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RELIABILITY COMPARISON OF TWO UNIT REDUNDANCY SYSTEMS UNDER THE LOAD REQUIREMENT

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This paper compares the reliability functions of the cold standby, hot standby, and load-sharing redundancy configurations, each of which is composed of two identical components for meeting a given system requirement. Thus far, no research has been done into the conditions that make one configuration more reliable than another because their reliability functions have no closed forms even when the component follows a Weibull lifetime distribution. In this paper, two analytical results are obtained given that the reliability of each configuration is expressed in terms of the design and operational loads of the component. First, higher reliability can be achieved in a cold standby configuration than in a load-sharing configuration if the increase in the component reliability obtained from the reduction in the operational load is not significant. Second, a cold standby configuration exhibits better reliability and carries a higher load than a hot standby configuration if the design load can be increased with a less decrease in the component reliability.

Keywords: cold standby, hot standby, load-sharing, step-stress model

1. INTRODUCTION

In the early design stage, an engineer rates a system to carry a certain amount of load during operation. Design for reliability is then considered to ensure that the system sustains its design rated load for a fixed mission time under the operating condition. When very high reliability is necessary, it is a common practice to implement redundancy in a system design. Two common types of redundancy are hot standby and cold standby redundancies. The hot standby is the same type of redundancy as the simple parallel system, where a redundant component is operating simultaneously with an identical primary component. In the cold standby case, a redundant component stands by to be called upon if a primary component fails. While both the hot and cold standby redundancies assume the independent components, load-sharing is another form of redundancy in which the failure of one component affects the likelihood of failure for the other component. When a given number of redundancies can combine in different ways for meeting the specified load requirement, an engineer confronts a fundamental question of selecting a more reliable configuration [9]. For example, when designing a reliable IT service system or an uninterruptible power system

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so as to supply a given demand, different options are available to combine the components such as the cold standby, the hot standby, and the load-sharing configurations [5,7,12,17,18].

Previous research on design for reliability has typically considered the problem of allocating the redundancy for a given system configuration [16]. For example, the redundancy level was determined to maximize reliability of a series system for the cold standby [3,32,35] and the hot standby [14,19,20], whereas the optimal redundancy allocation strategy was developed for a load-sharing system [10] and a series system with [38] and without repair [40]. However, litter research has been done to compare the reliability functions of different redundancy configurations. Li [18] compared the hot standby and the cold standby configurations each of which is composed of two identical components following an exponential distribution. Yun and Nakagawa [37] compared the hot standby and the cold standby configurations when common causes and component failures are competing with exponential distributions. Mettas and Vassiliou [23] numerically compared the hot standby and the load-sharing configurations for two components following a Weibull lifetime distribution.

At present, there is no research done for comparing the load-sharing and the cold standby configurations even for a simple system of two identical components following a Weibull lifetime distribution. This is probably because their reliability functions have no closed form expressions involving the derivation of the distribution of a sum of random variables [21]. Without using a mathematical model, therefore, Bazovsky [6] discussed that a load-sharing arrangement will exhibit a higher reliability than a cold standby arrangement because components change their failure rates in the former, where the same components undergo higher operational loads all the time in the latter.

The purpose of this paper is to compare the reliability functions of different redundancy configurations each composed of two identical components. Section 2 gives the problem statement. In Section 3, the system reliability is expressed in terms of the component reliability depending on its design and operational loads for each of the cold standby, the load-sharing, and the hot standby configurations. Analytical results are presented in Section 4 to rank the reliability functions of the three configurations when each component follows a Weibull lifetime distribution. We explain how a redundancy configuration that maximizes the system reliability depends on the change of the component reliability in terms of the design rated load as well as the operational load. Finally, conclusions are given in Section 5.

2. PROBLEM STATEMENT

We consider an engineer for selecting a more reliable redundancy configuration that combines two identical components so as to meet the system load requirement denoted by L. In the previous literature, not much research has been done to compare the reliability functions of different redundancy configurations.

A cold standby configuration was compared with a hot standby configuration for the exponential component lifetime distribution, under the implicit assumption that the hot standby is over-designed carrying $2L$ if both components survive and L if one component fails [18,37]. Also, the reliability functions of the load-sharing and the hot standby configurations were numerically compared for the Weibull case, under the assumption that the hot standby is under-designed carrying L if both components survive and $L/2$ if one component fails [23]. At present, no research has been done for comparing the load-sharing and the cold standby configurations even for a simple case of two identical components following a Weibull distribution because their reliability functions have no closed form expressions.

This paper intends to compare the reliability functions of the cold standby, the hot standby, and the load-sharing redundancy configurations. All of the previous

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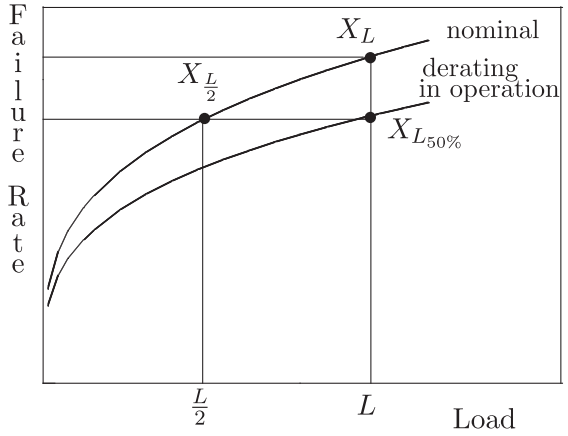


FIGURE 1. Failure rate in terms of design and operational loads.

TABLE 1. Design and operational loads of components under different redundancy configurations

	Design load	Operational load
Cold standby	L	L
Load-sharing	L	$L/2$, L
Hot standby		
(1) Over-designed	L	L
(2) Under-designed	$L/2$	$L/2$

research [18,23,37] have expressed the reliability function of a given redundancy configuration in terms of the fixed component reliability, without referring to its particular design or operational load. For comparing different configurations, however, we must express the reliability function of a given redundancy configuration in terms of the design and operational loads of components. This is because the component reliability strongly depends on its design and operational loads [6,13,15], as further explained below.

It is clear from Table 1 that the effect of different design loads of components must be quantified for comparing the cold standby and the hot standby configurations. Specifically, if $X_{L/2}$ and X_L denote the lifetimes of two components developed by a given engineer having design rated loads of $L/2$ and L, respectively, then the difference between $X_{L/2}$ and X_L depends on the ability of the engineer to increase the component design load. Similarly, different operational loads must be considered to compare the cold standby and the load-sharing configurations. That is, if $X_{L_{50\%}}$ denotes the lifetime of a component operating under the half capacity of its design load L, then the relation between $X_{L_{50\%}}$ and X_L depends on the effect of different operational loads on the same component. Furthermore, when comparing the load-sharing and the hot standby configurations, the trade-off between the design uprating and the operational derating must be considered to relate $X_{L_{50\%}}$ with $X_{L/2}$. In general, it is known that when developing a component to carry L in field, an engineer rates the component to carry L/α in the design assuming that it operates in field at $100\alpha\%$ of its design rated load where $0 < \alpha \leq 1$ [6,25]. It is noted that $X_{L_{50\%}}$ and $X_{L/2}$ are the lifetimes of two components with $\alpha = 0.5$ and $\alpha = 1$, respectively. When comparing the load-sharing and the hot standby configurations later in Section 4, we will consider a special case of $X_{L/2} = X_{L_{50\%}}$ as illustrated in Figure 1. In general, however, the component

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reliability can depend on the value of α, and the optimal value of α that maximizes reliability can be smaller than one if the increase in the failure rate from the design uprating is smaller than the decrease in the failure rate from the operational derating [15].

3. RELIABILITY OF A GIVEN REDUNDANCY CONFIGURATION

In this section, we reconsider the reliability functions of the cold standby, the load-sharing, and the hot standby configurations presented in the literature so as to describe how the reliability function of a given redundancy configuration is expressed in the design and operational loads of components, assuming that each configuration is composed of two identical components.

3.1. Cold Standby System

To arrange two components in a standby configuration, an engineer must rate each component in the ordinal design to carry the load L, and the component is operating at 100% of its design value for the entire mission time. Let X_L be the component lifetime. Let $\overline{F}(t;L)$ and $f(t;L)$ be the reliability and density functions of X_L, respectively. The corresponding system reliability function is well expressed in the literature [4] as follows:

$$R_{\text{cold}}(t) = \overline{F}(t;L) + \int_0^t f(x;L)\overline{F}(t-x;L)\,dx, \quad t>0. \tag{1}$$

While Eq. (1) is obtained for a case of a perfectly reliable sensing and switch-over device, it can be modified to consider the reliability of a failure sensing and switch-over device.

3.2. Load-Sharing System

Two alternate approaches have been taken in the literature for analyzing the reliability function of a load-sharing system. The first one has employed the strength degradation model [22,33,34,39,41], whereas the second one has assumed a given lifetime distribution with either selecting a tampered failure rate model (TFRM) or a cumulative exposure model (CEM) for modeling the effect of the load history on the lifetime distribution [11,18,21,28–30,36]. While most of the literature has considered an exponential component lifetime distribution [13,18,28–30,36], only numerical examples were given for calculating the reliability of a system composed of two components when components follow any other lifetime distributions because the reliability function has no closed form [11,21,39]. To be consistent with the case of a cold standby configuration, we take the second approach and describe the reliability function of a load-sharing system incorporating both the TFRM and the CEM simultaneously.

To arrange two components in a load-sharing system, each component must be rated in the design at L. Then, the component is operating at 50% of its design rated load when both components survive and at 100% after one component fails. Let $X_{L_{50\%}}$ denotes the lifetime of a component operating under the half capacity of its design rated load L. Let $\overline{F}(t;L_{50\%})$ and $f(x;L_{50\%})$ be the reliability and density functions of $X_{L_{50\%}}$, respectively. Also, let $X_{L_{50\%},x,L}$ denotes the lifetime of a component operating under half capacity for a fixed time $[0,x)$ and operating under full capacity for time $[x,\infty)$ for a fixed value of $x>0$. Let $\overline{F}(t;L_{50\%},x,L)$ denotes the reliability function of $X_{L_{50\%},x,L}$. Then, we can express the

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reliability function of a load-sharing configuration by

$$R_{\text{share}}(t) = [\overline{F}(t; L_{50\%})]^2 + 2\int_0^t f(x; L_{50\%})\overline{F}(t; L_{50\%}, x, L)\, dx, \quad t > 0. \tag{2}$$

The first term in Eq. (2) is the probability of both components surviving for the entire mission time. The second term indicates the probability of one component failing at random time x, and the other component surviving for the entire mission time while changing its load at random time x from 50% to 100%, where $x < t$. Although the previous literature expresses the system reliability using $\overline{F}(t; L_{50\%})$ as a baseline [1,2,21], we use $\overline{F}(t; L)$ as a baseline below so as to be consistent with the case of a cold standby configuration.

First, $\overline{F}(t; L_{50\%})$ in Eq. (2) is expressed in terms of $\overline{F}(t; L)$ in Eq. (1) using a constant stress model. Two commonly used constant stress models are the proportional hazard model (PHM) or the accelerated failure time model (AFTM) [24]. Under the PHM [27], the load acts multiplicatively on the failure rate such that the failure rates of the same component operating under two different loads are proportional with each other. Let $h(t; L)$ and $h(t; L_{50\%})$ denote the failure rate function of $\overline{F}(t; L)$ and $\overline{F}(t; L_{50\%})$, respectively. Then, it is given by

$$\phi h(t; L_{50\%}) = h(t; L), \quad t > 0, \tag{3}$$

where $\phi > 1$ is the multiplicative factor that relates the failure rate under full capacity with the failure rate under half capacity for a given design rated load. It follows from Eq. (3) that

$$\overline{F}(t; L_{50\%}) = [\overline{F}(t; L)]^{1/\phi}, \quad t > 0,\ \phi > 1. \tag{4}$$

In the case of AFTM, the load acts multiplicatively on the lifetime assuming that the components experience the same failure mechanism at different operational loads [27]. Thus, we have $X_{L_{50\%}} = \psi X_L$, where $\psi > 1$ is the accelerated factor from the operational load $L/2$ to L for a given design load of L. It follows that

$$\overline{F}(t; L_{50\%}) = \Pr\left(X_L > \frac{t}{\psi}\right) = \overline{F}\left(\frac{t}{\psi}; L\right), \quad t > 0,\ \psi > 1. \tag{5}$$

A specific form of ψ and ϕ is usually considered for modeling the reliability of a load-sharing configuration such as the power law or the log-linear functions [22–24,28].

Second, we express $\overline{F}(t; L_{50\%}, x, L)$ in Eq. (2) in terms of $\overline{F}(t; L)$ in Eq. (1) using a step-stress model. Two commonly used step-stress models are the TFRM [8] and the CEM [26]. Let $h_{\text{TFR}}(t; L_{50\%}, x, L)$ and $\overline{F}_{\text{TFR}}(t; L_{50\%}, x, L)$ be the failure rate and reliability functions of $X_{L_{50\%},x,L}$ under the TFR, respectively. Since

$$h_{\text{TFR}}(t; L_{50\%}, x, L) = \begin{cases} \dfrac{1}{\phi} h(t; L) & \text{if } t \le x, \\ h(t; L) & \text{if } t > x, \end{cases}$$

we obtain

$$\overline{F}_{\text{TFR}}(t; L_{50\%}, x, L) = [\overline{F}(x; L)]^{(1/\phi)-1}\overline{F}(t; L), \quad t > x,\ \phi > 1. \tag{6}$$

Alternatively, the reliability function of $X_{L_{50\%},x,L}$ under the CEM is given [1,26] by

$$\overline{F}_{\text{CE}}(t; L_{50\%}, x, L) = \overline{F}(t - x + x^e; L), \tag{7}$$

where x^e is the solution of $\overline{F}(x^e; L) = \overline{F}(x; L_{50\%})$ with $x^e < x$. While x in Eq. (6) or Eq. (7) denotes the fixed accelerated test time in the step-stress model, x in a load-sharing system is a random failure time at which one of the two components fails.

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Substituting Eqs. (6) and (7) in Eq. (2), we obtain the reliability of a load-sharing system under the TFR and the CEM by

$$R_{\mathrm{TFR}}(t) = \begin{cases} \dfrac{2}{2-\phi}\overline{F}(t;L) - \dfrac{\phi}{2-\phi}[\overline{F}(t;L)]^{2/\phi} & \text{if } \phi \neq 2, \\ \overline{F}(t;L) + \overline{F}(t;L)\int_0^t h(x;L)\,dx & \text{if } \phi = 2, \end{cases} \tag{8}$$

and

$$R_{\mathrm{CE}}(t) = \left[\overline{F}\left(\frac{t}{\psi};L\right)\right]^2 + 2\int_0^t \frac{1}{\psi} f\left(\frac{x}{\psi};L\right)\overline{F}(t-x+x^e;L)\,dx, \quad t>0,\ \psi>1, \tag{9}$$

respectively. Equation (9) has a closed form solution only if the component follows an exponential lifetime distribution [28], whereas a numerical calculation is considered for the Weibull case [21].

3.3. Hot Standby System

For a hot standby configuration, we consider two different design schemes. In the over-designed hot standby configuration, each component has the lifetime X_L, and the corresponding system reliability is well given in the literature [4,18] by

$$R_{\mathrm{over}}(t) = 2\overline{F}(t;L) - [\overline{F}(t;L)]^2, \quad t>0. \tag{10}$$

In the under-designed hot standby configuration [23], each component has the lifetime $X_{L/2}$. Let $\overline{F}(t;L/2)$ be the reliability function of $X_{L/2}$. The corresponding system reliability is expressed by

$$R_{\mathrm{under}}(t) = 2\overline{F}\left(t;\frac{L}{2}\right) - \left[\overline{F}\left(t;\frac{L}{2}\right)\right]^2, \quad t>0. \tag{11}$$

The under-designed hot standby configuration is possible in many operations where the degradation of system performance is accepted [6]. One example is a system of two 40 kVA generators supplying 80 kVA initially at full capacity; however, when one generator fails, the other generator supplies only 40 kVA after immediately disconnecting nonessential loads to avoid tripping the surviving generator.

To relate $\overline{F}(t;L/2)$ in Eq. (11) with $\overline{F}(t;L)$ in Eq. (1), we must consider the effect of the design rated load on the component reliability. Previously, NASA [25] has proposed a power law model with an exponential distribution to explain the effect of different design thrust values on the engine reliability. The model is given by $\lambda_1 = \varphi\lambda_2$, where $\varphi = (L_1/L_2)^b$, and λ_1 and λ_2 are the failure rates at design load of L_1 and L_2, respectively. We generalize this model by assuming that $X_{L/2} = \varphi X_L$, where $\varphi > 1$ is a factor for scaling the lifetime depending on the ability of the engineer to produce a component from design load L to $L/2$. The value of φ gets smaller if the engineer is more capable of increasing the design load with a lesser decrease in the component reliability. Thus, we obtain

$$\overline{F}\left(t;\frac{L}{2}\right) = \overline{F}\left(\frac{t}{\varphi};L\right), \quad t>0.$$

4. RELIABILITY COMPARISONS

We compare the cold standby with the load-sharing configurations in Section 4.1, and the cold standby with the hot standby configurations in Section 4.2. Then, these results are combined in Section 4.3 for comparing all of the three redundancy configurations.

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4.1. Effect of Operational Rating

Lemma 1 compares the cold standby and the load-sharing configurations under the TFRM for the general component lifetime distribution.

LEMMA 1: *Let $R_{\text{cold}}(t)$ and $R_{\text{TFR}}(t)$ be the reliability functions of the cold standby and the load-sharing systems given in Eqs. (1) and (8), respectively. Let ϕ denotes the multiplicative factor for the TFRM defined in Eq. (8).*

i. If the component has a constant failure rate (CFR) distribution, then $R_{\text{cold}}(t) \lesseqgtr R_{\text{TFR}}(t)$ whenever $\phi \gtreqless 2$.

ii. If the component has either the increasing failure rate (IFR) or the decreasing failure rate (DFR) distributions, then a value of ϕ, denoted by ϕ_b, exists such that $R_{\text{cold}}(t) \lesseqgtr R_{\text{TFR}}(t)$ whenever $\phi \gtreqless \phi_b$. We have $\phi_b > 2$ for an IFR and $\phi_b < 2$ for a DFR.

PROOF: We prove only the IFR case; other cases can be proved similarly. Fix $t > 0$. If $\phi = 2$, then $R_{\text{cold}}(t) > R_{\text{TFR}}(t)$ because $\overline{F}(t-x;L)\overline{F}(x;L) > \overline{F}(t;L)$ for the IFR case. This means that $R_{\text{cold}}(t) > R_{\text{TFR}}(t)$ whenever $\phi < 2$. As $R_{\text{cold}}(t)$ is independent of ϕ and $R_{\text{TFR}}(t)$ increases with ϕ, a value of ϕ exists such that $R_{\text{cold}}(t) = R_{\text{TFR}}(t)$ if $\phi > 2$. This completes the proof. ■

One can see from Lemma 1 that a higher reliability can be achieved in a load-sharing arrangement than in a cold standby arrangement only if an appreciable gain is obtained in the component reliability by operating the component at one-half of its design load. The gain in the component reliability, measured by ϕ, is compared with a threshold. Since the threshold value gets larger for the IFR than the DFR distributions, it is more likely to choose a cold standby than a load-sharing arrangement when the component has an IFR. Intuitively, this is because a component surviving in a load-sharing system is required to operate the entire mission time, where the same component in a cold standby system is subject to a shorter mission time.

Lemma 2 compares the cold standby system with the load-sharing system under the CEM for the Weibull case.

LEMMA 2: *Let $R_{\text{cold}}(t)$ and $R_{\text{CE}}(t)$ be the reliability function of a cold standby and a load-sharing system given in Eqs. (1) and (9), respectively. Assume that each component in the system follows a Weibull lifetime distribution with the scale parameter changing by the operational loads and the shape parameter keeping constant. Let ψ be the acceleration factor for the CEM given in Eq. (9). We have $R_{\text{CE}}(t) \gtreqless R_{\text{cold}}(t)$ whenever $\psi \gtreqless 2$.*

PROOF: Let $\overline{F}(t;L) = \exp[-(\lambda t)^\beta], t > 0, \lambda, \beta > 0$. Equations (1) and (9) are expressed by

$$R_{\text{cold}}(t) = \exp[-(\lambda t)^\beta] + \int_0^t \lambda^\beta \beta x^{\beta-1} \exp[-(\lambda x)^\beta] \exp[-\lambda^\beta (t-x)^\beta]\, dx$$

and

$$R_{\text{CE}}(t) = \exp\left[-2\left(\frac{\lambda t}{\psi}\right)^\beta\right] + 2\int_0^t \left(\frac{\lambda}{\psi}\right)^\beta \beta x^{\beta-1} \exp\left[-\left(\frac{\lambda x}{\psi}\right)^\beta\right] \times \exp\left\{-\left[\lambda(t-x) + \frac{\lambda}{\psi}x\right]^\beta\right\} dx, \tag{12}$$

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respectively. If $\psi = 2$, then Eq. (12) can be obtained by

$$R_{\mathrm{CE}}(t) = \exp\left[-2\left(\frac{\lambda}{2}t\right)^{\beta}\right] + 2\int_0^{t/2} G(x,t)\,dx,$$

where $G(x,t) = \lambda^{\beta}\beta x^{\beta-1}\exp[-(\lambda x)^{\beta}]\exp[-\lambda^{\beta}(t-x)^{\beta}]$. By expressing

$$\int_{t/2}^{t} G(x,t)\,dx = \int_0^{t/2} G(t-x,t)\,dx,$$

one can demonstrate that

$$\int_0^{t/2} G(x,t) - G(t-x,t)\,dx = \exp[-(\lambda t)^{\beta}] - \exp\left[-2\left(\frac{\lambda}{2}t\right)^{\beta}\right].$$

Therefore, if $\psi = 2$, then $R_{\mathrm{cold}}(t) = R_{\mathrm{CE}}(t)$, implying that the difference of the first failures between the two configurations is exactly the same as the difference of the second failures between the two configurations, irrespective of the shape parameter values. Otherwise, if $\psi \neq 2$, then let $\lambda^* = 2\lambda/\psi$. If $\psi \lessgtr 2$, then $\lambda^* \gtrless \lambda$. Equation (12) can be expressed by

$$R_{\mathrm{CE}}(t) = \exp\left[-2\left(\frac{\lambda^*}{2}t\right)^{\beta}\right] + 2\int_0^{t}\left(\frac{\lambda^*}{2}\right)^{\beta}\beta x^{\beta-1}\exp\left[-\left(\frac{\lambda^*}{2}x\right)^{\beta}\right] \times \exp\left\{-\left[\lambda^*(t-x) + \frac{\lambda^*}{2}x\right]^{\beta}\right\} dx$$

from which we conclude that $R_{\mathrm{CE}}(t) \lessgtr R_{\mathrm{cold}}(t)$ if $\psi \lessgtr 2$. ■

Lemma 2 shows that the reliability of the cold standby system is equivalent to that of the load-sharing system under the CEM if the component lifetime under half capacity is exactly twice its lifetime under full capacity. One may recall from Lemma 1 that the threshold value to choose the load-sharing system over the cold standby system depends on the IFR or the DFR property of the component distribution if the TFRM is used; however, the threshold value is two if the CEM is used, irrespective of the component aging property. This is probably because the load acts multiplicatively on the failure rate in the TFRM but on the lifetime in the CEM.

We now compare the reliability functions of the TFRM and the CEM for the Weibull case.

LEMMA 3: *Let $R_{\mathrm{TFR}}(t)$ and $R_{\mathrm{CE}}(t)$ be the reliability functions of a load-sharing system under the TFRM and the CEM, respectively. Assume that each component in the system follows a Weibull lifetime distribution with the scale parameter depending on the operational loads and the shape parameter, denoted by β, keeping constant. If $\beta \gtrless 1$, then $R_{\mathrm{CE}}(t) \gtrless R_{\mathrm{TFR}}(t)$.*

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PROOF: We first recall that Eqs. (4) and (5) are the same for a Weibull distribution, where $\phi = \psi^{\beta}$. Then, Eqs. (6) and (7) are expressed by

$$\overline{F}_{\mathrm{TFR}}(t; L_{50\%}, x, L) = \exp\left[-\frac{1}{\phi}(\lambda x)^{\beta}\right] \exp\left[-(\lambda t)^{\beta} + (\lambda x)^{\beta}\right] \quad \text{and}$$

$$\overline{F}_{\mathrm{CE}}(t; L_{50\%}, x, L) = \exp\left[-\lambda^{\beta}\left(t - x + \frac{x}{\psi}\right)^{\beta}\right], \quad t > x,$$

respectively. The result follows because

$$\lambda^{\beta}\left(t - x + \frac{x}{\psi}\right)^{\beta} \lessgtr \left(\frac{\lambda x}{\psi}\right)^{\beta} + (\lambda t)^{\beta} - (\lambda x)^{\beta} \quad \text{if } \beta \gtrless 1.$$

■

Wang and Fei [31] proved that the two step-stress models of the CEM and the TFRM are identical when the underlying distribution is exponential. Lemma 3 states that for the Weibull case, which of the CEM and the TFRM exhibit higher reliability depends on the component aging property. In the following, Lemmas 1–3 are combined in Proposition 1, under the assumption that the component follows a Weibull lifetime distribution.

PROPOSITION 1: *Let $R_{\mathrm{cold}}(t)$, $R_{\mathrm{TFR}}(t)$, and $R_{\mathrm{CE}}(t)$ be the system reliability function defined in Eqs. (1), (8), and (9), respectively. Assume that each component in the system follows a Weibull lifetime distribution where the scale parameter increases with the operational load; however, the shape parameter, denoted by $\beta > 0$, is constant. Let ψ be the acceleration factor for the CEM.*

i. Let $\beta = 1$. If $\psi \gtrless 2$, then $R_{\mathrm{cold}}(t) \lessgtr R_{\mathrm{TFR}}(t) = R_{\mathrm{CE}}(t)$.

ii. Let $\beta > 1$.
   a. If $\psi \leq 2$, then $R_{\mathrm{cold}}(t) \geq R_{\mathrm{CE}}(t) > R_{\mathrm{TFR}}(t)$.
   b. If $\psi > 2$, then we define ψ_b as the value of ψ such that $R_{\mathrm{cold}}(t) = R_{\mathrm{TFR}}(t)$. If $\psi > \psi_b$, then $R_{\mathrm{cold}}(t) < R_{\mathrm{TFR}}(t) < R_{\mathrm{CE}}(t)$. Otherwise, if $\psi < \psi_b$, then $R_{\mathrm{TFR}}(t) < R_{\mathrm{cold}}(t) < R_{\mathrm{CE}}(t)$.

iii. Let $\beta < 1$.
   a. If $\psi \geq 2$, then $R_{\mathrm{cold}}(t) \leq R_{\mathrm{CE}}(t) < R_{\mathrm{TFR}}(t)$.
   b. If $\psi < 2$, then we define ψ_b as the value of ψ such that $R_{\mathrm{cold}}(t) = R_{\mathrm{TFR}}(t)$. If $\psi > \psi_b$, then $R_{\mathrm{CE}}(t) < R_{\mathrm{cold}}(t) < R_{\mathrm{TFR}}(t)$. Otherwise, if $\psi < \psi_b$, then $R_{\mathrm{CE}}(t) < R_{\mathrm{TFR}}(t) < R_{\mathrm{cold}}(t)$.

PROOF: The result follows directly from Lemmas 1–3. ■

Proposition 1 shows that the decision of which configuration gives higher reliability depends on (i) the component aging property, (ii) how much gain is achieved in the component reliability when operating the components under half capacity of its design load, and (iii) which of the CEM or the TFRM is selected for the step-stress model. If component reliability increases significantly more under half capacity than under full capacity, then a load-sharing system establishes higher reliability than a cold standby system, irrespective of selecting the CEM or the TFRM. Specifically, the acceleration factor, denoted by ψ, should be greater than two if $\beta \leq 1$, and greater than ψ_b, which is larger than two if $\beta > 1$. On

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the other hand, if the increase in the component reliability obtained by the reduction of the operational load is negligible, then the cold standby system gives a higher reliability than the load-sharing system. Specifically, the value of ψ should be smaller than two if $\beta \geq 1$ and smaller than ψ_b, which is smaller than two if $\beta < 1$. Finally, if the increase in the component reliability obtained from the reduction of the operational load is neither negligible nor significant (i.e., if $\psi_b < \psi < 2$), then the reliability of the cold standby system is between the two load-sharing models. It is clear that the threshold value gets smaller if reliability of a sensing and switching device is incorporated in Eq. (1).

4.2. Effect of Design Rating

Proposition 2 compares the cold standby and the hot standby configurations for a general component lifetime distribution.

PROPOSITION 2: *Let X_L and Y_L denote the lifetimes of two identical and independent components operating at design load L. For a given mission time $t > 0$, let $R_{\text{over}}(t) = \Pr(\max(X_L, Y_L) > t)$, $R_{\text{cold}}(t) = \Pr(X_L + Y_L > t)$, and $R_{\text{under}}(t) = \Pr(\max(\varphi X_L, \varphi Y_L) > t)$, where $\varphi > 1$.*

i. If $\varphi \geq 2$, then $R_{\text{over}}(t) < R_{\text{cold}}(t) < R_{\text{under}}(t)$.

ii. If $\varphi < 2$, then a value of φ, denoted by φ_b, exists such that $R_{\text{cold}}(t) = R_{\text{under}}(t)$. If $\varphi > \varphi_b$, then $R_{\text{over}}(t) < R_{\text{cold}}(t) < R_{\text{under}}(t)$. If $\varphi < \varphi_b$, then $R_{\text{over}}(t) < R_{\text{under}}(t) < R_{\text{cold}}(t)$.

PROOF: Because $X_L + Y_L$ is stochastically larger than $\max(X_L, Y_L)$, we have $R_{\text{over}}(t) < R_{\text{cold}}(t)$. Since $R_{\text{under}}(t)$ increases in φ and $R_{\text{under}}(t) > R_{\text{cold}}(t)$ when $\varphi = 2$, we conclude $R_{\text{under}}(t) > R_{\text{cold}}(t)$ whenever $\varphi \geq 2$. ■

PROOF: The result follows directly from the fact that $R_{\text{under}}(t)$ increases in φ. ■

Li [18] showed that the over-designed hot standby system achieves lower reliability than the cold standby system. Similarly, one may theorize that an under-designed hot standby system will exhibit higher reliability than a cold standby system because the former carries a lower load than the latter. However, Proposition 2 shows that such a conjecture is true only if the component lifetime decreases more than half when the design rated load is double rating (i.e., if $\varphi > \varphi_b$). To illustrate, we consider the component following a Weibull distribution. While the value of φ_b depends on the mission time, one can show that, if $\varphi < g(\beta)$, where $g(\beta) = [1 - (1/\beta)(1/2)^{(1/\beta)+1}]^{-1}$, then a cold standby system exhibits a longer mean lifetime than an under-designed hot standby system. As $g(\beta)$ has a maximum value of 1.36 at $\beta = 0.693$, the cold standby system achieves a longer lifetime and carries a higher load on average than the under-designed hot standby system if the component lifetime is decreased less than approximately three-fourths (i.e., 1/1.36) when the design rated load is double rated. It is noted that in Proposition 2, the component failures in the hot standby configuration are assumed to be independent which is rarely the case in practice because the components are subjected to the same system load.

4.3. Effect of Design and Operational Ratings

Proposition 3 is given under the assumption that $X_{L/2} = X_{L_{50\%}}$, as illustrated in Section 2.

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PROPOSITION 3: *Let* $R_{\text{over}}(t)$, $R_{\text{cold}}(t)$, $R_{\text{CE}}(t)$, $R_{\text{TFR}}(t)$, *and* $R_{\text{under}}(t)$ *be as defined in Propositions 1 and 2. Assume that two components in each system follow a Weibull lifetime distribution where the scale parameter increases with the operational load; however, the shape parameter, denoted by* $\beta > 0$, *is constant. Let* ψ *in Proposition 1 be equivalent to* φ *in Proposition 2.*

i. Let $\beta = 1$.
   a. If $\varphi = 2$, *then* $R_{\text{over}}(t) < R_{\text{cold}}(t) = R_{\text{CE}}(t) = R_{\text{TFR}}(t) < R_{\text{under}}(t)$.
   b. If $\varphi > 2$, *then* $R_{\text{over}}(t) < R_{\text{cold}}(t) < R_{\text{CE}}(t) = R_{\text{TFR}}(t) < R_{\text{under}}(t)$.
   c. If $\varphi < 2$, *then let* φ_l *be the value of* φ *such that* $R_{\text{cold}}(t) = R_{\text{under}}(t)$. *If* $\varphi > \varphi_l$, *then* $R_{\text{over}}(t) < R_{\text{CE}}(t) = R_{\text{TFR}}(t) < R_{\text{cold}}(t) < R_{\text{under}}(t)$. *Otherwise, if* $\varphi < \varphi_l$, *then* $R_{\text{over}}(t) < R_{\text{CE}}(t) = R_{\text{TFR}}(t) < R_{\text{under}}(t) < R_{\text{cold}}(t)$.

ii. Let $\beta > 1$.
   a. If $\varphi = 2$, *then* $R_{\text{over}}(t) < R_{\text{TFR}}(t) < R_{\text{CE}}(t) = R_{\text{cold}}(t) < R_{\text{under}}(t)$.
   b. If $\varphi < 2$, *then let* φ_l *be the value of* φ *such that* $R_{\text{cold}}(t) = R_{\text{under}}(t)$. *If* $\varphi > \varphi_l$, *then* $R_{\text{over}}(t) < R_{\text{TFR}}(t) < R_{\text{CE}}(t) < R_{\text{cold}}(t) < R_{\text{under}}(t)$. *Otherwise, if* $\varphi < \varphi_l$, *then* $R_{\text{over}}(t) < R_{\text{TFR}}(t) < R_{\text{CE}}(t) < R_{\text{under}}(t) < R_{\text{cold}}(t)$.
   c. If $\varphi > 2$, *then there exists* φ_b *such that* $R_{\text{TFR}}(t) = R_{\text{cold}}(t)$ *depending on the given value of* $t > 0$ *and* $\beta < 1$. *If* $\varphi > \varphi_b$, *then* $R_{\text{over}}(t) < R_{\text{cold}}(t) < R_{\text{TFR}}(t) < R_{\text{CE}}(t) < R_{\text{under}}(t)$. *Otherwise, if* $\varphi < \varphi_b$, *then* $R_{\text{over}}(t) < R_{\text{TFR}}(t) < R_{\text{cold}}(t) < R_{\text{CE}}(t) < R_{\text{under}}(t)$.

iii. Let $\beta < 1$.
   a. If $\varphi = 2$, *then* $R_{\text{over}}(t) < R_{\text{cold}}(t) = R_{\text{CE}}(t) < R_{\text{TFR}}(t) < R_{\text{under}}(t)$.
   b. If $\varphi > 2$, *then* $R_{\text{over}}(t) < R_{\text{cold}}(t) < R_{\text{CE}}(t) < R_{\text{TFR}}(t) < R_{\text{under}}(t)$.
   c. If $\varphi < 2$, *then there exists* φ_b *such that* $R_{\text{TFR}}(t) = R_{\text{cold}}(t)$ *and* φ_l *such that* $R_{\text{cold}}(t) = R_{\text{under}}(t)$ *where* $\varphi_l < \varphi_b$. *If* $\varphi > \varphi_b$, *then* $R_{\text{over}}(t) < R_{\text{CE}}(t) < R_{\text{cold}}(t) < R_{\text{TFR}}(t) < R_{\text{under}}(t)$. *If* $\varphi_l < \varphi < \varphi_b$, *then* $R_{\text{over}}(t) < R_{\text{CE}}(t) < R_{\text{TFR}}(t) < R_{\text{cold}}(t) < R_{\text{under}}(t)$. *Otherwise, if* $\varphi < \varphi_l$, *then* $R_{\text{over}}(t) < R_{\text{CE}}(t) < R_{\text{TFR}}(t) < R_{\text{under}}(t) < R_{\text{cold}}(t)$.

PROOF: Part (a) is obtained directly from Propositions 1 and 2. To prove part (b), it is sufficient to show that $R_{\text{under}}(t) > R_{\text{CE}}(t)$ when $\beta = 1$ and $\varphi > 2$. Since Eq. (11) is given by

$$R_{\text{under}}(t) = 2\exp\left[-\left(\frac{\lambda}{\varphi}t\right)^{\beta}\right] - \exp\left[-2\left(\frac{\lambda}{\varphi}t\right)^{\beta}\right], \quad t > 0,\ \varphi > 1,\ \lambda, \beta > 0, \tag{13}$$

if $\beta = 1$, then it follows from Eq. (12) that

$$R_{\text{under}}(t) - R_{\text{CE}}(t) = 2\left\{\exp\left(-\frac{\lambda t}{\varphi}\right) - \exp\left(-\frac{2\lambda t}{\varphi}\right) - \frac{1}{2-\varphi}\left[\exp(-\lambda t) - \exp\left(-\frac{2\lambda t}{\varphi}\right)\right]\right\}.$$

We have $R_{\text{under}}(t) > R_{\text{CE}}(t)$ because if $g(x)$ is a convex function of x, then $h(x_1, x_2) = (g(x_1) - g(x_2))/(x_1 - x_2)$ is increasing in x_1 for every fixed x_2. Part (c) follows because $R_{\text{CE}}(t) = R_{\text{over}}(t)$ at $\varphi = 1$, and $R_{\text{CE}}(t)$ increases with φ. ■

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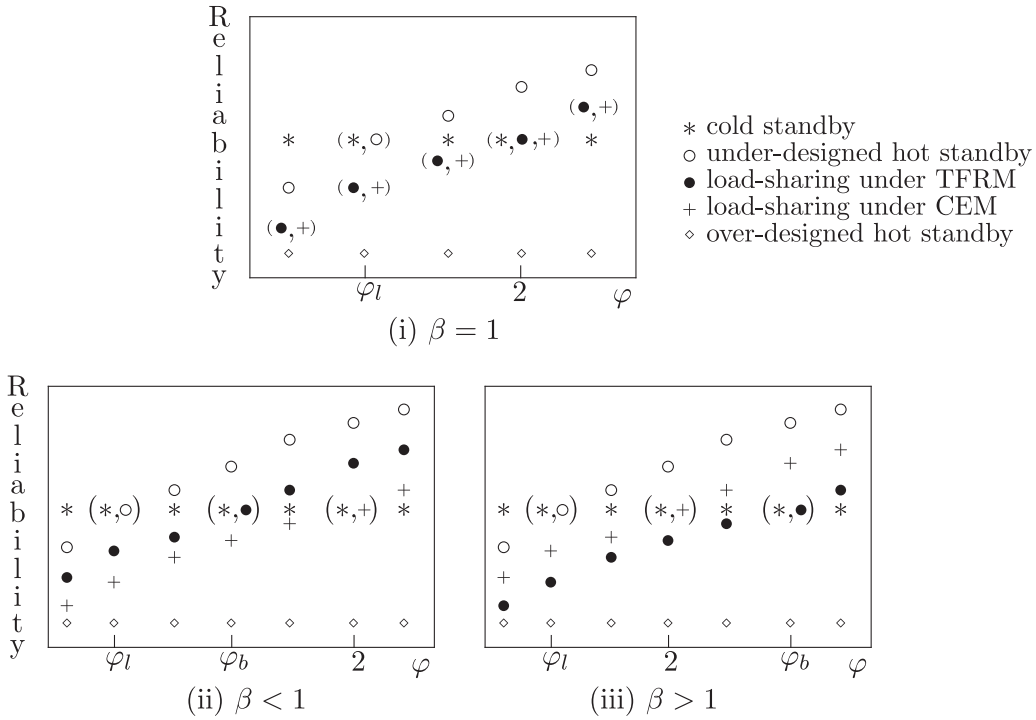


FIGURE **2.** Illustration of Proposition 3.

PROOF: It is sufficient to show that $R_{\mathrm{CE}}(t) < R_{\mathrm{under}}(t)$ from Propositions 1 and 2. As $\lambda(t-x) + \lambda x/\varphi > (\lambda/\varphi)(t-x) + \lambda x/\varphi = \lambda t/\varphi$, we can express this by

$$\begin{aligned} R_{\mathrm{CE}}(t) &= \exp\left[-2\left(\frac{\lambda t}{\varphi}\right)^{\beta}\right] + 2\int_0^t \left(\frac{\lambda}{\varphi}\right)^{\beta} \beta x^{\beta-1} \exp\left[-\left(\frac{\lambda}{\varphi}x\right)^{\beta}\right] \\ &\quad \times \exp\left\{-\left[\lambda(t-x) + \frac{\lambda}{\varphi}x\right]^{\beta}\right\} dx \\ &< \exp\left[-2\left(\frac{\lambda t}{\varphi}\right)^{\beta}\right] + 2\int_0^t \left(\frac{\lambda}{\varphi}\right)^{\beta} \beta x^{\beta-1} \exp\left[-\left(\frac{\lambda}{\varphi}x\right)^{\beta}\right] \exp\left[-\left(\frac{\lambda}{\varphi}t\right)^{\beta}\right] \\ &= 2\exp\left[-\left(\frac{\lambda}{\varphi}t\right)^{\beta}\right] - \exp\left[-2\left(\frac{\lambda}{\varphi}t\right)^{\beta}\right] = R_{\mathrm{under}}(t). \end{aligned}$$

■

PROOF: The proof is complete if we show $R_{\mathrm{TFR}}(t) < R_{\mathrm{under}}(t)$. Using $\phi = \varphi^{\beta}$ with Eqs. (8) and (13), it is expressed by

$$\begin{aligned} R_{\mathrm{under}}(t) - R_{\mathrm{TFR}}(t) &= 2\left\{\exp\left[-\left(\frac{\lambda t}{\varphi}\right)^{\beta}\right] - \exp\left[-2\left(\frac{\lambda t}{\varphi}\right)^{\beta}\right]\right\} \\ &\quad - \frac{2}{2-\varphi^{\beta}}\left\{\exp\left[-(\lambda t)^{\beta}\right] - \exp\left[-2\left(\frac{\lambda t}{\varphi}\right)^{\beta}\right]\right\}, \end{aligned}$$

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which is greater than zero irrespective of the value of β and φ because if $g(x)$ is a convex function of x, then $h(x_1, x_2) = (g(x_1) - g(x_2))/(x_1 - x_2)$ is increasing in x_1 for every fixed x_2. ■

Figure 2 illustrates Proposition 3 from which the following observations are made. First, a cold standby configuration establishes the highest reliability among the three configurations if the acceleration factor is smaller than a threshold value (i.e., if $\varphi < \varphi_l$). Second, if $\varphi > \varphi_l$, then one may choose the under-designed hot standby configuration whenever the degradation of the performance is accepted. If the degradation of the performance is not acceptable, then the load-sharing system is selected when $\varphi > 2$ for $\beta \leq 1$, and $\varphi > \varphi_b$ for $\beta > 1$. Otherwise, the choice depends on which of the CEM or the TFRM is used for modeling the effect of the operational load change.

5. CONCLUSION

When different system configurations are possible in the early design stage to arrange a fixed number of redundancy components, how to select a design configuration is one of the important problems for practitioners and researchers. In this study, the reliability functions of three different redundancy configurations were compared after introducing the accelerated stress model for changing the component reliability in terms of the design rated load as well as the operational load. The results include two findings that are useful for practitioners. First, a load-sharing arrangement is selected over a cold standby arrangement if the component reliability increases significantly when derating in operation. If the component reliability increase from the reduction of the operational load is small enough, then the cold standby system can be more reliable than the load-sharing system. Otherwise, the choice between the cold standby system and the load-sharing system depends on which of the CEM and the TFRM are selected for the step-stress model. Second, the cold standby system exhibits better reliability and carries a higher load than the under-designed hot standby system if the engineer is capable of increasing the component design load without increasing its failure rate significantly.

To compare the three redundancy configurations, we assumed that each component in a given system configuration is designed to sustain the rated load value. It is possible for certain types of components to achieve high reliability by using a design uprating for the operational derating; as such, future research will focus on how this effects the system configuration.

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