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List Improper Colourings of Planar Graphs

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A graph G is m-choosable with impropriety d, or simply $(m,d)^*$-choosable, if for every list assignment L, where $|L(v)| \geqslant m$ for every $v \in V(G)$, there exists an L-colouring of G such that each vertex of G has at most d neighbours coloured with the same colour as itself. We show that every planar graph is $(3,2)^*$-choosable and every outerplanar graph is $(2,2)^*$-choosable. We also propose some interesting problems about this colouring.

1. Introduction

All graphs considered in this paper are simple, finite and undirected. A graph G is *m-colourable with impropriety d*, or simply *$(m,d)^*$-colourable*, if the vertices of G can be coloured with m colours so that each vertex has at most d neighbours of the same colour as itself. The smallest m for which G is $(m,d)^*$-colourable is called the *d-improper chromatic number* of G and is denoted by $\chi^*(G,d)$. An $(m,0)^*$-colouring is an ordinary (proper) m-colouring and $\chi^*(G,0)$ is the chromatic number of G.

This improper colouring was introduced by Cowen, Cowen and Woodall [3]. For the class of planar graphs, and the class of outerplanar graphs, they have found all pairs (m,d) such that every graph in the class is $(m,d)^*$-colourable. In particular, they showed that every outerplanar graph is $(2,2)^*$-colourable and every planar graph is $(3,2)^*$-colourable. They also showed, without using the four-colour theorem, that every planar graph is $(4,1)^*$-colourable. Cowen, Goddard, and Jesurum [4] extended the study of this colouring to the graphs of higher genera. More references about this and related improper colourings can be found in Frick [6], Woodall [18], and Jensen and Toft [8, Problem 3.4].

A *list assignment* of G is a function L which assigns to each vertex $v \in V(G)$ a list of colours $L(v)$. An *L-colouring with impropriety d*, or simply *$(L,d)^*$-colouring*, is a mapping

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λ, which assigns to each vertex $v \in V(G)$ a colour $\lambda(v)$ from $L(v)$ such that v has at most d neighbours coloured with $\lambda(v)$. For $m \in \mathbf{N}$, the graph is *m-choosable with impropriety d*, or simply $(m,d)^*$*-choosable*, if there exists an $(L,d)^*$-colouring for every list assignment L with $|L(v)| \geqslant m$ for each $v \in V(G)$. The smallest m for which G is $(m,d)^*$-choosable is called the *d-improper list chromatic number* of G and is denoted by $\chi_l^*(G,d)$. Note that $(m,0)^*$-choosability is the same as m-choosability (a concept introduced by Erdős, Rubin, and Taylor [5] and independently by Vizing [14]). Kratochvíl and Tuza [9] observed that every planar graph with no triangles is 4-choosable, and Voigt [16] showed that not every such graph is 3-choosable. On other hand, Thomassen [11, 12] proved that every planar graph is 5-choosable and every planar graph with no triangles and quadrangles is 3-choosable. The author [13] has proved a Grötzsch-type theorem for colouring with impropriety one. Improper list colourings were also investigated in a more general sense by Borowiecki, Drgas-Burchardt, and Mihók [1] and Borowiecki, Broere, and Mihók [2]. The authors of the papers mentioned considered hereditary properties of graphs and generalized chromatic and list chromatic numbers.

We show that every planar graph is $(3,2)^*$-choosable and every outerplanar graph is $(2,2)^*$-choosable. At the end, we propose some problems about improper colourings.

In the paper, we will frequently use the following definitions. For $v \in V(G)$, let $N(v)$ be the set of its neighbours. For a graph G and a cycle or a path H of G, each edge $e \in E(G) \setminus E(H)$ with both ends in $V(H)$ is a *chord* of H. Denote by $S(u,v)$ the set comprising vertices u, v, and all cut-vertices of G which lies on every path between u and v. Let C be a cycle or a facial walk of a planar graph G. For two adjacent vertices u and v of C, we say that v is the *next-vertex* of u or u is the *previous-vertex* of v on C if v immediately follows u in the clockwise orientation of C. If C is a cycle of G, then $\mathrm{Int}(C)$ is the graph of all vertices and edges inside or on C. For any two vertices u and v from C, denote by $C[u,v]$ the path on C between u and v which contains the next-vertex of u. For an improper colouring of a graph G, the number of neighbours of $v \in V(G)$ coloured with the same colour as itself is called the *impropriety* of v and is denoted by $\mathrm{im}(v)$. A list assignment L of a planar graph G (with an outer walk C) is called *good* whenever $|L(v)| \geqslant 2$ for every $v \in V(C)$, and $|L(v)| \geqslant 3$ otherwise.

2. Colourings with impropriety two

Lemma 2.1. *Let G be a planar connected graph, C be its outer walk, and L be a good list assignment of G. For some vertices $u,v \in V(C)$, let λ be an $(L,2)$-colouring of $S(u,v)$. Then λ can be extended to an $(L,2)^*$-colouring of G so that $\mathrm{im}(u) \leqslant 1$, $\mathrm{im}(v) \leqslant 1$, and moreover $\mathrm{im}(u) = 0$, $\mathrm{im}(v) \leqslant 1$ if $\lambda(u) \neq \lambda(v)$ and $uv \in E(C)$.*

Proof. Assume that the lemma is false and G is a counterexample with $|V(G)|$ as small as possible. We will show that there exists such an λ (as stated in the lemma) and so obtain a contradiction.

We first claim that *G is 2-connected*. Suppose it is false. Denote by $v_0(=u), v_1, \ldots, v_n, v_{n+1}(=v)$ vertices of $S(u,v)$ where $v_{i+1}(i = 0, \ldots, n-1)$ is the first cut-vertex different from v_i on any path from v_i to v_{n+1}; and denote by B_i the block that contains v_{i-1} and v_i. By

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the minimality of G, extend λ to every B_i so that $\mathrm{im}_{B_i}(v_{i-1}) \leqslant 1$ and $\mathrm{im}_{B_i}(v_i) \leqslant 1$. Notice that the extension of λ is also possible in the required way if u and v belong to the same block B_1. If G is not yet coloured, then there exists a block B of G whose vertices have not yet been coloured except for a vertex of attachment, say w_1. Let $w_2 \in V(B)$ be a vertex adjacent to w_1 such that edge w_1w_2 lies on the outer walk of B and let $\lambda(w_2) \in L(w_2)\setminus\lambda(w_1)$. By the minimality of G, colour B so that $\mathrm{im}_B(w_1) = 0$ and $\mathrm{im}_B(w_2) \leqslant 1$. Now repeat this procedure until G is coloured. Note that we obtain $\mathrm{im}(u) \leqslant 1$ and $\mathrm{im}(v) \leqslant 1$ or $\mathrm{im}(u) = 0$ and $\mathrm{im}(v) \leqslant 1$ if $uv \in E(C)$ and $\lambda(u) \neq \lambda(v)$.

Thus, we may assume that G is 2-connected and so $S(u,v) = \{u,v\}$. Since $G \neq K_2$, C is a cycle. We may assume that each facial cycle of G, except perhaps C, is a triangle; otherwise we add some edges in order to obtain this situation. If u and v are adjacent and $uv \in E(C)$, we shall then assume that v is the next-vertex of u on C. Denote by u^* and v^* the previous-vertex of u and the next-vertex of v on C, respectively.

We will consider several cases below. In these cases (except Case 1(a)) we shall construct a connected graph G' by removing some vertices from G. Thus, G' will be smaller than G. The graph G' with its vertices u' and v', corresponding list assignment L', and its L'-colouring λ is constructed such that the assumptions of the lemma are fulfilled. Thus, by the minimality of G, G' is colourable in the required way. Let $S' = S_{G'}(u',v')$ and let C' be the outer walk of G'.

Case 1: u and v are adjacent.

(a) uv is a chord of C.
Let $G_1 = \mathrm{Int}(C[u,v]\cup\{vu\})$ and $G_2 = \mathrm{Int}(C[v,u]\cup\{uv\})$. Note that uv lies on the outer cycles of both of these graphs. Now extend λ to G_1 and G_2, by the minimality of G, as follows. If $\lambda(u) = \lambda(v)$ colour both of these graphs so that $\mathrm{im}_{G_i}(u) = \mathrm{im}_{G_i}(v) = 1$, $i = 1,2$. Otherwise, colour G_1 so that $\mathrm{im}_{G_1}(u) = 0$, $\mathrm{im}_{G_1}(v) \leqslant 1$ and colour G_2 so that $\mathrm{im}_{G_2}(u) \leqslant 1$, $\mathrm{im}_{G_2}(v) = 0$.

(b) $uv \in E(C)$ and $\lambda(u) \neq \lambda(v)$.
Let $G' = G \setminus \{u\}$. Define $L'(x) = L(x) \setminus \{\lambda(u)\}$ for every $x \in N(u) \setminus V(C)$ and $L'(x) = L(x)$ otherwise. The vertices u^* and v play the role of u' and v'. Assign to every $x \in S' \setminus \{v'\}$ a colour $\lambda(x) \in L(x) \setminus \{\lambda(u)\}$ and, by the minimality of G, extend λ to G'. Note that $\mathrm{im}(u) = 0$ and $\mathrm{im}(v) \leqslant 1$.

(c) $uv \in E(C)$ and $\lambda(u) = \lambda(v)$.
Let $G' = G \setminus \{u,v\}$. For every $x \in V(G')$ define $L'(x) = L(x) \setminus \{\lambda(u)\}$ whenever $x \in [N(u) \cup N(v)] \setminus V(C)$ and $L'(x) = L(x)$ otherwise. If $u^* \neq v^*$ then let u^* and v^* play the role of u' and v'. Otherwise, $u^* = v^*$, let w^* be the next-vertex of u^* on C' and let u^* and w^* play the role of u' and v'. Assign to every not coloured vertex x from S' a colour $\lambda(x) \in L(x) \setminus \{\lambda(u)\}$ and, by the minimality of G, extend λ to G'. Note that $\mathrm{im}(u) = \mathrm{im}(v) = 1$ since u is the only neighbour of v with the same colour $\lambda(v)$, and vice versa.

Case 2: u and v are not adjacent.
Let $P^* = w_0w_1\cdots w_k$ be a chordless (in G) path where $w_0 = u$, $w_k = v$, and $w_i \in C[u,v]$ for $i = 1,\ldots,k-1$. Clearly, P^* exists and is uniquely determined. Choose the largest

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$s \in [0, k-1]$ so that $\lambda(u) \in L(w_i)$ for $i = 0, \ldots, s$. If $s = k-1$ and $\lambda(u) = \lambda(v)$, then reset $s = k$. Let $G_i = \mathrm{Int}(C[w_i, w_{i+1}] \cup \{w_{i+1}w_i\})$ for $i = 0, \ldots, k-1$. Observe that, if $w_iw_{i+1} \in E(C)$, then G_i is isomorphic to the graph K_2.

(a) $s = k$.
For $i = 1, \ldots, k-1$, assign colour $\lambda(u)$ to w_i. Obtain G' by removing the vertices of G_i, $i = 0, \ldots, k-1$. For every $x \in V(G')$, define $L'(x) = L(x) \setminus \{\lambda(u)\}$ whenever $x \in [\cup_{i=0}^{k} N(w_i)] \setminus V(C)$ and $L'(x) = L(x)$ otherwise. If $u^* \neq v^*$ then let u^* and v^* play the role of u' and v'. Otherwise, $u^* = v^*$, let w^* be the next-vertex of u^* on C' and let w^* and u^* play the role of u' and v'. Assign to every not coloured vertex x from S' a colour $\lambda(x) \in L(x) \setminus \{\lambda(u)\}$ and, by the minimality of G, extend λ to G'. Furthermore, the colouring of each G_i for $i = 0, \ldots, k-1$ can be completed by Case 1(c). Note that $\mathrm{im}(u) = \mathrm{im}(v) = 1$ since w_1 and w_{k-1} are the only neighbours of u and v, respectively, coloured with the same colour.

(b) $s < k$ and $w_sw_{s+1} \in E(C)$.
Note that $\lambda(u) \notin L(w_{s+1})$ (when $w_{s+1} \neq v$) because of the definition. In a similar way to Case 2(a): colour vertices w_i for $i = 1, \ldots, s$ with colour $\lambda(u)$ and obtain G' by deleting the vertices of G_i for $i = 0, \ldots, s-1$. For every $x \in V(G')$, let $L'(x) = L(x) \setminus \{\lambda(u)\}$ whenever $x \in [\cup_{i=0}^{s} N(w_i)] \setminus V(C)$ and $L'(x) = L(x)$ otherwise. The vertices u^* and v correspond to the special vertices u' and v' of G'. For every $x \in S' \setminus \{v'\}$, let $\lambda(x) \in L(x) \setminus \{\lambda(u)\}$ and afterwards extend λ to G'.

(c) $s < k$ and $w_sw_{s+1} \notin E(C)$.
Proceed as in the previous case. Also, delete the vertices of the subgraph G_s with the exception of w_{s+1}. In addition, we have to take care about the colouring of G_s to ensure $\mathrm{im}(w_{s+1}) \leqslant 2$. After extending λ to G', we obtain that $\mathrm{im}_{G'}(w_s) = 1$. We ensure $\mathrm{im}(w_{s+1}) \leqslant 2$ by colouring G_s using Case 1(b) so that $\mathrm{im}_{G_s}(w_{s+1}) = 0$ and $\mathrm{im}_{G_s}(w_s) \leqslant 1$. Moreover, $\mathrm{im}(w_{s+1}) \leqslant 1$ if $w_{s+1} = v$. □

The following theorem is an easy consequence of the above lemma.

Theorem 2.2.

(a) *Every outerplanar graph is* $(2,2)^*$*-choosable.*
(b) *Every planar graph is* $(3,2)^*$*-choosable.* □

These bounds are sharp in the sense that if we replace $(2,2)$ by $(1,2)$ in (a) and replace $(3,2)$ by $(2,2)$ in (b), we obtain false statements; for counterexamples see [3]. One may ask whether every planar graph is $(3,1)^*$-choosable and every outerplanar graph is $(2,1)^*$-choosable. The answers to these two questions are negative and some counterexamples can be found in [3].

Here we want to note that we have found a short proof of Theorem 2.2(a), but, since the validity of this assertion follows from the above lemma, we have not included it.

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3. Some problems

For some integer $d \geqslant 0$, let g_d^1 (or g_d^2) be the smallest integer, if it exists at all, such that every planar graph of girth at least g_d^1 (or g_d^2) is $(2,d)^*$-colourable (or $(2,d)^*$-choosable). If such integer does not exist, then let $g_d^1 = \infty$ (or $g_d^2 = \infty$).

Problem 3.1. *Find* g_d^1 *and* g_d^2 *for every* $d \geqslant 1$.

Obviously, we have $g_d^1 \leqslant g_d^2$ for all d, $g_0^1 = g_0^2 = \infty$ and $g_{d_1}^i \leqslant g_{d_2}^i$ for $i = 1,2$ and $d_1 \geqslant d_2$. Below we give some bounds for g_d^1 and g_d^2.

Proposition 3.2. $6 \leqslant g_1^1 \leqslant g_1^2 \leqslant 12$ *and* $5 \leqslant g_d^1 \leqslant g_d^2 \leqslant 8$ *for* $d \geqslant 2$.

Proof. Let us first show the lower bounds. We will construct a graph G^* from graphs G_1 and G_2 (see Figure 1) of girth 5 which is not $(2,1)^*$-colourable. This will imply $g_1^1 \geqslant 6$. Take a copy of G_1 (called the *frame*) and in each boundary of its four faces glue a copy of G_2. Graph G^* has 44 vertices. Observe that any $(2,1)^*$-colouring of G_2 has at least three vertices of the outer cycle with impropriety one. At least one of them has impropriety one because it is joined with a vertex of the same colour not lying on the outer cycle. Further, any colouring of G_1 has two pairs of vertices such that, in each pair, the vertices are adjacent and coloured by the same colour. By these facts, in any $(2,1)^*$-colouring of G^*, each frame vertex has impropriety one. Now take two copies of G^*, in each of them select one frame vertex, and identify these two vertices. The resulting graph is planar, of order 87 and with girth 5, and it is not $(2,1)^*$-colourable.

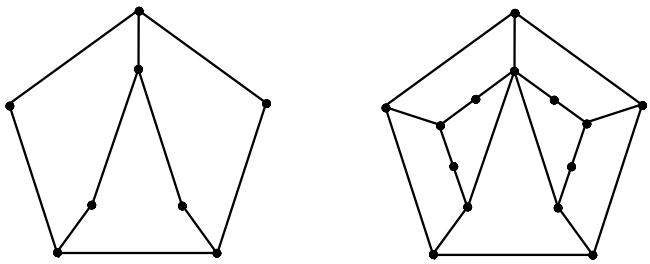


Figure 1 Graphs G_1 and G_2

Now we shall construct for any $d \geqslant 2$ a planar graph of girth 4 which is not $(2,d)^*$-colourable. Hence, $g_d^1 \geqslant 5$ for any $d \geqslant 2$. For this purpose, let $C^*(m)$ be a graph composed of a 4-cycle (called the *main cycle*) and an additional vertex (called the *central vertex*) connected by m 2-paths to each vertex of the cycle. Clearly, $C^*(m)$ is planar and we will assume that the main cycle is the outer cycle of this graph. Now take a copy of $C^*(2d+1)$. In this graph choose arbitrarily $\lceil \frac{d+1}{2} \rceil$ pairwise edge-disjoint facial 4-cycles incident with the central vertex. Observe that any $(2,d)^*$-colouring of this graph assigns the same colour to the vertices of the main cycle and the central vertex. Furthermore, there is a vertex on one of the chosen facial 4-cycles, adjacent to the central vertex and coloured differently from it. Hence, if in each of these facial 4-cycles we glue a copy of $C^*(2d+1)$, we obtain a

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planar graph of girth 4 which is not $(2,d)^*$-colourable. (No doubt, there exist such graphs of smaller order.)

Now we will prove the upper bounds. We will use the following two properties to colour (by induction) graphs with sufficiently large girth.

(1) every graph of girth at least 12 has a vertex u of degree 1 or two adjacent vertices v, w of degree 2.

(2) every graph of girth at least 8 has a vertex u of degree 1 or a face f with all incident vertices of degree 3, or a path P of length at least one with end-vertices of degree 2 and all inner vertices of degree 3 (the end-vertices of P are allowed to coincide).

After the proof of (1) and (2), it is not hard to show that $g_d^2 \leqslant 12$ for $d \geqslant 1$ and $g_d^2 \leqslant 8$ for $d \geqslant 2$. For the first bound, let $G' = G \setminus \{u\}$ or $G' = G \setminus \{v, w\}$, colour G' by induction, and after that extend the colouring of G'. Similarly, for the second bound, let $G' = G \setminus \{u\}$, or $G' = G \setminus V(f)$, or $G' = G \setminus V(P)$, colour G' by induction and, since $d \geqslant 2$ is assumed, it is easy to extend the colouring of G' to G.

Now we shall prove the properties (1) and (2) using the technique of discharging. Let G be a planar graph of girth at least 12 or 8, depending on which property, (1) or (2), is considered. Recall that we can rewrite Euler's polyhedral formula as

$$\sum_{v \in V(G)} (4 - \deg(v)) + \sum_{f \in F(G)} (4 - r(f)) = 8,$$

(where $r(f)$ is the length of the boundary of the face f). We assign a charge of $4 - \deg(v)$ to each vertex $v \in V(G)$ and a charge of $4 - r(f)$ to each face $f \in F(G)$. Thus, the total charge of G is 8 and so it is positive. Now we shall redistribute the charge in the following way. Redistribute a charge of 2 from each vertex of degree 2 to its incident faces so that each one receives a charge of 1. Also redistribute a charge of 1 from each vertex of degree 3 so that every incident face receives a charge of $1/3$. After this redistribution, denote by $c(x)$ the charge of every $x \in V(G) \cup F(G)$.

If (1) is not true for G (assume that the girth of G is at least 12), then it is not hard to show that $c(x) \leqslant 0$ for every $x \in V(G) \cup F(G)$. Now, it is obviously a contradiction since the total charge must be positive.

Suppose now that (2) is not true. Obviously, $c(v) \leqslant 0$ for every $v \in V(G)$. Suppose that there exists a face $f \in F(G)$ with $c(f) > 0$. Note that $c(f) = 4 - r(f) + n_2 + n_3/3$, where n_2 and n_3 are the numbers of vertices of degree two and three, respectively, incident with f. If we write $r(f) = n_2 + n_3 + n_4$, where n_4 is the number of vertices of degree at least 4 which are incident with f, then $c(f) = 4 - (2/3)n_3 - n_4 > 0$. By the assumptions, we obtain that $1 \leqslant n_4 \leqslant 3$. Since the girth of G is at least 8 and G does not satisfy (2), it follows that: if $n_4 = 1$, then $n_3 \geqslant 6$; if $n_4 = 2$, then $n_3 \geqslant 4$; and if $n_4 = 3$, then $n_3 \geqslant 2$. In all of these cases, $c(f) < 0$, a contradiction. This implies that, for every $f \in V(G)$, $c(f) \leqslant 0$. Thus the total charge of G is not positive, which is obviously not possible. This proves (2). □

In 1993 Voigt [15] showed that in general the chromatic number of a planar graph is not equal to its list chromatic number. We formulate a similar question for list improper colourings.

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Problem 3.3. *Let d be a positive integer. Is it true that, for every planar graph G, $\chi^*(G,d) = \chi_l^*(G,d)$?*

Gutner [7], Mirzakhani [10] and, independently, Voigt and Wirth [17] found 3-colourable planar graphs with list chromatic number 5. Thus, they solved the first part of Problem 2.13 in the book of Jensen and Toft [8]. Thus, we may ask the following.

Problem 3.4. *Is every 3-colourable planar graph $(3,1)^*$-choosable?*

In our notation, Thomassen [11] proved that every planar graph is $(5,0)^*$-choosable. We showed that every planar graph is $(3,2)^*$-choosable. Cowen, Cowen, and Woodall [3] found, for any $d \geqslant 0$, which are not $(2,d)^*$-colourable (and so not $(2,d)^*$-choosable) planar graphs. Clearly, if a graph is $(m,d)^*$-choosable, then it is also $(m',d')^*$-choosable whenever $m' \geqslant m$ and $d' \geqslant d$. So, in order to determine all pairs (m,d) for which all planar graphs are $(m,d)^*$-choosable, it is necessary (and sufficient) to solve the next problem.

Problem 3.5. *Is every planar graph $(4,1)^*$-choosable?*

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