, i.e. it changes position on the plane by rotating about edges. The result is Figure 1. In current terminology the tetrahedron is an *orbifold* for the pattern. Readers interested in the tiling will find information not immediately relevant to this Note at [2].

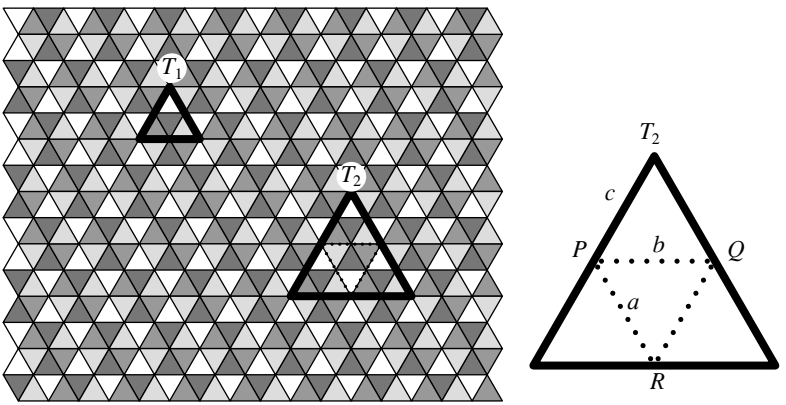


FIGURE 1

By virtue of the rolling, any triangle such as T_1 is the net of the tetrahedron. (The tetrahedron is chiral. The effect on the tiling of switching from one form to the other would be that, having fixed the positions of one pair of colours, the other two would be reversed.)

https://doi.org/10.1017/mag.2016.22 Published online by Cambridge University Press

The point of this Note is to draw attention to the following:

We can make a bigger tetrahedron, using T_2 as the net, and by rolling that, generate the same tiling. The same applies by whatever factor k we multiply the edge-length to obtain a bigger tetrahedron T_k.

The following proof depends only on the property of a periodic tiling that a given symmetry element operates on all scales. In our case that element is the half-turn symmetry, of which every vertex is a centre.

Imagine we have assembled the bigger tetrahedron so that it sits within the dotted outline. We fix one vertex at P. Looking towards P, if we roll the tetrahedron anticlockwise about edge a, we know we shall achieve a matching patch because that is the way we built the model. Likewise if we roll it clockwise about edge b. But we also know, by virtue of the half-turn symmetry of the tiling, that, if we roll it clockwise about c, that is three times from the bottom left position, thus bringing into play once more the face which produced that print, we shall also find a match. We can continue, duplicating vertically opposite triangles till we have printed a complete regular hexagon of edge-length 2. We can repeat for vertices Q and R, producing the (irregular) hexagon shown by the inner dashed line in Figure 2.

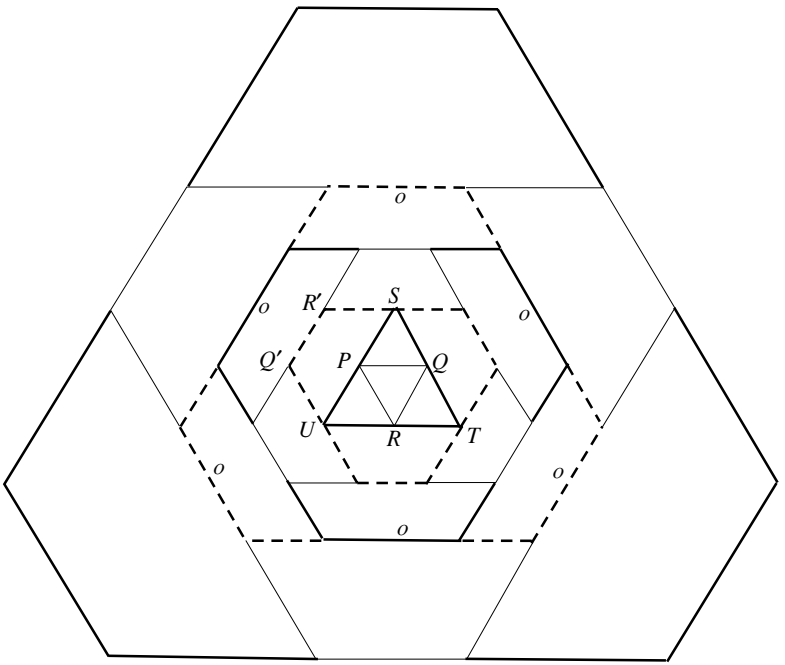


FIGURE 2: The repeated sequence:

- - - - - - - - - - - - - - - —————————— ——————————

shows how the initial region grows by the addition of half-hexagons (overlapping where marked by an 'o')

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We shall call the regular trapezium which results from halving a regular hexagon a 'half-hexagon'. By virtue of the half-turn symmetry of the tiling any half-hexagon centred on the edge of, and lying within, the region already claimed, implies a matching half-hexagon outside. Thus $SQRU$ implies $UQ'R'S$. We proceed outwards, always claiming the biggest possible external half-hexagons. The figure shows how additions are made to alternate sides of the hexagonal region with each iteration. This guarantees that the whole plane can be covered. At no step in the argument have we specified k. So, by construction, we have confirmed the general proposition.

References

1. H. Steinhaus, *Mathematical Snapshots*, (3rd American edition), Oxford (1983).
2. www.magicmathworks.org/steinhaustiling

doi:10.1017/mag.2016.22

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100.14 Optimum choice of interpolation points in Newton's divided difference formula

The widely-used Newton's divided difference interpolation formula is given by

$$f(x) = (x - x_0)[01] + (x - x_0)(x - x_1)[012] + \dots$$
$$\dots + (x - x_0)(x - x_1)\dots(x - x_{n-1})[012\dots n] + R_n, \tag{1}$$

where the truncation error R_n is given by

$$\frac{(x - x_0)(x - x_1)\dots(x - x_n)}{(n + 1)!} f^{(n+1)}(\xi) \tag{2}$$

with

$$x_0 < \xi < x_n \tag{3}$$

and the $[012\dots n]$ are divided differences, defined recursively (see [1]). (Note that the above applies only if the points x_i are arranged in increasing order.)

Let us define

$$\prod_n = (x - x_0)(x - x_1)\dots(x - x_n). \tag{4}$$

Then, if we have available a table of known values of f_i at X_i where

$$i = -k, -k + 1, \dots, \ell - 1, \ell \tag{5}$$

and wish to approximate $f(x)$ at a given value of x, then we have a wide choice of points $x_0, x_1, \dots, x_n$ from among the tabular points $X_{-k}, X_{-k+1}, \dots, X_\ell$. These points $x_0, x_1, \dots, x_n$ need not, and usually should not, be consecutive.

https://doi.org/10.1017/mag.2016.22 Published online by Cambridge University Press