Combinatorics, Probability and Computing (1997), 6, 497–506. Printed in the United Kingdom
© 1997 Cambridge University Press

The Zero-Free Intervals for Chromatic Polynomials of Graphs

CARSTEN THOMASSEN

Mathematical Institute, Technical University of Denmark, DK-2800 Lyngby, Denmark
(e-mail: C.Thomassen@mat.dtu.dk)

Received 29 September 1995; revised 31 October 1996

The maximal zero-free intervals for chromatic polynomials of graphs are precisely $(-\infty, 0)$, $(0, 1)$, $(1, 32/27]$. We also investigate the distribution of zeros of chromatic polynomials in various classes of graphs closed under minors. For example, the zeros of chromatic polynomials of graphs of tree-width at most k consist of 0, 1 and a dense subset of the interval $(32/27, k]$.

1. Introduction

If G is a graph, then $P(G, \lambda)$ denotes the chromatic polynomial of G, that is, when λ is a nonnegative integer, then $P(G, \lambda)$ is the number of colourings of the vertex set in colours $1, 2, \ldots, \lambda$. Clearly, 0 and 1 are roots of $P(G, \lambda)$ when G has at least one edge. It is not difficult to see that $(-\infty, 0)$ and $(0, 1)$ are zero-free intervals for $P(G, \lambda)$; see, for example, [11]. Jackson [6] obtained the intriguing result that $(1, 32/27]$ is also zero-free and that each interval of the form $(32/27, 32/27 + \epsilon)$, $\epsilon > 0$, contains a root of some chromatic polynomial. We prove that Jackson's result is best possible in the stronger sense that each interval of the form (λ_1, λ_2), where $32/27 \leqslant \lambda_1 < \lambda_2$, contains a root of some chromatic polynomial, a question raised by Jensen and Toft [7, page 225]. Woodall [18] proved that, for each family $\mathscr{F}$ of graphs that is closed under minors and does not include all graphs, there exists a positive real number c such that $P(G, \lambda) > 0$ whenever G is in $\mathscr{F}$ and $\lambda > c$. We show that, for the family of graphs having no K_5-minor, the best constant c is the same as that for planar graphs. Birkhoff and Lewis [3] proved that, for this class, $c \leqslant 5$ and conjectured that the best constant is 4. We prove that the zeros of chromatic polynomials of planar graphs contain a dense subset of the interval $(32/27, 3)$ and conjecture that 3 can be replaced by 4. Woodall [18] raised the question of whether there is an analogous result for the class of d-degenerate graphs where d is any fixed natural number. We answer this in the negative for each $d \geqslant 2$. For analogous results and problems on planar triangulations, see [16] and [7]. Zeros in the complex plane have been investigated by Read and Royle [10]; see also [4].

https://doi.org/10.1017/S0963548397003131 Published online by Cambridge University Press

2. Chromatic roots close to a prescribed real number $> 32/27$

We shall make use of the following technical lemmas, which follow by standard methods in elementary group theory.

Lemma 2.1. *Let a, b be real numbers such that $b < 0 < a$, and a/b is irrational. Then, for any real number c and any $\epsilon > 0$, there exist natural numbers s, t such that*

$$c < sa + tb < c + \epsilon.$$

Proof. The set D of real numbers of the form $sa + tb$, where s and t are integers, forms a subgroup of the additive group of real numbers. As a/b is irrational, D is not a cyclic group. Hence D is a dense subset of the real numbers and so there exist integers s and t such that

$$0 < sa + tb < \epsilon.$$

We may assume that $\epsilon < a, \epsilon < -b$. Then s and t have the same sign and hence there are natural numbers s_0, t_0 such that

$$-\epsilon < s_0 a + t_0 b < \epsilon.$$

Let s_1, t_1 be natural numbers such that

$$t_1 b < c < s_1 a.$$

Now there exists a natural number m such that either

$$t_1 b + m(s_0 a + t_0 b) \quad \text{or} \quad s_1 a + m(s_0 a + t_0 b_0)$$

is in the interval $(c, c + \epsilon)$. □

As log is a homeomorphism and an isomorphism of the multiplicative group of positive real numbers onto the additive group of real numbers, Lemma 2.1 implies the following.

Lemma 2.2. *Let a, b be real numbers such that*

$$0 < b < 1 < a$$

and $\log a / \log b$ is irrational. Then, for any positive real numbers c, ϵ, there exist natural numbers s, t such that

$$c < a^s b^t < c + \epsilon.$$

Moreover, s and t can be chosen such that they have prescribed parity.

Proof. The first part follows from Lemma 2.1. If, for example, we want s to be odd and t to be even, then we apply the first part of Lemma 2.2 with $a^2, b^2, c/a$ instead of a, b, c, respectively. □

Proposition 2.3. *Let λ_0, δ be real numbers, $\lambda_0 > 1, \delta > 0$. Assume that there exists a graph G_0 having an edge $e = xy$ such that*

$$|P(G_0, \lambda_0)| < (\lambda_0 - 1)\,|P(G_0/e, \lambda_0)|.$$

https://doi.org/10.1017/S0963548397003131 Published online by Cambridge University Press

Assume further that, if $\lambda_0 > 2$, *then*

$$P(G_0, \lambda_0)\, P(G_0/e, \lambda_0) < 0.$$

Then there exist real numbers λ_1, λ_2 *and a graph* H *such that*

$$\lambda_0 - \delta < \lambda_1 < \lambda_2 < \lambda_0 + \delta$$

and

$$P(H, \lambda_1)\, P(H, \lambda_2) < 0.$$

In particular, $P(H, \lambda)$ *has a root between* λ_1 *and* λ_2.

Proof. By replacing $(\lambda_0 - \delta, \lambda_0 + \delta)$ with an appropriate subinterval (if necessary) we may assume that, for all $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$, we have

$$P(G_0, \lambda) \neq 0, |P(G_0, \lambda)| < (\lambda - 1)\, |P(G_0/e, \lambda)|,$$

and, if $\lambda_0 > 2$, then

$$P(G_0, \lambda)\, P(G_0/e, \lambda) < 0.$$

Let s and t be natural numbers and let $H(s, t)$ be the graph obtained from s pairwise disjoint copies of G_0 and t pairwise disjoint copies of C_4, the cycle of length 4 formed by first selecting some edge $e = xy$ in each copy of C_4, and then identifying the edge xy in all the $s + t$ copies of G_0 and C_4. The resulting graph is called $H_1(s, t)$. Finally, $H(s, t)$ is obtained from $H_1(s, t)$ by deleting e. Then

$$\begin{aligned}
P(H(s,t),\lambda) &= P(H_1(s,t),\lambda) + P(H_1(s,t)/e,\lambda) \\
&= \lambda(\lambda-1)\Big(P(G_0,\lambda)/\lambda(\lambda-1)\Big)^s \Big(P(C_4,\lambda)/\lambda(\lambda-1)\Big)^t \\
&\quad + \lambda\Big(P(G_0/e,\lambda)/\lambda\Big)^s \Big(P(C_3,\lambda)/\lambda\Big)^t \\
&= \lambda^{1-s}(\lambda-1)\Big(P(G_0,\lambda)/(\lambda-1)\Big)^s (\lambda^2-3\lambda+3)^t \\
&\quad + \lambda^{1-s}\Big(P(G_0/e,\lambda)\Big)^s \Big((\lambda-1)(\lambda-2)\Big)^t \\
&= \lambda^{1-s}\Big(P(G_0,\lambda)/(\lambda-1)\Big)^s (\lambda^2-3\lambda+3)^t \Big[(\lambda-1) \\
&\quad + \Big((\lambda-1)\, P(G_0/e,\lambda)/P(G_0,\lambda)\Big)^s \cdot \Big((\lambda-1)(\lambda-2)/(\lambda^2-3\lambda+3)\Big)^t\Big].
\end{aligned}$$

We shall now apply Lemma 2.2. Assume first that $\lambda_0 > 2$. For all λ in $(\lambda_0 - \delta, \lambda_0 + \delta)$, we define

$$\begin{aligned}
a(\lambda) &= (1-\lambda)\, P(G_0/e,\lambda)/P(G_0,\lambda), \\
b(\lambda) &= (\lambda-1)(\lambda-2)/(\lambda^2-3\lambda+3) \quad \text{and} \\
c(\lambda) &= \lambda - 1.
\end{aligned}$$

https://doi.org/10.1017/S0963548397003131 Published online by Cambridge University Press

Clearly,

$$0 < b(\lambda) < 1 < a(\lambda)$$

for all λ in $(\lambda_0-\delta, \lambda_0+\delta)$.

We claim that there exists a λ_1 in $(\lambda_0-\delta, \lambda_0+\delta)$, so that $\log a(\lambda_1)/\log b(\lambda_1)$ is irrational. Otherwise, there would exist a rational number α such that $\log a(\lambda)/\log b(\lambda) = \alpha$, that is, $a(\lambda) = b(\lambda)^\alpha$ for all λ in $(\lambda_0-\delta, \lambda_0+\delta)$. Since $a(\lambda)$ is a rational function, α must be an integer and we have $a(\lambda) = b(\lambda)^\alpha$ for all λ except for the singularities. But, if λ is a large real number then $a(\lambda) < 0 < b(\lambda)$, a contradiction, which proves the claim that λ_1 exists. Note that the sign of $P(H(s,t),\lambda_1)$ is determined by the parity of s and the sign of $c(\lambda_1)+(-a(\lambda_1))^s b(\lambda_1)^t$.

Applying Lemma 2.2 with $a = a(\lambda_1)$, $b = b(\lambda_1)$, and $c = c(\lambda_1)$, there are sequences s_n and t_n of natural numbers such that

$$a^{s_n} b^{t_n} > \lambda_1 - 1,$$

$a^{s_n} t^{t_n} \to \lambda_0 - 1$ as $n \to \infty$, and s_n is odd for each $n \geqslant 1$. Similarly, there are sequences s_n' and t_n' of natural numbers (with s_n' odd for all n) such that

$$a^{s_n'} b^{t_n'} < \lambda_1 - 1$$

and $a^{s_n'} b^{t_n'} \to \lambda_1 - 1$ as $n \to \infty$.

We may assume that each of s_n, t_n, s_n', t_n' tends towards infinity as n tends towards infinity. Then both s_n/t_n and s_n'/t_n' tend towards $-\log b/\log a$ as n tends towards infinity.

Note that $P(H(s_n,t_n),\lambda_1)$ and $P(H(s_n',t_n'),\lambda_1)$ have different signs for all n.

By a previous argument, the function $\log a(\lambda)/\log b(\lambda)$ cannot be constant in $(\lambda_0-\delta, \lambda_0+\delta)$. Hence there exists a λ_2 in $(\lambda_0-\delta, \lambda_0+\delta)$ such that

$$\log a(\lambda_2)/\log b(\lambda_2) \neq \log a(\lambda_1)/\log b(\lambda_1).$$

Then $P(H(s_n,t_n),\lambda_2)$ and $P(H(s_n',t_n'),\lambda_2)$ have the same sign for n sufficiently large since, as n tends to infinity, $s_n \log a(\lambda_2) + t_n \log b(\lambda_2)$ and $s_n' \log a(\lambda_2) + t_n' \log b(\lambda_2)$ both tend to infinity or minus infinity, depending on the sign of $\log a(\lambda_2)/\log b(\lambda_2) - \log a(\lambda_1)/\log b(\lambda_1)$.

Since $P(H(s_n,t_n),\lambda_1)$ and $P(H(s_n',t_n'),\lambda_1)$ have different signs for all n, it follows that, for n sufficiently large, either $H(s_n,t_n)$ or $H(s_n',t_n')$ satisfies the conclusion of Proposition 2.3.

If $1 < \lambda_0 < 2$, we argue similarly, except that we apply Lemma 2.2 where s is even and t is odd, and we define

$$b(\lambda) = (\lambda-1)(2-\lambda)/(\lambda^2-3\lambda+3). \qquad \square$$

It is easy to prove that, for any connected graph G with n vertices and any real $\lambda \leqslant 1$, we have

$$|P(G,\lambda)| \geqslant |\lambda(\lambda-1)^{n-1}|,$$

with equality holding when G is a tree. We now extend this.

By Jackson's result [6], $P(G,\lambda) \neq 0$ for $1 \leqslant \lambda \leqslant 32/27$. Hence Proposition 2.3 implies by induction on the number of edges of G.

Theorem 2.4. *If G is a graph with n vertices and $1 \leqslant \lambda \leqslant 32/27$, then*

$$|P(G,\lambda)| \geqslant \lambda(\lambda-1)^{n-1}.$$

https://doi.org/10.1017/S0963548397003131 Published online by Cambridge University Press

Theorem 2.5. *If $\lambda_0 > 32/27$, $\epsilon > 0$, then there exists a graph G such that $P(G, \lambda)$ has a root in $(\lambda_0 - \epsilon, \lambda_0 + \epsilon)$.*

Proof. By Proposition 2.3 it suffices to describe a graph G_0 having an edge e such that

$$|P(G_0, \lambda_0)| < (\lambda_0 - 1)\,|P(G_0/e, \lambda_0)| \tag{2.1}$$

and (if $\lambda_0 > 2$)

$$P(G_0, \lambda_0)\,P(G_0/e, \lambda_0) < 0. \tag{2.2}$$

If k is a natural number such that

$$2 \leqslant k - 1 < \lambda_0 < k,$$

then the complete graph K_{k+1} can play the role of G_0.

If $\lambda_0 < 2$, then we do not need condition (2.2). Now (2.1) is satisfied when $G_0 = K_3$ and $3/2 < \lambda_0 < 2$.

Consider now the case where G_0 denotes $K_{2,3}$, the complete bipartite graph whose colour classes have 2 and 3 vertices, respectively. We have

$$P(K_{2,3}, \lambda) = \lambda(\lambda - 1)\,(\lambda^3 - 5\lambda^2 + 10\lambda - 7),$$
$$P(K_{2,3}/e, \lambda) = \lambda(\lambda - 1)\,(\lambda - 2)^2.$$

It is now easy to verify that (2.1) is satisfied for $1.36 \leqslant \lambda_0 < 3/2$ when $G_0 = K_{2,3}$.

To dispose of the remaining case where λ_0 is in the interval $(32/27, 1.36)$, we shall use a result of Jackson [6]. Jackson defines a *double subdivision* of an edge $e = xy$ as the operation of replacing e by two paths of length 2. In [6, Lemma 11], he says that there exists a (smallest) graph G_1 obtained from a triangle C_3 by a sequence of double subdivisions such that $P(G_1, \lambda_0) > 0$. Clearly, $P(C_3, \lambda_0) < 0$. So G_1 is obtained from a graph G_0 by a double subdivision of an edge $e = xy$. The minimality of G_1 implies that $P(G_0, \lambda_0) < 0$.

Now,

$$0 < P(G_1, \lambda_0) = P(G_0, \lambda_0)\,(\lambda_0 - 2)^2 + P(G_0/e, \lambda_0)\,(\lambda_0 - 1)^2.$$

Since $P(G_0, \lambda_0) < 0$, this implies that

$$P(G_0/e, \lambda_0)\,(\lambda_0 - 1)^2 > |P(G_0, \lambda_0)\,|(\lambda_0 - 2)^2,$$

and hence

$$P(G_0/e, \lambda_0)\,(\lambda_0 - 1) > |P(G_0, \lambda_0)|.$$

This last inequality holds since

$$\lambda_0 < (5 - \sqrt{5})/2 = 1.38\ldots.$$ □

3. Special families of graphs

If $\mathscr{F}$ is a family of graphs, then we let $c(\mathscr{F})$ denote the supremum of the roots of the chromatic polynomials of graphs in $\mathscr{F}$. If $c(\mathscr{F})$ is well-defined and $c(\mathscr{F})$ is (respectively, is not) the root of a chromatic polynomial in $\mathscr{F}$, we say that the interval $(c(\mathscr{F}), \infty)$

https://doi.org/10.1017/S0963548397003131 Published online by Cambridge University Press

(respectively, $[c(\mathscr{F}), \infty)$) is the *upper zero-free interval for* $\mathscr{F}$. Woodall [16] proved that the family of bipartite graphs has no upper zero-free interval. Perhaps it has no zero-free intervals in $(32/27, \infty)$ at all.

The situation changes for graph families closed under minors. A *minor* of a graph G is any graph obtained from G by deleting and contracting edges (and removing isolated vertices). A family $\mathscr{F}$ of graphs is *closed under minors* if, for every graph G in $\mathscr{F}$ and every minor G' of G, G' is in $\mathscr{F}$. Mader [9] proved that, for every natural number k, there exists a (smallest) natural number $f(k)$ such that every graph of minimum degree greater than $f(k)$ has a K_k as a minor. Kostochka [8] and Thomason [12] proved that there exist constants c_0, c_1 such that

$$c_1 k \sqrt{\log k} < f(k) < c_0 k \sqrt{\log k}.$$

It follows that, if $\mathscr{F}$ is not the class of all graphs, then there exists a smallest number $d = d(\mathscr{F})$ such that every graph G in $\mathscr{F}$ is d-degenerate, which means that every subgraph of G contains a vertex of degree at most d. With this notation Woodall [18] proved the following.

Theorem 3.1. (Woodall [18]) *If $\mathscr{F}$ is a family of graphs closed under minors such that $\mathscr{F}$ does not contain all graphs, then for every graph G in $\mathscr{F}$ and for every $\lambda > d(\mathscr{F})$,*

$$P(G, \lambda) > 0.$$

Although Woodall's proof is short and elegant we shall here derive Theorem 3.1 from a reduction formula which may be of independent interest.

Let v be a vertex of degree d in a graph G. For each $i (1 \leqslant i \leqslant d)$, we let $P_i(\lambda)$ denote the sum of chromatic polynomials of the graphs obtained from G, by selecting i neighbours of v which are pairwise nonadjacent and then contracting the i edges joining these i neighbours to v. With this notation we have the following.

Theorem 3.2.

$$P(G, \lambda) = (\lambda - d)\, P(G - v, \lambda) + \sum_{i=1}^{d} (i-1)\, P_i(\lambda).$$

Proof. For each natural number $i, 1 \leqslant i \leqslant d$, and each natural number λ, let $Q_i(\lambda)$ denote the number of λ-colourings of $G - v$ such that the neighbours of v have precisely i distinct colours. $Q_i(\lambda)$ may be expressed as a sum of chromatic polynomials. Then

$$P(G - v, \lambda) = \sum_{i=1}^{d} Q_i(\lambda) \quad \text{and} \tag{3.1}$$

$$P(G, \lambda) = \sum_{i=1}^{d} (\lambda - i)\, Q_i(\lambda). \tag{3.2}$$

We now claim that

$$\sum_{i=1}^{d} (d-i)\, Q_i(\lambda) = \sum_{i=1}^{d} (i-1)\, P_i(\lambda). \tag{3.3}$$

As both the RHS (right-hand side) and LHS of (3.3) are polynomials, it suffices to verify (3.3) when λ is a natural number. Consider any colouring of $G - v$ in λ colours such that

https://doi.org/10.1017/S0963548397003131 Published online by Cambridge University Press

the neighbours of v have q colours and the colour classes have $m_1, m_2, \ldots, m_q$ vertices, respectively. The number of times that this colouring is counted on the RHS of (3.3) is

$$\sum_{j=1}^{q} (m_j - 1) = d - q.$$

This is clearly also the number of times it is counted on the LHS of (3.3). Hence (3.3) holds. By (3.1), (3.2) and (3.3),

$$\begin{aligned} P(G,\lambda) &= \sum_{i=1}^{d} (\lambda - i)\, Q_i(\lambda) \\ &= \sum_{i=1}^{d} (d-i)\, Q_i(\lambda) + (\lambda - d)\, P(G-v,\lambda) \\ &= (\lambda - d)\, P(G-v,\lambda) + \sum_{i=1}^{d} (i-1)\, P_i(\lambda). \end{aligned}$$

□

If $\mathscr{F}$ denotes the class of graphs having no K_k as a minor, then Theorem 3.1 and the remark preceding it says that $\mathscr{F}$ has an upper zero-free interval and that this includes $(c_0 k\sqrt{\log k}, \infty)$. We now consider some other specific families of graphs closed under minors.

Proposition 3.3. *Let $\mathscr{F}$ denote a family of graphs closed under minors and let $\mathscr{F}_{max}$ denote the maximal graphs in $\mathscr{F}$, i.e., if G is in $\mathscr{F}_{max}$ and x and y are non-adjacent vertices in G, then $G \cup \{xy\}$ is not in $\mathscr{F}$. If $\mathscr{F}_{max}$ has an upper zero-free interval, then so has $\mathscr{F}$, and the two intervals are identical.*

Proof. It suffices to prove that if λ is a fixed real number such that $P(G,\lambda) > 0$ for each G in $\mathscr{F}_{\max}$, then $P(G,\lambda) > 0$ for each G in $\mathscr{F}$. Suppose (*reductio ad absurdum*) that this is false and let G be a counterexample with n vertices. Add q edges, say, to G such that we obtain a graph in $\mathscr{F}_{\max}$. We choose G such that

(i) n is minimum and

(ii) q is minimum subject to (i).

G cannot be in $\mathscr{F}_{\max}$. So let e be one of the q edges mentioned above. Then

$$P(G,\lambda) = P(G \cup \{e\},\lambda) + P((G \cup \{e\})/e,\lambda).$$

But this is positive by (i) and (ii), a contradiction. □

Let $G_1, G_2, \ldots, G_m$ be pairwise disjoint connected graphs with no separating complete graph, *i.e.* if $S \subseteq V(G_i)$ and S induces a complete graph, then $G - S$ is connected. We now form a new connected graph G_0 as follows. Select a complete subgraph in G_1 and a complete subgraph in G_2 with the same number of vertices and identify the two complete subgraphs. Call the resulting graph $G_1 \cup G_2$. Repeat the construction with $G_1 \cup G_2$, G_3 instead of G_1, G_2, *etc*. We say that $(G_1, G_2, \ldots, G_m)$ is a *simplicial decomposition* of the resulting graph G_0 and that $G_1, G_2, \ldots, G_m$ are the *simplicial summands*. (For more details, see, for instance, Diestel [5].) It is easy to see that the roots of $P(G_0,\lambda)$ are the union of the roots of $P(G_i,\lambda)\, i = 1, 2, \ldots, m$. A graph has *tree-width at most k* if it is a subgraph of a graph that has

https://doi.org/10.1017/S0963548397003131 Published online by Cambridge University Press

a simplicial decomposition whose simplicial summands are complete graphs with at most $k+1$ vertices. As $P(K_{k+1}, \lambda) = \lambda(\lambda-1)\dots(\lambda-k)$, we get the following.

Theorem 3.4. *If k is a natural number, $k \geqslant 2$, then the set M_k of roots of the chromatic polynomials of graphs of tree-width at most k consists of* $0, 1$ *and a dense subset of the interval* $(32/27, k]$.

Proof. Clearly M_k contains 0 and 1. The proof of Theorem 2.5 shows that M_k contains a dense subset of $(32/27, k]$. Theorem 3.1 or Proposition 3.3 combined with the remarks preceding Theorem 3.4 show that M_k contains no number greater than k. □

Theorem 3.4 shows, in particular, that the upper zero-free interval for the graphs of tree-width at most k is (k, ∞). The method may be repeated for any family $\mathscr{F}$ of graphs closed under minors provided the simplicial summands of the graphs in $\mathscr{F}_{\max}$ have a sufficiently simple structure. Diestel [5, pages 180–181] has collected such families and we consider a couple of examples.

Theorem 3.5. *The following three classes of graphs have the same upper zero-free interval*:
(1) *the class of planar graphs*
(2) *the class of maximal planar graphs*
(3) *the class of graphs having no K_5 as a minor*.

Proof. For classes (1) and (2) we apply Proposition 3.3. By a classical theorem of Wagner [14], the maximal graphs in (3) have a simplicial decomposition whose simplicial summands are the graphs in (2) and possibly one more graph, namely a cubic graph on 8 vertices. As all cubic graphs on at most 16 vertices have chromatic polynomials with no root in $(3, \infty)$ (see [4, 10]), and since K_4 is planar and $P(K_4, 3) = 0$, it follows that the upper zero-free intervals for (2) and (3) are identical. □

Birkhoff and Lewis [3] proved that the upper zero-free interval in Theorem 3.5 contains $[5, \infty)$ and conjectured that it contains $[4, \infty)$.

Conjecture 3.6. *The set of roots of the chromatic polynomials of planar graphs consists of* $0, 1$, *and a dense subset of* $(32/27, 4)$.

The graphs used in the proof of Theorem 2.5 show that the roots in Conjecture 3.6 are dense in $(32/27, 3)$. Conjecture 3.6 does not extend to maximal planar graphs [17].

Theorem 3.7. *If $\mathscr{F}$ is the family of graphs having no $K_{3,3}$ as a minor, then the upper zero-free interval for $\mathscr{F}$ is $I \backslash \{4\}$, where I is the interval in Theorem* 3.5.

Proof. By a result of Wagner [15], the maximal graphs having no $K_{3,3}$-minor have simplicial decompositions whose simplicial summands are either maximal planar or isomorphic to K_5. □

https://doi.org/10.1017/S0963548397003131 Published online by Cambridge University Press

Let S denote a surface obtained from the sphere by adding a finite number of handles or cross-caps. Let $\chi(S)$ denote the largest chromatic number of a graph embedded in S. Let $d(S) = d(\mathscr{F})$, where $\mathscr{F}$ is the class of graphs that can be embedded in S. Then

$$\chi(S) = d(S)+1$$

for every surface other than the sphere and the Klein bottle. So Theorem 3.1 or 3.2 implies the following.

Theorem 3.8. *If S is a surface distinct from the sphere and the Klein bottle, then the upper zero-free interval for the graphs embeddable in S* is $(\chi(S)-1, \infty)$.

If S denotes the Klein bottle, then $d(S) = \chi(S) = 6$. Perhaps $(5, \infty)$ is the upper zero-free interval for the Klein bottle. It suffices to verify this for those triangulations of the Klein bottle in which all vertices have degree 6. These graphs are well understood [13], but their chromatic polynomials are not.

More generally, if $\mathscr{F}$ is a family of graphs closed under minors and $\chi(\mathscr{F})$ is the largest chromatic number of a graph in $\mathscr{F}$, then it would be interesting to decide if the upper zero-free interval for $\mathscr{F}$ always contains the interval $[\chi(\mathscr{F}), \infty)$. The case where $\mathscr{F}$ is the family of graphs having no K_k-minor is of particular interest. For this family, Hadwiger's conjecture says that $\chi(\mathscr{F}) = k-1$.

Biggs, Damerell and Sands [2] conjectured that, for each natural number k, there exists a natural number $g(k)$ such that, for each k-regular graph G and each complex root z of $P(G, \lambda)$, z has modulus $\leqslant g(k)$. Brenti, Royle and Wagner [4] asked if 'k-regular' might be relaxed to 'maximum degree at most k'. However, this version would follow from the conjecture of [2] since every graph G of maximum degree k can be extended to a k-regular or $(k+1)$-regular graph H such that G is the union of blocks in H (*i.e.* all roots of $P(G, \lambda)$ are also roots of $P(H, \lambda)$).

Woodall [18] asked whether, for each fixed natural number k, the class of k-degenerate graphs has an upper zero-free interval. We conclude this paper with a negative answer.

Theorem 3.9. *Let $\lambda_0 > 2$ be any real number that is not an integer. Then there exists a 2-degenerate graph G such that*

$$P(G, \lambda_0) < 0.$$

Proof. Let K be any graph such that $P(K, \lambda_0) < 0$. (K may be chosen to be a complete graph.) If v is a vertex of degree d, then we replace v by a set $A(v)$ of d vertices $v_1, v_2, \ldots, v_d$. Each edge uv in K is replaced by an edge from $A(u)$ to $A(v)$ such that the resulting graph K' becomes 1-regular. Now let m be any natural number. For each vertex v of degree d and each $i = 1, 2, \ldots, d-1$, we add m vertices of degree 2 joined to precisely v_i and v_{i+1}. Call the resulting graph $K(m)$. Each λ-colouring of K gives rise to a λ-colouring of K' where all vertices of $A(v)$ have the colour of v for each v in K. The contribution of these colourings to $P(K(m), \lambda)$ is $P(K, \lambda)\cdot(\lambda-1)^{(2q-n)m}$, where $n = |V(K)|$ and $q = |E(K)|$. Any other λ-colouring of K' can be extended to $K(m)$ in at most $(\lambda-1)^{(2q-n-1)m}(\lambda-2)^m$ ways. So, for

https://doi.org/10.1017/S0963548397003131 Published online by Cambridge University Press

each fixed number λ, $P(K(m),\lambda)/(\lambda-1)^{(2q-n)\,m}$ tends to $P(K,\lambda)$ as m tends to infinity. This also holds for $\lambda=\lambda_0$, and hence $P(K(m),\lambda_0)<0$ for m sufficiently large. □

Note that, if K is bipartite, then $K(m)$ is also bipartite. So Woodall's question has a negative answer even for bipartite 2-degenerate graphs. However, if G is a bipartite graph in which all vertices in one of the colour classes have degree 2, then $P(G,\lambda)>0$ for $\lambda\geqslant 2$. This follows easily from Theorem 3.2.

Also note that, if K is planar, then $K(m)$ can be chosen to be planar. In [11] it is mentioned that there are planar graphs K such that $P(K,\lambda)<0$ for some λ close to 4. So, Woodall's conjecture [18] that the chromatic polynomials of planar k-chromatic graphs are positive in the interval $[k,\infty)$ fails for $k=3$.

Note added

Paul Seymour has proved Theorem 3.9 independently. James Oxley has proved a result analogous to Theorem 3.2 (also implying Theorem 3.1) in *Quart. J. Math.* **29** (1978) 11–22.

References

[1] Beraha, S. and Kahane, J. (1979) Is the four-color conjecture almost false? *J. Combin. Theory Ser. B* **27** 1–12.

[2] Biggs, N. L., Damerell, R. M. and Sands, D. A. (1972) Recursive families of graphs. *J. Combin. Theory Ser. B* **12** 123–131.

[3] Birkhoff, G. D. and Lewis, D. C. (1946) Chromatic polynomials. *Trans. Amer. Math. Soc.* **60** 355–451.

[4] Brenti, F., Royle, G. F. and Wagner, D. G. (1994) Location of zeros of chromatic and related polynomials of graphs. *Can. J. Math.* **46** 55–80.

[5] Diestel, R. (1990) *Graph decompositions*. Clarendon Press, Oxford.

[6] Jackson, B. (1993) A zero-free interval for chromatic polynomials of graphs. *Combinatorics, Probability and Computing* **2** 325–336.

[7] Jensen, T. and Toft, B. (1995) *Graph Coloring Problems*. Wiley, New York.

[8] Kostochka, A. (1982) A lower bound for the Hadwiger number of a graph as a function of the average degree of its vertices. *Discret. Analyz. Novosibirsk* **38** 37–58.

[9] Mader, W. (1967) Homomorphieeigenschaften und mittlere Kantendichte von Graphen. *Math. Ann.* **174** 265–268.

[10] Read, R. C. and Royle, G. F. (1991) Chromatic roots of families of graphs. In *Graph Theory, Combinatorics, and Applications* (Alavi, Chartrand, Oellermann and Schwenk, eds), Wiley, New York, pp. 1009–1029.

[11] Read, R. C. and Tutte, W. T. (1988) Chromatic polynomials. In *Selected Topics in Graph Theory*, volume 3 (L. W. Beineke and R. J. Wilson, eds), Academic Press, pp. 15–42.

[12] Thomason, A. (1984) An extremal function for complete subgraphs. *Math. Proc. Camb. Phil. Soc.* **95** 261–265.

[13] Thomassen, C. (1991) Tilings of the torus and the Klein bottle and vertex-transitive graphs on a fixed surface. *Trans. Amer. Math. Soc.* **323** 605–635.

[14] Wagner, K. (1937) Über eine Eigenschaft der ebenen Komplexe. *Math. Ann.* **114** 570–590.

[15] Wagner, K. (1937) Über eine Erweiterung eines Satzes von Kuratowski. *Deutsche Math.* **2** 280–285.

[16] Woodall, D. R. (1977) Zeros of chromatic polynomials. In *Combinatorial Surveys, Proc. Sixth British Combinatorial Conference* (P. Cameron, ed.), Academic Press, London, pp. 199–223.

[17] Woodall, D. R. (1992) A zero-free interval for chromatic polynomials. *Discrete Math.* **101** 333–341.

[18] Woodall, D. R. The largest zero of the chromatic polynomial. *Discrete Math.*, to appear.

https://doi.org/10.1017/S0963548397003131 Published online by Cambridge University Press