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HARMONISING HARMONY

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Abstract. The term 'harmony' refers to a condition that the rules governing a logical constant ought to satisfy in order to endow it with a proper meaning. Different characterizations of harmony have been proposed in the literature, some based on the inversion principle, others on normalization, others on conservativity. In this paper we discuss the prospects for showing how conservativity and normalization can be combined so to yield a criterion of harmony equivalent to the one based on the inversion principle: We conjecture that the rules for connectives obeying the inversion principle are *conservative over normal deducibility*. The plausibility of the conjecture depends in an essential way on how normality is characterized. In particular, a normal deduction should be understood as one which is irreducible, rather than as one which does not contain any maximal formula.

§1. Introduction. It is quite uncontroversial that the natural deduction rules for paradoxical connectives, such as $\bullet$ (Read, 2010), or the more traditional λ (see Prawitz, 1965; Tennant, 1982):

$$\frac{\neg\lambda}{\lambda}\,\lambda\mathrm{I} \qquad\qquad \frac{\lambda}{\neg\lambda}\,\lambda\mathrm{E}$$

satisfy the inversion principle: "A proof of the conclusion of an elimination is already 'contained' in the proofs of the premisses when the major premiss is inferred by introduction" (Prawitz, 1971, pp. 246–247, see also Lorenzen, 1955; Prawitz, 1965; Schroeder-Heister, 2007; Moriconi & Tesconi, 2008). The inversion principle suggests the idea that consecutive applications of the introduction rule followed immediately by the elimination rule constitute a redundancy. This can be made explicit by defining a reduction to cut such redundancies away:

$$\dfrac{\dfrac{\begin{array}{c}\mathscr{D}\\ \neg\lambda\end{array}}{\lambda}}{\neg\lambda} \quad \lambda\text{-Red} \quad \rhd \quad \begin{array}{c}\mathscr{D}\\ \neg\lambda\end{array}$$

Although the rules for paradoxical connectives satisfy the inversion principle, they extend in a nonconservative way deducibility relations satisfying reflexivity, monotonicity and transitivity.[1] Furthermore normalization fails for the natural deduction systems containing these rules.

Dummett (1981, 1991) introduces the concept of 'harmony' when he discusses the reasons for revising or even rejecting parts of our linguistic practices. Lack of harmony is presented as one such reason. When Dummett considers how the notion of harmony should apply to connectives, he alternatively hints at both conservativity and at the existence of appropriate reductions as possible ways of making the notion precise. Since reductions are an essential ingredients of the normalization process, some authors also consider the option

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[1] We assume deducibility relations to hold between sets, rather than multi-sets of formulas. Otherwise, deducibility should be taken to be closed under contraction as well.

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of developing an account of harmony based on normalization (although most of the times to discard it as inappropriate, e.g. Read, 2010).

The case of λ however shows that the three possible characterizations of harmony (harmony as inversion, harmony as normalization and harmony as conservativity) come apart.

Were harmony identified with either conservativity or normalization, the rules for the paradoxical connectives would not count as harmonious and thus paradoxical connectives would count as expressions whose meaning stands in need of revision. On the other hand, on an understanding of harmony based on the inversion principle, the rules for paradoxes would count as harmonious, and thus paradoxical connectives would belong to the part of our linguistic practices which are immune to criticism (or at least of criticism of this kind).

Here we are not interested in whether the latter view of paradoxical expressions (i.e. the one according to which there is nothing wrong with their meaning) can be given a thorough philosophical defence (see, e.g., Tranchini, 2015a,b). We limit ourselves to record that several authors (at least implicitly) adopt this view by choosing the inversion principle as the best candidate for an appropriate account of harmony (e.g. Hallnäs & Schroeder-Heister, 1990, although in the sequent calculus setting, more recently and in the natural deduction setting Read, 2010). For example Read claims: "Harmony is not normalization, nor is harmony conservative extension [...] Harmony is given by the inversion principle" (2010, p. 575).

Although we essentially agree with this view about paradoxical expressions, in the present note we wish to address another point. Namely whether, in spite of their divergence, it is possible to find a systematic relationship between the three characterizations of harmony.

In particular, we will provide grounds to believe that conservativity and normalization can be combined so to yield a criterion of harmony equivalent to that arising from the inversion principle. The rules of a connective satisfy the inversion principle if and only if they are *conservative over normal deducibility*.

The statement of this general result would require a prior formulation of the conditions at which a set of rules is said to satisfy the inversion principle. This is the object of ongoing debate (see Prawitz, 1979, Schroeder-Heister, 1984, Read, 2010, Francez & Dyckhoff, 2012 and Schroeder-Heister, 2014) and goes beyond the scope of the present paper. We will rather discuss two examples: One is that of the paradoxical λ, whose rules satisfy the inversion principle and which will be shown to be conservative over normal deducibility; the other is Prior's `tonk` (1960), whose rules do not to satisfy the inversion principle and which will be shown not to be conservative over normal deducibility.

From the discussion of `tonk` it will be clear that, for the conjecture to be plausible at all, the notion of a normal deduction must be given a somewhat unusual characterization.

After presenting the main feature of λ in Section 2, in Section 3 we show that the rules for this connective are conservative over normal deducibility. On the usual understanding of normal deduction, however, also the rules for `tonk` turn out to be conservative over normal deducibility. In Section 4 we thereby distinguish two ways of understanding "normal". It is then argued that, in presence of connectives not satisfying the inversion principle, the usual characterization should be replaced by the other one. In Section 5 we show that, on the revised notion of normal deduction, `tonk` does fail to be conservative over normal deducibiltiy, while λ does not. The last section contains some concluding remarks.

§2. Paradox: A simplified natural deduction presentation. We call `NM` the natural deduction system for the $\{\supset, \bot\}$-language fragment of minimal logic, whose rules are:

$$\frac{\begin{array}{c}[A]\\ B\end{array}}{A \supset B}\ \supset\mathrm{I} \qquad \frac{A \supset B \qquad A}{B}\ \supset\mathrm{E}$$

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Consecutive applications of the introduction rule followed immediately by the elimination rule constitute a redundancy of which one can get rid according to the following reduction:

$$\dfrac{\dfrac{\dfrac{\overset{n}{[A]}}{\mathscr{D}_1}}{\dfrac{B}{A \supset B}\supset\mathrm{I}\ (n)} \qquad \dfrac{\mathscr{D}_2}{A}}{B}\supset\mathrm{E} \quad \overset{\supset\text{-Red}}{\triangleright} \quad \begin{array}{c}\mathscr{D}_2\\ [A]\\ \mathscr{D}_1\\ B\end{array}$$

Negation is defined as follows: $\neg A =_{def} A \supset \bot$.

We call $\mathtt{NM}_\lambda$ the extension of NM to the $\{\supset, \bot, \lambda\}$-language fragment with the rules for λ. In $\mathtt{NM}_\lambda$ one can very easily produce a closed deduction of $\bot$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{1}{\lambda}}{\neg\lambda}\lambda\mathrm{E} \qquad \overset{1}{\lambda}}{\bot}\supset\mathrm{E}}{\neg\lambda}\supset\mathrm{I}\ (1) \qquad \dfrac{\dfrac{\dfrac{\overset{2}{\lambda}}{\neg\lambda}\lambda\mathrm{E} \qquad \overset{2}{\lambda}}{\bot}\supset\mathrm{E}}{\neg\lambda}\supset\mathrm{I}\ (2)}{\lambda}\lambda\mathrm{I}}{\bot}\supset\mathrm{E} \qquad (\boldsymbol{\Lambda})$$

Deducibility in $\mathtt{NM}_\lambda$ thus fails to be conservative over deducibility in NM, since $\bot$ cannot be established by means of $\supset$I and $\supset$E alone.

The deduction $\boldsymbol{\Lambda}$ is also a counterexample to normalization in $\mathtt{NM}_\lambda$. A *maximal formula occurrence* in a deduction is the occurrence of a formula which is the consequence of an application of an introduction rule and the major premise of an application of the elimination rule for the same connective.[2] A deduction is called *normal* if it contains no maximal formula occurrence. The deduction $\boldsymbol{\Lambda}$ is not normal since the major premise of the last application of $\supset$E is obtained by $\supset$I. By applying $\supset$-Red to $\boldsymbol{\Lambda}$ one obtains a deduction $\boldsymbol{\Lambda}'$ which is also not normal due to an occurrence of λ which is both the consequence of an application of λI and the premise of an application of λE:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{1}{\lambda}}{\neg\lambda}\lambda\mathrm{E} \qquad \overset{1}{\lambda}}{\bot}\supset\mathrm{E}}{\neg\lambda}\supset\mathrm{I}\ (1)}{\lambda}\lambda\mathrm{I}}{\neg\lambda}\lambda\mathrm{E} \qquad \dfrac{\dfrac{\dfrac{\overset{2}{\lambda}}{\neg\lambda}\lambda\mathrm{E} \qquad \overset{2}{\lambda}}{\bot}\supset\mathrm{E}}{\neg\lambda}\supset\mathrm{I}(2)}{\lambda}\lambda\mathrm{I}}{\bot}\supset\mathrm{E} \qquad (\boldsymbol{\Lambda}')$$

By an application of λ-Red this deductions reduces back to $\boldsymbol{\Lambda}$. No other reduction can be applied either to $\boldsymbol{\Lambda}$ or $\boldsymbol{\Lambda}'$. Therefore neither can be reduced to a normal one (Prawitz, 1965, Appendix B and Tennant, 1982).

We call the *degree* of a maximal formula occurrence the number of logical constants it contains. An application of $\supset$-Red to a deduction may result in a deduction containing new maximal formula occurrences. However, it is always possible to apply $\supset$-Red in such a way that in the resulting deduction all "new" maximal formula occurrences have a lower degree than the one cut away by the application of the reduction.[3] Therefore, in NM

[2] The major premise of an application of a rule is the one which corresponds, in the rule schema, to the premise in which the connective to be eliminated occurs.

[3] An application of $\supset$-Red introduces new maximal formula occurrences whose degree is not lower than the one cut away only when: (i) the deduction of the minor premise of the relevant application

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it is possible to devise a terminating normalization strategy working by induction on the number of maximal formulas occurrences of maximal degree.

On the other hand, there are deductions in $\mathtt{NM}_\lambda$ which contain only one maximal formula occurrence of the form λ and such that the application of λ-$\mathtt{Red}$ to them yields deductions containing a new maximal formula occurrence of the form $\neg\lambda$, and thus of a higher degree than the one cut away by the reduction (for example $\mathbf{\Lambda}'$ above). The presence of λ therefore makes it impossible to prove normalization.[4]

§3. Conservativity over normal deductions. In spite of the fact that normalization fails in $\mathtt{NM}_\lambda$, normal deductions in this system also have the peculiar structure of normal deduction in $\mathtt{NM}$.

A *track* is a sequence of formulas occurrences in a deduction such that (i) the first is an assumption of the deduction; (ii) all other members of the sequence are the consequence of an application of an inference rule of which the previous member is one of the premises; (iii) none of them is the minor premise of an application of $\supset$E.

In each track of a normal deduction in $\mathtt{NM}$, all eliminations precede the introductions. The two parts (either of which is possibly empty) of a track are separated by a minimal part. This is a formula which is both the consequence of an elimination and the premise of an introduction. Furthermore, each formula occurrence in the elimination part is a sub-formula of the preceding formula occurrence in the track, and each formula occurrence in the introduction part is a sub-formula of the next formula occurrence in the track.

From this it follows (almost) immediately that normal deductions in $\mathtt{NM}$ enjoy the sub-formula property: Each formula in a normal deduction is the sub-formula either of the conclusion or of one of the undischarged assumptions of the deduction.

3.1. The conservativity of λ. Prawitz (1965) observed that in an extension of $\mathtt{NM}$ with rules codifying an unrestricted set-comprehension principle, the tracks in normal deductions are still divided into an introduction and elimination part. This holds for normal deduction in $\mathtt{NM}_\lambda$ as well. The reason is the same as in $\mathtt{NM}$: In order for the consequence of an application of an introduction to act as the major premise of an application of an elimination, the deduction must be non-normal.

However, given the standard definition of sub-formula:

DEFINITION 3.1 (sub-formula).

- *For all A, A is a sub-formula of A;*
- *all sub-formulas of A and B are sub-formulas of $A \supset B$,*

the neat sub-formula relationships between the formula occurrences constituting a track are lost in $\mathtt{NM}_\lambda$. To wit, both in the left and right parts of $\mathbf{\Lambda}$, we need to pass through $\neg\lambda$

$\supset$E contains at least one maximal formula occurrence whose degree is not lower than the one of the maximal formula occurrence cut away by the application of $\supset$-$\mathtt{Red}$; and (ii) the relevant application of $\supset$I discharges more than one assumption. Choose among the maximal formula occurrences in a deduction in $\mathtt{NM}$ one of maximal degree which does not fulfil condition (i) above (such a formula occurrence can always be found). Let n be the degree of the chosen formula. By cutting away such a maximal formula occurrence with $\supset$-$\mathtt{Red}$, the number of maximal formula occurrences of degree n necessarily decreases by one.

[4] At least in presence of contraction, represented in the natural deduction setting by the possibility of discharging more than one copy of an assumption with a single application of $\supset$I . Without contraction, both $\supset$-$\mathtt{Red}$ and λ-$\mathtt{Red}$ make the size of the deduction (i.e. the number of applications of inference rules in a deduction) decrease. Therefore one can show normalization to terminate by induction on the size rather than on the number of redexes of maximal degree.

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(i.e. $\lambda \supset \bot$) in order to establish $\bot$ from λ. Thus, normal deductions in $\mathtt{NM}_\lambda$ do not enjoy the sub-formula property.

The reason for this is that the premise of λI is the formula $\neg\lambda$ which is more complex than its consequence λ; and, dually, in λE the consequence of the rule is more complex than the premise.

If we take, in the inferentialist spirit, the rules of a connective to codify semantic information, this situation is unsurprising. The rules for $\supset$ encode the fact that the semantic complexity of an implicational formula correspond to its syntactic complexity: The rules $\supset$I and $\supset$E give the meaning of an implication in terms of its sub-formulas.[5] On the other hand, the rules λI and λE give the meaning of λ in terms of the more complex formula $\neg\lambda$. Whereas the syntactic complexity of formulas in the $\{\supset, \bot, \lambda\}$-language fragment is well-founded, one could say that their semantic complexity is not.

This informal remark can be spelled out by defining the following notion, which in lack of a better name we call *pre-formula*. Intuitively, it reflects the semantic complexity of a formula, in the sense that the pre-formulas of a formula A are the formula A itself together with those formulas one has to understand in order to understand A.

DEFINITION 3.2 (Pre-formula).

- *For all A, A is a pre-formula of A;*
- *all pre-formulas of A and B are pre-formulas of* $A \supset B$*;*
- *all pre-formulas of* $\neg\lambda$ *are pre-formulas of* λ*.*

The seemingly inductive process by which pre-formulas are defined is clearly non-well-founded. However, this is not a reason to reject it as a definition.[6] Indeed, the notion of pre-formula turns out to be very useful in describing the structure of tracks in normal deductions in $\mathtt{NM}_\lambda$: The neat sub-formula relationship holding between the members of a track in normal deductions in NM are replaced by pre-formula relationships between members of a track in normal deductions in $\mathtt{NM}_\lambda$.

FACT 3.3 (The form of tracks). *Each track* $A_1 \dots A_{i-1}, A_i, A_{i+1}, \dots A_n$ *in a normal deduction in* $\mathtt{NM}_\lambda$ *contains a minimal formula* A_i *such that*

- *If* $i > 1$ *then* A_j *(for all* $1 \leq j < i$*) is the premise of an application of an elimination rule of which* A_{j+1} *is the consequence and thereby* A_{j+1} *is a pre-formula of* A_j*.*
- *If* $n > i$ *then* A_j *(for all* $i \leq j < n$*) is the premise of an application of an introduction rule of which* A_{j+1} *is the consequence and thereby* A_j *is a pre-formula of* A_{j+1}*.*

Proof. For a deduction to be normal, all applications of elimination rules must precede all applications of introduction rules in all its tracks: This warrants the existence of a minimal formula in each track. Since a track ends whenever it "encounters" the minor

[5] Of course the same is true if one takes only introduction rules as giving the meaning of the connective, and the elimination rules as consequences of such specifications.

[6] As observed by one of the referees, to see that there is nothing wrong with the notion of pre-formula one could first define the notion of *immediate pre-formula* as follows: (i) the immediate pre-formulas of $A \supset B$ are A and B; (ii) the immediate pre-formula of λ is $\neg\lambda$. The notion of pre-formula could then be introduced as the reflexive and transitive closure of the one of immediate pre-formula.

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premise of an application of $\supset$E, the pre-formula relationships between the members of a track hold. □

THEOREM 3.4 (Pre-formula property). *All formulas in a normal deduction in* $\mathtt{NM}_\lambda$ *are either pre-formulas of the conclusion or of some undischarged assumption.*

Proof. The proof of the theorem is by induction on the order of tracks, where the order of a track is defined as follows: The unique track to which the conclusion belongs is of order 0. A track is of order n if its last formula is the minor premise of an application of $\supset$E whose major premise belong to a track of order $n - 1$.

The proof follows exactly the pattern of the proof of the sub-formula property for NM given by Prawitz (1965, Ch. III, §2). □

We thus have the following:

COROLLARY 3.5. *If* Γ *and* A *are* λ*-free, then there is a normal deduction of* A *from* Γ *in* $\mathtt{NM}_\lambda$ *iff there is one in* NM.

Proof. This follows immediately from the theorem together with the fact that if λ does not occur in a formula than it is not a pre-formula of it (which can be established by induction on the degree of formulas). □

That is, normal deducibility in $\mathtt{NM}_\lambda$ is a conservative extension of normal deducibility in NM. More briefly, we will refer to this fact by saying that the rules for λ are conservative over normal deducibility (in NM).

A generalisation of this result would be that whenever the rules for a propositional connective satisfy the inversion principle, then they are conservative over normal deducibility (in NM). As observed at the end of section §1, this result depends on a precise formulation of the notion of harmony based on the inversion principle and goes beyond the scope of the present note. The above remarks can however be taken as evidence in favour of this claim.

As indicated in Section 1, our aim is that of providing grounds for the *equivalence* between the notion of harmony based on the inversion principle and the notion of harmony as conservativity over normal deducibility.

Therefore we now turn to the other direction of the equivalence: Does a connective whose rules do not obey the inversion principle conservatively extend normal deducibility in NM?

As already anticipated in Section 1, under the understanding of the notion of normality adopted so far, connectives not satisfying the inversion principle may still yield conservative extensions of normal deducibility. In Section 3.2 we show this by discussing a famous example. In Section 4, this situation will be taken as hinting towards the need of an alternative characterization of normal deductions.

***3.2. The conservativity of* `tonk`.** In a famous paper, Prior (1960) introduced the connective `tonk` governed by the following rules:

$$\frac{A}{A\ \mathtt{tonk}\ B}\ \mathtt{tonk}\mathrm{I} \qquad \frac{A\ \mathtt{tonk}\ B}{B}\ \mathtt{tonk}\mathrm{E}$$

The rules for `tonk` do *not* satisfy the inversion principle, as testified by the fact that there is no reduction procedure to cut away from a proof a formula occurrence which is the consequence of an application of `tonk`I and the premise of an application of `tonk`E.

In spite of the crucial difference as to the inversion principle between `tonk` and λ, the salient features of the system $\mathtt{NM}_\lambda$ considered in Section 3.1 carry over to $\mathtt{NM}_{\mathtt{tonk}}$, the extension of NM to the $\{\supset, \bot, \mathtt{tonk}\}$-language fragment with the rules for `tonk`.

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The notion of maximal formula occurrence and hence that of normal deduction can be naturally extended to $\mathtt{NM}_{\mathtt{tonk}}$ as well. As in the case of $\mathtt{NM}_{\lambda}$, normalization fails for $\mathtt{NM}_{\mathtt{tonk}}$. It is sufficient to consider the following deduction:

$$\dfrac{\dfrac{\dfrac{\overset{(1)}{p}}{p \supset p}\supset \mathrm{I}\,(1)}{(p \supset p)\ \mathtt{tonk}\ \bot}\ \mathtt{tonk}\mathrm{I}}{\bot}\ \mathtt{tonk}\mathrm{E} \qquad (\mathbf{\Pi})$$

The occurrence of $(p \supset p)\ \mathtt{tonk}\ \bot$ is maximal. Thus the deduction $\mathbf{\Pi}$ is not normal. Since there is no way of cutting away maximal formula occurrences having tonk as main connective, the deduction is not normal and does not reduce to a normal one. In other words, as $\mathbf{\Lambda}$ was a counterexample to normalization in $\mathtt{NM}_{\lambda}$, $\mathbf{\Pi}$ is a counterexample to normalization in $\mathtt{NM}_{\mathtt{tonk}}$.

Furthermore, Prawitz's analysis of the structure of normal deductions applies to $\mathtt{NM}_{\mathtt{tonk}}$ as well. Actually, it does in an even more straightforward way than in the case of $\mathtt{NM}_{\lambda}$, since there is now no need to introduce the notion of pre-formula.

Once the notion of sub-formula is extended in the obvious way to the $\{\supset, \bot, \mathtt{tonk}\}$-language fragment, the Fact, Theorem and Corollary of the previous section keep on holding when we replace λ with tonk, and pre-formula with sub-formula.

The validity of the Corollary amounts to the fact that the addition of tonk results in an extension of NM which *is* conservative over normal deducibility.

§4. From normality to irreducibility. The results of the Sections 3.1 and 3.2 seem to suggest that there is no hope of distinguishing between a connective satisfying the inversion principle, such as λ, from one not satisfying it, such as tonk, by looking at whether they yield a conservative extension of normal deducibility in NM. Thus, the prospects to establish the equivalence conjecture between harmony as inversion and harmony as conservativity over normal deducibility seem quite meagre.

We take this to be a wrong conclusion which is due to the wrong way of characterizing the notion of normal deduction when discussing systems such as $\mathtt{NM}_{\mathtt{tonk}}$.

It is true that the notion of normal deduction given above (a normal deduction is one containing no maximal formula) is the most usual one. However, we believe that there are strong reasons against its adoption in the case of systems containing connectives whose rules do not satisfy the inversion principle.

Our argument rests on the following (quite uncontroversial) assumption: The notion of normal deduction aims at grasping the intuitive idea of a deduction containing no redundancy. Keeping this in mind, let us consider whether it is always correct to expect a redundancy-free deduction to contain no maximal formula.

This is certainly the case in NM, where consecutive applications of the $\supset$I and $\supset$E rules do constitute redundancies. But what about a system containing the rules for tonk? The rules for tonk do not satisfy the inversion principle. This is tantamount to deny that we had already a deduction of the consequence of an application of the elimination rule, provided that the premise had been established by introduction. In other words, when we establish something passing through a complex formula governed by tonk, we are not making an unnecessary detour. The fact that the rules for tonk do not enjoy the inversion principle means exactly that in some (actually most) cases we can establish a deducibility claim not involving tonk only by appealing to its rules. This is the diametrical opposite of the claim that maximal formula occurrences having tonk as main connective constitute a redundancy. Rather, they are the most essential ingredient for establishing a wide range

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of deducibility claims. For example, in the deduction $\mathbf{\Pi}$, the maximal formula occurrence $(p \supset p)$ tonk $\bot$ is in no way redundant: Without passing through it, it would have been impossible to establish the conclusion $\bot$.

At first, it may look as if the situation in $\mathtt{NM}_\lambda$ is similar to the one in $\mathtt{NM}_{\mathtt{tonk}}$. It is only using the rules for λ that we can establish $\bot$. In the deduction $\mathbf{\Lambda}$ we have a maximal occurrence of $\neg\lambda$ and in the deduction $\mathbf{\Lambda'}$ we have a maximal occurrence of λ. Thus one may think that the same argument applies, yielding the following conclusion: Maximal formula occurrences containing λ do not always constitute redundancies, since they are necessary steps in order to deduce $\bot$. This is true only in part. Although in $\mathtt{NM}_\lambda$ it is not possible to establish $\bot$ without passing through some maximal formula occurrence containing λ, we have a way of eliminating each such maximal formula occurrence. What happens with $\mathbf{\Lambda}$ and $\mathbf{\Lambda'}$ is that, although we can get rid of *each* maximal formula occurrence occurring in them, we cannot get rid of *all* of them. Thus, each single maximal formula occurrence in $\mathtt{NM}_\lambda$ constitutes a redundancy that can be get rid of. This seems to be in the end the content of the claim that the rules for λ (and of $\supset$) enjoy the inversion principle.

The upshot of these considerations is that consecutive applications of an introduction and an elimination rule for a connective constitute a redundancy only if the rules satisfy the inversion principle. This speaks against the identification of non-normal deductions with deductions containing maximal formula occurrences, at least when the rules are not well-balanced. In particular, deductions in NM or $\mathtt{NM}_\lambda$ containing an application of λI followed by one of λE or of $\supset$I followed by one of $\supset$E should not count as normal, since we can always get rid of the maximal formula occurrences squeezed between two rule applications of this kind. On the other hand, a deduction in $\mathtt{NM}_{\mathtt{tonk}}$ whose only maximal formula occurrences have tonk as main connective should count as normal, since there is no way of getting rid of them.

The following alternative definition of normal deduction thus suggests itself: A deduction is normal if and only if no reductions can be applied to it, i.e. if and only if it is irreducible.

In the next section we will show that on the alternative understanding of 'normal', the rules for λ are still conservative over normal deducibility, whereas those for tonk are not, thereby providing grounds for the equivalence between harmony as inversion and harmony as conservativity over normal deducibility.

§5. Conservativity over irreducible deductions. How much of the results established in section §3 is preserved if we replace the notion of normal deduction adopted so far with the one of irreducible deduction?

Concerning the system NM and $\mathtt{NM}_\lambda$ nothing changes. As already observed in the previous Section, in both systems irreducible deductions just coincide with deduction not containing any maximal formula occurrence. Thus we have that normalization holds for NM also in the sense that every deduction reduces to an irreducible one. Analogously, the deduction $\mathbf{\Lambda}$ shows that in $\mathtt{NM}_\lambda$ normalization fails also in the sense that not every deduction reduces to an irreducible one.

Furthermore, irreducible deductions enjoy the sub-formula property in NM and the pre-formula property in $\mathtt{NM}_\lambda$. The latter result implies the following: If A is derivable from Γ by means of an irreducible deduction in $\mathtt{NM}_\lambda$ then, provided both A and Γ are λ-free, there is also an irreducible deduction of A from Γ in NM. In other words, $\mathtt{NM}_\lambda$ conservatively extends irreducible deducibility in NM.

On the other hand, in $\mathtt{NM}_{\mathtt{tonk}}$ things go very differently. Look again at the deduction $\mathbf{\Pi}$ above. Although it does contain a maximal formula occurrence, viz. $(p \supset p)$ tonk $\bot$,

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it is irreducible. More in general, whereas in $\mathtt{NM}_{\mathtt{tonk}}$ it is not possible to reduce any deduction to one which contains no maximal formula occurrence, it *is* possible to reduce every deduction to an irreducible one. In other words, when normal is equated with irreducible, normalization *does* hold in $\mathtt{NM}_{\mathtt{tonk}}$. To prove this fact it is enough to use the very same normalization strategy for NM (see footnote 3 above).

Furthermore, differently from what happens in NM and $\mathtt{NM}_{\lambda}$, irreducible deductions in $\mathtt{NM}_{\mathtt{tonk}}$ do not possess the same properties of deductions containing no maximal formula occurrence. This is exemplified by the deduction $\mathbf{\Pi}$: Although it is irreducible, eliminations do not precede introductions in its (only) track and clearly it lacks the sub-formula property. In turn, the deduction $\mathbf{\Pi}$ also shows that there may be an irreducible deduction of A from Γ with both A and Γ tonk-free in $\mathtt{NM}_{\mathtt{tonk}}$ without there being one in NM (e.g. $\mathbf{\Pi}$, where $\Gamma = \emptyset$ and $A = \bot$). In other words, irreducible deducibility in $\mathtt{NM}_{\mathtt{tonk}}$ does not conservatively extend irreducible deducibility in NM.

In Sections 2 and 3 we equated normal deductions with deduction not containing maximal formula occurrences. The notion of harmony based on the idea of conservativity over normal deductions was incapable of discriminating tonk from λ.

On the other hand, when normal is equated with irreducible we have a difference which can be summarized as follows: Although normalization does not hold for the system $\mathtt{NM}_{\lambda}$, normal deducibility in $\mathtt{NM}_{\lambda}$ conservatively extends normal deducibilty in NM; on the other hand, normal deducibility in $\mathtt{NM}_{\mathtt{tonk}}$ does not conservatively extends normal deducibilty in NM, in spite of the fact that normalization holds for $\mathtt{NM}_{\mathtt{tonk}}$.

Thus, provided that normal is equated to irreducible, the notion of harmony as conservativity over normal deducibility and the notion of harmony based on the inversion principle come to coincide, at least in the two examples here considered.

The possibility of generalizing these results are left for future work. We remark however that the connective λ here discussed can be viewed as a condensation of Russell's paradox in naive set theory (see Prawitz, 1965, appendix B) and as such its discussion is not wholly devoid of significance. Moreover, although we did not discussed the standard intutionistic connectives, it is obvious that the validity of the conjecture can be established in their case as well, using the same line of reasoning developed above for λ.

§6. Concluding remarks.

6.1. The notion of irreducible deduction is clearly relative to the set of reductions that one decides to adopt. Consequently, in a certain system, the notion of an irreducible deduction will be of some interest (by enjoying, e.g., some stronger or weaker variant of the sub-formula property) depending on the appropriateness of the chosen set of reductions.

It may look as if the notion of normal deduction as defined in Section 2, i.e. of deduction containing no maximal formula, is not subject to this criticism. However, this is not the case when the rules of a system allow to generate other kinds of redundancies than just maximal formulas.

A typical example is provided by $\mathtt{NM}^{\vee}$, the extension of NM to the $\{\supset, \bot, \vee\}$-language fragment with the following rules:

$$\frac{A}{A \vee B}\ \vee\mathrm{I}_1 \qquad \frac{B}{A \vee B}\ \vee\mathrm{I}_2 \qquad \frac{A \vee B \qquad \begin{matrix}[A] \\ C\end{matrix} \qquad \begin{matrix}[B] \\ C\end{matrix}}{C}\ \vee\mathrm{E}$$

Besides maximal formulas having $\vee$ as main connective, the indirect form of $\vee$E allows to generate redundancies of a new kind, namely when the consequence of the rule is the major premise of an elimination and at least one of the minor premises of the rule has been

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obtained by introduction. In this cases, the formula C may be neither a sub-formula of one of the undischarged assumptions nor of the conclusion of the deduction. Clearly, the occurrences of C would constitute a redundancy in that they are an unnecessary detour in the path from the assumptions to the conclusion of the deduction.

Although it is possible to introduce new transformations on deductions to get rid of redundancies of this kind (the so-called permutations), in the absence of these transformations irreducible deductions are devoid of interest, since they lack the sub-formula property.

However, the same is true of normal deductions as defined in Section 2 above, i.e. as deductions without maximal formula occurrences. To attain a notion of normal form enjoying the sub-formula property one has to replace the notion of track with that of path, and the notion of maximal formula with the one of maximal segment.

Furthermore, in natural deduction systems for classical logic, in order for normal deductions to enjoy (some weaker version of) the sub-formula property, even further transformations on deductions have to be considered, with the result that the only plausible notion of normal deduction is the one defined in terms of irreducibility (see, for instance, Stålmark, 1991, p. 130, def. iii).

6.2. At any rate the plausibility of our conjecture is dependent on the choice of the right set of reductions. For instance, the rules of λ would not be conservative over irreducible deductions in $\mathtt{NM}^{\vee}$, if this system were not equipped with the $\vee$-permutations. A counter-example is provided by the following deduction ($\mathscr{D}_1$ and $\mathscr{D}_2$ stand for the immediate sub-deductions of $\mathbf{\Lambda}$ above):[7]

$$\dfrac{\dfrac{p \vee q \qquad \begin{matrix}[\neg\lambda] & [\neg\lambda] \\ \mathscr{D}_1 & \mathscr{D}_1\end{matrix}}{\neg\lambda}\vee\mathrm{E} \qquad \begin{matrix}[\lambda] \\ \mathscr{D}_2\end{matrix}}{\bot}\supset\mathrm{E}$$

since $\bot$ does not follow from the disjunction of two atomic formulas in $\mathtt{NM}$.

On reflection, an even more trivial case can arise already in considering $\mathtt{NM}$ itself: if one 'forgets' about λ-$\mathtt{Red}$, i.e. one takes $\supset$-$\mathtt{Red}$ to be the only reduction associated to $\mathtt{NM}_{\lambda}$, the rules for λ would not be conservative over irreducible deducibility in $\mathtt{NM}$.

Cases of this kind, however, do not show the arbitrariness of our conjecture. Rather, they speak in favour of the adoption, in a given system, of all reductions that can be obtained from the inversion principle.

Although permutations are not usually thought of as immediate consequences of the inversion principle, in the end they are designed to get rid of formulas which are first introduced and then eliminated in the course of the deduction. Thus, it is undeniable that, at the very least, they stand in a close connection with the inversion principle (for recent results in this direction see Ferreira & Ferreira, 2009).

A full defence of this point would require a thorough investigation of the notion of transformation of deductions, in particular by addressing the questions of what in general is to count as such a transformation (along the lines of Prawitz, 1973), and of when are such transformation admissible (as pointed out by Widebäck, 2001; Došen, 2003, the set of transformations cannot be arbitrarily extended beyond the reductions of maximal formulas, permutations and expansions without trivializing the notion of identity of proof).

6.3. In the sequent calculus, the inversion principle holds between left and right rules for connectives and the role of normal deducibility is played by cut-free deducibility.

[7] I thank one of the referees for bringing this point to my attention.

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It should be stressed that the notion of cut-free deduction corresponds to the notion of normal deduction adopted in Sections 2 and 3 according to which a normal deduction is one containing no maximal formula occurrence.

To wit, both the rules of a connective like λ and the rules for a connective like tonk yield a conservative extension of cut-free deducibility, irrespective of whether these rules satisfy the inversion principle.

Take λ to be governed by the following left and right rules:

$$\frac{\Gamma \neg\lambda \Rightarrow \Delta}{\Gamma, \lambda \Rightarrow \Delta}\, \mathrm{L}\lambda \qquad \frac{\Gamma \Rightarrow \neg\lambda, \Delta}{\Gamma \Rightarrow \lambda, \Delta}\, \mathrm{R}\lambda$$

and tonk to be governed by the following left and right rules:

$$\frac{\Gamma, B \Rightarrow \Delta}{\Gamma, A \ \mathtt{tonk}\ B \Rightarrow \Delta}\, \mathrm{L}\mathtt{tonk} \qquad \frac{\Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow A \ \mathtt{tonk}\ B, \Delta}\, \mathrm{R}\mathtt{tonk}$$

Call $\mathtt{LK}_{\mathtt{tonk}}$ and $\mathtt{LK}_{\lambda}$ the extensions of the (cut-free) implicative fragment of a sequent calculus for classical logic LK, whose rules are:

$$\frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma', B \Rightarrow \Delta'}{A \supset B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, \mathrm{L}\supset \qquad \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \supset B, \Delta}\, \mathrm{R}\supset$$

together with identity, exchange, weakening and contraction (for the present scopes, one could equivalently consider an intuitionistic or minimal variant of the system).

The following hold:

FACT 6.1. *For Γ and Δ λ-free: $\Gamma \Rightarrow \Delta$ is deducible in LK iff it is deducible in $\mathtt{LK}_{\lambda}$.*

FACT 6.2. *For Γ and Δ tonk-free: $\Gamma \Rightarrow \Delta$ is deducible in LK iff it is deducible in $\mathtt{LK}_{\mathtt{tonk}}$.*

Proof. Given the rules for $\mathtt{LK}_{\lambda}$ (resp. $\mathtt{LK}_{\mathtt{tonk}}$), if there is no occurrence of λ (resp. tonk) in the consequence of a rule-application then there is none in the premises of the rule-application. Thus if the conclusive sequent of a deduction is λ-free (resp. tonk-free), the whole deduction is. □

Thus conservativity over LK (i.e. cut-free) deducibility—like conservativity over deductions without maximal formula occurrences—does not allow to distinguish between tonk and λ.

To recover the full analogy with the natural deduction setting one can consider $\mathtt{LK}^*$, $\mathtt{LK}^*_{\lambda}$ and $\mathtt{LK}^*_{\mathtt{tonk}}$, the systems extending (respectively) LK, $\mathtt{LK}_{\lambda}$ and $\mathtt{LK}_{\mathtt{tonk}}$ with the cut rule. Whereas for the rules for $\supset$ and λ opportune reductions can be defined to push applications of the cut rule towards the axioms, this cannot be done in the case of tonk rules. Consequently, although cut is neither eliminable in $\mathtt{LK}^*_{\mathtt{tonk}}$ nor in $\mathtt{LK}^*_{\lambda}$, this would be for different reasons: In $\mathtt{LK}^*_{\mathtt{tonk}}$ one would have deductions containing applications of the cut rule which cannot be further reduced; in $\mathtt{LK}^*_{\lambda}$ one would have deductions containing applications of the cut rule to which reductions can be applied, but that cannot be brought into cut-free form due to a loop arising in the process of reduction. By introducing the notion of irreducible deduction, it would be possible to show that whereas the rules for λ are conservative over irreducible deductions in $\mathtt{LK}^*$, the rules for tonk are not.

6.4. The discussion of λ and tonk offers the prospects of establishing more general results on the basis of a precise and general formulation of the inversion principle: Namely, that rules satisfying the inversion principle are exactly those that are conservative over

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normal deducibility in NM, provided that the notion of normal deduction is equated with that of irreducible deduction.

We observe however that the prospects for the equivalence between conservativity over normal deducibility and satisfaction of the inversion principle apply only to propositional connectives. The matter is very different in the case of quantifiers, at least for those of second-order logic. In particular, as remarked by Prawitz (1994), "from Gödel's incompleteness theorem we know that the addition to arithmetic of higher-order concepts may lead to an enriched system that is not a conservative extension of the original one in spite of the fact that some of these concepts are governed by rules that must be said to satisfy the requirement of harmony."

Thus, the hope for the equivalence between the notion of harmony based on the inversion principle and the one of conservativity over normal deducibility cannot but be restricted to the domain of connectives. However, we believe this could be a welcome result towards an harmonisation of the different conceptions of harmony.

6.5. Finally, the notion of harmony is often presented as two-fold. The inversion principle does not only warrant the existence of reductions, but also of expansions, that is procedures which permit to expand a deduction by replacing in it an occurrence of a logically complex formula with a deduction of it from itself (Francez & Dyckhoff, 2012, §3.2). Normalization is one side of the coin, the other side of which is the possibility of reducing the minimal part of the tracks of normal deductions to atomic formulas (Prawitz, 1971, §3.3.3). For Belnap (1962), conservativity is one side of the coin, the other side of which is uniqueness.

These three notions have been thoroughly investigated by Naibo & Petrolo (2015) under the names: weak deducibility of identicals, strong deducibility of identicals and uniqueness. Their primary aim was that of stressing the (mostly unnoticed) difference between the three notions. The possible relation between the twin notions of existence of reductions, normalization and conservativity suggests the possibility of finding a systematics of these notions as well.

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BIBLIOGRAPHY

Belnap, N. D. (1962). Tonk, plonk and plink. *Analysis*, **22**(6), 130–134.

Došen, K. (2003). Identity of proofs based on normalization and generality. *Bulletin of Symbolic Logic*, **9**, 477–503.

Dummett, M. (1981). *Frege. Philosophy of Language* (second edition). London: Duckworth.

Dummett, M. (1991). *The Logical Basis of Metaphysics*. London: Duckworth.

Ferreira, F. and Ferreira, G. (2009). Commuting conversions vs. the standard conversions of the "good" connectives. *Studia Logica*, **92**(1), 63–84.

Francez, N. and Dyckhoff, R. (2012). A note on harmony. *Journal of Philosophical Logic*, **41**(3), 613–628.

https://doi.org/10.1017/S1755020315000179 Published online by Cambridge University Press

Hallnäs, L. and Schroeder-Heister, P. (1990). A proof-theoretic approach to logic programming I. Clauses as rules. *Journal of Logic and Computation*, **1**(2), 261–283.
Lorenzen, P. (1955). *Einführung in die operative Logik und Mathematik*, Berlin: Springer.
Moriconi, E. and Tesconi, L. (2008). On inversion principles. *History and Philosophy of Logic*, **29**(2), 103–113.
Naibo, A. and Petrolo, M. (2015). Are uniqueness and deducibility of identicals the same? *Theoria*, **81**(2), 143–181.
Prawitz, D. (1965). *Natural Deduction. A Proof-Theoretical Study*, Stockholm: Almqvist & Wiksell. Reprinted in 2006 for Dover Publication.
Prawitz, D. (1971). Ideas and results in proof theory, In Fenstad J. E., editor, *Proceedings of the Second Scandinavian Logic Symposium*, Volume 63 of Studies in Logic and the Foundations of Mathematics, Amsterdam: North-Holland, pp. 235–307.
Prawitz, D. (1973). Towards a foundation of a general proof theory, In Suppes, P., Henkin, L., Joja, A., and Moisil, G. C., editors, *Proceedings of the Fourth International Congress for Logic, Methodology and Philosophy of Science, Bucharest, 1971*, Volume 74 of Studies in Logic and the Foundations of Mathematics, Amsterdam: North-Holland, pp. 225–250.
Prawitz, D. (1979). Proofs and the meaning and completeness of the logical constants, In Hintikka, J., Niiniluoto, I., and E. Saarinen, editors, *Essays on Mathematical and Philosophical Logic: Proceedings of the Fourth Scandinavian Logic Symposium and the First Soviet-Finnish Logic Conference, Jyväskylä, Finland, June 29–July 6, 1976*, Dordrecht: Kluwer, pp. 25–40.
Prawitz, D. (1994). Review: Michael Dummett, The Logical Basis of Metaphysics. *Mind*, **103**(411), 373–376.
Prior, A. N. (1960). The runabout inference-ticket. *Analysis*, **21**(2), 38–39.
Read, S. (2010). General-elimination harmony and the meaning of the logical constants. *Journal of Philosophical Logic*, **39**, 557–76.
Schroeder-Heister, P. (1984). A natural extension of natural deduction. *The Journal of Symbolic Logic*, **49**(4), 1284–1300.
Schroeder-Heister, P. (2007). Generalized definitional reflection and the inversion principle. *Logica Universalis*, **1**, 355–376.
Schroeder-Heister, P. (2014). Harmony in proof-theoretic semantics: A reductive analysis, In Wansing, H., editor, *Dag Prawitz on Proof and Meaning* Volume 7 of Outstanding Contributions to Logic, Heidelberg: Springer, pp. 329–358.
Stålmark, G. (1991). Normalization theorems for full first order classical natural deduction. *The Journal of Symbolic Logic*, **56**(1), 129–149.
Tennant, N. (1982). Proof and paradox *Dialectica*, **36**, 265–96.
Tranchini, L. (2015a). Proof-theoretic semantics, paradoxes, and the distinction between sense and denotation, *Journal of Logic and Computation*. DOI: 10.1093/logcom/exu028
Tranchini, L. (2015b). Paradox and inconsistency: Revising Tennant's distinction through Schroeder-Heister's assumption rules, In Lolli, G., Panza, M., and Venturi, G., editors, *From Logic to Practice*, Volume 308 of Boston Studies in the Philosophy and History of Science, Heidelberg: Springer, pp. 111–121.
Widebäck, F. (2001). *Identity of proofs*, Stockholm: Almqvist & Wiksell.

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