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Global attractivity in the delay logistic equation with variable parameters

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Abstract

Using a non-linear variation of constants formula, sufficient conditions are derived for the global asymptotic stability of the trivial solution of

$$\frac{dx(t)}{dt} = -r(t)\,(1+x(t))\,x(t-\tau(t)).$$

1. *Introduction*

The non-linear scalar autonomous delay differential equation with a single delay

$$\frac{dy(s)}{ds} = -\alpha y(s-1)\,(1+y(s)) \quad (s>0) \tag{1·1}$$

where $\alpha \in (0,\infty)$ has been studied by several authors and notably by Wright [**10**], Jones [**6**] and Kakutani and Markus [**7**]. One can show that (1·1) arises from a change of variables in

$$\frac{du(t)}{dt} = ru(t)\left(1-\frac{u(t-\tau)}{K}\right), \quad (r,K,\tau \in (0,\infty). \tag{1·2}$$

The autonomous delay logistic equation (1·1) has been of some interest in mathematical ecology especially to describe certain observed oscillations of laboratory populations where the parameter K is related to the carrying capacity of the environment. If K is t-dependent there is no positive equilibrium solution of (1·2) and hence there is no *a priori* reason for positive solutions of (1·2) to converge as $t\to\infty$ to constants. Several characteristics of equations such as (1·2) with variable K have been studied recently by Zhang and Gopalsamy [**11**, **12**]. The primary purpose of this article is to assume that K is a positive constant while r and τ are t-dependent in (1·2) and examine solutions of (1·2) with respect to their convergence to K as $t\to\infty$. Precisely we are concerned with the asymptotic behaviour of equations of the form

$$\frac{du(t)}{dt} = r(t)\,u(t)\left(1-\frac{u(t-\tau(t))}{K}\right). \tag{1·3}$$

If we let $$u(t) = K(1+x(t))$$

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in (1·3), we are then led to

$$\frac{dx(t)}{dt} = -r(t)\,(1+x(t))\,x(t-\tau(t)). \tag{1·4}$$

Due to the relation between (1·3) and (1·4), we are interested in only those solutions of (1·4) corresponding to initial conditions ϕ of the form

$$1+\phi(s) \geqslant 0, \quad 1+\phi(0) > 0, \quad s \in [-\sup_{u>0} \tau(u), 0], \tag{1·5}$$

where ϕ is assumed to be piecewise continuous on its domain of definition.

We shall also consider a formal generalization of (1·4) in the following form

$$\frac{dx(t)}{dt} = -r(t)\,(1+x(t))\,f(x(t-\tau(t))), \tag{1·6}$$

where f is not necessarily a linear function.

One can show by the method of steps (see El'sgol'ts and Norkin [3]) that solutions of (1·4) and (1·6) corresponding to initial conditions (1·5) are defined for all $t > 0$ when r and τ are bounded positive functions and when f in (1·6) satisfies appropriate assumptions (as in Section 3 below). It is not difficult to show that solutions of (1·4) and (1·6) satisfy $1+x(t) > 0$ for $t > 0$ under (1·5).

2. *Exponential convergence*

We shall first derive a set of sufficient conditions for the trivial solution of the autonomous equation

$$\frac{dx(t)}{dt} = -r_0(1+x(t))\,x(t-\tau_0), \quad r_0, \tau_0 \in (0,\infty) \tag{2·1}$$

to be globally exponentially asymptotically stable; that is, if x is any solution of (2·1) corresponding to a given initial function $\phi : [-\tau, 0] \to \mathbb{R}$, where $1+\phi(0) > 0$, then there exist positive numbers T, M, δ such that

$$|x(t)| \leqslant Me^{-\delta t} \quad \text{for } t \geqslant T \tag{2·2}$$

where M, T, δ can depend on ϕ, r_0, τ_0.

We recall that a solution x of (2·1) is said to be oscillatory if there exists a sequence $\{t_p\}$ such that $x(t_p) = 0$ for $p = 1, 2, \ldots$ and $t_p \to \infty$ as $p \to \infty$; a solution x is said to be non-oscillatory if there exists a $t_0 \geqslant 0$ such that $|x(t)| > 0$ for $t \geqslant t_0$.

LEMMA 2·1. *If u is a non-oscillatory solution of* (2·1) *then*

$$\lim_{t\to\infty} u(t) = 0. \tag{2·3}$$

Proof. Suppose u is eventually positive; it follows that $\dot{u}(t) < 0$ for all large t implying that

$$u(t) \to l \geqslant 0 \quad \text{as } t \to \infty. \tag{2·4}$$

Since u is bounded, it follows from (2·1) that $\dot{u}(t)$ is bounded for $t \geqslant 0$ and hence (by Barbalat's lemma [1])

$$\dot{u}(t) \to 0 \quad \text{as } t \to \infty. \tag{2·5}$$

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Thus we have from (2·1), (2·4) and (2·5) that

$$0 = -r_0(1+l)\,l. \tag{2·6}$$

Since $l \geqslant 0$ it follows that $l = 0$. Suppose now that u is eventually negative. Then $\dot{u}(t) > 0$ for all large t and therefore $u(t) \to m \leqslant 0$, as $t \to \infty$.

Similarly

$$0 = -r_0(1+m)\,m \quad (m \leqslant 0). \tag{2·7}$$

Since $1+m \geqslant 0$ we have to consider only the case $m = -1$; if $m = -1$ then for all large t

$$\dot{u}(t) = -r_0(1+u(t))\,u(t-\tau_0) < -r_0(1-\tfrac{1}{2})\tfrac{1}{2},$$

which implies that

$$u(t) < u(t^*) - r_0\tfrac{1}{4}(t-t^*), \tag{2·8}$$

leading to $u(t) \to -\infty$ as $t \to \infty$, and this is impossible since $1+u(t) > 0$. Thus we have from (2·7) that $m = 0$. This completes the proof.

LEMMA 2·2. *If u is an arbitrary solution of* (2·1), *then there exists a number $T_0 > 0$ such that*

$$\exp(-r_0\tau_0(e^{r_0\tau_0}-1)) \leqslant 1+u(t) \leqslant e^{r_0\tau_0} \quad \textit{for } t \geqslant T_0. \tag{2·9}$$

Proof. Suppose that u is an oscillatory solution and let $\{t_p\}$ be a sequence of zeros of u such that $t_p \to \infty$ as $p \to \infty$. Let t^* be a local maximum of u; we have from (2·1),

$$0 \leqslant \dot{u}(t^*) = -r_0(1+u(t^*))\,u(t^*-\tau_0)$$

and therefore $u(t^*-\tau_0) \leqslant 0$; as a consequence of this there exists $\xi^* \in [t^*-\tau_0, t^*)$ such that $u(\xi^*) = 0$. An integration of

$$\dot{u}(t) = -r_0(1+u(t))\,u(t-\tau_0) \tag{2·10}$$

over $[\xi^*, t^*]$ leads to

$$\log\left(\frac{1+u(t^*)}{1+u(\xi^*)}\right) = -r_0\int_{\xi^*}^{t^*} u(s-\tau_0)\,ds \leqslant r_0(t^*-\xi^*) \leqslant r_0\tau_0 \tag{2·11}$$

and hence

$$1+u(t^*) \leqslant e^{r_0\tau_0}. \tag{2·12}$$

Since $u(t^*)$ is an arbitrary local maximum, we can conclude that

$$1+u(t) \leqslant e^{r_0\tau_0} \quad (t \geqslant t_1), \tag{2·13}$$

where t_1 is the first zero of the oscillating solution.

Now let t^{**} be an arbitrary local minimum of u; as for the maximum we can derive that there exists $\xi^{**} \in [t^{**}-\tau_0, t^{**}]$ such that $u(\xi^{**}) = 0$. As before we have

$$\log\left(\frac{1+u(t^{**})}{1+u(\xi^{**})}\right) = -r_0\int_{\xi^{**}}^{t^{**}} u(s-\tau_0)\,ds \tag{2·14}$$

$$\geqslant -r_0(e^{r_0\tau_0}-1)\,(t^{**}-\xi^{**}) = -r_0\tau_0(e^{r_0\tau_0}-1) \tag{2·15}$$

implying that

$$1+u(t^{**}) \geqslant \exp(-r_0\tau_0(e^{r_0\tau_0}-1)).$$

Since $u(t^{**})$ is an arbitrary local minimum we can conclude that

$$1+u(t) \geqslant \exp(-r_0\tau_0(e^{r_0\tau_0}-1)) \quad \text{for } t \geqslant t_1. \tag{2·16}$$

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If u is an oscillating solution of (2·1), the conclusion (2·9) follows from (2·13) and (2·16). If u is a non-oscillatory solution of (2·1), then by Lemma 2·1 $1+u(t)\to 1$ as $t\to\infty$ and therefore there exists a $t_1\geqslant 0$ such that (2.9) holds. The proof is complete.

It is not difficult to see that the estimates in (2·9) can be improved sequentially; for instance using the lower bound in (2·9) one can derive a better estimate for the upper bound so that

$$1+u(t)\leqslant \exp\left(r_0\tau_0(1-\exp\left(-r_0\tau_0(e^{r_0\tau_0}-1)\right))\right) \tag{2·17}$$

for $t\geqslant T_1\geqslant T_0$. One can then use this upper estimate for the derivation of a revised lower estimate and this procedure can be repeated iteratively.

THEOREM 2·3. *Assume that* $r_0,\tau_0\in(0,\infty)$ *and satisfy*

$$r_0\tau_0\exp\left(r_0\tau_0(1-\exp\left(-r_0\tau_0(e^{r_0\tau_0}-1)\right))\right)<1. \tag{2·18}$$

Then the trivial solution of the non-linear equation (2·1) *is exponentially globally asymptotically stable.*

Proof. We rewrite Equation (2·1) in the form

$$\dot{x}(t)=-a(t)\,x(t-\tau_0) \tag{2·19}$$

where $a(t)=r_0(1+x(t))$ and define u as follows:

$$u=\sigma(t)=\int_{t_0}^{t}a(s)\,ds\quad(t_0\geqslant 0); \tag{2·20}$$

we note that by Lemmas 2·1 and 2·2

$$\sigma^{-1}(\cdot)\ \text{exists}\quad\text{and}\quad u(t)\to\infty\ \text{as}\ t\to\infty.$$

Furthermore
$$\sigma(t-\tau_0)=\int_{t_0}^{t-\tau_0}a(s)\,ds=u-\int_{\sigma^{-1}(u)-\tau_0}^{\sigma^{-1}(u)}a(s)\,ds$$

and
$$t-\tau_0=\sigma^{-1}\left(u-\int_{\sigma^{-1}(u)-\tau_0}^{\sigma^{-1}(u)}a(s)\,ds\right).$$

We now let
$$x(t)=x(\sigma^{-1}(u))=y(u)$$

so that
$$\frac{dy(u)}{du}=-y(u-\eta(u)) \tag{2·21}$$

where
$$\eta(u)=\int_{\sigma^{-1}(u)-\tau_0}^{\sigma^{-1}(u)}a(s)\,ds=\int_{t-\tau_0}^{t}a(s)\,ds$$
$$\leqslant r_0\tau_0\exp\left((r_0\tau_0)(1-\exp\left(-r_0\tau_0(e^{r_0\tau_0}-1)\right))\right). \tag{2·22}$$

We rewrite (2·21) in the form

$$\frac{dy(u)}{du}=-y(u)+(y(u)-y(u-\eta(u)))$$
$$=-y(u)+\eta(u)\,\dot{y}(\xi)\quad(\xi\in[u-\eta(u),u]) \tag{2·23}$$

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and note that for all u for which $y(u) \neq 0$,

$$\frac{d}{du}|y(u)| \leqslant -|y(u)| + \eta(u)\left(\sup_{s\in[u-2\eta(u),\,u]}|y(s)|\right)$$
$$\leqslant -|y(u)| + \eta^*\left(\sup_{s\in[u-2r_0\tau_0 e^{r_0\tau_0},\,u]}|y(s)|\right), \tag{2·24}$$

where η^* denotes the left-hand side of (2·18). Since $\eta \leqslant \eta^* < 1$, it follows from (2·21), (2·24) and a lemma due to Halanay ([**4**], p. 378) that there exist positive numbers M and α such that

$$|y(u)| \leqslant Me^{-\alpha u}, \tag{2·25}$$

which of course is valid when $y(u) = 0$; we have from (2·25) and (2·16)

$$|x(t)| \leqslant M\exp\left(-\alpha\int_{t_0}^{t} a(s)\,ds\right)$$
$$\leqslant M\exp\left(-\alpha(r_0\exp\left(-r_0\tau_0(e^{r_0\tau_0}-1)\right)(t-t_0))\right) \tag{2·26}$$

and this completes the proof.

We remark that it is possible to obtain an estimate for the positive number α but we do not need such an estimate in the following; the existence of a positive number α is sufficient for our subsequent needs.

THEOREM 2·4. *Let c be a positive number and let $\tau:[0,\infty)\to[0,\infty)$ be such that*

$$\tau(t)\to\tau_0 > 0 \quad \text{as } t\to\infty$$

and

$$c\tau_0\exp\left(c\tau_0(1-\exp\left(-c\tau_0(e^{c\tau_0}-1)\right))\right) < 1.$$

Then the trivial solution of

$$\frac{dw(t)}{dt} = -c(1+w(t))\,w(t-\tau(t)) \tag{2·27}$$

is exponentially asymptotically stable.

Proof. We note that only those solutions of (2·27) satisfying $1+w(t) > 0$ for $t \geqslant t_0$ are considered. It is easy to see that τ is bounded, and therefore solutions of (2·27) remain bounded uniformly as $t\to\infty$. We rewrite (2·27) in the form

$$\frac{dw(t)}{dt} = -c(1+w(t))\,w(t-\tau_0) + c(1+w(t))\,(w(t-\tau_0)-w(t-\tau(t))). \tag{2·28}$$

Our strategy of proof is to compare (2·28) with

$$\frac{dz(t)}{dt} = -c(1+z(t))\,z(t-\tau_0). \tag{2·29}$$

We know from Theorem 2·3 that the trivial solution of (2·29) is exponentially globally asymptotically stable. One can prove that the trivial solution of the linear variational system corresponding to (2·29) is also exponentially asymptotically stable (we explain some details of this in the course of the proof of the next theorem).

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By the non-linear variation of constants formula (see Shanholt[9], Hastings[5]) we can represent the solution of (2·28) in the form

$$w(t) = z(t) + c\int_{t_0}^{t} (T(t,s,x_s)X_0)(1+w(s))(w(s-\tau_0)-w(s-\tau(s)))\,ds \qquad (2\cdot30)$$

for $t \geqslant t_0 \geqslant 0$, where z denotes a solution of (2·29) with

$$z(s) = w(s) \quad (s \in [-\sup_{t\geqslant 0}\tau(t), 0])$$

and $T(t,s,x_s)X_0$ is the solution of

$$\frac{\partial(T(t,s,x_s)X_0)}{\partial t} = -c(T(t,s,x_s)X_0) \quad (t \geqslant s \geqslant 0)$$

$$T(s,s,x_s)X_0 = x_s = x(s+\theta) \quad (\theta \in [s-\tau_0, s])$$

associated with the linear variational system corresponding to (2·29). From the properties of (2·29) we know that there exist numbers $B_0, B_1, \beta > 0$ such that

$$|z(t)| \leqslant B_0 e^{-\beta(t-t_0)} \quad (t \geqslant t_0 \geqslant 0)$$

$$\|T(t,s,x_s)X_0\| \leqslant B_1 e^{-\beta(t-s)} \quad (t \geqslant s \geqslant t_0 \geqslant 0), \qquad (2\cdot31)$$

for sufficiently large t_0. We have, from (2·30) and (2·31),

$$|w(t)| \leqslant B_0 e^{-\beta(t-t_0)} + B_1\int_{t_0}^{t} e^{-\beta(t-s)}|\dot{w}(\xi(s))|\,|\tau_0-\tau(s)|\,ds \qquad (2\cdot32)$$

where $\xi(s)$ lies between $s-\tau_0$ and $s-\tau(s)$; $s \geqslant t_0$. Using the boundedness of solutions of (2·27), one can estimate $\dot{w}(\xi(s))$ and therefore

$$|w(t)| \leqslant B_0 e^{-\beta(t-t_0)} + B_2\int_{t_0}^{t} e^{-\beta(t-s)}|\tau-\tau(s)|\,ds \qquad (2\cdot33)$$

for some constant $B_2 > 0$ representing an upper bound of $\dot{w}(t)$ for $t \geqslant t_0$. An application of Gronwall's inequality to (2·33) leads to

$$|w(t)| \leqslant (B_0 e^{\beta t_0}) \exp\left(t\left(-\beta + B_2\frac{1}{t}\int_{t_0}^{t}|\tau-\tau(s)|\,ds\right)\right); \qquad (2\cdot34)$$

since $\tau(t) \to \tau_0$ as $t \to \infty$, for every $\epsilon > 0$ there exists a t_0 such that

$$\int_{t_0}^{t}|\tau-\tau(s)|\,ds < \epsilon t \quad (t > t_0),$$

and hence if t_0 is sufficiently large we have from (2·34)

$$|w(t)| \leqslant (B_0 e^{\beta t_0}) \exp(-(\beta - B_2\epsilon)t) \quad (t \geqslant t_0 \geqslant 0). \qquad (2\cdot35)$$

Since ϵ is arbitrary the exponential asymptotic stability of the trivial solution of (2·27) follows and this is true for all solutions of (2·27). The asymptotic stability is global and this completes the proof.

COROLLARY 2·5. *Let τ_0 be a positive constant and let r be a non-negative continuous function defined on $[0,\infty)$ such that*

$$\int^{\infty} r(s)\,ds = \infty, \quad \lim_{t\to\infty}\int_{t-t_0}^{t} r(s)\,ds = r^* \qquad (2\cdot36)$$

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where $$r^* \exp(r^*(1-\exp(-r^*(e^{r^*}-1)))) < 1. \tag{2·37}$$

Then positive solutions of

$$\frac{dx(t)}{dt} = -r(t)(1+x(t))\,x(t-\tau_0) \tag{2·38}$$

satisfy $$\lim_{t\to\infty} x(t) = 0. \tag{2·40}$$

Proof. As in the proof of Theorem 2·3, we introduce new variables u and δ where

$$u = \delta(t) = \int_{t_0}^{t} r(s)\,ds$$

and $$x(t) = x(\delta^{-1}(u)) = z(u),$$

so that $$\frac{dz(u)}{du} = -(1+z(u))\,z(u-\tilde{\delta}(u)),$$

where $\tilde{\delta}$ is defined by $$\tilde{\delta}(u) = \int_{t-\tau_0}^{t} r(s)\,ds.$$

Since $$\lim_{u\to\infty} \tilde{\delta}(u) = \lim_{t\to\infty} \int_{t-\tau_0}^{t} r(s)\,ds = r^*$$

the conclusion follows from Theorem 2·4.

Our next result is concerned with the asymptotic behaviour of solutions of equations of the form

$$\frac{dx(t)}{dt} = -r(t)(1+x(t))\,x(t-\tau(t)). \tag{2·41}$$

THEOREM 2·6. *Assume the following.*

(i) *r is a non-negative continuous function defined for $t \geqslant 0$ such that*

$$\int^{\infty} r(s)\,ds = \infty \quad \text{and} \quad \int^{\infty} r(s)\,e^{-as}\,ds < \infty \text{ for any } a > 0; \tag{2·42}$$

(ii) *τ is continuous real-valued for $t \geqslant 0$ such that there exists a positive constant τ_0 satisfying*

$$\int^{\infty} |\tau(s)-\tau_0|\,r(s)\,ds < \infty \quad \text{and} \quad \lim_{t\to\infty} \int_{t-\tau_0}^{t} r(s)\,ds = r^*, \tag{2·43}$$

where $$r^* \exp(r^*(1-\exp(-r^*)e^{r^*}-1))) < 1.$$

Then the trivial solution of (2·41) is globally asymptotically stable.

Proof. We rewrite (2·41) in the form

$$\frac{dx(t)}{dt} = -r(t)(1+x(t))\,x(t-\tau_0) + r(t)(1+x(t))(x(t-\tau_0)-x(t-\tau(t))) \tag{2·44}$$

and compare it with

$$\frac{dy(t)}{dt} = -r(t)(1+y(t))\,y(t-\tau_0). \tag{2·45}$$

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We know from the result of Corollary 2·5 that the trivial solution of (2·45) is exponentially globally asymptotically stable. The variational system associated with (2·45) is

$$\frac{d\psi(t)}{dt} = -r(t)\,y(t-\tau_0)\,\psi(t) - r(t)\,(1+y(t))\,\psi(t-\tau_0) \tag{2·46}$$

where y denotes any solution of (2·45). Since $y(t)\to 0$ as $t\to\infty$ exponentially, we have from the second condition of (2·42),

$$\int^{\infty} r(s)\,y(s)\,ds < \infty, \quad \text{implying} \quad \lim_{t\to\infty}\int_{t-\tau_0}^{t} r(s)\,y(s)\,ds = 0. \tag{2·47}$$

We now let

$$A(t) = r(t)\,y(t-\tau_0), \quad z(t) = \psi(t)\exp\left(\int_T^t A(s)\,ds\right) \quad (t \geqslant T \geqslant t_0 \geqslant 0),$$

and note that (2·46) simplifies to

$$\frac{dz(t)}{dt} = -Q(t)\,z(t-\tau_0) \tag{2·48}$$

where

$$Q(t) = r(t)\,(1+y(t))\exp\left(\int_{t-\tau_0}^{t} A(s)\,ds\right); \tag{2·49}$$

since $y(t)\to 0$ and $\int_{t-\tau_0}^{t} A(s)\,ds \to 0$ as $t\to\infty$ we have

$$\lim_{t\to\infty}\int_{t-\tau_0}^{t} Q(s)\,ds = r^*. \tag{2·50}$$

It follows from (2·50) and Theorem 2·3 that the trivial solution of (2·48) is exponentially globally asymptotically stable. We have from the non-linear variation of constants formula (cf. Brauer [**2**], Marlin and Struble [**8**]),

$$y(t) = x(t) + \int_t^{\infty} (T(t,s,x_s)\,X_0)\,r(s)\,(1+x(s))\,(x(s-\tau_0)-x(s-\tau(s)))\,ds \tag{2·51}$$

where x is any solution of (2·41). By the boundedness of all solutions of (2·41) we have

$$\left.\begin{aligned} &\left|\int_t^{\infty} T(t,s,x_s)\,X_0\,r(s)\,(1+x(s))\,(x(s-\tau_0)-x(s-\tau(s)))\,ds\right| \\ &\leqslant K_1\int_t^{\infty} \|T(t,s,x_s)\,X_0\|\,r(s)\,|\dot{x}(\xi(s))|\,|\tau_0-\tau(s)|\,ds \\ &\leqslant K\int_t^{\infty} r(s)\,|\tau_0-\tau(s)|\,ds \to 0 \quad \text{as } t\to\infty, \end{aligned}\right\} \tag{2·52}$$

where $\xi(s)$ lies between $s-\tau_0$ and $\tau(s)$ and K is a positive number such that

$$\|T(t,s,x_s)\,X_0\|\,|\dot{x}(\xi(s))| \leqslant K \quad \text{for } t \geqslant s \geqslant 0.$$

The conclusion follows from (2·51) and (2·52) and this completes the proof.

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3. *A generalized delay logistic equation*

In this section we consider the limiting behaviour of solutions of equations of the form

$$\frac{dx(t)}{dt} = r(t)\,x(t)f\left(1-\frac{x(t-\tau(t))}{K}\right), \tag{3·1}$$

which reduces to the commonly known delay logistic equation when $f(y) \equiv y$. We assume that r and τ are continuous functions defined on $[0, \infty)$ such that $r(t) > 0$, $0 \leqslant \tau(t) < \tau_0$; f is a continuous function on $(-\infty, \infty)$ such that $yf(y) > 0$ for $y \neq 0$. If we let

$$y(t) = \frac{x(t)}{K} - 1$$

then (3·1) simplifies to

$$\frac{dy(t)}{dt} = r(t)\,(1+y(t))\,f(-y(t-\tau(t))). \tag{3·2}$$

Under the standard type of initial conditions (i.e. $1+y(s) \geqslant 0$, $1+y(0) > 0$ for $s \in [-\sup_{t\geqslant 0}\tau(t), 0]$) it is easy to see that solutions of (3·2) satisfy $1+y(t) > 0$ for $t \geqslant 0$.

LEMMA 3·1. *Assume* $\int_0^\infty r(s)\,ds = \infty$. *Then every solution of* (3·2) *is either oscillatory or tends to zero as* $t \to \infty$ *monotonically.*

Proof. Let y be a non-oscillatory solution and suppose $y(t) > 0, y(t-\tau(t)) > 0$ for $t \geqslant T$. From (3·2) we note since $yf(y) > 0$ for $y \neq 0$, that

$$\frac{dy(t)}{dt} < 0 \quad \text{for } t \geqslant T$$

implying the existence of

$$\lim_{t\to\infty} y(t) = \alpha \geqslant 0. \tag{3·3}$$

Suppose now $\alpha > 0$; then $y(t) \geqslant \alpha$ and $-y(t) \leqslant -\alpha$ for $t \geqslant T$. Define

$$-m = \sup_{t\geqslant T} f(-y(t-\tau(t))). \tag{3·4}$$

It follows from (3·2) that

$$\frac{dy(t)}{dt} \leqslant -mr(t)\,(1+y(t)) \leqslant -m(1+\alpha)\,r(t), \tag{3·5}$$

which on integration leads to

$$y(t)-y(T) \leqslant -m(1+\alpha)\int_T^t r(s)\,ds \to -\infty \quad \text{as } t\to\infty, \tag{3·6}$$

showing that y can become negative for t sufficiently large, and this contradiction implies $\alpha = 0$. The convergence to zero of eventually negative solutions of (3·2) can be proved similarly.

LEMMA 3·2. *Assume*

$$\int_{t-\tau(t)}^t r(s)\,ds \quad \textit{is bounded for } t > 0. \tag{3·7}$$

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Then every oscillatory solution of (3·2) *is bounded for* $t \geqslant 0$, *and if* $y(t_K)$ *is a local maximum then*

$$y(t_K) \leqslant \exp\left(M \int_{t_K-\tau(t_K)}^{t_K} r(s)\,ds\right) - 1, \tag{3·8}$$

where

$$M = \sup_{y>-1} f(-y) = \sup_{-y<1} f(-y).$$

Proof. Let $\{J_K\}$ denote a sequence of non-overlapping intervals on $[0, \infty)$ such that y is zero at the end-points of any J_K and y is of the same sign in the interior of J_K. Let t_K denote s typical local maximum for y. This means that

$$f(-y(t_K-\tau(t_K))) = 0, \quad \text{which implies} \quad y(t_K-\tau(t_K)) = 0.$$

Assuming that $y > 0$ on $(t_K-\tau(t_K), t_K)$ an integration of (3·2) on $[t_K-\tau(t_K), t_K]$ leads to

$$\ln[1+y(t_K)] = \int_{t_K-\tau(t_K)}^{t_K} r(s) f(-y(s-\tau(s)))\,ds \leqslant M \int_{t_K-\tau(t_K)}^{t_K} r(s)\,ds \tag{3·9}$$

from which the result follows.

THEOREM 3·1. *The trivial solution of* (3·2) *is (locally) uniformly stable if*

$$\int_{t-\tau(t)}^{t} r(s)\,ds \to 0 \quad as\ t \to \infty. \tag{3·10}$$

Proof. Define $v(y) = y^2$, where y denotes any solution of (3·2) on $[t_1, t_2] \subset [0, \infty)$. We shall first show that

$$v(y(t)) > 0 \text{ on } [t_1, t_2] \quad \text{and} \quad \frac{d}{dt} v(y(t))|_{t=t_2} > 0 \tag{3·11}$$

implies $t_1 \geqslant t_2-\tau(t_2)$. Suppose that this is not the case. Then $t_1 < t_2-\tau(t_2)$ and hence

$$\left.\frac{dv(y(t))}{dt}\right|_{t=t_2} = 2y(t_2)\left.\frac{dy(t)}{dt}\right|_{t=t_2} = 2y(t_2)\,r(t_2)(1+y(t_2))f(-y(t_2-\tau(t_2))) > 0. \tag{3.12}$$

But

$$y(t_2)\,y(t_2-\tau(t_2)) > 0 \quad \text{for} \quad t_1 < t_2-\tau(t_2) < t_2$$

and hence (3·12) is impossible. Thus $t_1 \geqslant t_2-\tau(t_2)$.

Let $\epsilon > 0$ be given and let $M(\epsilon) = \sup\{|f(-y)|; |y| \leqslant \epsilon\}$. There exists a $T(\epsilon) > 0$ satisfying

$$\int_{t-\tau(t)}^{t} r(s)\,ds < \frac{\epsilon}{2(1+\epsilon)M(\epsilon)} \quad \text{for } t \geqslant T(\epsilon). \tag{3·13}$$

We first show that for any $\phi : [t_0-\tau, t_0] \to \mathbb{R}, t_0 \geqslant T(\epsilon), |\phi| < \frac{1}{2}\epsilon$,

$$|y(t, t_0, \phi)| < \epsilon \quad \text{for all } t \geqslant t_0. \tag{3·14}$$

Suppose that (3·14) does not hold. Then there exists a solution $y(t) = y(t, t_0, \phi)$ with $t_0 \geqslant T(\epsilon)$ and $|\phi| < \frac{1}{2}\epsilon$ satisfying $|y(t_3)| \geqslant \epsilon$ for some $t_3 > t_0$. We define

$$t_2 = \inf\{t > t_0 \,|\, |y(t)| > \epsilon\}, \quad t_1 = \sup\{t < t_2 \,|\, |y(t)| = \tfrac{1}{2}\epsilon\},$$

and note that

$$v(y(t_1)) = \tfrac{1}{4}\epsilon^2, \quad v(y(t_2)) = \epsilon^2,$$

$$\tfrac{1}{4}\epsilon^2 < v(t) < \epsilon^2 \quad \text{for } t \in (t_1, t_2). \tag{3·15}$$

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It follows from (3·15) that

$$\frac{dv}{dt}(y(t))|_{t=t_2} > 0$$

implying $t_1 \geqslant t_2 - \tau(t_2)$. On integrating (3·2) over (t_1, t_2), we have

$$\begin{aligned} \tfrac{1}{2}\epsilon = |y(t_2)| - |y(t_1)| &\leqslant \int_{t_1}^{t_2} r(s)(1+|y(s)|)|f(-y(s-\tau(s)))|\,ds \\ &\leqslant (1+\epsilon)M(\epsilon)\int_{t_2-\tau(t_2)}^{t_2} r(s)\,ds < \tfrac{1}{2}\epsilon \end{aligned} \tag{3·16}$$

which is a contradiction and hence (3·14) holds.

It is known that solutions of (3·2) depend continuously on the initial data, from which it follows that for any $t_0 \in [0, T(\epsilon)]$, there exists a $\delta(\epsilon)$ such that

$$\left.\begin{aligned} &\text{if} \quad \phi : [t_0 - \tau, t_0] \to \mathbb{R} \quad \text{and} \quad |\phi| < \delta(\epsilon) \\ &\text{then} \quad |y(t, t_0, \phi)| < \tfrac{1}{2}\epsilon \quad \text{for all } t \in [t_0, T(\epsilon)]. \end{aligned}\right\} \tag{3·17}$$

Combining (3·17) together with (3·14) one derives the assertion and the proof is complete.

We note here that the assumption (3·10) can be replaced by

$$r(t) \to 0 \quad \text{as } t \to \infty \tag{3·18}$$

and still the conclusion of Theorem 3·1 will hold. For instance we have

$$\begin{aligned} \tfrac{1}{2}\epsilon = |y(t_2)| - |y(t_1)| &\leqslant (1+\epsilon)\int_{t_2-\tau(t_2)}^{t_2} r(s)\,ds \\ &\leqslant (1+\epsilon)M(\epsilon)\tau r(\xi) < \tfrac{1}{2}\epsilon \end{aligned} \tag{3·19}$$

for some ξ with $t_2 - \tau(t_2) < \xi < t_2$, for t_2 large enough, by the mean value theorem of integral calculus, and (3·19) is not possible.

We recall that in the case of a logistic equation, the assumptions (3·18) and (3·19) have been weakened as in Section 2.

The next result is concerned with the global attractivity of the trivial solution of (3·2).

THEOREM 3·2. *Assume the following*:

(*a*)
$$\int_{t-\tau(t)}^{t} r(s)\,ds \to 0 \quad \text{as } t \to \infty; \tag{3·20}$$

(*b*)
$$\int_0^{\infty} r(s)\,ds = \infty. \tag{3·21}$$

Then the trivial solution of (3·2) *is globally attractive.*

Proof. It follows from Theorem 3·1 that the trivial solution of (3·2) is uniformly (locally) stable. We now show that every solution of (3·2) approaches the trivial solution as $t \to \infty$. By Lemma 3·1 and condition (*b*), every non-oscillatory solution tends to zero as $t \to \infty$. By Lemma 3·2 all oscillatory solutions are bounded on $(0, \infty)$.

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Let $y(t)$ be an oscillatory solution of (3·2) which does not tend to zero as $t \to \infty$. There exist $\epsilon > 0$ and sequences $\{t_n\}, \{t'_n\} \to \infty$ as $n \to \infty$ such that for each n, either $y(t_n) = 0$, $y(t'_n) = \epsilon$, $y'(t'_n) \geqslant 0$ and $0 < y(t) < \epsilon$ for $t_n < t < t'_n < t_{n+1}$ or $y(t_n) = 0$, $y(t'_n) = -\epsilon$, $y'(t'_n) \leqslant 0$ and $0 > y(t) > -\epsilon$ for $t_n < t < t'_n < t'_{n+1}$. We consider the former case since the latter case can be considered similarly.

On integrating (3·2) over (t_n, t'_n) we have

$$\epsilon = y(t'_n) - y(t_n) \leqslant \int_{t_n}^{t'_n} r(s)(1+y(s))f(-y(s-\tau(s)))\,ds$$

$$\leqslant K \int_{t_n}^{t'_n} r(s)\,ds \leqslant K \int_{t'_n-\tau(t'_n)}^{t'_n} r(s)\,ds < \epsilon \tag{3·22}$$

on using Lemma 3·2 for large n, where K denotes a bound for $(1+y(s))f(-y(s-\tau(s)))$ on $[-\tau, \infty)$, but (3·22) is impossible. Thus there exists no oscillatory solution of (3·2) which does not tend to zero.

We make a note that if we assume (3·18) instead of (a) in Theorem 3·1, the preceding argument leads to

$$\epsilon \leqslant Mr(\xi)(t'_n - t_n),$$

implying that

$$t'_n - t_n \geqslant \frac{\epsilon}{Mr(\xi)}$$

and hence from $r(t) \to 0$ as $t \to \infty$, we have $t'_n - t_n > \tau$ for large n.

As a consequence

$$t_n < t'_n - \tau \leqslant t'_n - \tau(t'_n) < t'_n$$

and

$$y'(t'_n) = r(t'_n)(1+y(t'_n))f(-y(t'_n-\tau(t'_n)) < 0$$

which is incompatible with $y'(t'_n) \geqslant 0$. Thus Theorem 3·2 will hold if (a) is replaced by (3·10).

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