Ergod. Th. & Dynam. Sys., (2026), **46**, 1755–1799 © The Author(s), 2026. Published by Cambridge University Press.
doi:10.1017/etds.2026.10283

Correlation number for potentials with entropy gaps and cusped Hitchin representations

LIEN-YUNG KAO† and GIUSEPPE MARTONE‡

† *Mathematics, The George Washington University, USA*
(e-mail: lkao@gwu.edu)
‡ *Sam Houston State University, USA*
(e-mail: gxm120@shsu.edu)

(*Received* 20 *March* 2025 *and accepted in revised form* 19 *January* 2026)

Abstract. We introduce a correlation number for two strictly positive locally Hölder continuous independent potentials with strong entropy gaps at infinity on a topologically mixing countable state Markov shift with big images and pre-images (BIP) property. We define in this way a correlation number for pairs of cusped Hitchin representations. Furthermore, we explore the connection between the correlation number and the Manhattan curve, along with several rigidity properties of this correlation number.

Key words: symbolic dynamics, topological dynamics
2020 Mathematics Subject Classification: 37D35 (Primary); 57M50 (Secondary)

Contents

1 Introduction 1755
2 Background 1761
3 The correlation number 1770
4 Preparing to count 1773
5 The global growth rate control 1778
6 Local counting, asymptotic estimates, and Theorems 1.1, 1.2, and 1.3 1782
7 Application to cusped Hitchin representations: Theorem 1.6 1791
Acknowledgements 1793
A Appendix. Proof of Theorem 2.7 1793
B Appendix. Proof of the saddle-point method 1795
References 1797

1. *Introduction*

In this article, we introduce a correlation number for pairs of well-behaved potentials over countable-state Markov shifts, defined through the study of their simultaneous orbital

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

distributions. These dynamical results let us define a correlation number for pairs of cusped Hitchin representations. This correlation number quantifies the similarities of their marked length spectra and captures some rigidity properties.

Our dynamical results extend Lalley's earlier work [25] to non-compact settings. We derive the geometric counterparts through the symbolic coding of cusped Hitchin representations. A similar approach was successfully employed in previous work by Bray *et al* [5], where we established counting and equidistribution results for cusped Hitchin representations. In this article, we obtain a comparison of the growth rate of the marked length spectra of a pair of cusped Hitchin representations.

The primary motivation for this work stems from (higher rank) Teichmüller theory, especially from the study of geometrically meaningful diverging sequences of (cusped) Hitchin representations. Motivated by results of Dai and Martone [13], we wish to study the behavior of the correlation number along these sequences and interpret its limit. For example, given two sequences of hyperbolic structures converging to distinct points in the same stratum of the augmented Teichmüller space, as a consequence of Theorem 1.6, we can now define the correlation number of the limiting points and it would be interesting to compare it with the correlation number along the diverging sequences.

In the following subsections, we present our dynamical results first and then their applications to geometry.

1.1. *General thermodynamical results.* See §2 for precise definitions. Let Σ^+ be a topologically mixing (one-sided) countable state Markov shift with big images and pre-images (BIP) property and consider two strictly positive, locally Hölder continuous, and independent potentials $f, g\colon \Sigma^+ \to \mathbb{R}$. Let $\mathcal{A}$ denote the countable alphabet for Σ^+. Following [5], we say that a potential f has a *strong entropy gap at infinity* if the series

$$Z_1(f,s) = \sum_{a\in\mathcal{A}} e^{-s\sup\{f(\underline{x})\colon \underline{x}\in\Sigma^+ \text{ and } x_1=a\}}$$

has a finite critical exponent $d(f) > 0$ and diverges when $s = d(f)$. For the rest of this introduction, assume that f and g have strong entropy gaps at infinity, $\|f - g\|_\infty$ is finite, and set $\mathbf{f} = (f, g)$.

We fix $\xi, m > 0$: ξ is a (small) precision in our estimates, while m is an *admissible slope*, defined below in terms of the *Manhattan curve of* $\mathbf{f}$. For any $t > 0$, let $I^2_{\xi,m}(t)$ denote the open rectangle $(t, t+\xi) \times (mt, mt+\xi)$ and for any $n \in \mathbb{Z}_{>0}$, define

$$\mathcal{M}(n,t) = \mathcal{M}(n,t;\mathbf{f},\xi,m) := \{\underline{x} \in \Sigma^+\colon\ \underline{x} \in \mathrm{Fix}^n,\ S_n\mathbf{f}(\underline{x}) \in I^2_{\xi,m}(t)\},$$

where Fix^n is the set of *n-periodic words* of Σ^+ and $S_n\mathbf{f}$ denotes the pair of nth ergodic sums of $\mathbf{f}$, that is,

$$S_n\mathbf{f}(\underline{x}) := \left(\sum_{k=1}^{n} f(\sigma^{k-1}(\underline{x})), \sum_{k=1}^{n} g(\sigma^{k-1}(\underline{x}))\right).$$

The first goal of this article is to study the growth rate as t goes to infinity of

$$M(t;\mathbf{f},m,\xi) := \sum_n \frac{1}{n}\#\mathcal{M}(n,t;\mathbf{f},m,\xi).$$

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

THEOREM 1.1. *For any admissible slope m, $M(t; \mathbf{f}, m, \xi)$ grows exponentially. Specifically, the limit*

$$\alpha_{\mathbf{f}}(m) := \lim_{t\to\infty} \frac{1}{t} \log M(t; \mathbf{f}, m, \xi)$$

exists and, moreover, $0 < \alpha_{\mathbf{f}}(m) < \infty$.

In what follows, we examine properties of the exponential growth rate $\alpha_{\mathbf{f}}(m)$ of $M(t; \mathbf{f}, m, \xi)$, which we call the *correlation number* of $\mathbf{f}$ and m. A *cylinder* p is a subset of Σ^+ consisting of words sharing an initial string of length $|p| \in \mathbb{N}$. Our second result provides a more precise asymptotic growth estimate for

$$M_p(t; \mathbf{f}, m, \xi) := \sum_n \frac{1}{n} \#\mathcal{M}_p(n, t; \mathbf{f}, m, \xi),$$

where, for any cylinder p, we write $\mathcal{M}_p(n, t; \mathbf{f}, m, \xi) := \mathcal{M}(n, t; \mathbf{f}, m, \xi) \cap p$. Notably, we obtain a finer asymptotic estimate when we restrict to a cylinder.

THEOREM 1.2. *For any admissible slope m and any cylinder p, there exist constants $C_1(p)$ and $C_2(p)$ such that*

$$C_1(p) \le \lim_{t\to\infty} \frac{t^{3/2}}{e^{\alpha_{\mathbf{f}}(m)t}} M_p(t; \mathbf{f}, m, \xi) \le C_2(p)$$

and

$$\lim_{|p|\to\infty} \frac{C_1(p)}{C_2(p)} = 1.$$

By restricting our focus to the counting problem on cylinders, Theorem 1.2 provides an estimate for the *local* asymptotic expansion of the orbital distribution with respect to $\mathbf{f}$ and m. Unlike in the compact case, the *global* asymptotic expansion does not immediately follow from the local version. While we expect that the global asymptotic expansion also holds, we would need a stronger version of Lemma 5.4 to establish it. See Remark 6.4 for more details. However, let us note that the local asymptotic expansion suffices for our main applications.

Next, we explicitly relate the growth rate $\alpha_{\mathbf{f}}(m)$ to the Manhattan curve, an important dynamical object associated with orbital distribution problems for $\mathbf{f}$, first introduced by Burger in [7]. Let P denote the *topological pressure* associated with Σ^+. We recall (see §2.5 for more details) that the *Manhattan curve of* $\mathbf{f}$ is the curve

$$\mathcal{C}(\mathbf{f}) = \{(a, b) \in \mathbb{R}^2 : P(-af - bg) = 0,\ a \ge 0,\ b \ge 0,\ a + b > 0\}.$$

By the analyticity properties of the pressure function P and [5, Theorem C], we know that the Manhattan curve $\mathcal{C}(\mathbf{f})$ can be parameterized by $(s, q(s))$, where $q = q(s)$ is a real analytic function for $s \in [0, \delta_f]$. Here, δ_f is the unique value for which $P(-\delta_f f) = 0$. The slope of the normal to the Manhattan curve at the point $(s, q(s))$ is given by $m(s) = -1/q'(s)$. We then say that $m = m(s)$ is an *admissible slope* and we denote by $\mathcal{S}(\mathbf{f})$ the set of admissible slopes. In particular, note that $m_* = \delta_f/\delta_g \in \mathcal{S}(\mathbf{f})$ (see Theorem 2.14).

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

Given an admissible slope m, we denote by (a_m, b_m) the point on $\mathcal{C}(\mathbf{f})$ where the slope of the normal equals m. With this notation, we can define the quantity

$$H_{\mathbf{f}}(m) := a_m + m \cdot b_m.$$

The following result shows that $H_{\mathbf{f}}(m)$ equals the correlation number of $\mathbf{f}$ and m.

THEOREM 1.3. *If $m \in \mathcal{S}(\mathbf{f})$ is an admissible slope, then*

$$\alpha_{\mathbf{f}}(m) = H_{\mathbf{f}}(m).$$

Using the convexity of the Manhattan curve and the above results, we derive a rigidity result for the special admissible slope $m_* = \delta_f/\delta_g$.

COROLLARY 1.4. (Correlation number rigidity) *For $m_* = \delta_f/\delta_g$, we have*

$$\alpha_{\mathbf{f}}(m_*) = H_{\mathbf{f}}(m_*) \leq \delta_f.$$

Moreover, the equality holds if and only if $m_* S_n f(\underline{x}) = S_n g(\underline{x})$ for all $\underline{x} \in \mathrm{Fix}^n$.

Another interesting dynamical growth rate associated with $\mathbf{f}$ is $h^{\mathrm{BS}}_{(\alpha,\beta)}(\mathbf{f})$, the (α, β)-*Bishop–Steger entropy*, defined for $\alpha, \beta > 0$ as

$$h^{\mathrm{BS}}_{(\alpha,\beta)}(\mathbf{f}) := \lim_{t\to\infty} \frac{1}{t} \log \sum_n \frac{1}{n} \#\{\underline{x} \in \mathrm{Fix}^n : \alpha S_n f(\underline{x}) + \beta S_n g(\underline{x}) \leq t\}.$$

Our last result in the symbolic setting relates the Bishop–Steger entropy with the correlation number.

THEOREM 1.5. *Let $m \in \mathcal{S}(\mathbf{f})$ be an admissible slope. Then,*

$$\frac{H_f(m)}{\alpha + m\beta} \leq h^{\mathrm{BS}}_{(\alpha,\beta)}(\boldsymbol{f}),$$

with equality if and only if m is the unique slope satisfying $a_m/b_m = \alpha/\beta$.

1.2. *Application to cusped Hitchin representations.* We apply the results above to pairs of cusped Hitchin representations. See §7 for details. Let Γ be a torsion-free geometrically finite Fuchsian group which is not convex-cocompact. The recurrent portion of the geodesic flow on $T^1(\mathbb{H}^2/\Gamma)$ can be coded by a topologically mixing countable states Markov shift Σ^+ with BIP. In particular, there is a map $G\colon \mathrm{Fix}^n \to \Gamma$ such that if $\gamma \in \Gamma$ is hyperbolic, then it is conjugated to $G(\underline{x})$ for some n and a unique up to shift $\underline{x} \in \mathrm{Fix}^n$.

A representation $\rho\colon \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is *cusped Hitchin* if there exists a continuous, ρ-equivariant, positive map from the limit set of Γ to the space of complete flags in $\mathbb{R}^d$. Using a Lie theoretic construction, a cusped Hitchin representation together with a choice of a non-zero positive linear combination ϕ of the simple roots of $\mathsf{SL}(d, \mathbb{R})$ determines a length function $\ell^{\phi}_{\rho}\colon \Gamma \to \mathbb{R}_{\geq 0}$, which is constant on the conjugacy class $[\gamma] \in [\Gamma]$ of an element $\gamma \in \Gamma$. Then, for any cusped Hitchin representation $\rho\colon \Gamma \to \mathsf{SL}(d, \mathbb{R})$ and any such ϕ, there exists a strictly positive locally Hölder continuous potential τ^{ϕ}_{ρ} with a strong entropy gap at infinity and such that

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

$$\tau_\rho^\phi(\underline{x}) = \ell_\rho^\phi(G(\underline{x}))$$

for any $\underline{x} \in \mathrm{Fix}^n$. Moreover, given another cusped Hitchin representation $\eta\colon \Gamma \to \mathsf{SL}(d,\mathbb{R})$, then $\|\tau_\rho^\phi - \tau_\eta^\phi\|_\infty$ is finite and τ_ρ^ϕ and τ_η^ϕ are independent unless $\eta = \rho, (\rho^{-1})^\top$, see Lemma 7.3.

Recall that for a cusped Hitchin representation ρ and a non-zero positive linear combination ϕ of the simple roots of $\mathsf{SL}(d,\mathbb{R})$, we can define $\delta_\phi(\rho)$, the *ϕ-topological entropy*, as $\delta_{\tau_\rho^\phi}$, the entropy of the corresponding potential τ_ρ^ϕ. Furthermore, for a pair of cusped Hitchin representations (ρ,η) and a given ϕ, we can define the Manhattan curve $\mathcal{C}^\phi(\rho,\eta)$ by $\mathcal{C}(\tau_\rho^\phi, \tau_\eta^\phi)$. (See [5, Corollaries 1.3 and 1.4] for more details.) Thus, we can apply our results from symbolic dynamics to obtain the following theorem.

THEOREM 1.6. *Let $\rho,\eta\colon \Gamma \to \mathsf{SL}(d,\mathbb{R})$ be cusped Hitchin representations such that $\eta \neq \rho, (\rho^{-1})^\top$ and let ϕ be a non-zero positive linear combination of the simple roots of $\mathsf{SL}(d,\mathbb{R})$. Let m be an admissible slope for the Manhattan curve $\mathcal{C}^\phi(\rho,\eta)$. Then,*

$$\lim_{t\to\infty} \frac{1}{t} \log\{[\gamma] \in [\Gamma]\colon (\ell_\rho^\phi(\gamma), \ell_\eta^\phi(\gamma)) \in I^2_{\xi,m}(t)\} =: H^\phi_{(\rho,\eta)}(m) = a_m + m \cdot b_m,$$

where $H^\phi_{(\rho,\eta)}(m)$ is the correlation number of (ρ,η) and ϕ, and (a_m, b_m) is the point $\mathcal{C}^\phi(\rho,\eta)$ where the slope of the normal equals m.

The following rigidity result follows immediately from Theorem 1.6, Corollary 1.4, and [5, Corollary 1.5].

COROLLARY 1.7. (Correlation number rigidity II) *Under the same assumptions as in Theorem 1.6, for $m_* = \delta_\phi(\rho)/\delta_\phi(\eta)$, we have*

$$H^\phi_{(\rho,\eta)}(m_*) \leq \delta_\phi(\rho).$$

Moreover, equality holds if and only if $m_*\ell_\rho^\phi(\gamma) = \ell_\eta^\phi(\gamma)$ for all $\gamma \in \Gamma$.

Similarly, we can define the *(α,β)-Bishop–Steger entropy* of ϕ, ρ, and η as $h^\phi_{(\alpha,\beta)}(\rho,\eta) := h^{\mathrm{BS}}_{(\alpha,\beta)}(\tau_\rho^\phi, \tau_\eta^\phi)$. Then, Theorem 1.5 can be rewritten as follows.

Theorem 1.5*. *Let m be an admissible slope. Then,*

$$\frac{H^\phi_{(\rho,\eta)}(m)}{\alpha + m\beta} \leq h^\phi_{(\alpha,\beta)}(\rho,\eta),$$

with equality if and only if m is the unique slope satisfying $a_m/b_m = \alpha/\beta$.

1.3. *Outline of the paper and main steps.* The approach in this paper combines ideas from [5, 25]. The results in [5] rely on a renewal theorem, which is not available when studying simultaneous orbital distribution problems. For this reason, we follow Lalley's strategy from the compact case [25] and use Fourier analysis to study the asymptotic behavior of the simultaneous orbital distribution of a pair of potentials.

Lalley's method involves two key steps. The first step consists of converting the counting problem into estimates on thermodynamical quantities. The second step uses Fourier

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

analysis on these quantities to derive the relevant asymptotic expansions. The simultaneous orbital growth problem studied in this paper is a refinement of the single-orbital growth problem treated in [5]. To extend Lalley's method to the non-compact setting, we build upon the framework in [5] to establish more delicate estimates that we can then use to perform the two steps as above.

In what follows, we outline the main steps of each section. Section 2 gathers essential background results and definitions from the thermodynamic formalism and the entropy gaps at infinity introduced in [5]. In §3, we begin our study of the correlation number $H_{\mathbf{f}}(m)$ and establish the rigidity results given in Corollary 1.4 and Theorem 1.5. Section 4 reformulates the orbit counting problem by linking it to the transfer operator, enabling the application of tools from thermodynamic formalism. In §5, we derive *a priori* estimates for the counting problem over countable Markov shifts, giving an upper bound for the simultaneous orbital growth. Section 6 uses Fourier analysis and the saddle point method to establish a local asymptotic expansion, providing a lower bound for the simultaneous orbital growth and thereby proving Theorems 1.1, 1.2, and 1.3. In §7, we review key aspects of the theory of cusped Hitchin representations and establish Lemma 7.3, which allows us to apply our main dynamical results in this context, leading to the proof of Theorem 1.6. Finally, in Appendices A and B, we provide proofs of Theorem 2.7 and Proposition 6.11, which are known to experts, but we were not able to find in the available literature.

1.4. *Historical remarks.* Orbital distribution has long been a central theme in ergodic geometry. A key early milestone in this field is the prime orbit theorem (see, for example, the foundational works of Huber, Margulis, Lalley, Parry, and Pollicott [18, 27, 29, 31]), which establishes a connection between the growth rate of closed orbits and topological entropy. Lalley's pioneering work [25, 26] introduces probabilistic perspectives to study orbital distribution problems within compact settings. Building on this foundation, along with earlier contributions by the authors [5, 19, 20], our work extends Lalley's results to a large class of countable Markov shifts.

Lalley's results in [25] initiated further investigations into correlation numbers, which Sharp [43] later explored through a different approach. Lalley's subsequent work [26] focused on counting closed orbits within specific homology classes, drawing inspiration from Phillips and Sarnak [33], and Katsuda and Sunada [22]. This problem was further studied by Pollicott and Sharp [34, 42] using a zeta function approach. Babillot and Ledrappier [2] later provided a unified framework for these two closely related counting problems in compact hyperbolic flows.

Schwartz and Sharp [41] used Lalley's work [25] to define and examine the correlation of length spectra of two hyperbolic structures on a closed surface. Pollicott and Sharp [35, 36] further analyzed a related correlation number, defined relative to word length. Glorieux [17] studied correlation numbers with slope from the perspective of globally hyperbolic maximally Cauchy-compact anti-de Sitter manifolds. Dai and the second author [13] extended Schwartz and Sharp's approach to pairs of Hitchin representations. Chow and Oh [11, 12] recently developed a new approach and generalized these results to tuples of Anosov groups.

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

Cusped Hitchin representations are examples of positive representations in the sense of Fock and Goncharov [16]. Canary, Zhang, and Zimmer [9] establish several of their geometric properties, which extend results on Anosov representations. Bray *et al* developed dynamical aspects of this theory in [5, 6]. We refer to [8] for a survey on (cusped) Hitchin representations.

In the non-convex cocompact setting, constrained counting problems remain less developed. Epstein [15] studied the counting of orbits in given homology classes for finite-volume hyperbolic manifolds, while Babillot and Peigné [3] extended these results to certain infinite-volume hyperbolic manifolds with cusps. To the best of our knowledge, no other results on correlation numbers exist in non-convex cocompact settings, apart from Theorems 1.1 and 1.2. Moreover, these two results partially answer open question no. 7 listed on [32, p. 11].

Theorem 1.3 generalizes [43, Theorem 1] and [17, Theorem 3.25] (with slopes) for convex cocompact Fuchsian representations, as well as [13, Theorem 6.14] for Hitchin representations. Corollary 1.4 draws inspiration from the intersection number rigidity result in [5, Corollary 1.5]. Finally, Theorem 1.5 studies the relation between the correlation number and (generalized) Bishop–Steger entropy, extending [17, Theorem 3.15] for convex cocompact Fuchsian representations to cusped Hichin representations.

2. *Background*

2.1. *Markov shifts.*

A *countable state Markov shift* is the data of a countable *alphabet* $\mathcal{A}$, a *transition matrix* $\mathbb{T} = (t_{ab}) \in \{0, 1\}^{\mathcal{A}\times\mathcal{A}}$, the set of *words*

$$\Sigma^+ = \{\underline{x} = (x_n) \in \mathcal{A}^{\mathbb{Z}_{>0}} : t_{x_n x_{n+1}} = 1\},$$

and the *left-shift* $\sigma : \Sigma^+ \to \Sigma^+$ defined by $\sigma((x_n)) = (x_{n+1})$. The countable state Markov shift is: (i) *topologically mixing* if for any letters $a, b \in \mathcal{A}$, there exists $N = N(a, b)$ such that for any $n > N$, there exists a word $\underline{x} \in \Sigma^+$ with $x_1 = a$ and $x_n = b$; (ii) has *BIP*, short for *big images and pre-images property*, if there exists a finite subset $\mathcal{B}$ of the alphabet $\mathcal{A}$ such that for any $a \in \mathcal{A}$, there exist letters $b_p, b_s \in \mathcal{B}$ such that $t_{b_p a} = t_{a b_s} = 1$. For the rest of this section, we fix a topologically mixing, countable state Markov shift with BIP which, with a slight abuse of notation, we denote by Σ^+. Furthermore, we equip Σ^+ with the metric $d(\underline{x}, \underline{y}) = e^{-\inf\{n:\, x_n \neq y_n\}}$.

For any $k \in \mathbb{Z}_{>0}$, a *k-cylinder* is a non-empty subset $p \subset \Sigma^+$ defined by the property

$$\underline{x}, \underline{y} \in p \iff x_i = y_i \quad \text{for all } i = 1, \dots, k.$$

Denote by Λ_k the set of k-cylinders.

2.2. *Potentials, entropy gaps, and pressure.*

A *potential* is a continuous function $f : \Sigma^+ \to \mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ equipped with the norm $|\cdot|$. The *nth ergodic sum* of f is the potential

$$S_n f(\underline{x}) = \sum_{i=0}^{n-1} f(\sigma^i(\underline{x})).$$

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

A potential $f\colon \Sigma^+ \to \mathbb{K}$ is *locally Hölder continuous* (or *β-locally Hölder continuous with constant A*) if there exist $A, \beta > 0$ such that

$$|f(\underline{x}) - f(\underline{y})| < Ad(\underline{x}, \underline{y})^\beta$$

whenever $x_1 = y_1$. We denote by $\mathcal{F}_\beta$ (respectively $\mathcal{F}_\beta(\mathbb{C})$) the space of β-locally Hölder continuous potentials valued in $\mathbb{R}$ (respectively $\mathbb{C}$) equipped with the norm

$$\|\phi\|_\beta = \|\phi\|_\infty + \mathrm{Lip}(\phi) \quad \text{where } \mathrm{Lip}\,(\phi) := \sup\left\{\frac{|\phi(\underline{x}) - \phi(\underline{y})|}{d(\underline{x}, \underline{y})^\beta} \colon \underline{x} \neq \underline{y},\ x_1 = y_1\right\}.$$

Then, let $\mathcal{F}_\beta^b(\mathbb{C})$ denote the subspace of bounded potentials and let $\mathcal{B}(\mathcal{F}_\beta^b(\mathbb{C}))$ be the space of bounded operators on $\mathcal{F}_\beta^b(\mathbb{C})$ equipped with the operator norm

$$\|\mathcal{L}\|_{\mathrm{op}} = \sup\left\{\frac{\|\mathcal{L}(\phi)\|_\beta}{\|\phi\|_\beta} \colon \phi \in \mathcal{F}_\beta^b(\mathbb{C})\right\}.$$

Note that $(\mathcal{F}_\beta^b(\mathbb{C}), \|\cdot\|_\beta)$ and $(\mathcal{B}(\mathcal{F}_\beta^b(\mathbb{C})), \|\cdot\|_{\mathrm{op}})$ are Banach spaces.

Let $f\colon \Sigma^+ \to \mathbb{R}$ be locally Hölder continuous and set $S(f, a) = \sup\{f(\underline{x})\colon \underline{x} \in \Sigma^+, x_1 = a\}$. Then, recall from the introduction that f has a *strong entropy gap at infinity* if the series

$$Z_1(f, s) = \sum_{a \in \mathcal{A}} e^{-sS(f,a)}$$

has a finite critical exponent $d(f) > 0$ and diverges when $s = d(f)$.

Let $\mathcal{M}_\sigma$ denote the space of shift-invariant probability measures on Σ^+ and, for any $\mu \in \mathcal{M}_\sigma$, let $h_\sigma(\mu)$ denote the measure-theoretic entropy of σ with respect to μ. The *(topological) pressure* of the locally Hölder continuous potential f is

$$P(f) = \sup\left\{h_\sigma(\mu) + \int_{\Sigma^+} f\, d\mu \colon \mu \in \mathcal{M}_\sigma \text{ and } -\int_{\Sigma^+} f\, d\mu < \infty\right\}.$$

An *equilibrium state* for a potential f is a σ-invariant Borel probability measure μ on Σ^+ such that

$$P(f) = h_\sigma(\mu) + \int_{\Sigma^+} f\, d\mu.$$

A *Gibbs state* for f is a Borel probability measure ν on Σ^+ for which there exists a constant $Q > 1$ such that

$$\frac{1}{Q} \leq \frac{e^{S_n f(\underline{x}) - nP(f)}}{\nu(p)} \leq Q$$

for any n-cylinder p and any $\underline{x}$ in p.

The Gibbs states and equilibrium states of potentials with good regularity (if they exist) are closely related. Specifically, we have the following result.

THEOREM 2.1. (Mauldin and Urbański [30, Theorem 2.2.9], Sarig [40, Theorem 4.9]) *Let Σ^+ be a topologically mixing, countable state Markov shift with BIP. If f is a real-valued, locally Hölder continuous potential with finite pressure, then f admits a unique*

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

shift-invariant Gibbs state μ_f. *Moreover, if* $-\int f\,d\mu_f < \infty$, *then* μ_f *is the unique equilibrium state for f.*

Two locally Hölder continuous potentials f and g are *(Livšic) cohomologous* if there exists a locally Hölder continuous potential h such that $f - g = h - h \circ \sigma$. The real-valued potential f is *strictly positive* if there exists a constant $B > 0$ such that $f(\underline{x}) > B$ for all $\underline{x} \in \Sigma^+$.

Remark 2.2. Several of the results from [5] used in this work hold more generally for *eventually positive* potentials. However, in this paper, we will need the stronger hypothesis of strict positivity, as in [5, Theorem C]. We note that, by [5, Lemma 3.2], any eventually positive locally Hölder continuous potential with a strong entropy gap at infinity is cohomologous to a strictly positive potential with the same properties.

A potential $f\colon \Sigma^+ \to \mathbb{R}$ is *arithmetic* if the additive subgroup of $\mathbb{R}$ generated by $\{S_n f(\underline{x})\colon \underline{x} \in \mathrm{Fix}^n,\ \ n \in \mathbb{N}\}$ is cyclic. Two potentials are *independent* if $af + bg$ is arithmetic only if $a = b = 0$. Note that if f and g are independent, then each one of them is non-arithmetic.

Assumptions 2.3. From now on through the paper, we assume that Σ^+ is a topologically mixing, countable state Markov shift with BIP, and let $f, g\colon \Sigma^+ \to \mathbb{R}$ denote strictly positive locally Hölder continuous potentials with strong entropy gaps at infinity such that $\|f - g\|_\infty$ is finite. We often write $\mathbf{f}$ for the pair (f, g).

Remark 2.4.

(1) When clear from context, our notation will not highlight the dependence on the fixed pair of potentials $\mathbf{f}$.

(2) Assuming that $\|f - g\|_\infty$ is finite and f has a strong entropy gap at infinity guarantees that the same property holds for g. However, we explicitly assume that g has a strong entropy gap at infinity to ease exposition.

2.3. *Transfer operators.* We are interested in certain weighted sums of f and g that have finite pressure. More specifically, we will consider weights in the set

$$\begin{aligned} D = D(\mathbf{f}) := (&\{(z_1, z_2) \in \mathbb{R}^2\colon z_1 \le 0, z_2 \le 0, z_1 + z_2 < 0\} \\ &\cup \{(z_1, z_2) \in \mathbb{R}^2\colon z_1 > 0, z_2 < 0, z_1 c(f) + z_2(c(f) - A) < 0\} \\ &\cup \{(z_1, z_2) \in \mathbb{R}^2\colon z_1 < 0, z_2 > 0, z_1(c(g) - A) + z_2 c(g) < 0\}) \\ &\cap \{(z_1, z_2) \in \mathbb{R}^2\colon d(-z_1 f - z_2 g) < 1\}, \end{aligned} \tag{2.1}$$

where $A > 0$ is such that $\|f - g\|_\infty = A < \infty$, $c(f) = \inf_{\Sigma^+} f > 0$, and $c(g) = \inf_{\Sigma^+} g > 0$. Notice that for any $(z_1, z_2) \in D$, the potential $-(z_1 f + z_2 g)$ is locally Hölder continuous, strictly positive, and with strong entropy gap at infinity. In particular, [5, Lemma 3.3(1)] implies that $D \subset \{(z_1, z_2) \in \mathbb{R}^2\colon P(z_1 f + z_2 g) < \infty\}$. We now list some important properties of the weights in D.

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

PROPOSITION 2.5. *For any $(z_1, z_2) \in D$, the following holds.*

(1) *The potential $z_1 f + z_2 g$ has a unique equilibrium state $\mu_{z_1 f + z_2 g}$.*
(2) *The potentials f, g are in $L^n(\mu_{z_1 f + z_2 g})$ for all $n \in \mathbb{N}$.*
(3) *D is an open set in $\mathbb{R}^2$.*

Proof. The first assertion follows from the assumption $d(-z_1 f - z_2 g) < 1$ together with [5, Lemma 3.4]. For the second assertion, since $-(z_1 f + z_2 g)$ has a strong entropy gap at infinity, the series

$$Z_1(-z_1 f - z_2 g, s) = \sum_{a \in \mathcal{A}} e^{-sS(-z_1 f - z_2 g, a)}$$

converges for all $s > d(-(z_1 f + z_2 g))$. Since we assume $(z_1, z_2) \in D$, we can find $\epsilon > 0$ small enough such that $1 - \epsilon > d(-(z_1 f + z_2 g))$, and thus,

$$\sum_{a \in \mathcal{A}} e^{-(1-\epsilon)S(-z_1 f - z_2 g, a)} < \infty.$$

Then,

$$\begin{aligned}
\infty > \sum_{a \in \mathcal{A}} e^{-(1-\epsilon)S(-z_1 f - z_2 g, a)} &= \sum_{a \in \mathcal{A}} e^{-S(-z_1 f - z_2 g, a)} e^{\epsilon S(-z_1 f - z_2 g, a)} \\
&= \sum_{a \in \mathcal{A}} e^{-S(-z_1 f - z_2 g, a)} \sum_{n \geq 1} \frac{\epsilon^n}{n!} (S(-z_1 f - z_2 g, a))^n \\
&= \sum_{n \geq 1} \sum_{a \in \mathcal{A}} \frac{\epsilon^n}{n!} (S(-z_1 f - z_2 g, a))^n e^{-S(-z_1 f - z_2 g, a)},
\end{aligned}$$

where we can swap the series because $(z_1, z_2) \in D$, $-z_1 f - z_2 g$ is strictly positive; thus, guaranteeing absolute convergence. It follows that for all $n \in \mathbb{N}$,

$$\sum_{a \in \mathcal{A}} (S(-z_1 f - z_2 g, a))^n e^{-S(-z_1 f - z_2 g, a)} < \infty.$$

Now, for all $a \in \mathcal{A}$, let $[a]$ denote the cylinder of words starting with the letter a and, for any locally Hölder continuous potential h, let $I(h, a)$ denote the infimum of $\{h(\underline{x}) \colon \underline{x} \in [a]\}$. By the Gibbs property of $\mu_{z_1 f + z_2 g}$, we know that there exists $Q > 1$ such that

$$e^{-I(-z_1 f - z_2 g, a)} = e^{S(z_1 f + z_2 g, a)} \geq \frac{e^{P(z_1 f + z_2 g)}}{Q} \mu_{z_1 f + z_2 g}([a]).$$

Hence,

$$\begin{aligned}
0 &< \int |-z_1 f - z_2 g|^n \, d\mu_{z_1 f + z_2 g} \\
&\leq \frac{Q}{e^{P(z_1 f + z_2 g)}} \sum_{a \in \mathcal{A}} (S(-z_1 f - z_2 g, a))^n e^{-I(-z_1 f - z_2 g, a)} \\
&\leq \frac{Q}{e^{P(z_1 f + z_2 g)}} \sum_{a \in \mathcal{A}} (S(-z_1 f - z_2 g, a))^n e^{-S(-z_1 f - z_2 g, a) + A} < \infty,
\end{aligned}$$

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

where we used that the potential $-z_1 f - z_2 g$ is locally Hölder continuous to see that

$$|I(-z_1 f - z_2 g, a) - S(-z_1 f - z_2 g, a)| < A$$

for some uniform constant $A > 0$. From here, we deduce that $f, g \in L^n(\mu_{z_1 f + z_2 g})$ for all $n \in \mathbb{N}$.

The last assertion is an immediate consequence of the formula

$$d(-z_1 f - z_2 g) = \frac{d(f)}{-(z_1 + z_2)}$$

established in the proof of [5, Theorem C*]. □

Consider a locally Hölder continuous potential $h\colon \Sigma^+ \to \mathbb{K}$. The *transfer operator* $\mathcal{L}_h$ of h (also known as the *Ruelle–Perron–Frobenius operator*) is defined by

$$\mathcal{L}_h \phi(\underline{x}) = \sum_{\underline{y} \in \sigma^{-1}(\underline{x})} e^{h(\underline{y})} \phi(\underline{y}),$$

where $\phi\colon \Sigma^+ \to \mathbb{K}$ is a *bounded* locally Hölder continuous potential. The eigenvalues and eigenvectors of the transfer operator are related to the pressure and equilibrium states as follows.

THEOREM 2.6. (Mauldin and Urbański [30, Corollary 2.7.5], Sarig [40, Theorem 4.9]) *If $u : \Sigma^+ \to \mathbb{R}$ is locally Hölder continuous, $P(u) < +\infty$ and $\sup u < +\infty$, then there exist unique probability measures μ_u and ν_u on Σ^+ and a positive function $h_u : \Sigma^+ \to \mathbb{R}$ so that*

$$\mu_u = h_u \nu_u, \quad \mathcal{L}_u h_u = e^{P(u)} h_u \quad \text{and} \quad \mathcal{L}_u^* \nu_u = e^{P(u)} \nu_u.$$

Moreover, h_u is bounded away from both 0 and $+\infty$, and μ_u is an equilibrium state for u.

We will be interested in complex analytic perturbations of the transfer operator. To this end, consider the complex domain

$$\overline{D} = \overline{D}(\mathbf{f}) := \{(z, w) \subset \mathbb{C} \times \mathbb{C} : \ (\Re z, \Re w) \in D\},$$

which contains $D = D(\mathbf{f})$.

THEOREM 2.7. (Holomorphicity) *The map $\overline{D} \to \mathcal{B}(\mathcal{F}_\beta^b(\mathbb{C}))$ defined by $(z, w) \mapsto \mathcal{L}_{zf+wg}$ is holomorphic.*

Proof. The proof follows from [39, Proposition 2(3)] with some minor modifications and simplifications. For the sake of completeness and readability, we present the argument in Appendix A. □

We recall the well-known analytic perturbation theorem (cf. for example, [21], [31, Proposition 4.6], [40, Theorem 5.6], or [24, Theorem]).

THEOREM 2.8. (Analytic perturbation theorem) *Let X be a complex, (respectively real) Banach space and $\mathcal{B}(X)$ be the space of bounded linear operators acting on X endowed with the operator norm. If $\mathcal{L}_0 \in \mathcal{B}(X)$ has a simple isolated eigenvalue, then there is an*

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

open neighborhood V of $\mathcal{L}_0$ such that every $\mathcal{L} \in V$ has an eigenvalue $\lambda_{\mathcal{L}}$ close to λ_0. The map $\lambda\colon V \to \mathbb{C}$ (respectively $\mathbb{R}$) is analytic, $\mathcal{L}$ does not have other eigenvalues near λ_0, and there is another analytic map $u : V \to X$ such that $u_{\mathcal{L}}$ is an eigenvector of $\mathcal{L}$ for $\lambda_{\mathcal{L}}$.

The following corollary is a consequence of the analytic perturbation theorem discussed above, which can be viewed as a complex version of the Ruelle–Perron–Frobenius theorem. See, for example, Sarig [40, Theorem 5.8] for the standard Ruelle–Perron–Frobenius theorem for the unperturbed transfer operator in the countable setting. This corollary plays a key role in our local asymptotic estimates in §6.

COROLLARY 2.9. *Let $\mathbf{z} = (z_1, z_2) \in D$ and for any $\boldsymbol{\Theta} = (\theta_1, \theta_2) \in \mathbb{R}^2$, let $w(\boldsymbol{\Theta}) := (z_1 + i\theta_1)f + (z_2 + i\theta_2)g$.*

(1) *The map $\boldsymbol{\Theta} \mapsto \mathcal{L}_{w(\boldsymbol{\Theta})}$ is analytic.*

(2) *There exists $\epsilon > 0$ such that for all $\|\boldsymbol{\Theta}\| < \epsilon$, there are functions $P(w(\boldsymbol{\Theta}))$, $h(w(\boldsymbol{\Theta}))$, and $\nu_{w(\boldsymbol{\Theta})}$ satisfying*

$$\mathcal{L}_{w(\boldsymbol{\Theta})} h(w(\boldsymbol{\Theta})) = e^{P(w(\boldsymbol{\Theta}))} h(w(\boldsymbol{\Theta})) \quad \text{and} \quad \mathcal{L}^*_{w(\boldsymbol{\Theta})} \nu_{w(\boldsymbol{\Theta})} = e^{P(w(\boldsymbol{\Theta}))} \nu_w(\boldsymbol{\Theta}),$$

where $\mathcal{L}^*_{w(\boldsymbol{\Theta})}$ is the dual operator of $\mathcal{L}_{w(\boldsymbol{\Theta})}$ and $\nu_{w(\boldsymbol{\Theta})}$ is a Radon complex measure.

(3) *For all $\|\boldsymbol{\Theta}\| < \epsilon$, the maps $\boldsymbol{\Theta} \mapsto P(w(\boldsymbol{\Theta}))$ and $\boldsymbol{\Theta} \mapsto h(w(\boldsymbol{\Theta}))$ are analytic, and $\boldsymbol{\Theta} \mapsto \nu_{w(\boldsymbol{\Theta})}$ is weak* analytic, in the sense that for any $\phi \in \mathcal{F}^b_\beta(\mathbb{C})$, $\boldsymbol{\Theta} \mapsto \nu_{w(\boldsymbol{\Theta})}(\phi)$ is analytic.*

(4) *For all $\|\boldsymbol{\Theta}\| < \epsilon$, there exist $K_1 > 0$ and $r_1 \in (0, 1)$ such that for any $\phi \in \mathcal{F}^b_\beta(\mathbb{C})$,*

$$\left\| e^{-nP(w(\boldsymbol{\Theta}))} \mathcal{L}^n_{w(\boldsymbol{\Theta})} \phi - h(w(\boldsymbol{\Theta})) \int \phi \, d\nu_{w(\boldsymbol{\Theta})} \right\|_\beta \le K_1 r_1^n \|\phi\|_\beta.$$

(5) *If f and g are independent, then there exists $\delta > 0$ such that for any $\phi \in \mathcal{F}^b_\beta(\mathbb{C})$,*

$$\lim_{n\to\infty} (1+\delta)^n \|e^{-nP(w(\mathbf{0}))} \mathcal{L}^n_{w(\boldsymbol{\Theta})} \phi\|_\beta = 0$$

for all $\boldsymbol{\Theta} \neq (0, 0)$ and this holds uniformly for $\boldsymbol{\Theta}$ in a compact set.

Proof. This corollary is standard once the analytic perturbation theorem and the complex Ruelle–Perron–Frobenius theorem are available (cf. [25, Appendix 1] for the case of subshifts of finite type). The first assertion follows from Theorem 2.7. Since $\mathbf{z} \in D$, we know that $\mathcal{L}_{w(\mathbf{0})}$ is a bounded linear operator with a simple isolated eigenvalue of $e^{P(w(\mathbf{0}))}$ (cf. [40, Theorem 5.8]). Therefore, the second and third assertions are direct consequences of Theorem 2.8. The fourth assertion follows from the proof of [25, Proposition 5]. Finally, the last assertion is established using the proof of [25, Proposition 6], where the complex Ruelle–Perron–Frobenius theorem is provided by [23, Theorem 2.14] in this context. □

The (*asymptotic*) *covariance* of f and g with respect to the shift-invariant measure μ is given by

$$\mathrm{Cov}(f, g, \mu) := \lim_{n\to\infty} \frac{1}{n} \int S_n\left(f - \int f \, d\mu\right) S_n\left(g - \int g \, d\mu\right) d\mu$$

and the (*asymptotic*) *variance* of f with respect to μ is $\mathrm{Var}(f, \mu) := \mathrm{Cov}(f, f, \mu)$.

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

COROLLARY 2.10. *For any* $\mathbf{z} = (z_1, z_2) \in D$, *let* $\mu = \mu_{z_1 f + z_2 g}$ *be the unique equilibrium state of* $z_1 f + z_2 g$. *Then:*

(1) $(z_1, z_2) \mapsto P(z_1 f + z_2 g) := \mathbb{P}(\mathbf{z})$ *is analytic in* D;
(2) $(\partial/\partial z_1)\mathbb{P}(\mathbf{z}) = \int f \, d\mu$ *and* $(\partial/\partial z_2)\mathbb{P}(\mathbf{z}) = \int g \, d\mu$;
(3) $(\partial^2/\partial z_1^2)\mathbb{P}(\mathbf{z}) = \mathrm{Var}(f, \mu)$, $(\partial^2/\partial z_2^2)\mathbb{P}(\mathbf{z}) = \mathrm{Var}(g, \mu)$, *and* $(\partial^2/\partial z_1 \partial z_2)\mathbb{P}(\mathbf{z}) = (\partial^2/\partial z_2 \partial z_1)\mathbb{P}(\mathbf{z}) = \mathrm{Cov}(f, g, \mu)$;
(4) *if* f *and* g *are independent, then* $\nabla^2 \mathbb{P}$ *is positive definite in* D.

Proof. The first assertion follows from the analytic perturbation theorem (Theorem 2.8). Specifically, for any $\mathbf{z} \in D$, we know that the transfer operator $\mathcal{L}_{z_1 f + z_2 g}$ is a bounded linear operator on $\mathcal{B}(\mathcal{F}_\beta^b)$, with a simple, isolated eigenvalue $e^{\mathbb{P}(\mathbf{z})}$ (cf. [**40**, Theorem 5.8]). By applying Theorem 2.7, we see that the map $\mathbf{z} \mapsto \mathcal{L}_{z_1 f + z_2 g}$ is real-analytic for $\mathbf{z} \in D$, and the analyticity of $\mathbf{z} \mapsto \mathbb{P}(\mathbf{z})$ then follows directly from Theorem 2.8.

The second and third assertions are consequences of Proposition 2.5 parts (1) and (2), and the formulas for the first and second derivatives of the pressure given in [**30**, Propositions 2.6.13 and 2.6.14].

For the final assertion, we need to show that $(\partial^2/\partial z_1^2)P(z_1 f + z_2 g) > 0$ and that $\det(\nabla^2 \mathbb{P}) > 0$. Indeed, $(\partial^2/\partial z_1^2)P(z_1 f + z_2 g) > 0$ follows from [**39**, Theorem 3 and Remark 2 on p. 635]. Specifically, since $f \in L^2(\mu)$ by Proposition 2.5(2), [**39**, Theorem 3 and Remark 2 on p. 635] state that $\mathrm{Var}(f, \mu) = 0$ if and only if f is cohomologous to a constant. However, since f and g are independent, we know that neither f (nor g) can be cohomologous to a constant. Therefore, $\mathrm{Var}(f, \mu) > 0$. Now, $\det(\nabla^2 \mathbb{P}) > 0$ follows from the non-degeneracy of the variance using a standard argument from probability theory. Namely, the positivity of $\det(\nabla^2 \mathbb{P})$ follows, via a short computation, from the positivity of the variance of the function $g - (\mathrm{Cov}(f, g, \mu))/(\mathrm{Var}(f, \mu))f$. □

2.4. *The Legendre transform.* Recall (see Assumptions 2.3) that f and g are strictly positive locally Hölder continuous potentials with strong entropy gaps at infinity and $\|f - g\|_\infty$ is finite. We assume further that f and g are independent. For $\mathbf{f} = (f, g)$, we consider the domain D defined in (2.1).

Part (4) in Corollary 2.10 implies that the map $\mathbb{P} : D \to \mathbb{R}$ is convex. The *Legendre transform* of $\mathbb{P}$ is defined for all $\mathbf{x} \in \nabla \mathbb{P}(D)$ as

$$\mathbb{P}^*(\mathbf{x}) = \sup_{\mathbf{z} \in D}(\langle \mathbf{x}, \mathbf{z} \rangle - \mathbb{P}(\mathbf{z})),$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^2$.

PROPOSITION 2.11. *For any* $\mathbf{x} \in \nabla \mathbb{P}(D)$, *the following properties hold:*

(1) $\mathbb{P}^*$ *is analytic on* $\nabla \mathbb{P}(D)$;
(2) $\nabla \mathbb{P}^* \circ \nabla \mathbb{P} = \mathrm{id}$ *on* D;
(3) $\nabla \mathbb{P} \circ \nabla \mathbb{P}^* = \mathrm{id}$ *on* $\nabla \mathbb{P}(D)$;
(4) $\nabla \mathbb{P}(\mathbf{z}) = \mathbf{x} \iff \nabla \mathbb{P}^*(\mathbf{x}) = z \iff \nabla^2 \mathbb{P}^*(\mathbf{x}) = (\nabla^2 \mathbb{P}(\mathbf{z}))^{-1}$;
(5) $\nabla \mathbb{P}(\mathbf{z}) = \mathbf{x} \implies \mathbb{P}^*(\mathbf{x}) + \mathbb{P}(\mathbf{z}) = \langle \mathbf{x}, \mathbf{z} \rangle$;
(6) $\nabla \mathbb{P}(\mathbf{z}) \neq \mathbf{x} \implies \mathbb{P}^*(\mathbf{x}) + \mathbb{P}(\mathbf{z}) > \langle \mathbf{x}, \mathbf{z} \rangle$;
(7) $\nabla \mathbb{P}(\mathbf{z}) = \mathbf{x} \implies \mathbb{P}^*(\mathbf{x}) = -h_{\mu_{\langle \mathbf{z}, \mathbf{f} \rangle}}(\sigma)$.

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

Proof. Since $\mathbb{P}$ is analytic and strictly convex in D, all but the last assertion are classical properties of the Legendre transform (cf. [37, Example 11.9] for a proof).

The last assertion follows from the derivative formula for the pressure, namely Corollary 2.10(2). Recall that $\mu = \mu_{z_1 f + z_2 g} = \mu_{\langle \mathbf{z}, \mathbf{f} \rangle}$ is the equilibrium state of $z_1 f + z_2 g$; thus,

$$P(z_1 f + z_2 g) = h_\mu(\sigma) + \int (z_1 f + z_2 g)\, d\mu.$$

Then, since by assumption

$$\mathbf{x} = \nabla \mathbb{P}(\mathbf{z}) = (\partial_{z_1} P(z_1 f + z_2 g), \partial_{z_2} P(z_1 f + z_2 g)) = \left(\int f \, d\mu, \int g \, d\mu \right),$$

we apply part (5) and compute

$$\mathbb{P}^*(\mathbf{x}) = \langle \mathbf{x}, \mathbf{z} \rangle - \mathbb{P}(\mathbf{z}) = \int z_1 f \, d\mu + \int z_2 g \, d\mu - P(z_1 f + z_2 g) = -h_\mu(\sigma). \qquad \square$$

LEMMA 2.12. *For any $\mathbf{x} \in \mathbb{R}^2$ and any $t > 0$ such that $\mathbf{x}/t \in \nabla \mathbb{P}(D)$, the following holds.*

(1) $(d/dt)(-t\mathbb{P}^*(\mathbf{x}/t)) = \mathbb{P}(\nabla \mathbb{P}^*(\mathbf{x}/t))$;

(2) $(d^2/dt^2)(-t\mathbb{P}^*(\mathbf{x}/t)) = -(1/t)(\nabla^2 \mathbb{P}^*(\mathbf{x}/t))\langle \mathbf{x}/t, \mathbf{x}/t \rangle < 0.$

In particular, the map $t \mapsto -t\mathbb{P}^*(\mathbf{x}/t)$ is concave down.

Proof. For part (1), we compute

$$\begin{aligned} \frac{d}{dt}(-t\mathbb{P}^*(\mathbf{x}/t)) &= -\mathbb{P}^*(\mathbf{x}/t) + (-t)\langle \nabla \mathbb{P}^*(\mathbf{x}/t), -\mathbf{x}/t^2 \rangle \\ &= -\mathbb{P}^*(\mathbf{x}/t) + \langle \nabla \mathbb{P}^*(\mathbf{x}/t), \mathbf{x}/t \rangle \\ &= \mathbb{P}(\nabla \mathbb{P}^*(\mathbf{x}/t)), \end{aligned}$$

where the last equality follows from Proposition 2.11(5).

For part (2), we take another derivative of the above equation. Then,

$$\begin{aligned} \frac{d^2}{dt^2}(-t\mathbb{P}^*(\mathbf{x}/t)) &= \frac{d}{dt}\mathbb{P}(\nabla \mathbb{P}^*(\mathbf{x}/t)) \\ &= \nabla \mathbb{P}(\nabla \mathbb{P}^*(\mathbf{x}/t)) \cdot \nabla^2 \mathbb{P}^*(\mathbf{x}/t) \cdot (-\mathbf{x}/t^2) = -\frac{1}{t}(\nabla^2 \mathbb{P}^*(\mathbf{x}/t))\langle \mathbf{x}/t, \mathbf{x}/t \rangle. \end{aligned}$$

The second derivative is smaller than zero because $\nabla^2 \mathbb{P}^*$ is positive definite (as by Corollary 2.10, $\nabla^2 \mathbb{P}$ is positive definite). $\square$

2.5. *The Manhattan curve.* The *Manhattan curve* of $\mathbf{f}$ is given by

$$\mathcal{C}(\mathbf{f}) = \mathcal{C}(f, g) = \{(a, b) \in \mathbb{R}^2_{\geq 0} : \ P(-af - bg) = 0, \ a + b > 0\},$$

and the *extended Manhattan curve of* $\mathbf{f}$ is

$$\mathcal{C}^*(\mathbf{f}) = \{(a, b) \in \mathcal{D}(\mathbf{f}) : \ P(-af - bg) = 0\},$$

where $\mathcal{D}(\mathbf{f}) = -D(\mathbf{f})$ and $D(\mathbf{f})$ is defined in (2.1).

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

Remark 2.13. We will focus primarily on the Manhattan curve $\mathcal{C}(\mathbf{f})$; however, it will be convenient at times to consider an open neighborhood of $\mathcal{C}(\mathbf{f})$ in $\mathcal{C}^*(\mathbf{f})$.

Before we list important properties of the Manhattan curve, we recall that under our assumptions on the potentials f and g (see Assumptions 2.3), there exist $\delta_f, \delta_g > d(f) = d(g) > 0$, which are the unique numbers satisfying Bowen's formula $P(-\delta_f f) = P(-\delta_g g) = 0$ (cf. [5, Corollary 1.2]). Recall that we are assuming that f and g are independent.

THEOREM 2.14. (Bray *et al*, [5, Theorem C*]) *Let $f, g : \Sigma^+ \to \mathbb{R}$ be strictly positive locally Hölder continuous potentials with strong entropy gaps at infinity and such that $\|f - g\|_\infty$ is finite. Then:*

(1) *$(\delta_f, 0)$ and $(0, \delta_g)$ are on $\mathcal{C}(\mathbf{f})$;*

(2) *$\mathcal{C}(\mathbf{f})$ is a closed subcurve of the analytic curve $\mathcal{C}^*(\mathbf{f})$;*

(3) *$\mathcal{C}^*(\mathbf{f})$ is convex and strictly convex unless*

$$S_n f(\underline{x}) = \frac{\delta_g}{\delta_f} S_n g(\underline{x})$$

for all $\underline{x} \in \mathrm{Fix}^n$. In particular, $\mathcal{C}^*(\mathbf{f})$ is strictly convex provided f and g are independent;

(4) *the slope of the normal to $\mathcal{C}^*(\mathbf{f})$ at $(a, b) \in \mathcal{C}^*(\mathbf{f})$ is*

$$m = \frac{\int g \, d\mu_{-af-bg}}{\int f \, d\mu_{-af-bg}},$$

where μ_{-af-bg} is the equilibrium state of the potential $-af - bg$.

By Theorem 2.14, we can parameterize $\mathcal{C}^*(\mathbf{f})$ by $(s, q(s))$ for $q = q(s)$ an analytic function and s in an open interval containing $[0, \delta_f]$. We define the set of *admissible slopes* as

$$\mathcal{S}(\mathbf{f}) := \{m : m \text{ is the slope of the normal to } \mathcal{C}^*(\mathbf{f}) \text{ for some point } (a, b) \in \mathcal{C}^*(\mathbf{f})\}.$$

Notation 2.15. When the potentials are independent, by the strict convexity of $\mathcal{C}^*(\mathbf{f})$, for any $m \in \mathcal{S}(\mathbf{f})$, there exists a unique point $(a_m, b_m) \in \mathcal{C}^*(\mathbf{f})$ such that the slope of the normal at (a_m, b_m) is m. Then, define

$$t_m := \left(\int f \, d\mu_{-a_m f - b_m g} \right)^{-1}, \quad \mathbf{m} := (1, m), \quad \mathbf{x}_m := \mathbf{m}/t_m, \quad \mathbf{z}_m := (-a_m, -b_m).$$

COROLLARY 2.16. *The set of admissible slopes $\mathcal{S}(\mathbf{f})$ is a non-empty open interval. Moreover, for any $m \in \mathcal{S}(\mathbf{f})$, the following holds.*

(1) *$\mathbb{P}(\mathbf{z}_m) = 0$ and, in particular, $\mathbf{z}_m \in D$.*

(2) *$\nabla \mathbb{P}^*(\mathbf{x}_m) = \mathbf{z}_m$ and $\nabla \mathbb{P}(\mathbf{z}_m) = \mathbf{x}_m$.*

(3) *$\mathbb{P}^*(\mathbf{x}_m) = -h_{\mu_{-a_m f - b_m g}}(\sigma)$.*

(4) *$\mathbb{P}(\nabla \mathbb{P}^*(\mathbf{x}_m)) = 0$ and thus, $t = t_m$ realizes the maximum of $t \mapsto -t\mathbb{P}(\mathbf{m}/t)$.*

Proof. Theorem 2.14(2) implies that $\mathcal{S}(\mathbf{f})$ is an open interval containing the interval with extremes given by the slopes of the normal at the points $(\delta_f, 0)$ and $(0, \delta_g)$.

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

Item (1) follows from the definition of $\mathcal{C}^*(\mathbf{f})$ and the observation (cf. [5, Lemma 3.3(2)]) that $P(\langle \mathbf{z}_m, \mathbf{f}\rangle) = 0$ implies $d(-\langle \mathbf{z}_m, \mathbf{f}\rangle) < 1$. For item (2), we notice that $\mathbf{x}_m = (1/t_m, m/t_m) = (\int f\, d\mu, \int g\, d\mu)$, where $\mu = \mu_{-a_m f - b_m g}$. Thus, by part (4) in Proposition 2.11, it is sufficient to show

$$\nabla \mathbb{P}(-a_m, -b_m) = \left(\int f\, d\mu, \int g\, d\mu \right).$$

However, this follows from the formula for the derivative of the pressure given in Corollary 2.10(2). The third assertion follows from Proposition 2.11 and the second assertion. The last assertion is an immediate consequence of assertions (1) and (2) and part (2) in Lemma 2.12. □

3. *The correlation number*

Let Σ^+ be a topologically mixing countable state Markov shift with BIP. Fix a pair $\mathbf{f} = (f, g)$ of strictly positive locally Hölder continuous independent potentials on Σ^+ with strong entropy gaps at infinity such that $\|f - g\|_\infty$ is finite.

Definition 3.1. Let $m \in \mathcal{S}(\mathbf{f})$ be an admissible slope for the Manhattan curve $\mathcal{C}(\mathbf{f})$. The *correlation number* of $\mathbf{f}$ and m is

$$H_{\mathbf{f}}(m) \colon = a_m + b_m \cdot m,$$

where $(a_m, b_m) \in \mathcal{C}(\mathbf{f})$ is the unique point such that the slope of the normal to $\mathcal{C}(\mathbf{f})$ at (a_m, b_m) is m.

Remark 3.2.

(1) It is unclear at this point why we call $H_{\mathbf{f}}(m)$ the correlation number. Theorem 1.3 justifies this definition by showing that $H_{\mathbf{f}}(m)$ equals the exponential growth rate $\alpha_{\mathbf{f}}(m)$ of $M(t; \mathbf{f}, m, \xi)$.

(2) There is a useful interpretation of $H_{\mathbf{f}}(m)$ via the suspension flow $\phi^f : \Sigma^f \to \Sigma^f$, which we do not use in this paper, but is still worth mentioning. Suppose $\Psi^g : \Sigma^f \to \mathbb{R}$ is the symbolic reparameterization function of g with respect to the suspension flow $\phi^f : \Sigma^f \to \Sigma^f$ (cf. [20, Definition 3.9] or [4, Theorem 10.3]). Then, for any $m \in \mathcal{S}(\mathbf{f})$, one can check that

$$H_{\mathbf{f}}(m) = h_{\hat{\mu}}(\phi^f),$$

where $\hat{\mu}$ is the equilibrium state of $-b_m \Psi^g$ and $h_{\hat{\mu}}(\phi^f)$ is the measure-theoretic entropy of $\hat{\mu}$. Indeed, observe that $P_{\phi^f}(-b_m \Psi^g) = a_m$, where P_{ϕ^f} is the pressure function with respect to the suspension flow, and thus,

$$a_m = P_{\phi^f}(-b_m \Psi^g) = h_{\hat{\mu}}(\phi^f) - b_m \int \Psi^g\, d\hat{\mu}.$$

Then, recall that, in this case, $\hat{\mu}$ is the lift of $\mu = \mu_{-a_m f - b_m g}$ (cf. [4, Corollary 10.2]) and hence, $\int \Psi^g\, d\hat{\mu} = (\int g\, d\mu)/(\int f\, d\mu) = m$.

3.1. *Properties of the correlation number.*

In this subsection, we establish some of the properties of $H_{\mathbf{f}}(m)$ including Corollary 1.4 and Theorem 1.5 from the introduction.

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

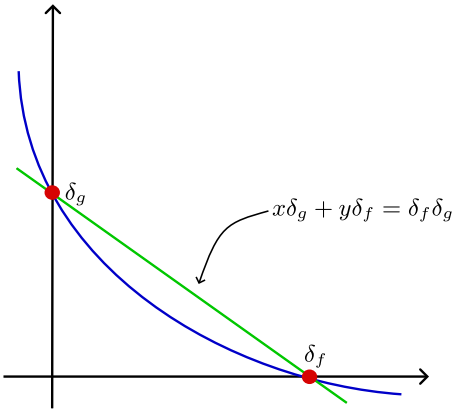


FIGURE 1. The Manhattan curve and the secant between its intercepts with the axes (colour online).

COROLLARY 1.4. (Correlation number rigidity) *For $m_* = \delta_f/\delta_g$, we have*

$$H_{\mathbf{f}}(m_*) \leq \delta_f. \tag{3.1}$$

Moreover, the equality holds if and only if $m_* S_n f(\underline{x}) = S_n g(\underline{x})$ for all $\underline{x} \in \mathrm{Fix}^n$.

Proof. Since the Manhattan curve $\mathcal{C}(\mathbf{f})$ is convex and analytic, the point $(a, b) \in \mathcal{C}(\mathbf{f})$ is below or on the secant passing through the points $(\delta_f, 0)$ and $(0, \delta_g)$. Hence,

$$a\delta_g + b\delta_f \leq \delta_f\delta_g,$$

and the equality holds if and only if $\mathcal{C}(\mathbf{f})$ is a straight line. See Figure 1.

We obtain (3.1) by specializing at the point $(a_{m_*}, b_{m_*}) \in \mathcal{C}(\mathbf{f})$ since

$$H_{\mathbf{f}}(m_*) = a_{m_*} + m_* b_{m_*} \leq \frac{\delta_f\delta_g}{\delta_g} = \delta_f.$$

The rigidity statement follows from Theorem 2.14(3) as $\mathcal{C}(\mathbf{f})$ is a straight line segment if and only if

$$S_n f(\underline{x}) = \frac{\delta_g}{\delta_f} S_n g(\underline{x})$$

for all $\underline{x} \in \mathrm{Fix}^n$. □

Recall that for any $\alpha, \beta > 0$, the (α, β)-Bishop–Steger entropy of $\mathbf{f}$ is defined as

$$h^{\mathrm{BS}}_{(\alpha,\beta)}(\mathbf{f}) := \lim_{T\to\infty} \frac{1}{T} \log \sum_n \frac{1}{n} \#\{\underline{x} \in \mathrm{Fix}^n : \alpha S_n f(\underline{x}) + \beta S_n g(\underline{x}) \leq T\}.$$

THEOREM 1.5. *Let $m \in \mathcal{S}(\mathbf{f})$ be an admissible slope. Then,*

$$\frac{H_f(m)}{\alpha + m\beta} \leq h^{\mathrm{BS}}_{(\alpha,\beta)}(\boldsymbol{f}),$$

and the equality holds if and only if m is the unique slope satisfying $a_m/b_m = \alpha/\beta$.

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

Proof. We ease notation by writing h_{BS} for $h^{\mathrm{BS}}_{(\alpha,\beta)}(\mathbf{f})$. From [5, Theorem A], we know that $P(-h_{\mathrm{BS}}(\alpha f + \beta g)) = 0$, which implies $(a_{\mathrm{BS}}, b_{\mathrm{BS}}) := (\alpha h_{\mathrm{BS}}, \beta h_{\mathrm{BS}}) \in \mathcal{C}(\mathbf{f})$. Observe that $a_{\mathrm{BS}}/b_{\mathrm{BS}} = \alpha/\beta$ and denote by m_{BS} the slope of the normal to $\mathcal{C}(\mathbf{f})$ at $(a_{\mathrm{BS}}, b_{\mathrm{BS}})$. We claim that for any $m \in \mathcal{S}(\mathbf{f})$,

$$\frac{a_m + m b_m}{\alpha + m\beta} \leq \frac{a_{m_{\mathrm{BS}}} + m_{\mathrm{BS}} b_{m_{\mathrm{BS}}}}{\alpha + m_{\mathrm{BS}}\beta} = h_{\mathrm{BS}}.$$

The last equality follows directly from the definition of h_{BS}. To prove the inequality, note that $\mathcal{C}(\mathbf{f})$ can be parameterized as $(s, q(s))$, where $q(s)$ is an analytic function, and we denote the slope of the normal to $\mathcal{C}(\mathbf{f})$ at $(s, q(s))$ by $m(s)$ so that $q'(s) = -1/m(s)$. Now, consider the function

$$F(s) = \frac{s + m(s)q(s)}{\alpha + m(s)\beta} = \frac{s - q(s)/q'(s)}{\alpha - \beta/q'(s)} = \frac{sq'(s) - q(s)}{\alpha q'(s) - \beta}.$$

Differentiating, we get

$$F'(s) = \frac{q''(s)(\alpha q(s) - \beta s)}{(\alpha q'(s) - \beta)^2}.$$

Since $\mathcal{C}(\mathbf{f})$ is strictly convex and real analytic, we know that $q''(s) > 0$. Moreover, $F'(0) > 0$, while $F'(\delta_f) < 0$, so $F(s)$ achieves its maximum when $\alpha q(s) = \beta s$, which implies $s/q(s) = \alpha/\beta$. By the strict convexity of $\mathcal{C}(\mathbf{f})$, this solution is unique, occurring at $(a_{\mathrm{BS}}, b_{\mathrm{BS}}) = (\alpha h_{\mathrm{BS}}, \beta h_{\mathrm{BS}})$. This completes the proof of the claim and the theorem. □

We conclude this section by establishing technical properties of $H_{\mathbf{f}}(m)$, which will be useful for the estimates carried out in the next two sections.

LEMMA 3.3. *Let $f, g\colon \Sigma^+ \to \mathbb{R}$ be strictly positive locally Hölder continuous potentials with strong entropy gaps at infinity such that $\|f - g\|_\infty$ is finite. Set $\mathbf{f} = (f, g)$. Then, for every $m \in \mathcal{S}(\mathbf{f})$:*

(1) $H_{\mathbf{f}}(m) = m \cdot H_{(g,f)}(1/m)$;
(2) $H_{\mathbf{f}}(m) = -t_m \mathbb{P}^*(\mathbf{x}_m)$;
(3) $(\boldsymbol{x}_m)^\top \cdot \nabla^2 \mathbb{P}^*(\boldsymbol{x}_m) \cdot (\boldsymbol{x}_m) > 0$, *where* $\mu = \mu_{-a_m f - b_m g}$.

Proof. For the first assertion, given $m \in \mathcal{S}(\mathbf{f})$, consider $(a_m, b_m) \in \mathcal{C}(\mathbf{f})$, so that $P(-a_m f - b_m g) = 0$, $H_{\mathbf{f}}(m) = a_m + b_m m$, and

$$m = \frac{\int g \, d\mu_{-a_m f - b_m g}}{\int f \, d\mu_{-a_m f - b_m g}}.$$

Similarly, for $m' \in \mathcal{S}(g, f)$, there is $(b_{m'}, a_{m'}) \in \mathcal{C}(g, f)$, that is, $P(-b_{m'} g - a_{m'} f) = 0$, and

$$m' = \frac{\int f \, d\mu_{-a_{m'} f - b_{m'} g}}{\int g \, d\mu_{-a_{m'} f - b_{m'} g}}.$$

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

Moreover, $H_{(g,f)}(m') = b_{m'} + a_{m'}m'$. By Theorem 2.14(4), we see that $m \in \mathcal{S}(\mathbf{f})$ if and only if $1/m \in \mathcal{S}(g, f)$, and that when $m' = 1/m$, we have $(a_{m'}, b_{m'}) = (b_m, a_m)$. Thus,

$$H_{(g,f)}\left(\frac{1}{m}\right) = b_m + a_m\frac{1}{m} = \frac{1}{m}(a_m + mb_m) = \frac{1}{m}H_{\mathbf{f}}(m).$$

For item (2), by Corollary 2.16(2), we know $\nabla\mathbb{P}(-a_m, -b_m) = \mathbf{x}_m$ and

$$\mathbb{P}^*(\mathbf{x}_m) = -h_{-a_m f - b_m g}(\sigma).$$

Hence, using the fact that $\mu_{-a_m f - b_m g}$ is the equilibrium state of $-a_m f - b_m g$, we compute

$$\begin{aligned} -t_m\mathbb{P}^*(\mathbf{x}_m) &= \frac{h_{-a_m f - b_m g}(\sigma)}{\int f\, d\mu_{-a_m f - b_m g}} \\ &= \frac{P(-a_m f - b_m g) - \int(-a_m f - b_m g)\, d\mu_{-a_m f - b_m g}}{\int f\, d\mu_{-a_m f - b_m g}} \\ &= a_m + b_m\frac{\int g\, d\mu_{-a_m f - b_m g}}{\int f\, d\mu_{-a_m f - b_m g}} = a_m + b_m m \\ &= H_{\mathbf{f}}(m). \end{aligned}$$

The last assertion follows from Proposition 2.11(4) as $\nabla\mathbb{P}^*(\mathbf{x}_m) = (-a_m, -b_m)$ and

$$\nabla^2\mathbb{P}^*(\mathbf{x}_m) = (\nabla^2\mathbb{P}(-a_m, -b_m))^{-1} = \frac{1}{\det\nabla^2\mathbb{P}}\begin{pmatrix}\mathrm{Var}(g,\mu) & \mathrm{Cov}(f,g,\mu)\\ \mathrm{Cov}(f,g,\mu) & \mathrm{Var}(f,\mu)\end{pmatrix}^{-1}$$

is positive definite (as by Corollary 2.10(4), $\nabla^2\mathbb{P}$ is positive definite). □

Remark 3.4. Using the same suspension flow interpretation as in Remark 3.2, one can relate the term in Lemma 3.3(3) with the second derivative of the pressure over the suspension flow. Explicitly,

$$\overline{\mathbb{P}}''_m := t_m^3 \cdot (\mathbf{x}_m)^\top \cdot \nabla^2\mathbb{P}^*(\mathbf{x}_m) \cdot \mathbf{x}_m = \left(\frac{d^2}{dz^2}P_{\phi^f}|_{z=-b_m}(z\Psi^g)\right).$$

The convenient notation $\overline{\mathbb{P}}''_m$ will be used in §6.2.

4. *Preparing to count*

Before presenting the setup and results, we give a brief outline of this section. The goal here is to convert the given orbital counting problem into estimates on thermodynamical quantities. In particular, Lalley's method counts periodic orbits by reducing this problem to studying preimages of sample points in the shift space.

To start the setup of this section, we fix a topologically mixing, countable state Markov shift Σ^+ with BIP and let $f, g\colon \Sigma^+ \to \mathbb{R}$ be strictly positive locally Hölder continuous potentials with strong entropy gaps at infinity such that $\|f - g\|_\infty$ is finite. We let $\mathbf{f} = (f, g)$ and use the notation from §§2 and 3.

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

For an admissible slope $m \in \mathcal{S}(\mathbf{f})$ and a precision $\xi > 0$, we set $I^2_{\xi,m}(t) = (t, t+\xi) \times (mt, mt+\xi)$ and define

$$M(n,t) := M(n,t;\mathbf{f},m,\xi) = \#\{\underline{x} \in \mathrm{Fix}^n \colon S_n\mathbf{f}(\underline{x}) \in I^2_{\xi,m}(t)\},$$

where $S_n\mathbf{f}(\underline{x}) := (S_n f(\underline{x}), S_n g(\underline{x}))$. The goal of this section is to obtain some *a priori* estimates on $M(n,t)$ by relating it to the transfer operator $\mathcal{L}_{-a_m f - b_m g}$. Our strategy is similar to the one from [5, §5], with some key differences which we single out below.

4.1. *Preimages of sample points and a priori estimates.* Fix an admissible slope $m \in \mathcal{S}(\mathbf{f})$, $k \in \mathbb{Z}_{>0}$, and a non-periodic word $\underline{z}_p \in \Sigma^+$ in each k-cylinder $p \in \Lambda_k$. For every $n \in \mathbb{Z}_{>0}$ and $\xi > 0$, define

$$\mathcal{W}(n,p,t;\mathbf{f},m,\xi) = p \cap \sigma^{-n}(\underline{z}_p) \cap \{\underline{x} \in \Sigma^+ \colon S_n\mathbf{f}(\underline{x}) \in I^2_{\xi,m}(t)\} \quad \text{and}$$
$$W(n,p,t) = W(n,p,t;\mathbf{f},m,\xi) := \#\mathcal{W}(n,p,t;\mathbf{f},m,\xi).$$

Propositions 4.1 and 4.6 relate $M(n,t)$ to $W(n,p,t)$, which is advantageous because $W(n,p,t)$ is itself related to the transfer operator via the *Laplace–Fourier transform* (defined below).

First, we introduce the following convenient notation. For any $\mathbf{y} \in \mathbb{R}^2$, define

$$M(n,\mathbf{y}) := M(n,\mathbf{y};\mathbf{f}) = \sum_{\underline{x} \in \mathrm{Fix}^n} \mathbf{1}_{\{\underline{x} \colon S_n\mathbf{f}(\underline{x}) = \mathbf{y}\}}(\underline{x})$$

and for any open bounded rectangle $U \subset \mathbb{R}^2$, let

$$M(n,U) = \#\{\underline{x} \in \mathrm{Fix}^n \colon S_n\mathbf{f}(\underline{x}) \in U\} = \int_U M(n,\mathbf{y})\,d\mathbf{y}.$$

Note that with this notation, $M(n,t) = M(n, I^2_{\xi,m}(t))$. Define $W(n,p,\mathbf{y})$ and $W(n,p,U)$ analogously and let

$$W(n,U) = \sum_{p \in \Lambda_k} W(n,p,U).$$

Finally, for any open rectangle $U = (a,b) \times (c,d)$ and sufficiently small $\epsilon > 0$, consider the open rectangles

$$U_{\pm\epsilon} := (a \mp \epsilon, b \pm \epsilon) \times (c \mp \epsilon, d \pm \epsilon).$$

PROPOSITION 4.1. [5, Lemma 5.1(ii)–(iv)] *Suppose that Σ^+ is a topologically mixing, countable state Markov shift with BIP, then $f, g \colon \Sigma^+ \to \mathbb{R}$ are strictly positive locally Hölder continuous potentials with strong entropy gaps at infinity such that $\|f - g\|_\infty$ is finite.*

(1) *For any $p \in \Lambda_k$ and $n \geq k$, there exists a bijection $\Psi^n_p \colon \mathrm{Fix}^n \cap p \to \sigma^{-n}(\underline{z}_p) \cap p$.*
(2) *There exists a sequence $(\epsilon_k)_{k=1}^\infty$ such that $\lim_k \epsilon_k = 0$ and if $\underline{y} \in \mathrm{Fix}^n \cap p$ and $n \geq k$, then*

$$\|S_n\mathbf{f}(\underline{y}) - S_n\mathbf{f}(\Psi^n_p(\underline{y}))\| \leq \epsilon_k.$$

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

(3) *For all $n \geq k$ and all open rectangles $U \subset \mathbb{R}^2$,*

$$W(n, U_{-\epsilon_k}) \leq M(n, U) \leq W(n, U_{\epsilon_k}). \tag{4.1}$$

Proof. Item (1) is [5, Lemma 5.1(ii)], item (2) is an immediate consequence of [5, Lemma 5.1(iii)], and item (3) follows from item (2). □

We will need to complement the results in Proposition 4.1 with an exponential control of the quantities $\sum_{p\in\Lambda_k} W(n, p, t)$ and $M(n, t)$ for $k \in \mathbb{Z}_{>0}$ and $n < k$, similar to [5, Lemma 5.1(iv)]. However, in the current setting, we will need a different approach which relies on Lemmas 4.2 and 4.4.

LEMMA 4.2. *Let $f, g\colon \Sigma^+ \to \mathbb{R}$ be strictly positive locally Hölder continuous potentials with strong entropy gaps at infinity such that $\|f - g\|_\infty$ is finite. If $m \geq 1$ is an admissible slope, then*

$$d(f) < H_{\mathbf{f}}(m).$$

Remark 4.3. There exist potentials f and g for which every admissible slope is strictly less than 1.

Proof. By [5, Lemma 3.3], it suffices to show that $P(-H_{\mathbf{f}}(m) f) < \infty$. Recall that, by definition, there exists a point (a_m, b_m) on the Manhattan curve $\mathcal{C}(\mathbf{f})$ such that $H_{\mathbf{f}}(m) = a_m + m \cdot b_m$. Since $m \geq 1$ and the pressure function is increasing by Corollary 2.10(2), we see that

$$P(-H_{\mathbf{f}}(m) f) = P(-a_m f - m \cdot b_m f) \leq P(-a_m f - b_m f).$$

Then, since $\|f - g\|_\infty = C < \infty$, and P is increasing and linear on constants, we see that

$$P(-a_m f - b_m f) \leq P(-a_m f - b_m g + b_m C) = P(-a_m f - b_m g) + b_m C = b_m C < \infty,$$

where we used the fact that $P(-a_m f - b_m g) = 0$ in the last equality. □

LEMMA 4.4. *Let $f, g\colon \Sigma^+ \to \mathbb{R}$ be strictly positive locally Hölder continuous potentials with strong entropy gaps at infinity such that $\|f - g\|_\infty$ is finite. Suppose $m \geq 1$. Then, for all $k \in \mathbb{N}$ and $b \in (d(f), H_{\mathbf{f}}(m))$, there exists $C = C(k, b, \xi) > 0$ such that for all $n < k$ and $t > 0$, we have*

$$\sum_{p\in\Lambda_k} W(n, p, t) \leq Ce^{bt} \quad \text{and} \quad M(k, t) \leq Ce^{bt}.$$

Remark 4.5. In this proof, we will use [5, Lemma 5.1(iv)]. In that result, one is only interested in the case $b < \delta_f$. However, its proof holds more generally for any $b > d(f)$ (see also [5, Lemma 3.1]).

Proof. We notice that, by [5, Lemma 5.1(iv)], for all $b > d(f)$, there exists $C = C(k, b, \xi)$ such that

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

$$\sum_{p\in\Lambda_k} W(n,\,p,\,t) \le \sum_{p\in\Lambda_k} \sum_{\underline{y}\in\sigma^{-n}(\underline{z}_p)} \mathbf{1}_p(\underline{y})\mathbf{1}_{\{S_n f(\underline{y})\le t+\xi\}}(\underline{y}) \le Ce^{bt}.$$

As $m \ge 1$, we have that $H_{\mathbf{f}}(m) > d(f)$ by Lemma 4.2, and hence, we may choose any $b \in (d(f),\, H_{\mathbf{f}}(m))$. The same argument shows that for all $b \in (d(f),\, H_{\mathbf{f}}(m))$, there exists $C = C(k, b, \xi)$ such that

$$M(k,t) \le Ce^{bt}. \qquad \square$$

We are now ready to bound $\sum_{p\in\Lambda_k} W(n,\,p,\,t)$ and $M(k,t)$ for any $k \in \mathbb{Z}_{>0}$ and $n < k$.

PROPOSITION 4.6. (Negligible parts) *Let $f, g\colon \Sigma^+ \to \mathbb{R}$ be strictly positive locally Hölder continuous potentials with strong entropy gaps at infinity such that $\|f - g\|_\infty$ is finite. Then, for any $k > 0$, there exist constants $\zeta = \zeta(m)$, $C = C(k, m, \xi) > 0$ such that for all $n < k$,*

$$\sum_{p\in\Lambda_k} W(n,\,p,\,t) \le Ce^{(H_{\mathbf{f}}(m)-\zeta)t} \quad \text{and} \quad M(k,t) \le Ce^{(H_{\mathbf{f}}(m)-\zeta)t}.$$

Proof. For $m \ge 1$, a stronger version is given by Lemma 4.4. Thus, we assume $m < 1$ and consider $H_{(g,f)}(m')$ with $m' = 1/m > 1$. Applying Lemma 4.4 to the pair $(g,\, f)$, if we pick ζ small enough such that $H_{(g,f)}(m') - m'\zeta \in (d(g),\, H_{(g,f)}(m'))$, then we know that there exists $C = C(k, m, \xi)$ such that

$$\begin{aligned}&\sum_{p\in\Lambda_k} W(n,\,p,\,\widetilde{t};\,(g,\,f),\,m',\,\xi)\\ &\quad = \#(p\cap\sigma^{-n}(\underline{z}_p)\cap\{\underline{x}\colon\ S_n(g,\,f)(\underline{x}) \in I^2_{\xi,m'}(\widetilde{t})\}) \le Ce^{(H_{(g,f)}(m')-m'\zeta)\widetilde{t}}\end{aligned}$$

for all $n < k$ and $\widetilde{t} \ge 0$. Therefore, setting $\widetilde{t} = mt$, we have

$$\begin{aligned}\sum_{p\in\Lambda_k} W(n,\,p,\,t;\mathbf{f},\,m,\,\xi) &= \#(p\cap\sigma^{-n}(\underline{z}_p)\cap\{\underline{x}\colon\ S_n\mathbf{f}(\underline{x}) \in I^2_{\xi,m}(t)\})\\ &= \sum_{p\in\Lambda_k} W\left(n,\,p,\,mt;\,(g,\,f),\,\frac{1}{m},\,\xi\right)\\ &\le Ce^{(H_{(g,f)}(1/m)-(1/m)\zeta)mt}\\ &= Ce^{(mH_{(g,f)}(1/m)-\zeta)t}\\ &= Ce^{(H_{\mathbf{f}}(m)-\zeta)t},\end{aligned}$$

where the last equality follows from Lemma 3.3(1). Applying the same argument to $M(k,t)$, we obtain

$$M(k,t) \le Ce^{(H_{\mathbf{f}}(m)-\zeta)t},$$

which ends the proof. $\square$

The *Laplace–Fourier transform* of a continuous function $F\colon \mathbb{R}^2 \to \mathbb{R}$ is

$$\widehat{F}(\mathbf{z}) = \int_{\mathbb{R}^2} e^{\langle \mathbf{z},\mathbf{x}\rangle} F(\mathbf{x})\, d\mathbf{x}. \tag{4.2}$$

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

Write $\widehat{W}(n, p, \mathbf{z})$, $\widehat{W}(n, \mathbf{z})$, and $\widehat{M}(n, \mathbf{z})$ for the Laplace–Fourier transforms of the functions $\mathbf{x} \mapsto W(n, p, \mathbf{x})$, $\mathbf{x} \mapsto W(n, \mathbf{x})$, $\mathbf{x} \mapsto M(n, \mathbf{x})$, respectively. The key observation is that $\widehat{W}(n, p, \mathbf{z})$ is related to the transfer operator as follows:

$$\begin{aligned}\widehat{W}(n, p, \mathbf{z}) &= \int_{\mathbb{R}^2} e^{\langle \mathbf{z}, \mathbf{x} \rangle} \, W(n, p, \mathbf{x}) \, d\mathbf{x} \\ &= \int_{\mathbb{R}^2} e^{\langle \mathbf{z}, \mathbf{x} \rangle} \sum_{\underline{y} \in \sigma^{-n}(\underline{z}_p)} \mathbf{1}_p(\underline{y}) \mathbf{1}_{\{\underline{y}: \, S_n \mathbf{f}(\underline{y}) = \mathbf{x}\}}(\underline{y}) \, d\mathbf{x} \\ &= \sum_{\underline{y} \in \sigma^{-n}(\underline{z}_p)} e^{\langle \mathbf{z}, S_n \mathbf{f}(\underline{y}) \rangle} \mathbf{1}_p(\underline{y}) \\ &= (\mathcal{L}^n_{\langle \mathbf{z}, \mathbf{f} \rangle} \mathbf{1}_p)(\underline{z}_p). \end{aligned} \tag{4.3}$$

Assume $\mathbf{z} \in D(\mathbf{f})$ so that $\langle \mathbf{z}, \mathbf{f} \rangle$ is a locally Hölder continuous potential with finite pressure. By Proposition 2.5(1), $\mu_{\langle \mathbf{z}, \mathbf{f} \rangle}$ is the unique invariant Gibbs measure and equilibrium state for $\langle \mathbf{z}, \mathbf{f} \rangle$, and we denote by $Q > 1$ its Gibbs constant.

LEMMA 4.7. *Consider $k \in \mathbb{N}$ and $\mathbf{z} \in D(\mathbf{f})$. For all $n \geq k$, and $p \in \Lambda_k$, we have that*

$$\widehat{W}(n, p, \mathbf{z}) \leq Q \mu_{\langle \mathbf{z}, \mathbf{f} \rangle}(p) e^{n \mathbb{P}(\mathbf{z})},$$

where $\mu_{\langle \mathbf{z}, \mathbf{f} \rangle}$ is the equilibrium state for $\langle \mathbf{z}, \mathbf{f} \rangle$ with Gibbs constant Q.

Proof. Notice that

$$\begin{aligned} e^{-n \mathbb{P}(\mathbf{z})} \cdot \widehat{W}(n, p, \mathbf{z}) = e^{-n \mathbb{P}(\mathbf{z})} (\mathcal{L}^n_{\langle \mathbf{z}, \mathbf{f} \rangle} \mathbf{1}_p)(\underline{z}_p) &= \sum_{\underline{y} \in \sigma^{-n}(\underline{z}_p)} e^{\langle \mathbf{z}, S_n \mathbf{f}(\underline{y}) \rangle - n \mathbb{P}(\mathbf{z})} \mathbf{1}_p(\underline{y}) \\ &= \sum_{\underline{y} \in \sigma^{-n}(\underline{z}_p) \cap p} e^{\langle \mathbf{z}, S_n \mathbf{f}(\underline{y}) \rangle - n \mathbb{P}(\mathbf{z})}. \end{aligned}$$

Since $n \geq k$, we have $\sigma^{-n}(\underline{z}_p) \cap p \neq \emptyset$. Then, for any $\underline{y} \in \sigma^{-n}(\underline{z}_p) \cap p$, there exists an n-cylinder $p_{\underline{y}}$ such that $\underline{y} \in p_{\underline{y}} \subset p$. Moreover, $p_{\underline{y}_1} = p_{\underline{y}_2}$ implies $\underline{y}_1 = \underline{y}_2$. Thus, we can repeatedly use the Gibbs property to see that

$$\sum_{\underline{y} \in \sigma^{-n}(\underline{z}_p) \cap p} e^{\langle \mathbf{z}, S_n \mathbf{f}(\underline{y}) \rangle - n \mathbb{P}(\mathbf{z})} \leq Q \sum_{\underline{y} \in \sigma^{-n}(\underline{z}_p) \cap p} \mu_{\langle \mathbf{z}, \mathbf{f} \rangle}(p_{\underline{y}}) \leq Q \mu_{\langle \mathbf{z}, \mathbf{f} \rangle}(p),$$

which concludes the proof. □

Finally, we apply Lemma 4.7 to estimate the quantity

$$W(n, U) := \sum_{p \in \Lambda_k} W(n, p, U).$$

LEMMA 4.8. *Assume $\mathbf{z} \in D(\mathbf{f})$, U is an open bounded rectangle in $\mathbb{R}^2$, and $n \geq k$. Let $\mathbf{x} \in \mathbb{R}^2$ be such that $\langle \mathbf{z}, \mathbf{y} \rangle \geq \langle \mathbf{z}, \mathbf{x} \rangle$ for all $\mathbf{y} \in U$. Then,*

$$W(n, U) \leq Q e^{n \mathbb{P}(\mathbf{z}) - \langle \mathbf{z}, \mathbf{x} \rangle}.$$

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

Proof. The statement follows from the proof of [25, Proposition 1]. Indeed,

$$\begin{aligned} W(n, U) &\leq \sum_{p\in\Lambda_k} \int_U e^{\langle \mathbf{z},\mathbf{y}\rangle - \langle \mathbf{z},\mathbf{x}\rangle} W(n, p, \mathbf{y})\, d\mathbf{y} \\ &\leq \sum_{p\in\Lambda_k} e^{-\langle \mathbf{z},\mathbf{x}\rangle} \int_{\mathbb{R}^2} e^{\langle \mathbf{z},\mathbf{y}\rangle} W(n, p, \mathbf{y})\, d\mathbf{y} \\ &\leq \sum_{p\in\Lambda_k} \widehat{W}(n, p, \mathbf{z}) e^{-\langle \mathbf{z},\mathbf{x}\rangle} \leq Q e^{n\mathbb{P}(\mathbf{z}) - \langle \mathbf{z},\mathbf{x}\rangle}, \end{aligned}$$

where we apply Lemma 4.7 to obtain the last inequality. □

5. *The global growth rate control*

The first goal of this section is to establish Proposition 5.1, that is, we find an *a priori* upper bound on the number of preimages of a fixed non-periodic word $\underline{z}_p$ in a cylinder p for which the nth ergodic sum of $\mathbf{f}$ is comparable to $t(1, m)$. It is important for our purposes that these upper bounds are global, in the sense that they do not depend on p. The second goal is to establish Lemma 5.3, which is a type of large-deviation estimate. There, we single out a set of preimages of the $\underline{z}_p$ whose growth is relatively slow and thus negligible.

To begin the setup of this section, we fix a topologically mixing, countable state Markov shift Σ^+ with BIP and strictly positive locally Hölder continuous potentials $f, g\colon \Sigma^+ \to \mathbb{R}$ with strong entropy gaps at infinity such that $\|f - g\|_\infty$ is finite. Set $\mathbf{f} = (f, g)$ and fix $\xi > 0$. Recall that for an admissible slope $m \in \mathcal{S}(\mathbf{f})$ in Notation 2.15, we defined the quantities $(a_m, b_m) = -\mathbf{z}_m$, t_m, $\mathbf{m}$, $\mathbf{x}_m = \mathbf{m}/t_m$, and that we define the correlation number as $H_{\mathbf{f}}(m) = a_m + b_m m$. Fix $k \in \mathbb{N}$, and for each k-cylinder $p \in \Lambda_k$, choose a non-periodic word $\underline{z}_p \in p$.

In this section, we obtain growth rate estimates which are independent on the choice of a cylinder.

PROPOSITION 5.1. (*A priori* estimate) *For any admissible slope $m \in \mathcal{S}(\mathbf{f})$, there exists a constant $C = C(k, m, \xi)$ such that for all $p \in \Lambda_k$, we have*

$$\sum_{n\geq 1} \sum_{\underline{y}\in\sigma^{-n}(\underline{z}_p)} \mathbf{1}_p(\underline{y}) \mathbf{1}_{\{t\leq S_n f(\underline{y})\leq t+\xi\}}(\underline{y}) \mathbf{1}_{\{mt\leq S_n g(\underline{y})\leq mt+\xi\}}(\underline{y}) \leq C e^{t H_f(m)} \mu_{-a_m f - b_m g}(p),$$

where $\mu_{-a_m f - b_m g}$ is the equilibrium state for $-a_m f - b_m g$.

Proposition 5.1 follows from Lemma 5.2, which is a similar *a priori* estimate for one potential.

LEMMA 5.2. (Technical lemma) *Suppose $f\colon \Sigma^+ \to \mathbb{R}$ is a strictly positive locally Hölder continuous potential with a strong entropy gap at infinity. Let $\delta_f > d(f)$ be the unique constant such that $P(-\delta_f f) = 0$. Then, there exists a constant $D > 0$ such that for all $p \in \Lambda_k$,*

$$\sum_{n\geq 1} \sum_{\underline{y}\in\sigma^{-n}(\underline{z}_p)} \mathbf{1}_p(y) \mathbf{1}_{\{S_n f(\underline{y})\leq t\}}(\underline{y}) \leq D e^{t\delta} \mu_{-\delta f}(p), \tag{5.1}$$

where $\mu_{-\delta f}$ is the equilibrium state of $-\delta f$.

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

Proof. By [5, Lemma 5.1(iv)], it is enough to consider $n > k$. Notice that by the Gibbs property of $\mu_{-\delta f}$, we know that

$$\begin{aligned}
&\sum_{n>k}\sum_{\underline{y}\in\sigma^{-n}(\underline{z}_p)} \mathbf{1}_p(\underline{y})\mathbf{1}_{\{S_n f(\underline{y})\le t\}}(\underline{y})\\
&\quad=\sum_{n>k}\sum_{\underline{y}\in\sigma^{-n}(\underline{z}_p)} \mathbf{1}_p(\underline{y})\mathbf{1}_{\{S_{n-k} f(\sigma^k(\underline{y}))\le t-S_k f(\underline{y})\}}(\underline{y})\\
&\quad\le\sum_{n>k}\sum_{\underline{y}\in\sigma^{-n}(\underline{z}_p)} \mathbf{1}_p(\underline{y})\mathbf{1}_{\left\{S_{n-k} f(\sigma^k(\underline{y}))\le t+\frac{\log \mu_{-\delta f}(p)+\log Q}{\delta}\right\}}(\underline{y})\\
&\quad\le\sum_{n>k}\sum_{\underline{w}\in\sigma^{k-n}(\underline{z}_p)} \mathbf{1}_{\left\{S_{n-k} f(\underline{w})\le t+\frac{\log \mu_{-\delta f}(p)+\log Q}{\delta}\right\}}(\underline{w})\\
&\quad=\sum_{m\ge 1}\sum_{\underline{w}\in\sigma^{-m}(\underline{z}_p)} \mathbf{1}_{\left\{S_m f(\underline{w})\le t+\frac{\log \mu_{-\delta f}(p)+\log Q}{\delta}\right\}}(\underline{w})\\
&\quad\le \widetilde{C}e^{\delta\left(t+\frac{\log \mu_{-\delta f}(p)+\log Q}{\delta}\right)}=\widetilde{C}Qe^{\delta t}\mu_{-\delta f}(p),
\end{aligned}$$

where Q is the Gibbs constant of $\mu_{-\delta_f}$ and the last inequality follows from [5, Lemma 4.3]. □

Proof of Proposition 5.1. Let us simplify notation by writing $a = a_m$, $b = b_m$, $H = H_{\mathbf{f}}(m) = a + bm$, and $\mu = \mu_{-af-bg}$. We first claim that there exists E such that

$$\sum_{n\ge 1}\sum_{\underline{y}\in\sigma^{-n}(\underline{z}_p)} \mathbf{1}_p(\underline{y})\mathbf{1}_{\{S_n f(\underline{y})\le t\}}(\underline{y})\mathbf{1}_{\{|S_n(g-mf)(\underline{y})|\le\xi\}}(\underline{y}) \le Ce^{tH}\mu(p).$$

Let $\varphi = (1/H)(af + bg)$ and observe that φ is a strictly positive locally Hölder continuous potential satisfying $P(-H\varphi) = 0$. Using Lemma 5.2, we know that there exists a constant $D > 0$ such that for all $p \in \Lambda_k$,

$$\sum_{n\ge 1}\sum_{\underline{y}\in\sigma^{-n}(\underline{z}_p)} \mathbf{1}_p(\underline{y})\mathbf{1}_{\{S_n\varphi(\underline{y})\le t\}}(\underline{y}) \le De^{tH}\mu(p), \tag{5.2}$$

where we note that μ is the equilibrium state for $-H\varphi = -af - bg$. Since $a = H - bm$, we have

$$\varphi = \frac{1}{H}(af + bg) = f + \frac{b}{H}(g - mf).$$

Hence, we know that for any n and $\underline{y} \in \Sigma^+$,

$$\begin{aligned}
&\mathbf{1}_{\{S_n f(\underline{y})\le t\}}(\underline{y})\mathbf{1}_{\{|S_n(g-mf)(\underline{y})|\le\xi\}}(\underline{y})\\
&\quad\le \mathbf{1}_{\{S_n f(\underline{y})+(b/H)S_n(g-mf)(\underline{y})\le t+(b/H)\xi\}}(\underline{y}) = \mathbf{1}_{\{S_n\varphi(\underline{y})\le t+(b/H)\xi\}}(\underline{y}).
\end{aligned}\tag{5.3}$$

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

By (5.2) and (5.3), we finally obtain

$$\sum_{n\geq 1}\sum_{\underline{y}\in\sigma^{-n}(\underline{z}_p)} \mathbf{1}_p(\underline{y})\mathbf{1}_{\{S_n f(\underline{y})\leq t\}}(\underline{y})\mathbf{1}_{\{|S_n(g-mf)(\underline{y})|\leq\xi\}}(\underline{y}) \leq De^{(b\xi/H)}e^{tH}\mu(p) =: Ee^{tH}\mu(p),$$

which completes the proof of the claim. Then, we get

$$\begin{aligned}
&\sum_{n\geq 1}\sum_{\underline{y}\in\sigma^{-n}(\underline{z}_p)} \mathbf{1}_p(\underline{y})\mathbf{1}_{\{t\leq S_n f(\underline{y})\leq t+\xi\}}(\underline{y})\mathbf{1}_{\{mt\leq S_n g(\underline{y})\leq mt+\xi\}}(\underline{y})\\
&\leq \sum_{n\geq 1}\sum_{\underline{y}\in\sigma^{-n}(\underline{z}_p)} \mathbf{1}_p(\underline{y})\mathbf{1}_{\{t\leq S_n f(\underline{y})\leq t+\xi\}}(\underline{y})\mathbf{1}_{\{|S_n g(\underline{y})-mS_n f(\underline{y})|\leq(m+1)\xi\}}(\underline{y})\\
&\leq \sum_{n\geq 1}\sum_{\underline{y}\in\sigma^{-n}(\underline{z}_p)} \mathbf{1}_p(\underline{y})\mathbf{1}_{\{S_n f(\underline{y})\leq t+\xi\}}(\underline{y})\mathbf{1}_{\{|S_n g(\underline{y})-mS_n f(\underline{y})|\leq(m+1)\xi\}}(\underline{y})\\
&\leq \underbrace{De^{\left(\frac{b(m+1)\xi}{H}+H\xi\right)}}_{C} e^{tH}\mu(p),
\end{aligned}$$

which finishes the proof of the proposition. □

Recall the notation $I^2_{\xi,m}(t) = (t, t+\xi)\times(mt, mt+\xi)$ and consider the set

$$\mathcal{W}(p,t) := \{\underline{y}\in\Sigma^+ \colon \underline{y}\in p,\ \sigma^n(\underline{y}) = \underline{z}_p,\ S_n\mathbf{f}(\underline{y})\in I^2_{\xi,m}(t) \text{ for some } n\geq 1\}.$$

Since we assume that $\underline{z}_p$ is not periodic, if $\underline{y}\in\mathcal{W}(p,t)$, then there exists a unique positive integer $n(\underline{y})$ such that $\sigma^{n(\underline{y})}(\underline{y}) = \underline{z}_p$. Now, for any $\epsilon>0$, define

$$\mathcal{W}(p,t,\geq\epsilon) := \left\{\underline{y}\in\mathcal{W}(p,t)\colon \left|\frac{n(\underline{y})}{t} - t_m\right|\geq\epsilon\right\} \quad \text{and}$$
$$\mathcal{W}(p,t,<\epsilon) := \left\{\underline{y}\in\mathcal{W}(p,t)\colon \left|\frac{n(\underline{y})}{t} - t_m\right|<\epsilon\right\}.$$

Observe that these sets are related to the quantities $W(n,p,t)$ introduced in §4 and, in particular,

$$\#\mathcal{W}(p,t) = \sum_{n\geq 1} W(n,p,t).$$

We now derive a type of large deviation result which shows that the contribution to the growth of $\mathcal{W}(p,t)$ by the elements $\underline{y}$ for which $n(\underline{y})/t$ is far from t_m is negligible.

LEMMA 5.3. (Negligible part) *Suppose $m\in\mathcal{S}(\mathbf{f})$ is an admissible slope for $\mathbf{f}$. For any $\epsilon>0$, there exist constants $C_2 = C_2(k,m,\xi,\epsilon)$, $\eta=\eta(k,m,\xi,\epsilon)>0$ such that*

$$\sum_{p\in\Lambda_k}\mathcal{W}(p,t,\geq\epsilon) = \sum_{p\in\Lambda_k}\sum_{n\colon |(n/t)-t_m|\geq\epsilon} W(n,p,t) \leq C_2 e^{t(H_{\mathbf{f}}(m)-\eta)}.$$

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

Proof. When $n < k$, the statement follows from a repeated application of Proposition 4.6. Thus, assume $n \geq k$ and let $\mathbf{m} = (1, m)$. By Lemma 2.12(2), we know that the function $t \mapsto -t\mathbb{P}^*(\mathbf{m}/t)$ is concave down and realizes its maximum at $t = t_m$. Let $t^1 = t_m + \epsilon$, $t^0 = t_m - \epsilon$, and $\mathbf{z}^i = \nabla\mathbb{P}^*(\mathbf{m}/t^i)$ for $i = 0, 1$. By Proposition 2.11(4), we have

$$\mathbb{P}(\mathbf{z}^1) < 0 = \mathbb{P}\left(\nabla\mathbb{P}^*\left(\frac{\mathbf{m}}{t_m}\right)\right) < \mathbb{P}(\mathbf{z}^0)$$

and $t^i\mathbb{P}(\mathbf{z}^i) - \langle \mathbf{m}, \mathbf{z}^i\rangle < H_{\mathbf{f}}(m)$ by Proposition 2.11(6) and Lemma 3.3(2).

Consider $U_t = (1, 1 + (\xi/t)) \times (m, m + (\xi/t))$, so that $t \cdot U_t = I^2_{\xi,m}(t)$. Recall that Lemma 4.8 shows that for all $n \geq k$ and for a fixed $\mathbf{z} \in D$, if $\mathbf{x} \in \mathbb{R}^2$ is such that $\langle \mathbf{z}, \mathbf{y}\rangle \geq \langle \mathbf{z}, \mathbf{x}\rangle$ for all $\mathbf{y} \in U_t$, then

$$W(n, U_t) \leq Qe^{n\mathbb{P}(\mathbf{z}) - \langle \mathbf{z}, \mathbf{x}\rangle}.$$

Notice that the linear functional $\mathbf{y} \mapsto \langle \mathbf{z}, \mathbf{y}\rangle$ for $\mathbf{y} \in \overline{U}_t$ reaches its extreme values at a vertex of $\overline{U}_t$.

We let $\mathbf{x}^i(t)$ be the vertex of $\overline{U}_t$ such that $\langle \mathbf{z}^i, \mathbf{y}\rangle \geq \langle \mathbf{z}^i, \mathbf{x}^i(t)\rangle$ for all $\mathbf{y} \in U_t$. We also notice that $|\mathbf{x}^i(t) - \mathbf{m}| \leq \sqrt{2}(\xi/t)$. With this notation, we have

$$W(n, I^2_{\xi,m}(t)) \leq Qe^{n\mathbb{P}(\mathbf{z}^1) - t\langle \mathbf{z}^1, \mathbf{x}^1(t)\rangle}$$

and given that $\mathbb{P}(\mathbf{z}^1) < 0$, we obtain

$$\begin{aligned}\sum_{n \geq tt^1} W(n, I^2_{\xi,m}(t)) &\leq \sum_{n \geq tt_1} Qe^{n\mathbb{P}(\mathbf{z}^1) - t\langle \mathbf{z}^1, \mathbf{x}^1(t)\rangle} \\ &\leq Q \cdot \frac{e^{tt^1\mathbb{P}(\mathbf{z}^1) - t\langle \mathbf{z}^1, \mathbf{x}^1(t)\rangle}}{1 - e^{\mathbb{P}(\mathbf{z}^1)}} = Q'e^{t(t^1\mathbb{P}(\mathbf{z}^1) - \langle \mathbf{z}^1, \mathbf{x}^1(t)\rangle)} \\ &\leq \overline{Q}'e^{t(H_{\mathbf{f}}(m) - \eta_1)}.\end{aligned}$$

Similarly,

$$\begin{aligned}\sum_{n \leq tt^0} W(n, I^2_{\xi,m}(t)) &\leq \sum_{n \leq tt^0} Qe^{n\mathbb{P}(\mathbf{z}^0) - t\langle \mathbf{z}^0, \mathbf{x}^0(t)\rangle} \\ &\leq Q \cdot \frac{e^{\mathbb{P}(\mathbf{z}^0)}(e^{tt^0\mathbb{P}(\mathbf{z}^0)} - 1)e^{-t\langle \mathbf{z}^0, \mathbf{x}^0(t)\rangle}}{e^{\mathbb{P}(\mathbf{z}^0)} - 1} = Q''e^{t(t^0\mathbb{P}(\mathbf{z}^0) - \langle \mathbf{z}^0, \mathbf{x}^0(t)\rangle)} \\ &\leq \overline{Q}''e^{t(H_{\mathbf{f}}(m) - \eta_2)}.\end{aligned}$$

□

LEMMA 5.4. *There exists a constant $C_3 = C_3(k, m, \xi) > 0$ independent of p such that*

$$\sum_{p \in \Lambda_k} \sum_{n \geq 1} \frac{1}{n} W(n, p, t) \leq C_3 \frac{e^{H_{\mathbf{f}}(m)t}}{t},$$

where $W(n, p, t) = \#(p \cap \sigma^{-n}(\underline{z}_p) \cap \{\underline{y} \colon S_n\mathbf{f}(\underline{y}) \in I^2_{\xi,m}(t)\})$.

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

Proof. This follows from Proposition 5.1 and Lemma 5.3 via the following estimate:

$$\begin{aligned}\sum_{p\in\Lambda_k}\sum_{n\geq 1}\frac{1}{n}W(n,p,t) &= \sum_{p\in\Lambda_k}\sum_{\underline{y}\in\mathcal{W}(p,t)}\frac{1}{n(\underline{y})}\\ &= \sum_{p\in\Lambda_k}\sum_{\underline{y}\in\mathcal{W}(p,t,\geq\epsilon)}\frac{1}{n(\underline{y})}+\sum_{p\in\Lambda_k}\sum_{\underline{y}\in\mathcal{W}(p,t,<\epsilon)}\frac{1}{n(\underline{y})}\\ &\leq C_2 e^{t(H_{\mathbf{f}}(m)-\eta)}+\sum_{p\in\Lambda_k}\sum_{\underline{y}\in\mathcal{W}(p,t,<\epsilon)}\frac{1}{t(t_m-\epsilon)}\\ &\leq C_2 e^{t(H_{\mathbf{f}}(m)-\eta)}+\frac{Ce^{tH_{\mathbf{f}}(m)}}{t(t_m-\epsilon)}\sum_{p\in\Lambda_k}\mu_{-a_m f-b_m g}(p)\\ &\leq \frac{C_3 e^{tH_{\mathbf{f}}(m)}}{t},\end{aligned}$$

where C_2 is the constant from Lemma 5.3, C is the constant from Proposition 5.1, and $C/t_m-\varepsilon$ is uniformly bounded for t sufficiently large (for instance, when $t_m>2\varepsilon$). Hence, a constant C_3 in the last inequality exists. □

6. *Local counting, asymptotic estimates, and Theorems 1.1, 1.2, and 1.3*

We begin by outlining the structure of this section. Our goal is to combine the global estimates derived in §5 with the local asymptotic expansion formula in Theorem 6.1 to establish the main results of the paper, namely Theorems 1.1, 1.2, and 1.3.

Let Σ^+ be a topologically mixing, countable state Markov shift with BIP and let $f,g\colon\Sigma^+\to\mathbb{R}$ be strictly positive locally Hölder continuous potentials with strong entropy gaps at infinity such that $\|f-g\|_\infty$ is finite. Set $\mathbf{f}=(f,g)$. Fix an admissible slope $m\in\mathcal{S}(\mathbf{f})$ and recall Notation 2.15 for $(a_m,b_m)=-\mathbf{z}_m$, t_m, $\mathbf{m}$, and $\mathbf{x}_m=\mathbf{m}/t_m$. Furthermore, as in Theorem 2.6, we let $h_{-a_m f-b_m g}$ and $\nu_{-a_m f-b_m g}$ denote the eigenfunction and eigenmeasure of the transfer operator $\mathcal{L}_{-a_m f-b_m g}$, respectively. Throughout this section, we fix $k\in\mathbb{N}$, a k-cylinder $p\in\Lambda_k$, and a non-periodic word $\underline{z}_p\in p$.

The main goal of this section is to establish the following local estimate.

THEOREM 6.1. (Local estimate) *Let $m\in\mathcal{S}(\mathbf{f})$ and consider $(a_m,b_m)\in\mathcal{C}(\mathbf{f})$. Then, as $t\to\infty$,*

$$\sum_{n\geq 1}\frac{1}{n}W(n,p,t)\sim\frac{e^{tH_{\mathbf{f}}(m)}}{t^{(3/2)}}(2\pi\overline{\mathbb{P}}''_m)^{-(1/2)}\cdot C_p(m)\cdot\int_0^\xi e^{a_m t}\,dt\cdot\int_0^\xi e^{b_m t}\,dt,$$

where $\overline{\mathbb{P}}''_m:=t_m^3\cdot(\boldsymbol{x}_m)^\top\cdot\nabla^2\mathbb{P}^*(\boldsymbol{x}_m)\cdot\boldsymbol{x}_m$ and $C_p(m)=h_{-a_m f-b_m g}(\underline{z}_p)\nu_{-a_m f-b_m g}(p)$.

Remark 6.2. The notation $\overline{\mathbb{P}}''_m$ is used to simplify the right-hand side in Theorem 6.1. One can also see Remark 3.4 for another interpretation of $\overline{\mathbb{P}}''_m$ as the second derivative of the pressure over the suspension flow.

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

6.1. *The proofs of Theorems 1.1, 1.2, and 1.3.* Theorem 6.1 is the last ingredient needed to establish Theorems 1.1, 1.2, and 1.3 from the introduction. In this subsection, we record the proofs of these results while assuming Theorem 6.1.

THEOREM 6.3. (Theorem 1.1 and Theorem 1.3) *For any $m \in \mathcal{S}(\mathbf{f})$, we have*

$$\lim_{t\to\infty} \frac{1}{t} \log M(t; \mathbf{f}, m, \xi) = H_{\mathbf{f}}(m).$$

Proof. Fix $k \in \mathbb{N}$. For $n \geq k$, we apply Proposition 4.1(3) with $U = I^2_{\xi,m}(t)$ to deduce that for any $p \in \Lambda_k$,

$$W(n, p, t; \mathbf{f}, m, \xi - \epsilon_k) \leq \#\mathcal{M}_p(n, t; \mathbf{f}, m, \xi) \leq W(n, p, t; \mathbf{f}, m, \xi + \epsilon_k), \tag{6.1}$$

where $(\epsilon_k)_k$ is the sequence from Proposition 4.1(2). Then, we apply Proposition 4.6 to see that there exists $\zeta = \zeta(m) > 0$ and $C_1 = C_1(k, m, \xi) > 0$ such that for any $n < k$,

$$\#\mathcal{M}(n, t; \mathbf{f}, m, \xi) \leq C_1 e^{t(H_{\mathbf{f}}(m)-\zeta)}. \tag{6.2}$$

Moreover, Lemma 5.4 and Theorem 6.1 show that there exists $C_2 = C_2(m, p, \xi)$ and $C_3 = C_3(k, m, \xi)$ such that

$$\frac{e^{tH_{\mathbf{f}}(m)}}{t^{3/2}}(C_2 + o(1)) = \sum_{n\geq 1} \frac{1}{n} W(n, p, t) \leq \sum_{p\in\Lambda_k} \sum_{n\geq 1} \frac{1}{n} W(n, p, t) \leq C_3 \frac{e^{tH_{\mathbf{f}}(m)}}{t}.$$

Thus, recalling that $M(t; \mathbf{f}, m, \xi) = \sum_{p\in\Lambda_k} \sum_{n\geq 1} (1/n)\#\mathcal{M}_p(n, t; \mathbf{f}, m, \xi)$, we obtain

$$\lim_{t\to\infty} \frac{1}{t} \log M(t; \mathbf{f}, m, \xi) = H_{\mathbf{f}}(m). \qquad \square$$

Remark 6.4. We expect the asymptotic growth rate of $\sum_{p\in\Lambda_k} \sum_{n\geq 1} (1/n) W(n, p, t)$ to be $e^{tH_{\mathbf{f}}(m)}/t^{3/2}$, up to a constant, as suggested by Theorem 1.2. The key missing step for establishing the global version of Theorem 1.2 is to improve the current upper bound in Lemma 5.4 from $e^{tH_{\mathbf{f}}(m)}/t$ to $e^{tH_{\mathbf{f}}(m)}/t^{3/2}$.

Proof of Theorem 1.2. By Theorem 6.1, we know

$$\sum_{n\geq 1} \frac{1}{n} W(n, p, t; \mathbf{f}, m, \xi) = \frac{e^{tH_{\mathbf{f}}(m)}}{t^{3/2}} C(m, p, \xi)(1 + o(1)),$$

where $C(m, p, \xi) = (2\pi \overline{\mathbb{P}}''_m)^{-1/2} \cdot C_p(m) \cdot \int_0^\xi e^{a_m t}\, dt \cdot \int_0^\xi e^{b_m t}\, dt$. Moreover, by (6.1) and (6.2), we get

$$\begin{aligned} &\frac{e^{tH_{\mathbf{f}}(m)}}{t^{3/2}} C(m, p, \xi - \epsilon_k)(1 + o(1)) \\ &\quad \leq \sum_{n\geq 1} \frac{1}{n} \#\mathcal{M}_p(n, t; \mathbf{f}, m, \xi) \leq \frac{e^{tH_{\mathbf{f}}(m)}}{t^{3/2}} C(m, p, \xi + \epsilon_k)(1 + o(1)). \end{aligned}$$

To conclude the proof, recall that $\epsilon_k \to 0$ as $k \to \infty$ by Proposition 4.1(2). $\square$

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

6.2. *Proof of Theorem 6.1.* This proof consists of three steps. First, for any $\epsilon > 0$, we break the sum into two parts:

$$\sum_{n\geq 1} \frac{1}{n} W(n,\, p,\, t) = \sum_{n:\ |(n/t)-t_m|\geq \epsilon} \frac{1}{n} W(n,\, p,\, t) + \sum_{n:\ |(n/t)-t_m|<\epsilon} \frac{1}{n} W(n,\, p,\, t).$$

By Lemma 5.3, for any $\epsilon > 0$,

$$\limsup_{t\to\infty} e^{-tH_{\mathbf{f}}(m)} \cdot t^{3/2} \left(\sum_{n:\ |(n/t)-t_m|\geq \epsilon} \frac{1}{n} W(n,\, p,\, t) \right) \leq \limsup_{t\to\infty} C_2 t^{3/2} e^{-t\eta} = 0.$$

Therefore, for any $\epsilon > 0$, it suffices to study the asymptotic behavior for $t \to \infty$ of

$$\sum_{n:\ |(n/t)-t_m|<\epsilon} \frac{1}{n} W(n,\, p,\, t).$$

Since the characteristic function of the rectangle $(0, \xi) \times (0, \xi)$ can be approximated by a non-negative smooth function u with compact support, it is enough to show that

$$\lim_{\epsilon\to 0} \lim_{t\to\infty} e^{-tH_{\mathbf{f}}(m)}\, t^{3/2} \left(\sum_{n:\ |(n/t)-t_m|<\epsilon} \frac{1}{n} \int_{\mathbb{R}^2} u(\mathbf{y} - t\mathbf{m}) W(n,\, p,\, \mathbf{y})\, d\mathbf{y} \right)$$
$$= (2\pi \overline{\mathbb{P}}''_m)^{-(1/2)} C_p(m) \int_{\mathbb{R}^2} u(\mathbf{y})\, e^{-\langle \mathbf{z}_m, \mathbf{y}\rangle}\, d\mathbf{y}. \tag{6.3}$$

We will need Lemma 6.5 below, whose proof is postponed to §6.3. For $N \in \mathbb{N}$, denote by $C_c^N(\mathbb{R}^2)$ the space of N-differentiable functions on $\mathbb{R}^2$ with compact support.

LEMMA 6.5. *Let K be a compact neighborhood of $\mathbf{x}_m$ in $\nabla\mathbb{P}(D)$. Suppose $\mathbf{x} = \nabla\mathbb{P}(\mathbf{z}) \in K$ and $u \in C_c^N(\mathbb{R}^2)$ is non-negative, then as $n \to \infty$,*

$$\int u(\mathbf{y} - n\boldsymbol{x}) W(n,\, p,\, \mathbf{y})\, d\mathbf{y} \sim \frac{e^{-n\mathbb{P}^*(\boldsymbol{x})}}{2\pi n} (\det \nabla\mathbb{P}^*(\boldsymbol{x}))^{1/2} \cdot C_p(\mathbf{z}) \cdot \int_{\mathbb{R}^2} u(\mathbf{y}) e^{-\langle \mathbf{z}, \mathbf{y}\rangle}\, d\mathbf{y},$$

where $C_p(\mathbf{z}) = h_{\langle \mathbf{z}, \mathbf{f}\rangle}(\underline{z}_p) \nu_{\langle \mathbf{z}, \mathbf{f}\rangle}(p)$. Furthermore, the above convergence is uniform in K.

Remark 6.6. Let us outline the strategy for obtaining (6.3) assuming Lemma 6.5. The first step is to derive (6.4) below, which provides the asymptotic expansion for the left-hand side of (6.3) for points near $\mathbf{x}_m$. Next, we expand the left-hand side of (6.4), denoted by $(*)$, and terms such as $\overline{\mathbb{P}}''_m$ appearing in (6.3) arise naturally. Using this expansion, together with Lemma 6.7, a Gaussian-type distribution result, we can rewrite and bound the left-hand side of (6.4), that is, $(*)$, in terms of $\overline{\mathbb{P}}''_m$ and the other quantities in (6.3). Equation (6.3) then follows by taking the limits of the upper and lower bounds for $(*)$.

We now begin with the first step. We plan to apply Lemma 6.5 to points $\mathbf{x}$ near $\mathbf{x}_m = \mathbf{m}/t_m$ (cf. Notation 2.15), in particular, to those of the form $\mathbf{x} = \mathbf{m}/(n/t)$ whenever $|n/t - t_m| < \epsilon$ for small ϵ. To do this, we need to ensure that these points lie in $\nabla\mathbb{P}(D)$. To see this, note first that D is an open set (by Proposition 2.5(3)), and that $\nabla\mathbb{P}$ is a homeomorphism on D. The latter follows from the inverse function theorem and because $\nabla^2\mathbb{P}$ is positive definite (Corollary 2.10(4)). Consequently, $\nabla\mathbb{P}(D) \subset \mathbb{R}^2$ is open.

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

Since $\mathbf{x}_m$ lies in the open set $\nabla\mathbb{P}(D)$, there exists $\epsilon > 0$ sufficiently small such that whenever $|n/t - t_m| < \epsilon$, the point $\mathbf{m}/(n/t)$ lies in a neighborhood of $\mathbf{x}_m$ contained in $\nabla\mathbb{P}(D)$. Thus, there exists $\mathbf{z}^* := \nabla\mathbb{P}^*(\mathbf{m}/n/t) \in D$.

Then, summing the estimate from Lemma 6.5 over the set $\{n : |n/t - t_m| < \epsilon\}$, for $t \to \infty$, we have

$$\begin{aligned}&\sum_{|n/t - t_m| < \epsilon} \frac{1}{n} \int u(\mathbf{y} - t\mathbf{m}) W(n, p, \mathbf{y})\, d\mathbf{y} \\ &\quad \sim \underbrace{\sum_{|(n/t) - t_m| < \epsilon} \frac{e^{-n\mathbb{P}^*(t\mathbf{m}/n)}}{2\pi n^2} \cdot (\det \nabla^2\mathbb{P}^*(t\mathbf{m}/n))^{1/2} \cdot \mathbf{Q}_p(t\mathbf{m}/n)}_{:= (*)}, \end{aligned} \tag{6.4}$$

where $\mathbf{Q}_p(t\mathbf{m}/n) := C_p(\mathbf{z}^*) \int_{\mathbb{R}^2} u(\mathbf{y}) e^{-\langle \mathbf{z}^*, \mathbf{y}\rangle}$. Now, the last step in the proof of Theorem 6.1 consists of using a standard probability trick, Lemma 6.7 below, to simplify the right-hand side of the above equation.

LEMMA 6.7. (Lalley [25, Lemma 6]) *Suppose F is continuous on a closed interval $[t_1, t_2]$ and $t_1 < t_m < t_2$. Let $\sigma > 0$. Then, for any $\epsilon > 0$ such that $(t_m - \epsilon, t_m + \epsilon) \subset [t_1, t_2]$, we have*

$$\lim_{t\to\infty} \sum_{n\,:\,|(n/t) - t_m| < \epsilon} \frac{1}{t^{1/2}} F\left(\frac{n}{t}\right) e^{-(t((n/t) - t_m)^2/2\sigma^2)} = F(t_m)(2\pi)^{1/2}\sigma. \tag{6.5}$$

Moreover, (6.5) holds uniformly for t_m and σ in a compact subset of $(0, \infty)$, and F in a compact subset of $C([t_1, t_2])$ with $t_1 < t_m < t_2$.

Recall that by Lemma 2.12(2) and Corollary 2.16(4), $s \mapsto -s\mathbb{P}^*(\mathbf{m}/s)$ is concave down for $s > 0$ and achieves its maximum uniquely at $s = t_m$. Expanding $-s\mathbb{P}^*(\mathbf{m}/s)$ near t_m, we have

$$\begin{aligned}-\frac{n}{t}\mathbb{P}^*\left(\mathbf{m}\frac{t}{n}\right) &= -t_m\mathbb{P}^*(\mathbf{m}/t_m) - t_m^{-1}(\mathbf{m}/t_m)^\top \cdot \nabla^2\mathbb{P}^*(\mathbf{m}/t_m) \cdot (\mathbf{m}/t_m) \cdot \frac{|n/t - t_m|^2}{2} \\ &\quad + R(\zeta, m)\left|\frac{n}{t} - t_m\right|^3 \\ &= \underbrace{-t_m\mathbb{P}^*(\mathbf{x}_m)}_{H_{\mathbf{f}}(m)} - \underbrace{t_m^{-1}(\mathbf{x}_m)^\top \cdot \nabla^2\mathbb{P}^*(\mathbf{x}_m) \cdot \mathbf{x}_m \cdot \frac{|n/t - t_m|^2}{2}}_{\text{main contribution}} + \underbrace{R(\zeta, m)\left|\frac{n}{t} - t_m\right|^3}_{\text{negligible term}},\end{aligned}$$

where $R(\zeta, m)$ is the remainder term from the Taylor expansion for some $\zeta \in ((n/t),\ t_m)$. Since the function $s \mapsto -s\mathbb{P}^*(\mathbf{m}/s)$ is analytic around $s = t_m$, there exists a constant $C_{m,\epsilon} > 0$ such that $|R(\zeta, x)| \le C_{m,\epsilon}$. Recalling that

$$t_m^{-1}(\mathbf{x}_m)^\top \cdot \nabla^2\mathbb{P}^*(\mathbf{x}_m) \cdot \mathbf{x}_m = t_m^{-4}\overline{\mathbb{P}}''_m,$$

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

for n such that $|(n/t) - t_m| < \epsilon$, we estimate

$$(*) \leq \sum_{|(n/t)-t_m|<\epsilon} \underbrace{e^{tH_{\mathbf{f}}(m) - tt_m^{-4}\overline{\mathbb{P}}''_m \cdot (|(n/t)-t_m|^2)/2 + tC_{m,\epsilon}\epsilon|(n/t)-t_m|^2}}_{=e^{tH_{\mathbf{f}}(m)} \cdot e^{-(t|(n/t)-t_m|^2)/2\sigma_+^2}} \cdot \underbrace{\frac{1}{2\pi n^2} \cdot (\det \nabla^2 \mathbb{P}^*(t\mathbf{m}/n))^{1/2} \cdot \mathbf{Q}_p(t\mathbf{m}/n)}_{=(1/t^2)F(n/t)} \tag{6.6}$$

$$(*) \geq \sum_{|(n/t)-t_m|<\epsilon} \underbrace{e^{tH_{\mathbf{f}}(m) - tt_m^{-4}\overline{\mathbb{P}}''_m \cdot (|(n/t)-t_m|^2)/2 - tC_{m,\epsilon}\epsilon|(n/t)-t_m|^2}}_{=e^{tH_{\mathbf{f}}(m)} \cdot e^{-(t|(n/t)-t_m|^2)/(2\sigma_-^2)}} \cdot \underbrace{1/2\pi n^2 \cdot (\det \nabla^2 \mathbb{P}^*(t\mathbf{m}/n))^{1/2} \cdot \mathbf{Q}_p(t\mathbf{m}/n)}_{=(1/t^2)F(n/t)},$$

where $F(s) = (1/2\pi s^2)(\det \nabla^2 \mathbb{P}^*(\mathbf{m}/s))^{1/2} \cdot \mathbf{Q}_p(\mathbf{m}/s)$ and $\sigma_\pm = (t_m^{-4}\overline{\mathbb{P}}''_m \mp 2\epsilon C_{m,\epsilon})^{-1/2}$.

By Lemma 6.7, we get

$$\lim_{t\to\infty} \frac{1}{F(t_m)} \sum_{|(n/t)-t_m|<\epsilon} t^{-1/2} e^{-(t|(n/t)-t_m|^2)/(2\sigma_+^2)} \cdot F(n/t) = \sqrt{2\pi}\sigma_+ = \sqrt{2\pi}(t_m^{-4}\overline{\mathbb{P}}''_m - 2\epsilon C_{m,\epsilon})^{-(1/2)} \tag{6.7}$$

and similarly,

$$\lim_{t\to\infty} \frac{1}{F(t_m)} \sum_{|(n/t)-t_m|<\epsilon} t^{-1/2} e^{-(t|(n/t)-t_m|^2)/(2\sigma_-^2)} \cdot F(n/t) = \sqrt{2\pi}\sigma_- = (t_m^{-4}\overline{\mathbb{P}}''_m + 2\epsilon C_{m,\epsilon})^{-1/2}.$$

Finally, we put all the estimates above together and obtain, for any ϵ small enough (depending on m),

$$\begin{aligned}
&\limsup_{t\to\infty} e^{-tH_{\mathbf{f}}(m)} \cdot t^{3/2} \left(\sum_{|(n/t)-t_m|\leq\epsilon} \frac{1}{n} \int u(\mathbf{y} - t\mathbf{m}) \cdot W(n, p, \mathbf{y})\, d\mathbf{y} \right) \\
&= \limsup_{t\to\infty} e^{-tH_{\mathbf{f}}(m)} \cdot t^{3/2} (*) \cdot (1 + o(1)) && \text{(by (6.4))} \\
&\leq \lim_{t\to\infty} e^{-tH_{\mathbf{f}}(m)} \cdot t^{3/2} \cdot \Bigg((1 + o(1)) \cdot t^{-3/2} e^{tH_{\mathbf{f}}(m)} \\
&\qquad \times \left(\sum_{|(n/t)-t_m|<\epsilon} t^{-1/2} e^{-(t|(n/t)-t_m|^2)/(2\sigma_+^2)} \cdot F(n/t) \right) \Bigg) && \text{(by (6.6))} \\
&= F(t_m)\sqrt{2\pi}\sigma_+. && \text{(by (6.7)).}
\end{aligned}$$

Similarly, we have

$$\liminf_{T\to\infty} e^{-tH_{\mathbf{f}}(m)} \cdot t^{3/2} \left(\sum_{|(n/t)-t_m|\leq\epsilon} \frac{1}{n} \int u(\mathbf{y} - t\mathbf{m}) \cdot W(n, p, \mathbf{y})\, d\mathbf{y} \right) \geq F(t_m)\sqrt{2\pi}\sigma_-.$$

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

Then, (6.3) follows by taking the limit

$$\lim_{\epsilon\to 0}\sigma_{\pm}=\lim_{\epsilon\to 0}(t_m^{-4}\overline{\mathbb{P}}''_m\mp 2\epsilon C_{m,\epsilon})^{-1/2}=t_m^{-2}\overline{\mathbb{P}}''_m,$$

which concludes the proof of Theorem 6.1.

6.3. *The proof of Lemma 6.5.* In this subsection, we aim to prove the following key asymptotic estimate.

Lemma 6.5. *Let K be a compact neighborhood of $\mathbf{x}_m$ in $\nabla\mathbb{P}(D)$. Suppose $\mathbf{x}=\nabla\mathbb{P}(\mathbf{z})\in K$ and $u\in C_c^N(\mathbb{R}^2)$ is non-negative. Then, as $n\to\infty$,*

$$\int u(\mathbf{y}-n\boldsymbol{x})W(n,\,p,\,\mathbf{y})\,d\mathbf{y}\sim\frac{e^{-n\mathbb{P}^*(\boldsymbol{x})}}{2\pi n}(\det\nabla\mathbb{P}^*(\boldsymbol{x}))^{1/2}\cdot C_p(\mathbf{z})\cdot\int_{\mathbb{R}^2}u(\mathbf{y})e^{-\langle z,\mathbf{y}\rangle}\,d\boldsymbol{y},$$

where $C_p(\mathbf{z})=h_{\langle\mathbf{z},\mathbf{f}\rangle}(\underline{z}_p)\nu_{\langle\mathbf{z},\mathbf{f}\rangle}(p)$. Furthermore, the above convergence is uniform in K.

In general, verifying Lemma 6.5 across the entire space $C_c^N(\mathbb{R}^2)$ is challenging. However, as noted by Babillot and Ledrappier [2, Lemma 2.4], it suffices to check Lemma 6.5 on a smaller, well-behaved function class $\mathcal{H}$. Specifically, the function class $\mathcal{H}$ is the linear span of functions of the form $\{e^{i\langle\mathbf{z},\cdot\rangle}h(\cdot):h\in\mathcal{H}^+,\,\mathbf{z}\in\mathbb{R}^2\}$, where a function $h:\mathbb{R}^2\to\mathbb{C}$ belongs to the class $\mathcal{H}^+$ if:

(1) h is real-valued and non-negative; and
(2) the Laplace–Fourier transform (see (4.2)) $\boldsymbol{\Theta}\mapsto\widehat{h}(-i\boldsymbol{\Theta})$ belongs to $C_c^N(\mathbb{R}^2)$.

Remark 6.8. The statement of Babillot and Ledrappier [2, Lemma 2.4] is formulated in terms of convergence of measures on $\mathbb{R}^2$. As explained immediately before [2, Lemma 2.4], by considering the Borel measures $\nu_n^{\mathbf{x}}$ on $\mathbb{R}^2$, defined (via the Riesz representation theorem) by

$$\nu_n^{\mathbf{x}}(G):=\frac{2\pi n\,e^{n\mathbb{P}^*(\mathbf{x})}}{C_p(\mathbf{z})(\det\nabla\mathbb{P}^*(\mathbf{x}))^{1/2}}\int_{\mathbb{R}^2}G(\mathbf{y}-n\mathbf{x})\,W(n,\,p,\,\mathbf{y})\,d\mathbf{y},$$

for any continuous compactly supported function G on $\mathbb{R}^2$, and by noting that the limiting measure λ (in the notation of [2, Lemma 2.4]) is given by

$$\lambda(G)=\int_{\mathbb{R}^2}G(\mathbf{y})\,e^{-\langle\mathbf{z},\mathbf{y}\rangle}\,d\mathbf{y},$$

the proof of [2, Lemma 2.4] applies verbatim in our setting.

The remainder of this section is devoted to deriving the following lemma.

Lemma 6.9. *Let K be a compact neighborhood of $\mathbf{x}_m$ in $\nabla\mathbb{P}(D)$. Suppose $\mathbf{x}=\nabla\mathbb{P}(\mathbf{z})\in K$ and $u\in\mathcal{H}$. Then, as $n\to\infty$,*

$$\int u(\mathbf{y}-n\boldsymbol{x})W(n,\,p,\,\mathbf{y})\,d\mathbf{y}\sim\frac{e^{-n\mathbb{P}^*(\boldsymbol{x})}}{2\pi n}(\det\nabla\mathbb{P}^*(\boldsymbol{x}))^{1/2}\cdot C_p(\mathbf{z})\cdot\int_{\mathbb{R}^2}u(\mathbf{y})e^{-\langle z,\mathbf{y}\rangle}\,d\boldsymbol{y},$$

where $C_p(\mathbf{z})=h_{\langle\mathbf{z},\mathbf{f}\rangle}(\underline{z}_p)\nu_{\langle\mathbf{z},\mathbf{f}\rangle}(p)$. Furthermore, the above convergence is uniform in K.

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

Remark 6.10. The proof of Lemma 6.9 is given in the next subsection. Roughly speaking, the proof relies on a perturbation approach together with the saddle-point method. We also use the complex Ruelle–Perron–Frobenius theorem to bound and estimate the terms in the perturbation.

6.3.1. *The saddle-point method and the proof of Lemma 6.9.* This subsection aims to prove Lemma 6.9. Before beginning the proof, we first introduce one of our main tools: the saddle-point method.

Let $F : \Omega := \{\|\mathbf{\Theta}\| < \epsilon\} \subset \mathbb{R}^2 \to \mathbb{R}$ be an analytic function such that $\nabla F(\mathbf{0}) = 0$ and $\nabla^2 F(\mathbf{0})$ is positive definite. As F is analytic $\Omega \subset \mathbb{R}^2$, we can extend F to a holomorphic function on $\Omega_{\mathbb{C}} = \{\mathbf{\Theta} \in \mathbb{C}^2 : \|\mathbf{\Theta}\| < \epsilon\}$. Here, we will abuse notation and continue to call the extended holomorphic function by $F : \Omega_{\mathbb{C}} \to \mathbb{C}$.

To fix our notation, we write $M_u = \sup\{|u(x)|\}$, $v(t) = O_F(t)$ if and only if there exist $t_0 > 0$ and two constants C_1 and C_2 only depending on F such that $C_1 t \le v(t) \le C_2 t$ for all $t > t_0$, and L_v is the Lipschitz constant of a Lipschitz function v, that is, $|v(z) - v(w)| \le L_v |z - w|$.

PROPOSITION 6.11. (Saddle-point method) *If $G : \mathbb{C}^2 \to \mathbb{C}$ is a Lipschitz function with compact support, then*

$$\int_{\Omega} G(i\mathbf{\Theta}) e^{nF(i\mathbf{\Theta})}\, d\mathbf{\Theta} = \frac{e^{nF(\mathbf{0})} G(\mathbf{0})}{n} \cdot \frac{2\pi}{\sqrt{\det \nabla^2 F(\mathbf{0})}} + \frac{e^{nF(\mathbf{0})}}{n}\left(L_G \cdot O_F\left(\frac{1}{\sqrt{n}}\right) + M_G \cdot O_F\left(\frac{1}{\sqrt{n}}\right)\right).$$

Proof. We include a proof of this statement in Appendix B because we were not able to find an appropriate reference. Our proof is adapted from [1, §5.2]. □

We establish Lemma 6.9 in two steps. First, Lemma 6.12 allows us to use the saddle-point method to derive one side of the statement of Lemma 6.9. The second step is Lemma 6.14, which relies on the complex Ruelle–Perron–Frobenius theorem (that is, Corollary 2.9).

LEMMA 6.12. *Let $\mathbf{z} \in D$ and $\mathbf{x} = \nabla \mathbb{P}(\mathbf{z})$. Suppose $u \in \mathcal{H}$ and define*

$$F(\mathbf{y}) := \langle \boldsymbol{x}, \boldsymbol{y} - \boldsymbol{z} \rangle + \mathbb{P}(\boldsymbol{z} - \boldsymbol{y})$$

and

$$G_p(\mathbf{y}) := \widehat{u}(\boldsymbol{y} - \boldsymbol{z}) C_p(\boldsymbol{z} - \boldsymbol{y}),$$

where $C_p(\mathbf{z} - \mathbf{y}) = h_{\langle \mathbf{z}-\mathbf{y}, \mathbf{f} \rangle}(\underline{z}_p) \nu_{\langle \mathbf{z}-\mathbf{y}, \mathbf{f} \rangle}(p)$. Then, there exists $\epsilon > 0$ such that F is analytic in $\Omega = \{\mathbf{\Theta} : \|\mathbf{\Theta}\| < \epsilon\}$, $F(\mathbf{0}) = -\mathbb{P}^*(\boldsymbol{x})$, $\nabla F(\mathbf{0}) = 0$, and $\nabla^2 F(\mathbf{0}) = \nabla^2 \mathbb{P}(\mathbf{z}) = (\nabla^2 \mathbb{P}^*(\mathbf{x}))^{-1}$ is positive definite. Moreover, G_p is a bounded Lipschitz continuous function in $\Omega_{\mathbb{C}}$.

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

Proof. The analyticity and the formulas for the derivatives of F follow from Corollary 2.10. As $u \in \mathcal{H}$, we know $\widehat{u} \in C_c^N(\mathbb{R}^2)$. Corollary 2.9 gives us the analyticity of $\mathbf{y} \mapsto C_p(\mathbf{z} - \mathbf{y})$ provided $\mathbf{y} \in \Omega_{\mathbb{C}}$. Hence, G_p is bounded Lipschitz in $\Omega_{\mathbb{C}}$. □

An immediate consequence of Proposition 6.11 and Lemma 6.12 is the following corollary.

COROLLARY 6.13. *Let $\Omega = \{\|\boldsymbol{\Theta}\| < \epsilon\} \subset \mathbb{R}^2$, and let F and G_p be as in Lemma 6.12. Then, as $n \to \infty$,*

$$\int_{\Omega} G_p(i\boldsymbol{\Theta}) e^{nF(i\boldsymbol{\Theta})}\, d\boldsymbol{\Theta} \sim e^{-n\mathbb{P}^*(\boldsymbol{x})} \left(\frac{2\pi}{n}\right) (\det \nabla^2 \mathbb{P}^*(\boldsymbol{z}))^{1/2} \widehat{u}(-\boldsymbol{z}) \cdot C_p(\boldsymbol{z}). \tag{6.8}$$

The following lemma shows that the asymptotic growth of $\int_{\mathbb{R}^2} u(\mathbf{y} - n\mathbf{x}) W(n, p, \mathbf{y})\, d\mathbf{y}$ is captured by integrating the transfer operator around zero after applying the Laplace–Fourier transform. The proof is based on estimates of the transfer operator.

LEMMA 6.14. *Let $\boldsymbol{z} \in D$ and $\boldsymbol{x} = \nabla \mathbb{P}(\mathbf{z})$. Suppose $u \in \mathcal{H}$, then as $n \to \infty$,*

$$\int_{\Omega} \widehat{u}(i\boldsymbol{\Theta} - \boldsymbol{z}) C_p(\boldsymbol{z} - i\boldsymbol{\Theta}) e^{n\langle \boldsymbol{x}, i\boldsymbol{\Theta} - \boldsymbol{z}\rangle + n\mathbb{P}(\boldsymbol{z} - i\boldsymbol{\Theta})}\, d\boldsymbol{\Theta} \sim (2\pi)^2 \int_{\mathbb{R}^2} u(\mathbf{y} - n\boldsymbol{x}) W(n, p, \mathbf{y})\, d\mathbf{y}$$

where $C_p(\mathbf{z} - i\boldsymbol{\Theta}) = h_{\langle \mathbf{z} - i\boldsymbol{\Theta}, \mathbf{f}\rangle}(\underline{z}_p) \cdot \nu_{\langle \mathbf{z} - i\boldsymbol{\Theta}, \mathbf{f}\rangle}(p)$. Moreover, the convergence above is uniform for $\mathbf{x} \in \nabla \mathbb{P}(D)$ in any compact neighborhood of $\mathbf{x}$.

Proof. First, recall the Pancherel–Parseval identity

$$(2\pi)^2 \int_{\mathbb{R}^2} U(\mathbf{y}) V(\mathbf{y})\, d\mathbf{y} = \int_{\mathbb{R}^2} \widehat{U}(i\boldsymbol{\Theta}) \widehat{V}(-i\boldsymbol{\Theta})\, d\boldsymbol{\Theta}, \tag{6.9}$$

and that the Laplace–Fourier transform of $W(n, p, \mathbf{y})$ is related to the transfer operator by (4.3). Now, we apply 6.9 to the functions

$$U(\mathbf{y}) = u(\mathbf{y} - n\mathbf{x}) e^{-\langle \mathbf{y}, \mathbf{z}\rangle} \quad \text{and} \quad V(\mathbf{y}) = W(n, p, \mathbf{y}) e^{\langle \mathbf{y}, \mathbf{z}\rangle}$$

to get

$$\begin{aligned}(2\pi)^2 \int_{\mathbb{R}^2} u(\mathbf{y} - n\mathbf{x}) W(n, p, \mathbf{y})\, d\mathbf{y} &= \int_{\mathbb{R}^2} \widehat{u}(i\boldsymbol{\Theta} - \mathbf{z}) e^{n\langle \mathbf{x}, i\boldsymbol{\Theta} - \mathbf{z}\rangle} \widehat{W}(n, p, \mathbf{z} - i\boldsymbol{\Theta})\, d\boldsymbol{\Theta} \\ &= \int_{\mathbb{R}^2} \widehat{u}(i\boldsymbol{\Theta} - \mathbf{z}) e^{n\langle \mathbf{x}, i\boldsymbol{\Theta} - \mathbf{z}\rangle} (\mathcal{L}^n_{\langle \mathbf{z} - i\boldsymbol{\Theta}, \mathbf{f}\rangle} \mathbf{1}_p)(\underline{z}_p)\, d\boldsymbol{\Theta}.\end{aligned}$$

By Corollary 2.9(5), there exists $\delta > 0$ such that for any $\boldsymbol{\Theta} \neq \mathbf{0}$,

$$\lim_{n \to \infty} (1 + \delta)^n e^{-n\mathbb{P}(\mathbf{z})} \|\mathcal{L}^n_{\langle \mathbf{z} - i\boldsymbol{\Theta}, \mathbf{f}\rangle} \mathbf{1}_p\|_{\infty} = 0, \tag{6.10}$$

so we can find n large enough such that

$$\|e^{-n\mathbb{P}(\mathbf{z})} (1 + \delta)^n (\mathcal{L}^n_{\langle \mathbf{z} - i\boldsymbol{\Theta}, \mathbf{f}\rangle} \mathbf{1}_p)(\underline{z}_p)\| \leq 1.$$

By Corollary 2.9(4), we know that if $\boldsymbol{\Theta}$ is such that $\|\boldsymbol{\Theta}\| < \epsilon$, there exist $R > 0$ and $\eta \in (0, 1)$ independent of $\boldsymbol{\Theta}$ such that

$$\|e^{-n\mathbb{P}(\mathbf{z} - i\boldsymbol{\Theta})} \mathcal{L}^n_{\langle \mathbf{z} - i\boldsymbol{\Theta}, \mathbf{f}\rangle} \mathbf{1}_p - C_p(\mathbf{z} - i\boldsymbol{\Theta})\|_{\infty} \leq R\eta^n. \tag{6.11}$$

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

Now, we split the integral as

$$(2\pi)^2 \int_{\mathbb{R}^2} u(\mathbf{y}-n\mathbf{x})W(n,\,p,\,\mathbf{y})\,d\mathbf{y} = \int_{\mathbb{R}^2} \underbrace{\widehat{u}(i\boldsymbol{\Theta}-\mathbf{z})e^{n\langle \mathbf{x}, i\boldsymbol{\Theta}-\mathbf{z}\rangle}(\mathcal{L}^n_{\langle \mathbf{z}-i\boldsymbol{\Theta},\mathbf{f}\rangle}\mathbf{1}_p)(\underline{z}_p)}_{U(\boldsymbol{\Theta})}\,d\boldsymbol{\Theta}$$
$$= \int_{\Omega} U(\boldsymbol{\Theta})\,d\boldsymbol{\Theta} + \int_{\Omega^c} U(\boldsymbol{\Theta})\,d\boldsymbol{\Theta}.$$

Recall that $\mathbb{P}^*(\mathbf{x}) = \langle \mathbf{x}, \mathbf{z}\rangle - \mathbb{P}(\mathbf{z})$ by Proposition 2.11(4), so we have

$$\begin{aligned}
&\left|\int_{\Omega^c} U(\boldsymbol{\Theta})\,d\boldsymbol{\Theta}\right| \\
&\quad\le \int_{K\cap\Omega^c} \widehat{u}(i\boldsymbol{\Theta}-\mathbf{z})e^{n(-\langle \mathbf{x},\mathbf{z}\rangle+\mathbb{P}(\mathbf{z}))}(1+\delta)^{-n}|e^{-n\mathbb{P}(\mathbf{z})}(1+\delta)^n(\mathcal{L}^n_{\langle \mathbf{z}-i\boldsymbol{\Theta},\mathbf{f}\rangle}\mathbf{1}_p)(\underline{z}_p)|\,d\boldsymbol{\Theta} \\
&\quad\le M\int_{K\cap\Omega^c} e^{-n\mathbb{P}^*(\mathbf{x})}(1+\delta)^{-n}\,d\boldsymbol{\Theta} \\
&\quad= Me^{-n\mathbb{P}^*(\mathbf{x})}(1+\delta)^{-n}\mathrm{Leb}(K),
\end{aligned} \tag{6.12}$$

where $\|\widehat{u}\|_\infty \le M$ and $\widehat{u}$ vanishes for $\boldsymbol{\Theta}$ outside of K. However, we can compute

$$\begin{aligned}
\int_{\Omega} U(\boldsymbol{\Theta})\,d\boldsymbol{\Theta} &= \int_{\Omega} \widehat{u}(i\boldsymbol{\Theta}-\mathbf{z})e^{n\langle \mathbf{x}, i\boldsymbol{\Theta}-\mathbf{z}\rangle}(\mathcal{L}^n_{\langle \mathbf{z}-i\boldsymbol{\Theta},\mathbf{f}\rangle}\mathbf{1}_p)(\underline{z}_p)\,d\boldsymbol{\Theta} \\
&= \underbrace{\int_{\Omega} \widehat{u}(i\boldsymbol{\Theta}-\mathbf{z})e^{n\langle \mathbf{x}, i\boldsymbol{\Theta}-\mathbf{z}\rangle + n\mathbb{P}(\mathbf{z}-i\boldsymbol{\Theta})}C_p(\mathbf{z}-i\boldsymbol{\Theta})\,d\boldsymbol{\Theta}}_{(*)} \\
&\quad + \int_{\Omega} \widehat{u}(i\boldsymbol{\Theta}-\mathbf{z})e^{n\langle \mathbf{x}, i\boldsymbol{\Theta}-\mathbf{z}\rangle}R(\mathbf{z},\mathbf{f},\boldsymbol{\Theta})\,d\boldsymbol{\Theta},
\end{aligned}$$

where $R(\mathbf{z},\mathbf{f},\boldsymbol{\Theta}) = \mathcal{L}^n_{\langle \mathbf{z}-i\boldsymbol{\Theta},\mathbf{f}\rangle}\mathbf{1}_p(\underline{z}_p) - e^{n\mathbb{P}(\mathbf{z}-i\boldsymbol{\Theta})}C_p(\mathbf{z}-i\boldsymbol{\Theta})$. Observe that by (6.11), we have

$$\begin{aligned}
\left|\int_{\Omega} \widehat{u}(i\boldsymbol{\Theta}-\mathbf{z})e^{n\langle \mathbf{x}, i\boldsymbol{\Theta}-\mathbf{z}\rangle}R(\mathbf{z},\mathbf{f},\boldsymbol{\Theta})\,d\boldsymbol{\Theta}\right| &\le M\left|\int_{\Omega} e^{n\langle \mathbf{x}, i\boldsymbol{\Theta}-\mathbf{z}\rangle}e^{n\mathbb{P}(\mathbf{z}-i\boldsymbol{\Theta})}e^{-n\mathbb{P}(\mathbf{z}-i\boldsymbol{\Theta})}R(\mathbf{z},\mathbf{f},\boldsymbol{\Theta})\,d\boldsymbol{\Theta}\right| \\
&\le MR\eta^n\left|\int_{\Omega} e^{n\langle \mathbf{x}, i\boldsymbol{\Theta}-\mathbf{z}\rangle}e^{n\mathbb{P}(\mathbf{z}-i\boldsymbol{\Theta})}\,d\boldsymbol{\Theta}\right| \\
&\le M'R\eta^n\int_{\Omega} e^{n\langle \mathbf{x}, -\mathbf{z}\rangle}e^{n\mathbb{P}(\mathbf{z})}\,d\boldsymbol{\Theta} \\
&\le \pi\epsilon^2 M'R\eta^n e^{-n\mathbb{P}^*(\mathbf{x})},
\end{aligned} \tag{6.13}$$

where we used Corollary 2.9(5) to obtain the third inequality. We notice that by Corollary 6.13, we know the exponential growth rate of

$$(*) := \int_{\Omega} \widehat{u}(i\boldsymbol{\Theta}-\mathbf{z})e^{n\langle \mathbf{x}, i\boldsymbol{\Theta}-\mathbf{z}\rangle + n\mathbb{P}(\mathbf{z}-i\boldsymbol{\Theta})}C_p(\mathbf{z}-i\boldsymbol{\Theta})\,d\boldsymbol{\Theta}$$

is $\mathbb{P}^*(\mathbf{x})$. Meanwhile, (6.12) and (6.13) show that the growth rate of these two terms are strictly less then $\mathbb{P}^*(\mathbf{x})$. Hence,

$$\lim_{n\to\infty} \frac{(2\pi)^2\int_{\mathbb{R}^2} u(\mathbf{y}-n\mathbf{x})W(n,\,p,\,\mathbf{y})\,d\mathbf{y}}{\int_{\Omega} \widehat{u}(i\boldsymbol{\Theta}-\mathbf{z})e^{n\langle \mathbf{x}, i\boldsymbol{\Theta}-\mathbf{z}\rangle + n\mathbb{P}(\mathbf{z}-i\boldsymbol{\Theta})}C_p(\mathbf{z}-i\boldsymbol{\Theta})\,d\boldsymbol{\Theta}} = 1. \qquad \square$$

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

We are now ready to present the proof of Lemma 6.9, which follows from Corollary 6.13 and Lemma 6.14.

Proof of Lemma 6.9. By Lemma 6.14, we know

$$\lim_{n\to\infty} \frac{(2\pi)^2 \int_{\mathbb{R}^2} u(\mathbf{y}-n\mathbf{x})W(n,\,p,\,\mathbf{y})\,d\mathbf{y}}{\int_\Omega \hat{u}(i\boldsymbol{\Theta}-\mathbf{z})e^{n\langle \mathbf{x}, i\boldsymbol{\Theta}-\mathbf{z}\rangle + n\mathbb{P}(\mathbf{z}-i\boldsymbol{\Theta})} C_p(\mathbf{z}-i\boldsymbol{\Theta})\,d\boldsymbol{\Theta}} = 1.$$

Moreover, Corollary 6.13 shows that

$$\lim_{n\to\infty} \frac{(e^{-n\mathbb{P}^*(\mathbf{x})}/2\pi n)(\det \nabla^2 \mathbb{P}^*(\mathbf{x}))^{1/2}\widehat{u}(-\mathbf{z})C_p(\mathbf{z})}{\int_\Omega \widehat{u}(i\boldsymbol{\Theta}-\mathbf{z})e^{n\langle \mathbf{x}, i\boldsymbol{\Theta}-\mathbf{z}\rangle + n\mathbb{P}(\mathbf{z}-i\boldsymbol{\Theta})} C_p(\mathbf{z}-i\boldsymbol{\Theta})\,d\boldsymbol{\Theta}} = 1.$$

Combining these two equations, we obtain

$$\lim_{n\to\infty} \frac{(e^{-n\mathbb{P}^*(\mathbf{x})}/2\pi n)(\det \nabla^2 \mathbb{P}^*(\mathbf{x}))^{1/2}\widehat{u}(-\mathbf{z})C_p(\mathbf{z})}{(2\pi)^2 \int_{\mathbb{R}^2} u(\mathbf{y}-n\mathbf{x})W(n,\,p,\,\mathbf{y})\,d\mathbf{y}} = 1. \qquad \square$$

7. *Application to cusped Hitchin representations: Theorem 1.6*

In this section, we recall results of Bray *et al* [5] and combine them with Theorems 1.1 and 1.3 to deduce Theorem 1.6 from the introduction.

Let $S = \mathbb{H}^2/\Gamma$ be a geometrically finite hyperbolic surface with limit set $\Lambda(\Gamma) \subseteq \mathbb{H}^2$. Then, Γ is a torsion-free geometrically finite Fuchsian group in $\mathsf{PSL}(2,\mathbb{R})$, which we assume is not convex cocompact. We connect these objects to the previous sections by coding the recurrent portion of the geodesic flow of S.

THEOREM 7.1. (Dal'Bo and Peigné [14], Ledrappier and Sarig [28], Stadlbauer [44]. See also [5, Theorem 9.3]) *Let Γ denote a torsion-free geometrically finite Fuchsian group which is not convex cocompact. There exists a topologically mixing countable state Markov shift Σ^+ with BIP which codes the recurrent portion of the geodesic flow on $T^1(\mathbb{H}^2/\Gamma)$. Moreover, there exists a map $G\colon \mathrm{Fix}^n \to \Gamma$ such that if $\gamma \in \Gamma$ is hyperbolic, then there exists $n \in \mathbb{Z}_{>0}$ and a unique (up to shift) $\underline{x} \in \mathrm{Fix}^n$ such that γ is conjugate to $G(\underline{x})$.*

Note that, depending on whether S has finite area or not, the construction of the shift Σ^+ is different; however, Theorem 7.1 holds in both settings. See [5, §9.4] for a more detailed discussion. In particular, when Γ is convex cocompact, Bowen and Series code the geodesic flow via a finite states Markov shift, and then the statement of Theorem 1.6 follows from [12, 13, 17, 25, 41, 43].

7.1. *Cusped Hitchin representations.*

To define cusped Hitchin representations, we will first need to discuss a notion of positivity on the space of complete flags.

A matrix $U \in \mathsf{SL}(d,\mathbb{R})$ is *unipotent and totally positive* with respect to a basis $u = (u_1, \ldots, u_d)$ of $\mathbb{R}^d$ if in the basis u, the matrix U is unipotent, upper triangular and the minors of U are strictly positive, unless they are forced to be zero due to the shape of the matrix. Recall that a (complete) flag F in $\mathbb{R}^d$ is a maximal nested sequence of subspaces of $\mathbb{R}^d$. That is, $F = (F^0, F^1, \ldots, F^{d-1}, F^d)$ with $\dim F^i = i$ and $F^{i-1} \subset F^i$ for all $i = 1, \ldots, d$. We denote by $\mathcal{F}_d$ the space of flags in $\mathbb{R}^d$. Two flags

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

F, G are *transverse* if $F^i \cap G^{d-i} = \{0\}$ for all $i = 0, \dots, d$. A basis $u = (u_1, \dots, u_d)$ is *consistent* with a pair of transverse flags F, G if $u_i \in F^i \cap G^{d-i+1}$ for all $i = 1, \dots, d$. Then, a k-tuple of flags $(F_1, \dots, F_k)$ is *positive* if there exist a basis $u = (u_1, \dots, u_d)$ consistent with F_1 and F_k, and unipotent and totally positive matrices $U_2, \dots, U_{k-1}$ such that $F_j = U_j \cdot \dots \cdot U_2 \cdot F_2$.

Given a torsion-free geometrically finite Fuchsian group, a representation $\rho\colon \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is *cusped Hitchin* if there exists a continuous, ρ-equivariant map $\xi\colon \Lambda(\Gamma) \to \mathcal{F}_d$ which sends any positive tuple flags in $\Lambda(\Gamma) \subset \partial\mathbb{H}^2 \cong \mathcal{F}_2$ to a positive tuple of flags in $\mathcal{F}_d$.

7.2. *Length functions and potentials.* Choose

$$\mathfrak{a} = \{x \in \mathbb{R}^d\colon x_1 + \dots + x_d = 0\} \quad \text{and} \quad \mathfrak{a}^+ = \{x \in \mathfrak{a}\colon x_1 \geq \dots \geq x_d\}$$

as a Cartan subspace and a closed positive Weyl chamber for the Lie algebra of $\mathsf{SL}(d, \mathbb{R})$, respectively. The Jordan projection is the map

$$\lambda\colon \mathsf{SL}(d, \mathbb{R}) \to \mathfrak{a}^+$$

which records the logarithms of the moduli of eigenvalues of elements of $\mathsf{SL}(d, \mathbb{R})$ in decreasing order. If $\gamma \in \Gamma$ is hyperbolic and ρ is a cusped Hitchin representation, then $(\lambda \circ \rho)(\gamma)$ lies in the interior of $\mathfrak{a}^+$ (see, for example, [16, Corollary 9.2] and [9, Theorem 1.4]). Thus, if Δ is the set of non-zero positive linear combinations of the simple roots for $\mathfrak{a}^+$, for any $\phi \in \Delta$, we define the *ϕ-length function of ρ* as

$$\ell_\rho^\phi(\gamma) = (\phi \circ \lambda \circ \rho)(\gamma) \geq 0$$

and observe that if γ is hyperbolic, then $\ell_\rho^\phi(\gamma) > 0$.

Bray *et al* [5, Theorem D] construct a potential on the shift space from Theorem 7.1 associated to a cusped Hitchin representation ρ and a choice of length function.

THEOREM 7.2. (Bray *et al* [5]) *Let Γ be a torsion-free, geometrically finite Fuchsian group which is not convex cocompact, and $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is a cusped Hitchin representation and $\phi \in \Delta$. Then, there exists a strictly positive locally Hölder continuous potential $\tau_\rho^\phi\colon \Sigma^+ \to \mathbb{R}$ with strong entropy gap at infinity such that for every $\underline{x} \in \mathrm{Fix}^n$,*

$$S_n \tau_\rho^\phi(\underline{x}) = \ell_\rho^\phi(G(\underline{x})).$$

Moreover, if $\eta\colon \Gamma \to \mathsf{SL}(d, \mathbb{R})$ is another cusped Hitchin representation, then $\|\tau_\rho^\phi - \tau_\eta^\phi\|_\infty$ is finite.

Proof. The proof follows from [5, Theorem D and Lemma 3.2]. □

In this setting, we can characterize algebraically when a pair of potentials is independent.

LEMMA 7.3. *If ρ, η are cusped Hitchin representations and $\eta \neq \rho, (\rho^{-1})^\top$, then τ_ρ^ϕ and τ_η^ϕ are independent.*

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

Proof. The Zariski closures of $\rho(\Gamma)$ and $\eta(\Gamma)$ are simple, center-free, and connected Lie groups by a theorem of Sambarino (see [38, Corollary 1.5 and Remark 6.5]). Then, we can apply [13, proof of Lemma 2.11] (see also [13, Lemma 6.12]) to obtain that the potentials τ_ρ^ϕ and τ_η^ϕ are independent. □

We are now ready to prove Thoerem 1.6.

Proof of Theorem 1.6. Theorems 7.1, 7.2, and Lemma 7.3 together imply that we can apply Theorems 1.1 and 1.3 to the potentials τ_ρ^ϕ and τ_ρ^η. □

Acknowledgements. L.-Y.K. was partially supported by the Simons Foundation (Gift No. 956047) and by the National Science Foundation under Award No. 2441413. Also, parts of this work were done during his visit to the National Center for Theoretical Sciences (NCTS) in Taiwan. He would like to thank NCTS for their support and hospitality. G.M. was partially supported by an AMS-Simons Research Enhancement Grant for PUI faculty and by a New Faculty Investment grant from Sam Houston State University. He would like to thank the Mathematics Department at Yale University where he completed part of this work. The authors would like to thank Harry Bray, Dick Canary, Xian Dai, François Ledrappier, and Hee Oh for helpful and insightful conversations. The authors used an AI tool (specifically, ChatGPT versions 4o and 5) to edit their writing for clarity and English grammar, and to help with LaTeX syntax. We thank the anonymous referee for helpful and thoughtful feedback on an earlier version of this manuscript.

A. *Appendix. Proof of Theorem 2.7*

THEOREM 2.7. *The map* $(z, w) \mapsto \mathcal{L}_{zf+wg}$ *is holomorphic in* $\mathcal{B}(\mathcal{F}_\beta^b(\mathbb{C}))$ *for all* $(z, w) \in \overline{D}$.

Proof. By [30, Lemma 2.6.1], we know $z \mapsto \mathcal{L}_{zf+wg}$ is holomorphic provided $z \mapsto \mathcal{L}_{zf+wg}$ is continuous. Using Hartog's theorem for Banach spaces (cf. [10, Theorem 14.27]), that is, separate holomorphicity implies joint holomorphicity, it is sufficient to show $z \mapsto \mathcal{L}_{zf+wg}$ is continuous for any $(z, w) \in \overline{D}$, as the same argument shows that $w \mapsto \mathcal{L}_{zf+wg}$ is continuous for any $(z, w) \in \overline{D}$.

To see this, we fix $(z, w) \in \overline{D}$ and consider $z_n = z + \epsilon_n$ such that $\overline{D} \ni (z_n, w) \to (z, w)$ as $n \to \infty$ in $\mathbb{C}^2$, that is, $\lim_{n\to\infty} |\epsilon_n| = 0$. We want to show that as $n \to \infty$,

$$\|\mathcal{L}_{z_n f+wg} - \mathcal{L}_{zf+wg}\|_{\mathrm{op}} \to 0,$$

where $\|\cdot\|_{\mathrm{op}}$ is the operator norm on $\mathcal{B}(\mathcal{F}_\beta^b(\mathbb{C}))$. The proof of this claim follows (with some minor modifications/simplifications) from Sarig [39, Proposition 2(3)]. We give a proof below for the sake of completeness.

Let $\mathcal{A}_{x_0} = \{a \in \mathcal{A} : t_{ax_0} = 1\}$. For any $\phi \in \mathcal{F}_\beta^b(\mathbb{C})$, fix $\underline{x}$, $\underline{y}$ such that $x_0 = y_0$. Write $U(\underline{x}) = (zf + wg)(\underline{x})$, $G_n(\underline{x}) = 1 - e^{\epsilon_n f(\underline{x})}$, and

$$R_n\phi(\underline{x}) := \mathcal{L}_{zf+wg}\phi(\underline{x}) - \mathcal{L}_{z_n f+wg}\phi(\underline{x}) = \sum_{a\in\mathcal{A}_{x_0}} e^{U(a\underline{x})} G_n(a\underline{x})\phi(a\underline{x}).$$

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

Hence,

$$\begin{aligned}|R_n\phi(\underline{x})| \leq & \|\phi\|_\beta \sum_{a\in\mathcal{A}_{x_0}} |e^{U(a\underline{x})}||G_n(a\underline{x})| \\ \leq & \|\phi\|_\beta \sum_{a\in\mathcal{A}_{x_0}} e^{(\Re z\cdot f+\Re w\cdot g)(a\underline{x})}|1-e^{\epsilon_n f(a\underline{x})}| \\ \leq & \|\phi\|_\beta \sum_{a\in\mathcal{A}_{x_0}} e^{(\Re z\cdot f+\Re w\cdot g)(a\underline{x})+|\epsilon_n|f(a\underline{x})}|e^{-|\epsilon_n|f(a\underline{x})}-e^{(\epsilon_n-|\epsilon_n|)f(a\underline{x})}| \\ = & \|\phi\|_\beta \sum_{a\in\mathcal{A}_{x_0}} e^{((\Re z+|\epsilon_n|)\cdot f+\Re w\cdot g)(a\underline{x})}|e^{-|\epsilon_n|f(a\underline{x})}-e^{(\epsilon_n-|\epsilon_n|)f(a\underline{x})}|.\end{aligned}$$

Since $\lim_{n\to\infty}|\epsilon_n|=0$, we know $(\Re z+|\epsilon_n|,\Re w)\in D$ when n is large enough. Thus, there exists $0<\epsilon<<1$ such that $(\Re z+\epsilon,\Re w)\in D$ and $|\epsilon_n|<\epsilon$ when n is large. Therefore, by the Gibbs property of $(\Re z+|\epsilon_n|)\cdot f+\Re w\cdot g$, we have

$$e^{((\Re z+|\epsilon_n|)\cdot f+\Re w\cdot g)(a\underline{x})}\leq e^{((\Re z+\epsilon)\cdot f+\Re w\cdot g)(a\underline{x})}\leq Q\cdot\mu_{(\Re z+\epsilon)\cdot f+\Re w\cdot g}([a]),$$

where $[a]$ is the cylinder generated by $a\in\mathcal{A}_{x_0}$. Set $F_n(\underline{x}):=e^{-|\epsilon_n|f(\underline{x})}-e^{(\epsilon_n-|\epsilon_n|)f(\underline{x})}$. Since $f\geq 0$, we can easily get $|F_n|\leq 2$ and $\lim_{n\to\infty}|F_n(\underline{x})|=0$. Hence,

$$\begin{aligned}|R_n\phi(\underline{x})|/\|\phi\|_\beta \leq & \sum_{a\in\mathcal{A}_{x_0}} Q\cdot\mu_{(\Re z+\epsilon)\cdot f+\Re w\cdot g}([a])|F_n(a\underline{x})| \\ \leq & Q\int_{\mathcal{A}_{x_0}}|F_n(\underline{x})|\,d\mu_{(\Re z+\epsilon)\cdot f+\Re w\cdot g}(\underline{x})<\infty.\end{aligned}$$

By the bounded convergence theorem, we have as $n\to\infty$,

$$\sup\left\{\frac{|R_n\phi(\underline{x})|}{\|\phi\|_\beta}:\ \phi\in\mathcal{F}_\beta^b(\mathbb{C})\right\}\to 0.$$

The second step is to show $\sup\{(\mathrm{Lip}(R_n\phi(\underline{x}))/\|\phi\|_\beta):\ \phi\in\mathcal{F}_\beta^b(\mathbb{C})\}\to 0$ as $n\to\infty$.

$$\begin{aligned}|R_n\phi(\underline{x})-R_n\phi(\underline{y})| \leq & \sum_{a\in\mathcal{A}_{x_0}}|e^{U(a\underline{x})}G_n(a\underline{x})\phi(a\underline{x})-e^{U(a\underline{y})}G_n(a\underline{y})\phi(a\underline{y})| \\ \leq & \sum_{a\in\mathcal{A}_{x_0}}|e^{U(a\underline{x})}|\cdot|(1-e^{U(a\underline{y})-U(a\underline{x})})|\cdot|G_n(a\underline{x})|\cdot|\phi(a\underline{x})| \\ & +\sum_{a\in\mathcal{A}_{x_0}}|e^{U(a\underline{y})}|\cdot|G_n(a\underline{x})-G_n(a\underline{y})|\cdot|\phi(a\underline{x})| \\ & +\sum_{a\in\mathcal{A}_{x_0}}|e^{U(a\underline{y})}|\cdot|G_n(a\underline{y})|\cdot|\phi(a\underline{x})-\phi(a\underline{y})|.\end{aligned}$$

Notice that if $U(a\underline{x})\neq U(a\underline{y})$, then

$$|(1-e^{U(a\underline{y})-U(a\underline{x})})|\leq\frac{|(1-e^{U(a\underline{y})-U(a\underline{x})})|}{|U(a\underline{y})-U(a\underline{x})|}(\mathrm{Lip}(U))d(a\underline{x},a\underline{y})^\beta<Kd(\underline{x},\underline{y})^\beta,$$

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

where, for example, $K = \max\{\mathrm{Lip}(U) \cdot \sup\{(|1 - e^{\delta}|/\delta) : \ |\delta| \le \mathrm{Lip}(U)\}, 2\}$. Hence,

$$\begin{aligned}
&\sum_{a\in\mathcal{A}_{x_0}} |e^{U(a\underline{x})}| \cdot |(1 - e^{U(a\underline{y})-U(a\underline{x})})| \cdot |G_n(a\underline{x})| \cdot |\phi(a\underline{x})| \\
&\quad \le K d(\underline{x}, \underline{y})^{\beta} \|\phi\|_{\beta} \sum_{a\in\mathcal{A}_{x_0}} |e^{U(a\underline{x})}| \cdot |G_n(a\underline{x})|, \\
&\sum_{a\in\mathcal{A}_{x_0}} |e^{U(a\underline{y})}| \cdot |G_n(a\underline{y})| \cdot |\phi(a\underline{x}) - \phi(a\underline{y})| \\
&\quad \le A e^{-1} d(\underline{x}, \underline{y})^{\beta} \|\phi\|_{\beta} \sum_{a\in\mathcal{A}_{x_0}} |e^{U(a\underline{y})}| \cdot |G_n(a\underline{y})|,
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{a\in\mathcal{A}_{x_0}} |e^{U(a\underline{y})}| \cdot |G_n(a\underline{x}) - G_n(a\underline{y})| \cdot |\phi(a\underline{x})| \\
&\quad \le \|\phi\|_{\beta} \sum_{a\in\mathcal{A}_{x_0}} |e^{U(a\underline{y})}| |e^{\epsilon_n f(a\underline{x})} - e^{\epsilon_n f(a\underline{y})}| \\
&\quad \le \|\phi\|_{\beta} \sum_{a\in\mathcal{A}_{x_0}} e^{\Re U(a\underline{y})} |e^{\epsilon_n (f(a\underline{x}))}| |1 - e^{\epsilon_n (f(a\underline{y})-f(a\underline{x}))}| \\
&\quad \le \|\phi\|_{\beta} (\epsilon_n K') d(\underline{x}, \underline{y})^{\beta} \sum_{a\in\mathcal{A}_{x_0}} e^{\Re U(a\underline{y})} |e^{\epsilon_n (f(a\underline{x}))}| \\
&\quad \le \epsilon_n K' \|\phi\|_{\beta} d(\underline{x}, \underline{y})^{\beta} \sum_{a\in\mathcal{A}_{x_0}} e^{((\Re z+\epsilon)\cdot f + \Re w \cdot g))(a\underline{x})} \\
&\quad \le \epsilon_n K' \|\phi\|_{\beta} d(\underline{x}, \underline{y})^{\beta} \left(Q \int d\mu_{(\Re z+\epsilon)\cdot f + \Re w \cdot g)} \right) \\
&\quad \le \epsilon_n \cdot K' Q \|\phi\|_{\beta} d(\underline{x}, \underline{y})^{\beta},
\end{aligned}$$

where, for example, $K' = \max\{\mathrm{Lip}(f) \cdot \sup\{|1 - e^{\delta}|/\delta : \ |\delta| \le \mathrm{Lip}(f)\}, 2\}$.

Hence,

$$|R_n\phi(\underline{x}) - R_n\phi(\underline{y})| \le (Ae^{-1} + K)\|\phi\|_{\beta} \sum_{a\in\mathcal{A}_{x_0}} |e^{U(a\underline{x})}| \cdot |G_n(a\underline{x})| + \epsilon_n \cdot K' Q \|\phi\|_{\beta}.$$

As we have proved in the first step: $\sum_{a\in\mathcal{A}_{x_0}} |e^{U(a\underline{x})}| \cdot |G_n(a\underline{x})| \to 0$ as $n \to \infty$, we have established the claim. □

B. *Appendix. Proof of the saddle-point method*

Let $F : \Omega = \{\boldsymbol{\Theta}\colon \|\boldsymbol{\Theta}\| < \epsilon\} \subset \mathbb{R}^2 \to \mathbb{R}$ be an analytic function such that $\nabla F(\mathbf{0}) = 0$ and $\nabla^2 F(\mathbf{0})$ is positive definite. By analyticity, we know F has an absolutely convergent Taylor expansion $F(\mathbf{x}) = F(\mathbf{0}) + \nabla^2 F(\mathbf{0})\langle \mathbf{x}, \mathbf{x}\rangle + R(\mathbf{x})$ for $\mathbf{x} \in \Omega$. Thus, F is analytic over

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

$\Omega_{\mathbb{C}} = \{\boldsymbol{\Theta} \in \mathbb{C}^2 : \|\boldsymbol{\Theta}\| < \epsilon\}$ with the same Taylor expansion. We will abuse notation and let F denote the corresponding analytic function over $\mathbb{C}^2$.

To fix our notation, we write $M_u = \sup\{|u(x)|\}$, $v(t) = O_F(t)$ if and only if there exists $t_0 > 0$ and two constants C_1 and C_2 only depending on F such that $C_1 t \le v(t) \le C_2 t$ for $t > t_0$, and L_v is the Lipschitz constant of a Lipschitz function v, that is, $|v(z) - v(w)| \le L_v |z - w|$.

LEMMA B.1. *Let $G : \mathbb{C}^2 \to \mathbb{C}$ be a Lipschitz function with compact support. Suppose $F : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is an analytic function such that $\nabla F(\mathbf{0}) = 0$ and $\nabla^2 F(\mathbf{0})$ is positive definite. Then,*

$$\int_\Omega G(i\boldsymbol{\Theta}) e^{nF(i\boldsymbol{\Theta})}\, d\boldsymbol{\Theta} = \frac{e^{nF(\mathbf{0})} G(\mathbf{0})}{n} \cdot \frac{2\pi}{\sqrt{\det \nabla^2 F(\mathbf{0})}} + \frac{e^{nF(\mathbf{0})}}{n}\left(L_G \cdot O_G\left(\frac{1}{\sqrt{n}}\right) + M_G \cdot O_F\left(\frac{1}{\sqrt{n}}\right)\right).$$

Proof. Let $\widetilde{\boldsymbol{\Theta}} = \sqrt{n}\boldsymbol{\Theta} \in \mathbb{R}^2$, then we have

$$\int_\Omega G(i\boldsymbol{\Theta}) e^{nF(i\boldsymbol{\Theta})}\, d\boldsymbol{\Theta} = \frac{e^{nF(\mathbf{0})}}{n} \int_{\{\|\widetilde{\boldsymbol{\Theta}}\| < \sqrt{n}\epsilon\}} G\left(\frac{i\widetilde{\boldsymbol{\Theta}}}{\sqrt{n}}\right) e^{n(F(i\widetilde{\boldsymbol{\Theta}}/\sqrt{n}) - F(\mathbf{0}))}\, d\widetilde{\boldsymbol{\Theta}}.$$

Since $R(\mathbf{x}) = F(\mathbf{x}) - F(\mathbf{0}) - \nabla^2 F(\mathbf{0})(\mathbf{x}, \mathbf{x})/2$, we have

$$\int_\Omega G(i\boldsymbol{\Theta}) e^{nF(i\boldsymbol{\Theta})}\, d\boldsymbol{\Theta} = \frac{e^{nF(\mathbf{0})}}{n} \int_{\{\|\widetilde{\boldsymbol{\Theta}}\| < \sqrt{n}\epsilon\}} G\left(\frac{i\widetilde{\boldsymbol{\Theta}}}{\sqrt{n}}\right) e^{-1/2\nabla^2 F(\mathbf{0})(\widetilde{\boldsymbol{\Theta}}, \widetilde{\boldsymbol{\Theta}}) + nR(i\widetilde{\boldsymbol{\Theta}}/\sqrt{n})}\, d\widetilde{\boldsymbol{\Theta}}$$

and, for $\|\widetilde{\boldsymbol{\Theta}}\| < \sqrt{n}\epsilon$, there exists $K > 0$ such that

$$\left| nR\left(\frac{i\widetilde{\boldsymbol{\Theta}}}{\sqrt{n}}\right)\right| \le \frac{K}{\sqrt{n}} \|\widetilde{\boldsymbol{\Theta}}\|^3 \le K\epsilon \|\widetilde{\boldsymbol{\Theta}}\|^2.$$

Since $|e^z - 1 - z| \le e^{|z|}(|z|^2/2)$, we can compute

$$\begin{aligned}
&\left| \int_{\{\|\widetilde{\boldsymbol{\Theta}}\| < \sqrt{n}\epsilon\}} \left| G\left(\frac{i\widetilde{\boldsymbol{\Theta}}}{\sqrt{n}}\right) e^{-1/2\nabla^2 F(\mathbf{0})(\widetilde{\boldsymbol{\Theta}}, \widetilde{\boldsymbol{\Theta}})} \left(e^{nR(i\widetilde{\boldsymbol{\Theta}}/\sqrt{n})} - 1 - nR\left(\frac{i\widetilde{\boldsymbol{\Theta}}}{\sqrt{n}}\right)\right)\right| |d\widetilde{\boldsymbol{\Theta}}| \right. \\
&\quad \le \frac{M_G}{2} \left| \int_{\{\|\widetilde{\boldsymbol{\Theta}}\| < \sqrt{n}\epsilon\}} e^{-1/2\nabla^2 F(\mathbf{0})(\widetilde{\boldsymbol{\Theta}}, \widetilde{\boldsymbol{\Theta}})} e^{|nR(i\widetilde{\boldsymbol{\Theta}}/\sqrt{n})|} \left| nR\left(\frac{i\widetilde{\boldsymbol{\Theta}}}{\sqrt{n}}\right)\right|^2 d\widetilde{\boldsymbol{\Theta}} \right| \\
&\quad \le \frac{M_G K^2}{2n} \left| \int_{\{\|\widetilde{\boldsymbol{\Theta}}\| < \sqrt{n}\epsilon\}} e^{-1/2\nabla^2 F(\mathbf{0})(\widetilde{\boldsymbol{\Theta}}, \widetilde{\boldsymbol{\Theta}}) + K\epsilon\|\widetilde{\boldsymbol{\Theta}}\|^2} \|\widetilde{\boldsymbol{\Theta}}\|^6 d\widetilde{\boldsymbol{\Theta}} \right| \\
&\quad \le C_1(F, \epsilon) \cdot \frac{M_G K^2}{2n},
\end{aligned}$$

where $C_1(F, \epsilon) := \int_{\mathbb{R}^2} e^{-(1/2)\nabla^2 F(\mathbf{0})(\widetilde{\boldsymbol{\Theta}}, \widetilde{\boldsymbol{\Theta}}) + K\epsilon\|\widetilde{\boldsymbol{\Theta}}\|^2} \|\widetilde{\boldsymbol{\Theta}}\|^6 d\widetilde{\boldsymbol{\Theta}}$ is finite when ϵ is small. So, we get

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

$$\int_{\{\|\boldsymbol{\Theta}\|<\epsilon\}} G(i\boldsymbol{\Theta})e^{nF(i\boldsymbol{\Theta})}\,d\boldsymbol{\Theta}$$
$$= \frac{e^{nF(\mathbf{0})}}{n}\left(\int_{\{\|\widetilde{\boldsymbol{\Theta}}\|<\sqrt{n}\epsilon\}} G\left(\frac{i\widetilde{\boldsymbol{\Theta}}}{\sqrt{n}}\right)e^{-1/2\nabla^2F(\mathbf{0})(\widetilde{\boldsymbol{\Theta}},\widetilde{\boldsymbol{\Theta}})}\left(1+nR\left(\frac{i\widetilde{\boldsymbol{\Theta}}}{\sqrt{n}}\right)\right)d\widetilde{\boldsymbol{\Theta}} + M_G O_F\left(\frac{1}{n}\right)\right).$$

Since G is Lipschitz with compact support, we have

$$\left|\int_{\{\|\widetilde{\boldsymbol{\Theta}}\|<\sqrt{n}\epsilon\}}\left|G\left(\frac{i\widetilde{\boldsymbol{\Theta}}}{\sqrt{n}}\right)-G(\mathbf{0})\right|e^{-(1/2)\nabla^2F(\mathbf{0})(\widetilde{\boldsymbol{\Theta}},\widetilde{\boldsymbol{\Theta}})}\,d\widetilde{\boldsymbol{\Theta}}\right|$$
$$\le \frac{L_G}{\sqrt{n}}\int_{\mathbb{R}^2}\|\widetilde{\boldsymbol{\Theta}}\|e^{-(1/2)\nabla^2F(\mathbf{0})(\widetilde{\boldsymbol{\Theta}},\widetilde{\boldsymbol{\Theta}})}\,d\widetilde{\boldsymbol{\Theta}} =: \frac{L_G}{\sqrt{n}}\cdot C_2(F)$$

and observe that $C_2(F)<\infty$. We also have

$$\left|\int_{\{\|\widetilde{\boldsymbol{\Theta}}\|<\sqrt{n}\epsilon\}}\left|G\left(\frac{i\widetilde{\boldsymbol{\Theta}}}{\sqrt{n}}\right)\cdot nR\left(\frac{i\widetilde{\boldsymbol{\Theta}}}{\sqrt{n}}\right)\right|e^{-(1/2)\nabla^2F(\mathbf{0})(\widetilde{\boldsymbol{\Theta}},\widetilde{\boldsymbol{\Theta}})}\,d\widetilde{\boldsymbol{\Theta}}\right|$$
$$\le M_G\int_{\mathbb{R}^2}\frac{K\|\widetilde{\boldsymbol{\Theta}}\|^3}{\sqrt{n}}e^{-(1/2)\nabla^2F(\mathbf{0})(\widetilde{\boldsymbol{\Theta}},\widetilde{\boldsymbol{\Theta}})}\,d\widetilde{\boldsymbol{\Theta}} =: \frac{KM_G}{\sqrt{n}}\cdot C_3(F)$$

and observe $C_3(F)<\infty$. Therefore, summing up all the estimates, we have

$$\int_{\{\|\boldsymbol{\Theta}\|<\epsilon\}} G(i\boldsymbol{\Theta})e^{nF(i\boldsymbol{\Theta})}\,d\boldsymbol{\Theta}$$
$$= \frac{e^{nF(\mathbf{0})}G(\mathbf{0})}{n}\left(\int_{\{\|\widetilde{\boldsymbol{\Theta}}\|<\sqrt{n}\epsilon\}} e^{-\frac{1}{2}\nabla^2F(\mathbf{0})(\widetilde{\boldsymbol{\Theta}},\widetilde{\boldsymbol{\Theta}})}\,d\widetilde{\boldsymbol{\Theta}}\right)$$
$$+\frac{e^{nF(\mathbf{0})}}{n}\left(L_G\cdot O_F\left(\frac{1}{\sqrt{n}}\right)+M_G\cdot O_F\left(\frac{1}{\sqrt{n}}\right)\right).$$

Lastly, it is routine to see that

$$\int_{\{\|\widetilde{\boldsymbol{\Theta}}\|<\sqrt{n}\epsilon\}} e^{-\frac{1}{2}\nabla^2F(\mathbf{0})(\widetilde{\boldsymbol{\Theta}},\widetilde{\boldsymbol{\Theta}})}\,d\widetilde{\boldsymbol{\Theta}} = \frac{2\pi}{\sqrt{\det\nabla^2F(\mathbf{0})}}+O_F(e^{-n\epsilon^2}).$$

□

References

[1] N. Anantharaman. Precise counting results for closed orbits of Anosov flows. *Ann. Sci. Éc. Norm. Supér. (4)* **33**(1) (2000), 33–56.

[2] M. Babillot and F. Ledrappier. Lalley's theorem on periodic orbits of hyperbolic flows. *Ergod. Th. & Dynam. Sys.* **18**(1) (1998), 17–39.

[3] M. Babillot and M. Peigné. Homologie des géodésiques fermées Sur des variétés hyperboliques avec bouts cuspidaux. *Ann. Sci. Éc. Norm. Supér. (4)* **33**(1) (2000), 81–120.

[4] H. Bray, R. Canary and L.-Y. Kao. Pressure metrics for deformation spaces of quasifuchsian groups with parabolics. *Algebr. Geom. Topol.* **23**(8) (2023), 3615–3653.

[5] H. Bray, R. Canary, L.-Y. Kao and G. Martone. Counting, equidistribution and entropy gaps at infinity with applications to cusped Hitchin representations. *J. Reine Angew. Math.* **791** (2022), 1–51.

[6] H. Bray, R. Canary, L.-Y. Kao and G. Martone. Pressure metrics for cusped Hitchin components. *Adv. Math.* **435** (2023), 109352.

[7] M. Burger. Intersection, the Manhattan curve, and Patterson–Sullivan theory in rank 2. *Int. Math. Res. Not. IMRN* **7** (1993), 217–225.

[8] R. Canary. Hitchin representations of Fuchsian groups. *EMS Surv. Math. Sci.* **9**(2) (2022), 355–388.

[9] R. Canary, T. Zhang and A. Zimmer. Cusped Hitchin representations and Anosov representations of geometrically finite Fuchsian groups. *Adv. Math.* **404** (2022), 108439.

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

[10] S. B. Chae. *Holomorphy and Calculus in Normed Spaces (Monographs and Textbooks in Pure and Applied Mathematics, 92)*. Marcel Dekker, Inc., New York, 1985; with an appendix by Angus E. Taylor.
[11] M. Chow and H. Oh. Jordan and Cartan spectra in higher rank with applications to correlations. *Preprint*, 2025, arXiv:2308.16329.
[12] M. Chow and H. Oh. Multiple correlations of spectra for higher rank Anosov representations. *Int. Math. Res. Not. IMRN* **6** (2025), Paper No. rnaf061.
[13] X. Dai and G. Martone. Correlation of the renormalized Hilbert length for convex projective surfaces. *Ergod. Th. & Dynam. Sys.* **43**(9) (2023), 2938–2973.
[14] F. Dal'bo and M. Peigné. Comportement asymptotique du nombre de géodésiques fermées sur la surface modulaire en courbure variable. *Astérisque* **238** (1996), 111–177.
[15] C. L. Epstein. Asymptotics for closed geodesics in a homology class, the finite volume case. *Duke Math. J.* **55**(4) (1987), 717–757.
[16] V. Fock and A. Goncharov. Moduli spaces of local systems and higher Teichmüller theory. *Publ. Math. Inst. Hautes Études Sci.* **103**(103) (2006), 1–211.
[17] O. Glorieux. Counting closed geodesics in globally hyperbolic maximal compact AdS 3-manifolds. *Geom. Dedicata.* **188** (2017), 63–101.
[18] H. Huber. Zur analytischen Theorie hyperbolischer Raumformen und Bewegungsgruppen. *Math. Ann.* **138**(1) (1959), 1–26.
[19] L.-Y. Kao. Manhattan curves for hyperbolic surfaces with cusps. *Ergod. Th. & Dynam. Sys.* **40**(7) (2020), 1843–1874.
[20] L.-Y. Kao. Pressure metrics and Manhattan curves for Teichmüller spaces of punctured surfaces. *Israel J. Math.* **240**(2) (2020), 567–602.
[21] T. Kato. *Perturbation Theory for Linear Operators (Classics in Mathematics)*. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition. Originally published as volume 132 in the series: Grundlehren der mathematischen Wissenschaften.
[22] A. Katsuda and T. Sunada. Homology and closed geodesics in a compact Riemann surface. *Amer. J. Math.* **110**(1) (1988), 145–155.
[23] M. Kesseböhmer and S. Kombrink. A complex Ruelle–Perron–Frobenius theorem for infinite Markov shifts with applications to renewal theory. *Discrete Contin. Dyn. Syst. Ser. S* **10**(2) (2017), 335–352.
[24] B. R. Kloeckner. Effective perturbation theory for simple isolated eigenvalues of linear operators. *J. Operator Theory* **81**(1) (2019), 175–194.
[25] S. Lalley. Distribution of periodic orbits of symbolic and Axiom A flows. *Adv. Appl. Math.* **8**(2) (1987), 154–193.
[26] S. Lalley. Closed geodesics in homology classes on surfaces of variable negative curvature. *Duke Math. J.* **58**(3) (1989), 795–821.
[27] S. Lalley. Renewal theorems in symbolic dynamics, with applications to geodesic flows, non-Euclidean tessellations and their fractal limits. *Acta Math.* **163**(1-2) (1989), 1–55.
[28] F. Ledrappier and O. Sarig. Fluctuations of ergodic sums for horocycle flows on $\mathbb{Z}^d$-covers of finite volume surfaces. *Discrete Contin. Dyn. Syst.* **22**(1–2) (2008), 247–325.
[29] G. Margulis. Applications of ergodic theory to the investigation of manifolds of negative curvature. *Funct. Anal. Appl.* **3**(4) (1969), 335–336.
[30] D. Mauldin and M. Urbański. *Graph Directed Markov Systems (Cambridge Tracts in Mathematics, 148)*. Cambridge University Press, Cambridge, 2003.
[31] W. Parry and M. Pollicott. Zeta functions and the periodic orbit structure of hyperbolic dynamics. *Astérisque* 187–188 (1990), 1–268.
[32] F. Paulin, M. Pollicott and B. Schapira. Equilibrium states in negative curvature. *Astérisque* **373** (2015), viii+281 pp.
[33] R. Phillips and P. Sarnak. Geodesics in homology classes. *Duke Math. J.* **55**(2) (1987), 287–297.
[34] M. Pollicott. Homology and closed geodesics in a compact negatively curved surface. *Amer. J. Math.* **113**(3) (1991), 379–385.
[35] M. Pollicott and R. Sharp. Correlations for pairs of closed geodesics. *Invent. Math.* **163**(1) (2006), 1–24.
[36] M. Pollicott and R. Sharp. Correlations of length spectra for negatively curved manifolds. *Comm. Math. Phys.* **319**(2) (2013), 515–533.
[37] R. T. Rockafellar and R. J.-B. Wets. *Variational Analysis (Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 317)*. Springer-Verlag, Berlin, 1998.
[38] A. Sambarino. Infinitesimal Zariski closures of positive representations. *J. Differ. Geom.* **128**(2) (2024), 861–901.
[39] O. Sarig. Continuous phase transitions for dynamical systems. *Comm. Math. Phys.* **267**(3) (2006), 631–667.
[40] O. Sarig. Lecture notes on thermodynamic formalism for topological Markov shifts. *Preprint*, 2009, available at: https://www.weizmann.ac.il/math/sarigo/sites/math.sarigo/files/uploads/tdfnotes.pdf.

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press

[41] R. Schwartz and R. Sharp. The correlation of length spectra of two hyperbolic surfaces. *Comm. Math. Phys.* **153**(2) (1993), 423–430.
[42] R. Sharp. Closed orbits in homology classes for Anosov flows. *Ergod. Th. & Dynam. Sys.* **13**(2) (1993), 387–408.
[43] R. Sharp. The Manhattan curve and the correlation of length spectra on hyperbolic surfaces. *Math. Z.* **228**(4) (1998), 745–750.
[44] M. Stadlbauer. The return sequence of the Bowen-series map for punctured surfaces. *Fund. Math.* **182**(3) (2004), 221–240.

https://doi.org/10.1017/etds.2026.10283 Published online by Cambridge University Press