Matter for Debate

Correct answer – dodgy method

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One of the great joys of teaching mathematics at tertiary level is that many of one's students go on to teach mathematics at secondary level and stay in touch, often with queries or knotty points that arise in school mathematics. Some years ago I received a letter from a former student of mine which contained the following query.

Example 1

"Why is there such a fuss about deriving formulae for the derivatives of products and quotients of functions? Why not use the following technique?"

Let $f(x) = u(x)v(v)$, so $\log f(x) = \log u(x) + \log v(x)$, and differentiating both sides we get

$$\frac{f'(x)}{f(x)} = \frac{u'(x)}{u(x)} + \frac{v'(x)}{v(x)} = \frac{u(x)v'(x) + v(x)u'(x)}{u(x)v(x)}$$

$$= \frac{u(x)v'(x) + v(x)u'(x)}{f(x)}; \text{ so } f'(x) = u(x)v'(x) + v(x)u'(x).$$

Similarly, if $f(x) = \dfrac{u(x)}{v(x)}$, with $v(x) \neq 0$, then $\log f(x) = \log u(x) - \log v(x)$, and differentiating both sides gives

$$\frac{f'(x)}{f(x)} = \frac{u'(x)}{u(x)} - \frac{v'(x)}{v(x)} = \frac{u(x)v'(x) - v(x)u'(x)}{u(x)v(x)}.$$

$$\text{So } f'(x) = \frac{u(x)}{v(x)}\left[\frac{v(x)u'(x) - u(x)v'(x)}{u(x)v(x)}\right] = \frac{v(x)u'(x) - u(x)v'(x)}{(v(x))^2}.$$

My immediate reaction was as follows:

1. You are using the derivative properties of the logarithmic function – how do you propose to establish these?
2. The logarithmic function is defined only for positive values; your method is not valid if $f(x)$, $u(x)$ or $v(x)$ is ever negative or zero.

Of course, the so-called logarithmic differentiation seems inevitable for functions such as $f(x) = x^x$ defined only for positive x. We have

$$\log f(x) = x \log x, \text{ so } \frac{f'(x)}{f(x)} = x \cdot \frac{1}{x} + \log x.$$

$$\text{So } f'(x) = f(x)(1 + \log x) = x^x(1 + \log x).$$

Similar methods apply to the function $g(x) = x^{1/x}$ for $x > 0$, but it is interesting that textbooks which use logarithmic differentiation rarely mention the above restrictions.

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It struck me that many mathematics courses, mathematics textbooks and student answers include methods which give the correct answers to problems but using techniques that are very dubious and would be difficult to justify. Of course, we must not be judgemental, because many great discoveries in mathematics came about in this manner and the proofs and justifications were supplied by later generations. The great Euler, for example, reasoned that because $f(x) = \sin x$ vanished at $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$, then it could be expressed as an infinite product,

$$x\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right)\ldots,$$

a correct result by very dubious means.

We now present a selection of correct mathematical results arrived at by dubious means. Our first example was the derivation of the formulae for derivatives of products and quotients of functions already referred to.

Example 2

Expression of repeating decimals as rational numbers, e.g. express $0.252525\ldots = 0.\overline{25}$ as a rational number. Let

$$x = 0.252525\ldots \text{ then } 100x = 25.252525\ldots .$$

Subtracting, we get

$$99x = 25 \text{ so } x = \frac{25}{99}$$

which is the correct answer, but the method is not valid. When multiplying a finite decimal by 100, we start at the right-hand side, but here there is no right-hand side to start at. Of course x is a limit and a correct procedure is to use the formula $s = \dfrac{a}{1-r}$, for an infinite geometric series which converges.

Example 3

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

Apply L'Hôpital's rule, $\lim\limits_{x \to 0} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to 0} \dfrac{f'(x)}{g'(x)}$, which applies in this case, and we get $\dfrac{\cos 0}{1} = 1$, which is the correct answer. The only trouble is, that to find the derivative of $\sin x$, one needs to know the value of $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$ in the first place, so the argument is circular. Also, the usual geometric proof that $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$ can be deemed to be dubious because it depends on a diagram. The only valid method appears to be to define $\sin x = \sum\limits_{k=0}^{\infty} \dfrac{(-1)^k x^{2k+1}}{(2k+1)!}$ as an infinite series.

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Example 4

A tangent to a circle and the radius at the point of contact are perpendicular to each other.

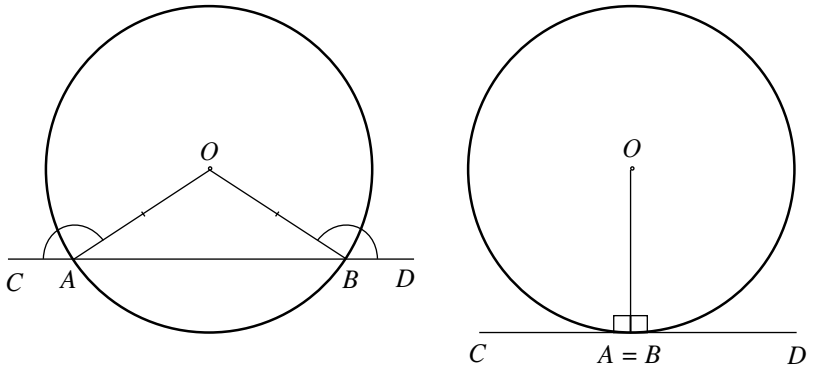


FIGURE 1

This is claimed because $|OA| = |OB|$ means that $\angle OAB = \angle OBA$ and $\angle OAC = \angle OBD$. This situation persists when A and B coincide at the point of tangency, so that $\angle OAC = \angle OBD = 90°$.

This 'proof' is contained in many textbooks and is referred to as the 'method of limits'. It is not valid, at least not without a great deal of further justification.

Example 5

Similarly the theorem that the angle between a tangent to a circle and a chord is equal to the angle in the alternate segment is often justified by this 'method of limits'. The angle between the tangent and the chord is deemed to be the limiting case of the angle on the chord. This result can also be viewed as a limiting case of external angles of cyclic quadrilaterals.

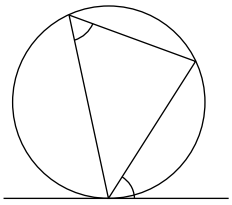


FIGURE 2

Example 6

Dissect a circular disc into n equal sectors and rearrange them as in Figure 3.

Given that the circumference of the circle is $2\pi r$, as n becomes large (infinite?) the area of the circular disc is given by $\pi r \cdot r = \pi r^2$. Another triumph for the 'method of limits'! [It probably can be patched up, but needs a lot of work!]

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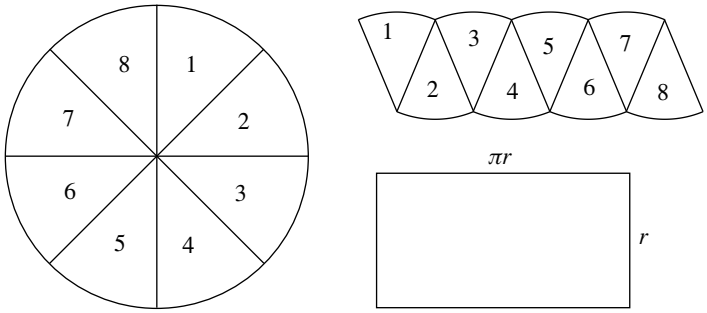


FIGURE 3

Example 7

If $ax^2 + bx + c = px^2 + qx + r$ for all real values of x and a, b, c, p, q, r are all real numbers, show that $a = p, b = q$ and $c = r$.

Put $x = 0$, and we get $c = r$ at once, so we may cancel x from both sides of the equation.

Thus $ax^2 + bx = px^2 + qx$, and dividing across by x, we get $ax + b = px + q$. Again, putting $x = 0$ we get $b = q$, so we may cancel x. Finally, $ax = px$, so we may cancel x to give $a = p$.

[This dubious method may be extended to polynomials of degree n.]

Example 8

And how about this one?

$$\begin{aligned}\sin(a + b)\sin(a - b) &= (\sin a + \sin b)(\sin a - \sin b)\\ &= \sin^2 a - \sin^2 b\end{aligned}$$

which happens to be the correct answer!

The question is, how is anyone, especially a struggling student, expected to recognise which methods in mathematics are valid and which are not? It is a skill, very difficult to teach, and most good mathematicians seem to pick it up by intuition.

I am sure many other examples are to be found in the classroom and in textbooks, and I would be interested to hear about them from readers. And isn't it ironic that examiners are allowed to award marks for an incorrect answer arising from a legitimate method (so-called 'method marks'), but not for a correct answer found by a fallacious method?

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