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Wavefronts and parallels in Euclidean space

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Given a smooth plane curve or surface in $\mathbb{R}^3$ its parallels consist of those curves or surfaces a fixed distance d down the normals in a fixed direction. Generically they have Legendre singularities. We are concerned here with the way in which these parallels change as we alter the distance d. (Alternatively the manner in which wavefronts change as they evolve from an initial smooth wavefront.)

This problem was considered in (1) by V. I. Arnold. In a very beautiful paper he describes the generic evolution of wavefronts but does not prove that for a generic initial wavefront in $\mathbb{R}^2$ or $\mathbb{R}^3$ the evolution is of the type described there. This we do here, using the tool of transversality. A more positive outcome of our investigation is that some of Arnold's generic forms do not occur (those corresponding to A_2 singularities). Using Arnold's notation the precise result we prove is the following:

THEOREM. *Let M be a smooth compact n-dimensional manifold, $n = 1$ or 2, and $\mathrm{Emb}^\infty(M, \mathbb{R}^{n+1})$ the space of smooth embeddings of M in the Euclidean space $\mathbb{R}^{n+1}$. For an open dense set of embeddings $i \in \mathrm{Emb}^\infty(M, \mathbb{R}^{n+1})$ the parallels of $i(M)$ have generic Legendre singularities apart from a discrete collection where the generic transitional forms of Arnold (1) occur. For $n = 1$ (the case of plane curves) there is only one transition and this of type A_3. For $n = 2$ (surfaces in $\mathbb{R}^3$) there are transitions of type A_4 and D_4 and A_3.*

(For pictures in the case $n = 2$ see the first three and fifth illustrations in Fig. 5 of (1). For $n = 1$ see our Fig. 1.)

As in Arnold's paper, these assertions describe the local nature of the parallels and not their self-intersections.

The method of proof is as follows. In § 1 we describe the mechanism for attacking the problem, using the family of distance-squared functions. We outline Arnold's methods and interpret his genericity conditions as transversality statements. In § 2 we prove a transversality theorem for plane curves and surfaces in $\mathbb{R}^3$ which we believe is of independent interest (and which has proved useful elsewhere). Roughly speaking the idea is as follows. At each point of the curve (for example) one can write the curve as the graph of a function, and the Taylor expansion of this function determines all of the infinitesimal information associated to the curve at this point. In § 3 we obtain (using the results of § 1) explicit conditions on the coefficients appearing in these Taylor expansions for the failure of Arnold's genericity conditions. The transversality theorem now asserts that if there are at least two conditions for curves, or at least three conditions for surfaces, then for an open dense set of embeddings these conditions are not satisfied. The main result, Theorem 1, then follows.

In what follows we shall assume that the reader is familiar with the paper of Arnold (1), and the terms used there, such as discriminant variety, simple singularity, Legendre

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singularity. We shall use the standard notations of singularity theory, e.g. those used in (6). One not so standard piece of notation: jet spaces with subscript zero will be those having no constants in the fibre.

1. *Submanifolds of Euclidean space*

Let $M \subset \mathbb{R}^n$ be an embedded submanifold of dimension p. One can study the extrinsic geometry of M by considering the family of distance squared functions

$$F_1 : M \times \mathbb{R}^n \to \mathbb{R},\ F_1(x, a) = \|x - a\|^2$$

(or its natural compactification discussed in (4) or (6)).

There is an associated map $F : M \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n$ whose first component is F_1 and whose second is the natural projection. The set of critical points of F is the normal bundle to M, and is denoted by Σ. The wavefronts associated with M are the sets $F(\Sigma) \cap \{t\} \times \mathbb{R}^n$ (where $t \geqslant 0$) say W_t. (When $p = n - 1$ the wavefronts W_t for $t > 0$ fall into two pieces). For t small (assuming M compact) the wavefronts are smooth. As t increases they acquire singularities which sweep out the caustic (or focal set) of M.

Another way of setting the wavefront up is as follows.

Define $G : M \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ by $G(x, a, t) = \|x - a\|^2 - t$. We have a jet extension $j_1^k G : M \times \mathbb{R}^n \times \mathbb{R} \to J^k(M, \mathbb{R})$ (where we include constants in the fibres of the jet bundle). The pull-back of the zero-valued singular jets projected onto $\mathbb{R}^n \times \mathbb{R}$ is $F(\Sigma)$ of course with the same natural projection onto $\mathbb{R}$. The jet extension $j_1^k G$ is transverse to an orbit (with zero constant) in $J^k(M, \mathbb{R})$ if and only if $j_1^k F$ is transverse to the corresponding orbit in $J_0^k(M, \mathbb{R})$. (All orbits in $J^k(M, \mathbb{R})$ will have zero constants in what follows.)

Now given Δ, the discriminant of a simple singularity at 0, and a function (our projection) $\pi : \Delta \to \mathbb{R}$ in (1) Arnold proves that provided $\partial\pi/\partial\lambda_1(0) \neq 0$, where the co-ordinate λ_1 corresponds to the invariant of minimal degree of the reflection group, one can reduce π to a normal form, namely $\pi = \lambda_1$.

To describe the evolution of the W_t we proceed as Arnold (1) does by first describing $F(\Sigma)$ for generic M and then proving that the natural projection $\pi : F(\Sigma) \to \mathbb{R}$ has a nice local normal form for generic M, using (1). In this paper we shall largely restrict to the cases $p = n - 1$ ($n = 2, 3$). To be able to apply Arnold's results at all one would need to restrict to the cases $p = 1$, $p = 2$ and $n < 7$ or $n < 6$. In this paper we shall only consider local phenomena, i.e. we will ignore the way in which the wavefronts self-intersect.

The first result we need is the result of Looijenga ((4) or (6)), which implies that the jet extension $j_1^k F_1 : M \times \mathbb{R}^n \to J_0^k(M, \mathbb{R})$ meets only the orbits of simple singularities in the dimensions given above and is transverse to these orbits. Thus the image of a small piece of Σ by F will be the discriminant variety of some simple singularity. We now want to prove that the projection $\pi : F(\Sigma) \to \mathbb{R}$ will be generic in the sense of Arnold (1). To do this we want to interpret the genericity conditions of Arnold as transversality statements. We do this as follows.

In sufficiently large jet spaces we want to associate with each of the simple singularities X a smooth manifold $N(X)$ which contains the orbit of X.

For $X = A_k$, $N(X) = \mathrm{cl}(A_{k-2})$ for $k \geqslant 3$, where cl denotes closure,
$X = D_k$, $N(X) = \mathrm{cl}(D_{k-1})$ for $k \geqslant 5$,

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$$X = E_6, \quad N(X) = \mathrm{cl}(D_4),$$
$$X = E_7, \quad N(X) = \mathrm{cl}(E_6),$$
$$X = E_8, \quad N(X) = \mathrm{cl}(E_7).$$

Our claim is that each of the $N(X)$ is a smooth manifold in a neighbourhood of the orbit of the singularity X. For example, in the case of E_6 we have a normal $g(x,y) = x^3 + y^4$, with transversal to the orbit

$$g + \alpha_3 y^3 + \beta_3 xy^2 + \alpha_2 y^2 + \beta_2 xy + \gamma_2 x^2 + \alpha_1 x + \beta_1 y + \alpha_0.$$

The set of D singularities is contained in the $\alpha_2 = \beta_2 = \gamma_2 = \alpha_1 = \beta_1 = \alpha_0 = 0$ space. We have a D_4 (resp. D_5) when $27\alpha_3^2 + 4\beta_3^3 \neq 0$ (resp. $= 0$ and $\alpha_3\beta_3 \neq 0$). Thus $\mathrm{cl}(D_4)$ cuts the transversal in the $\alpha_3\beta_3$ plane which is smooth.

There are two exceptional cases not covered above, namely D_4 and A_2. For D_4 the corresponding set $N(X) = \mathrm{cl}(A_3)$; this is not smooth but is the union of three co-dimension 5 manifolds in the transversal. For A_2, $N(X)$ consists of the jets with zero constant.

What is the geometric (transversality) interpretation of $\partial\pi/\partial\lambda_1(0) \neq 0$?

We separate into two cases.

Case I. Simple singularities of co-dimension $n+1$.

We first need a very simple fact.

LEMMA 1·1. *Let $f: \mathbb{R}^n, 0 \to \mathbb{R}, 0$ have 0 as a regular value and let $g: \mathbb{R}^{n-1}, 0 \to \mathbb{R}^n, 0$ be an immersion at 0 with $f \circ g$ identically zero. Then $\partial f/\partial x_n(0) \neq 0$ if and only if*

$$\det\left(\partial g_i(0)/\partial u_j\right)_{1 \leqslant i,j \leqslant n-1} \neq 0.$$

Proof. $\partial f(0)/\partial x_n = 0$ if and only if $T_0(f^{-1}(0))$ contains $(0, \ldots, 0, 1)$ if and only if $(0, \ldots, 0, 1)$ is in the image of $T_0 g$ if and only if

$$\det\left(\frac{\partial g_i(0)}{\partial u_j}\right)_{1 \leqslant i,j \leqslant n-1} = 0.$$

PROPOSITION 1·2. *If X is a simple singularity of co-dimension $n+1$ and (a) $j_1^k G \pitchfork X$ at (x,a,t) (equivalently $j_1^k F_1 \pitchfork X$ at (x,a)), (b) $j_1^k \tilde{G} \pitchfork N(X)$ at (x,a), where $\tilde{G}: M \times \mathbb{R}^n \to \mathbb{R}$ is defined by $\tilde{G}(y,b) = G(y,b,\|x-a\|^2)$, for all k, then $F(\Sigma)$ is locally the relevant discriminant with π equivalent (via a discriminant-preserving diffeomorphism) to the projection $\lambda \to \lambda_1$ described in* (1).

Proof. We do the case when X is of type A_{n+1}, the other cases being similar. By the transversal-implies-versal theorem we can reduce to the case when

$$G(x,a,t) = \sum_{i=1}^{k-1} x_i^2 + x^{n+2} + \sum_{p=1}^{n+1} w_p(a,t)\, x^{p-1}$$

and we are considering the germ at $(0,0,0) \in \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}$. Since $j_1^k \tilde{G}$ is transverse to $N(X)$ we find that

$$\det\left(\frac{\partial w_p}{\partial a_q}(0)\right)_{1 \leqslant p,q \leqslant n} \neq 0.$$

Now set $g = (w_1, \ldots, w_{n+1}): \mathbb{R}^n \times \{0\}, 0 \to \mathbb{R}^{n+1}, 0$, and $f = \pi \circ W^{-1}$, where

$$W = (w_1, \ldots, w_{n+1}): \mathbb{R}^{n+1}, 0 \to \mathbb{R}^{n+1}, 0$$

(which is a diffeomorphism by (*a*)). By Lemma 1·1 above we find that $\partial f(0)/\partial w_{n+1} \neq 0$, which is precisely Arnold's condition.

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Case II. Simple singularities of co-dimension $< n+1$.

Clearly there is no problem in the following case.

LEMMA 1·3. *If X is a simple singularity and $j_1^k\tilde{G} \pitchfork X$ (all k) then $F(\Sigma)$ is a discriminant cylinder with π equivalent (via a discriminant-preserving diffeomorphism) to a trivial projection.*

Proof. Since $G(x,a,t)$ is a versal unfolding of the singularity X, we have $F(\Sigma)$ diffeomorphic to a discriminant cylinder. By (1), corollary 2·4, it is enough to prove that π restricted to the edge of this cylinder is regular at $(a, \|x-a\|^2)$. However, $j_1^k G: M \times \mathbb{R}^{n+1} \times \mathbb{R} \to J^k(M, \mathbb{R})$ is transverse to X at $(x, a, \|x-a\|^2)$ and so is its restriction to $M \times \mathbb{R}^{n+1} \times \{\|x-a\|^2\}$. Consequently the projection of $(j_1^k G)^{-1}(X)$ to the $\mathbb{R}$ co-ordinate is regular at $(x, a, \|x-a\|^2)$ and hence the restriction to π to the edge is regular, whence the result.

One crucial fact to note is the following. The various strata of the discriminant set are obtained by pulling back orbits of $J^k(M, \mathbb{R})$ and projecting onto $\mathbb{R}^n \times \mathbb{R}$. When G is transverse to the orbit G is a versal unfolding of the corresponding singularity and this projection will be an immersion on the pull-backs of the orbits since the simple singularities are all Thom–Boardman strata of the unfolding.

It follows that given a stratum S in $\mathbb{R}^n \times \mathbb{R}$ corresponding to a singularity X the projection $\pi: S \to \mathbb{R}$ will have the same singularities as the projection onto $\mathbb{R}$ of $(j_1^k G)^{-1}(X)$, and it is enough to consider this latter projection.

Our problems begin when $j_1^k G \pitchfork X, j_1^k \tilde{G} \pitchfork N(X)$ but $j_1^k \tilde{G}$ is not transverse to X. This means that the projection (which we shall also denote by π) $(j_1^k G)^{-1}(X) \to \mathbb{R}$ has singularities (and hence $\pi: S \to \mathbb{R}$ has singularities.) The problems are (1) in general the $(j_1^k G)^{-1}(X)$ are not compact; (2) if we vary our embedding $(j_1^k G)^{-1}(X)$ will also vary and it is difficult to apply Thom's transversality lemma.

To overcome this second problem we are led through the following diversion.

PROPOSITION 1·4. *Let $(Y, 0)$ be the germ of a smooth manifold and $(M_i, 0)$ be germs of submanifolds, $i = 1, 2$, with $T_0 M_1 \subset T_0 M_2$ or $T_0 M_2 \subset T_0 M_1$. If $f_i: (Y, 0) \to (P_i, 0)$ are germs of submersions with $(f_i^{-1}(0), 0) = (M_i, 0)$ then $f_1: (M_2, 0) \to (P_1, 0)$, $f_2: (M_1, 0) \to (P_2, 0)$ are suspension contact equivalent. That is after adding a trivial factor $(\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ to f_1 or f_2 (say $\tilde{f}_2(x, u) = (f_2(x), u)$) f_1 and $\tilde{f}_2$ become contact equivalent.*

(See (3), p. 43, for the definition of contact equivalence.)

Proof. Given germs of manifolds $(M, 0)$, $(P, 0)$ and a smooth map germ $f: (M, 0) \to (P, 0)$ let $\mathscr{E}(M)$ denote the ring of smooth function germs $g: (M, 0) \to \mathbb{R}$ and $I(f) \lhd \mathscr{E}(M)$ the ideal of smooth germs generated by the co-ordinates of f. (In more invariant terms $I(f)$ is the pull-back via f of the maximal ideal $\mathfrak{m} \lhd \mathscr{E}(P)$ of smooth functions vanishing at $0 \in P$.)

By (3), p. 49, it is enough to prove that after adding a trivial factor say to f_2 to obtain $\tilde{f}_2$ as above then $I(f_1)$ and $I(\tilde{f}_2)$ are isomorphic via an isomorphism induced from a germ of a diffeomorphism $(M_1 \times \mathbb{R}^n, 0) \to (M_2, 0)$.

Write M_1 and M_2 locally as the images of the maps

$$(u_1, \ldots, u_r) \to (u_1, \ldots, u_r, h_{r+1}(u), \ldots, h_n(u))$$

$$(v_1, \ldots, v_s) \to (v_1, \ldots, v_t, g_{t+1}(v), \ldots, g_{t+n-s}(v), v_{t+1}, \ldots, v_s).$$

(using the same co-ordinate chart on the right).

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As a zero M_2 is given by $(w_{t+1}-g_{t+1}, \ldots, w_{t+n-s}-g_{t+n-s}) = f_2 = 0$, while M_1 is given by $(w_{r+1}-h_{r+1}, \ldots, w_n-h_n) = f_1 = 0$. Clearly $t \geqslant r$ if $T_0M_2 \supset T_0M_1$ say. Thus $f_2\colon M_1, 0 \to \mathbb{R}^{n-s}, 0$ is locally

$$f_2(u_1, \ldots, u_r) = ((h_{t+1}-g_{t+1})(u_1, \ldots, u_r, h_{r+1}, \ldots, h_t, h_{t+n-s+1}, \ldots, h_n), \ldots,$$
$$(h_{t+n-s}-g_{t+n-s})(u_1, \ldots, u_r, h_{r+1}, \ldots, h_t, h_{t+n-s+1}, \ldots, h_n)),$$

while $f_1\colon M_2, 0 \to \mathbb{R}^{n-r}, 0$ is

$$f_1(v_1, \ldots, v_s) = (v_{r+1}-h_{r+1}(v), \ldots, v_t-h_t(v), g_{t+1}(v)-h_{t+1}(v), \ldots, g_{t+n-s}(v)-h_{t+n-s}(v),$$
$$v_{t+1}-h_{t+n-s+1}(v), \ldots, v_s-h_n(v)),$$

where $v = (v_1, \ldots, v_r)$. Now

$$\langle v_{r+1}-h_{r+1}, \ldots, v_t-h_t, g_{t+1}-h_{t+1}, \ldots, g_{t+n-s}-h_{t+n-s}, v_{t+1}-h_{t+n-s+1}, \ldots, v_s-h_n\rangle$$
$$\cong \langle v'_{r+1}, \ldots, v'_t, v'_{t+1}, \ldots, v'_s, (g_{t+1}-h_{t+1})(v_1, \ldots, v_r, v'_{r+1}+h_{r+1}, \ldots, v'_s+h_s), \ldots,$$
$$(g_{t+n-s}-h_{t+n-s})(v_1, \ldots, v_r, v'_{r+1}+h_{r+1}, \ldots, v'_s+h_n)\rangle$$

(the isomorphism being induced from the obvious change of co-ordinates). Of course in turn this ideal equals

$$\langle v'_{r+1}, \ldots, v'_s, (g_{t+1}-h_{t+1})(v_1, \ldots, v_r, h_{r+1}, \ldots, h_n), \ldots,$$
$$(g_{t+n-s}-h_{t+n-s})(v_1, \ldots, v_r, h_{r+1}, \ldots, h_n)\rangle$$

(since $(g-h)(v_1, \ldots, v_r, v'_{r+1}+h_{r+1}, \ldots, v'_s+h_n)-(g-h)(v_1, \ldots, v_r, h_{r+1}, \ldots, h_n) = 0$ when $v'_{r+1} = \ldots = v'_s = 0$, this difference lies in $\langle v'_{r+1}, \ldots, v'_s\rangle$). Of course this is nothing but the ideal obtained from f_2 with some new variables $v'_{r+1}, \ldots, v'_s$ thrown in (both the ring and the ideal). The result now follows.

How does this result help us? We want to prove that generically the projection $\pi\colon (j^k_1G)^{-1}(X) \to \mathbb{R}$ has only Morse singularities. We can think of $Y = M \times \mathbb{R}^n \times \mathbb{R}$ and $M_1 = M \times \mathbb{R}^n \times \{0\}$, $M_2 = (j^k_1G)^{-1}(X)$. So if $\alpha\colon M \times \mathbb{R}^n \times \mathbb{R}, (x, a, t) \to Q, 0$ is a submersion with $\alpha^{-1}(0)$ and $(j^k_1G)^{-1}(X)$ coinciding as germs at (x, a, t) then $\alpha\colon M \times \mathbb{R}^n \times \{0\}$, $(x, a, t) \to Q, 0$ and π above are suspension contact equivalent. Thus we may choose $\alpha = \phi \circ j^k_1G$ where ϕ is some submersion with zero fibre the orbit of X. The point is that we now see that α has only $\Sigma^{1,0}$ or regular points if and only if π has only Morse singularities.

Of course one problem we will want to avoid is that since the $(j^k_1G)^{-1}(X)$ are open we may have Morse singularities accumulating (to a point on another stratum). Actually one can rule this out as follows. First we claim that if W is the big front (in Arnold's terminology) and $\pi\colon W \to \mathbb{R}$ is the time projection then π is proper provided M is compact. For W is the image of the closed set

$$\left\{(x, a, t) \in M \times \mathbb{R}^n \times \mathbb{R}\colon F_1(x, a) - t = \frac{\partial F_1}{\partial x_i}(x, a) = 0\right\}$$

under the natural projection to $\mathbb{R}^{n+1} \times \mathbb{R}$. If F_1 is bounded (and x is as well) then so is a, so the inverse image of compact sets of $\mathbb{R}$ under π are compact. In any compact set we find a finite number of zero-dimensional strata since from their normal forms we see they are isolated. However, if a stratum of dimension $\geqslant 1$ had an infinite number of critical points of π they would accumulate, and since they are all of Morse type (hence isolated again) they must accumulate to a point of a lower-dimensional

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stratum. But again the normal forms rule this out. (This leaves the possibility that as t increases to infinity the wavefront undergoes an infinite number of changes without the relevant t accumulating.)

2. *Monge–Taylor expansions*

For compact curves in $\mathbb{R}^2$ and surfaces in $\mathbb{R}^3$ rather than use jet spaces to prove genericity statements one can work more directly. It is natural to write these objects in Monge normal form, as the graph of a function f. One then expects the transversality conditions that one needs to prove genericity to appear as the non-vanishing of some combination of the coefficients appearing in the Taylor series of f (the Monge–Taylor expansion).

This has the advantage that one can interpret the transversality conditions geometrically (and indeed sometimes find that they automatically hold). Its major advantage for us, however, is that we do not see any way of using the general theory to obtain the results we require.

Our general idea is (in the surface case say) to consider the map which associates to each point of M its Monge normal form, with respect to some orthogonal coordinates, truncated to some degree. We then wish to prove a transversality statement. Of course unfortunately there is generally no canonical choice of orthogonal coordinates (certainly not at umbilics anyway!) and usually no global choice either. But these points are not important.

Now to details. We shall consider the surface case throughout; the curve case is similar but much easier. Let $M \subset \mathbb{R}^3$ be a compact surface, and cover M with finitely many open sets $\mathcal{U}_i$ each having a unit vector field, v_i; we also have the outward unit normal vector field denoted by n. On each $\mathcal{U}_i$ there is a third unit vector field w_i with the property that $v_i(p)$, $w_i(p)$, $n(p)$ form a positively oriented triple at each $p \in \mathcal{U}_i$ (this determines w_i uniquely).

If V^k denotes the vector space of polynomials of degree $\leqslant k$ and $\geqslant 2$ we have smooth maps $\mathcal{U}_i \to V^k$ defined as follows. At a point $p \in \mathcal{U}_i$ let the three coordinate axes x, y, z be chosen to coincide with $v_i(p)$, $w_i(p)$, $n(p)$ respectively. Writing M as $z = f(x, y)$, we associate to p the polynomial $j^k(f(x, y))$.

Of course the polynomial $j^k(f(x, y))$ depends heavily on the vector field $v_i(p)$, and is determined only up to the obvious action on V^k of SO(2), the special orthogonal group, via its variables. Suppose that $X \subset V^k$ is a submanifold invariant under the SO(2) action. We can ask if the $\mathcal{U}_i \to V^k$ is transverse to X; for points belonging to more than one $\mathcal{U}_i$ the condition is well defined. Indeed it is independent of the choice of the $\mathcal{U}_i$.

PROPOSITION 2·1. *Let $X \subset V^k$ be as above. For a dense set of embeddings of M in $\mathbb{R}^3$ the mappings $\mathcal{U}_i \to V^k$ are transverse to X.*

Proof. Consider the space of all polynomial mappings $\mathbb{R}^3 \to \mathbb{R}^3$ of degree d, P_d. The identity mapping I is in P_d and, given an open bounded region containing M, we may choose a neighbourhood $\mathcal{U}$ of I consisting of mappings which embed this region (and hence M). Since the U_i are embedded by mappings in $\mathcal{U}$ the vector fields v_i are mapped to non-zero fields which can be normalized, and used to determine a new set of w_i. Thus we have a smooth family

$$U_i \times \mathcal{U} \to V^k.$$

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We claim that this map is a submersion if $d \geqslant k$; it is clearly enough to do this at point $(p, 0) \in U_i \times \mathscr{U}$. At p write M locally as above $z = f(x, y)$. The deformation of M induced by the polynomial mapping $(x, y, z) \to (x, y, z + s\phi(x, y))$, where s is small and $2 \leqslant \deg \phi \leqslant k$, clearly gives the tangent vector $\phi(x, y)$ at the corresponding point of V^k, and our claim is proved.

Thus given any submanifold X as above by Thom's basic transversality lemma ((3), p. 49) for almost all P in $\mathscr{U}$ (in the sense of Lebesgue measure) $P(M)$ has the required property of yielding a map transverse to X.

In practice we shall always use the transversality conditions to ensure that we actually miss X. In general the conditions we shall be considering will result in our asking for transversality to algebraic subsets of V^k, so that X will be a smooth stratum of some stratification. We shall assume that we are dealing with closed semialgebraic sets (we can always take their closure without affecting dimensions), and we shall suppose it is furnished with a Whitney (A) regular stratification. With these hypotheses, and using the compactness of M, one easily finds that the set of embeddings satisfying the transversality conditions is open as well as dense. Note that since our deformation is polynomial we can ensure the same holds for analytic surfaces too.

3. *The examples*

Throughout we shall restrict attention to the evolution of the wavefront over some finite period of time so $|t| \leqslant K$ for some constant K. We shall find that if γ is the co-ordinate in $\mathbb{R}^2$ or $\mathbb{R}^3$ in the normal direction and V is the space of Monge normal forms our conditions will turn out most naturally to be given by the vanishing of polynomials in the coefficients of V and γ. The projection $V^k \times \mathbb{R} \to V$ will map semialgebraic sets to semialgebraic sets and the dimension of a variety can only drop under such a projection. So in the case of a curve (resp. a surface) it is enough to prove that in $V^k \times \mathbb{R}$ we have a set of co-dimension $\geqslant 3$ (resp. $\geqslant 4$) and then apply the transversality result to show that we can avoid its image in V^k.

(I) *Plane curves*

Generically we have A_1, A_2 and A_3 singularities to consider. We parametrize the curve (as in § 2) as (formally) $(s, as^2 + bs^3 + cs^4 + \ldots)$. Taking distance squared in the (β, γ) plane, we have as our unfolding

$$\begin{aligned} G(s, \beta, \gamma, t) &= (s - \beta)^2 + (as^2 + bs^3 + cs^4 + \ldots - \gamma)^2 - t \\ &= (\beta^2 + \gamma^2 - t) - 2\beta s + (1 - 2a\gamma)\, s^2 - 2b\gamma s^3 + (a^2 - 2\gamma c)\, s^4 + \ldots . \end{aligned}$$

$$\text{Condition for an } \begin{cases} A_{\geqslant 1} \text{ is} & t = \gamma^2, \quad \beta = 0, \\ A_{\geqslant 2} \text{ is in addition} & 2a\gamma = 1, \\ A_{\geqslant 3} \text{ is in addition} & 2b\gamma = 0, \\ A_{\geqslant 4} \text{ is in addition} & a^2 = 2\gamma c. \end{cases}$$

Concerning tangent vectors in the jet space

$$\partial/\partial\beta \text{ gives } 2\beta - 2s, \quad \partial/\partial\gamma \text{ gives } 2\gamma - 2as^2 - 2bs^3 + \ldots, \quad \partial/\partial t \text{ gives } -1.$$

Clearly $j_1^k G$ generically meets only $A_{\leqslant 3}$ orbits by § 2. It then follows that $j_1^k G$ is automatically transverse to the orbits. It is also easy to see that Arnold's conditions hold automatically. For example, for an A_2 we have, from $\partial/\partial\beta$ and $\partial/\partial\gamma$, $-2s$ and 2γ (mod $\langle s^2 \rangle$) which ensures transversality of $j_1^k \tilde{G}$ to A_2 itself. For an A_3 (when $b = 0$) we have transversality of $j_1^k \tilde{G}$ to $N(A_3)$.

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Thus there are no lips and beaks and, provided the curve has ordinary (A_3) vertices only, the wavefront evolution is generic with the only changes appearing at cusps of the caustic and being as illustrated in Fig. 1. Note in particular that a generic caustic implies generic wavefront evolution.

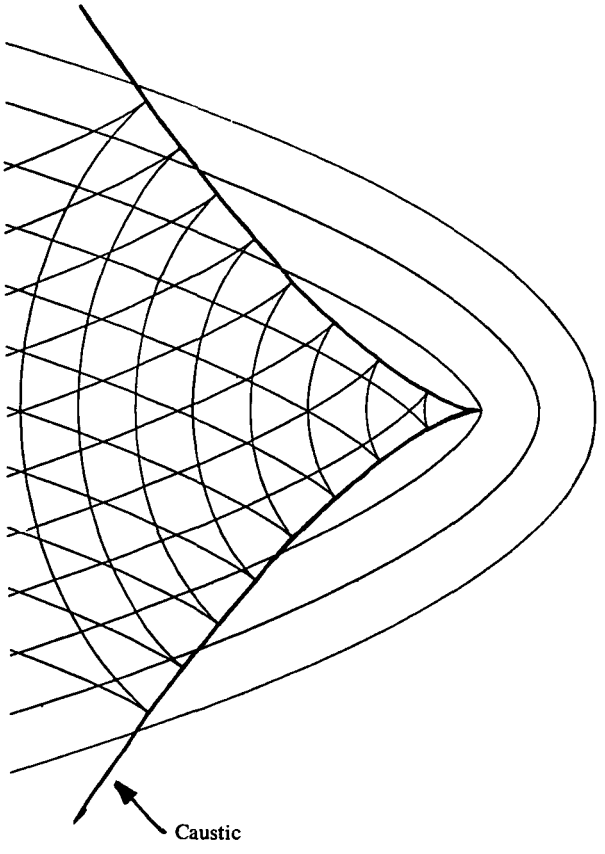


Fig. 1

(II) *Surfaces in Euclidean space*

Here to simplify matters we use the action of SO(2) to diagonalize the quadratic part of the Monge normal form. This will simplify the conditions for the various types of singularity and transversality to occur. If we work with the general quadratic form it is clear that we do get algebraic conditions, and the co-dimension of the bad sets will be the same (except in the umbilic case) as those in the restricted space below where we work with the form $\frac{1}{2}(\kappa x^2+\tau y^2)$. The reason for this is that if we consider the action of the special orthogonal group on this latter space, i.e. $(\theta,\kappa,\tau)\to\frac{1}{2}(\kappa(x\cos\theta+y\sin\theta)^2+\tau(y\cos\theta-x\sin\theta)^2)$ for $\kappa\neq\tau$, we have a local diffeomorphism onto the space of all binary forms which is algebraic.

Generically we have A_1, A_2, A_3, A_4 and D_4 singularities to consider. Parametrize the surface (as in § 2) as $(x,y,\frac{1}{2}(\kappa x^2+\tau y^2)+\psi(x,y))$ where $\psi\in\mathfrak{m}^3$ and $j^3\psi=a_0x^3+a_1x^2y+a_2xy^2+a_3y^3$. Taking distance squared in the (α,β,γ) space, we get

$$\begin{aligned}(x-\alpha)^2+(y-\beta)^2+(\tfrac{1}{2}(\kappa x^2+\tau y^2)+\psi(x,y)-\gamma)^2-t\\ =(\alpha^2+\beta^2+\gamma^2-t)-2\alpha x-2\beta y+(1-\gamma\kappa)x^2+(1-\gamma\tau)y^2+(\kappa x^2+\tau y^2)^2\\ +(\kappa x^2+\tau y^2)+\psi^2-2\psi\gamma.\end{aligned}$$

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$$\text{Condition for an } \begin{cases} A_{\geqslant 1} \text{ is } \quad t = \gamma^2, \quad \alpha = \beta = 0, \\ A_{\geqslant 2} \text{ is in addition } \quad 1 = \gamma\kappa \quad \text{or} \quad 1 = \gamma\tau, \\ A_{\geqslant 3} \text{ is in addition } \quad y|\psi \quad \text{or} \quad x|\psi, \\ \text{etc.}, \\ D_{\geqslant 4} \text{ is } \quad \gamma\kappa = \gamma\tau = 1, \\ D_{\geqslant 5} \text{ is in addition } \quad \psi \text{ has a repeated root.} \end{cases}$$

(The conditions for $A_{\geqslant 4}$, $A_{\geqslant 5}$ are not difficult to compute, just rather messy.) Concerning tangent vectors in the jet space

$$\partial/\partial\alpha \text{ gives } 2(\alpha - x),$$
$$\partial/\partial\beta \text{ gives } 2(\beta - y),$$
$$\partial/\partial\gamma \text{ gives } 2(\tfrac{1}{2}(\kappa x^2 + \tau y^2) + \psi - \gamma),$$
$$\partial/\partial t \text{ gives } -1.$$

Now for A_1 (or A_2) since we have $x, y, -2\gamma + \ldots$ from $\partial/\partial\alpha$, $\partial/\partial\beta$, $\partial/\partial\gamma$ we find $j_1^k \tilde{G} \pitchfork A_1$ (or A_2) and we have a trivial projection of the discriminant cylinder always. For an A_3 we suppose $1 = \gamma\kappa$ and $\psi = a_1x^2y + a_2xy^2 + a_3y^3$. Now the Jacobian ideal of an A_3 contains $\mathfrak{m}^3$ and

$$\frac{\partial G}{\partial x} = -4\gamma a_1 xy - 2\gamma a_2 y^2 + O(3) \quad (O(k) \text{ denotes a term in } \mathfrak{m}^k),$$

while

$$\frac{\partial G}{\partial y} = 2y(-\gamma\tau + 1) - 2\gamma(x_1x^2 + 2a_2xy + 3a_3y^2) + O(3).$$

We now ask for $j_1^k \tilde{G} \pitchfork A_3$. Since we can work modulo $\mathfrak{m}^3$ it is not difficult to see that we need $\{x, y, -2\gamma + (\kappa x^2 + \tau y^2), y^2, xy, 2\gamma a_1 xy - \gamma a_2 y^2, -2\gamma(ax^2 + 2a_2xy + 3a_3y^2)\}$ to span $\mathscr{E}/\mathfrak{m}^3$. Clearly this is so if and only if $a_1 \neq 0$. (Note that $j_1^k G \pitchfork A_3$ always, and $j_1^k \tilde{G} \pitchfork N(A_3)$ always.) We leave the case $a_1 = 0$ to one side for the time being and consider the case of an A_4. Here we write

$$\psi = (a_1x^2y + a_2xy^2 + a_3y^3) + (b_0x^4 + b_1x^3y + b_2x^2y^2 + b_3xy^3 + b_4y^4) + O(5).$$

The condition for an $A_{\geqslant 4}$ is in fact (assuming $1 = \gamma\kappa$) $4a_1^2 = (\kappa^3 - 8b_0)(\kappa - \tau)$.

If we have an A_4 the Jacobian ideal contains $\mathfrak{m}^4$, so we can work modulo quartic and higher terms. Moreover, as our germ starts off constant . y^2 we can work modulo y^3, y^2x, yx^2. From the Jacobian ideal we obtain the first four (row) vectors below. The vectors $\partial/\partial\alpha$, $\partial/\partial\beta$, $\partial/\partial\gamma$ give the last three

1	x	y	x^2	xy	y^2	x^3
0	0	0	0	$-4\gamma a_1$	$-2\gamma a_2$	$\kappa^2 - 8\gamma b_0$
0	0	$2(1-\gamma\tau)$	$-2\gamma a_1$	$-4\gamma a_2$	$-6\gamma a_3$	$-2\gamma b_1$
0	0	0	0	$2(1-\gamma\tau)$	0	$-2\gamma a_1$
0	0	0	0	0	$2(1-\gamma\tau)$	0
0	1	0	0	0	0	0
0	0	1	0	0	0	0
-2γ	0	0	κ	0	τ	0

These vectors span $\mathfrak{m}/\mathfrak{m}^4$ (the condition $j_1^k G \pitchfork A_4$) if and only if $2a_1a_2 - b_1(\kappa - \tau) \neq 0$. In addition $j_1^k \tilde{G} \pitchfork N(A_4)$ means that we can ignore the x^3 terms and ask that the remaining vectors span $\mathscr{E}/\mathfrak{m}^3$. Which is clearly the condition $a_1 \neq 0$.

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For the A singularities it only remains to discuss the case of an A_3 whose projection onto the t axis has a singularity. What is the condition for this singularity to be Morse? We here we find it convenient to use the discussion at the end of § 1. We want to show that the condition that we should have an A_3 with $j_1^k\tilde{G}$ failing to be transverse to the A_3 orbit *and* the projection failing to be Morse determines a set in V^k of co-dimension $\geqslant 3$. If we are to use Proposition 1·4 we must choose a submersion ϕ at each k jet j^kf of type A_3 with the property that $\phi^{-1}(0)$ is locally the A_3 orbit. Now the condition that a jet is of type $A_{\geqslant 3}$ is that its 2-jet is a perfect square L^2 and its cubic term has L as a factor. Thus to find ϕ we first project down to the 3-jet and then given

$$b_0+b_1x+b_2y+b_3x^2+b_4xy+b_5y^2+b_6x^3+b_7x^2y+b_8xy^2+b_9y^3$$

we define

$$\phi(b_0,\ldots,b_g)=(b_0,b_1,b_2,b_4^2-4b_3b_5,\ 8b_5^3b_6-4b_4b_5^2b_7+2b_4^2b_5b_8-b_4^3b_9).$$

(The first four components are self-explanatory. The fifth, however, is not the resultant of the quadratic and cubic parts of the jet, which does not in general give a submersion, but is instead the resultant of the repeated factor and cubic form. We assume we are working at a point with $b_5 \neq 0$.)

Now it is not difficult to see that the condition that the projection p composed with the jet extension is of type $\Sigma^{1,1}$ is algebraic (see (3), pp. 174–190, for example), and depends only on the 5-jet of the Monge–Taylor expansion of the surface. Of course the conditions for an A_3 with $j_1^k\tilde{G}$ not transverse to the A_3 orbit are linear as we saw above ($a_0=a_1=0$). So it is enough to produce a *single* Monge–Taylor expansion with $a_0=a_1=0$ whose associated map $\phi\circ j_1^k\tilde{G}$ *is* of type $\Sigma^{1,0}$. For in $V^k\times\mathbb{R}$ the closure of the set of points giving an A_3 with $j_1^k\tilde{G}$ not transverse to the A_3 orbit is algebraic of co-dimension 3, and it is not difficult to see that it is irreducible. By finding a point of this set of type $\Sigma^{1,0}$ we deduce that those of type $\Sigma^{1,1}$ are of co-dimension $\geqslant 4$. To do this we consider $z=\frac{1}{2}(2x^2+y^2)$ taking distance squared from $(\alpha,\beta,\gamma)=(0,0,\frac{1}{2})$. We shall set $\gamma=\frac{1}{2}+\delta$.

The 3-jet extension of the associated distance-squared functions expanded at $(-x,-y)$ is (using X, Y as the indeterminates) up to degree 2 in x and y and δ:

$$\begin{aligned}&(X+x-\alpha)^2+(Y+y-\beta)^2+(\tfrac{1}{2}(2x^2+4xX+2X^2+y^2+2yY+Y^2)-\tfrac{1}{2}-\delta)^2\\&\equiv\tfrac{1}{4}+\delta+\delta^2+\alpha^2+\beta^2-2\alpha x-2\beta y+\tfrac{1}{2}y^2+X(-2\alpha-4\delta x)+Y(-2\beta-2\delta y+y)\\&\quad+X^2(-2\delta+6x^2+y^2)+XY(4xy)+Y^2(-\delta+\tfrac{1}{2}+x^2+\tfrac{3}{2}y^2)\\&\qquad+X^3(4x)+X^2Y(2y)+XY^2(2x)+Y^3(y).\end{aligned}$$

Now the submersion ϕ with zero fibre the A_3 orbit is, composed with the above, and again modulo terms of degree 2, and constants

$$\begin{aligned}(x,y,\alpha,\beta,\delta)\mapsto(&\delta+\alpha^2+\beta^2+\delta^2-2\alpha x-2\beta y+\tfrac{1}{2}y^2,\ -2\alpha-4\delta x,\ -2\beta+y-2\delta y,\\&4\delta-8\delta^2-12x^2-2y^2,\ 4x-24\delta x).\end{aligned}$$

Adding the fourth component to -4 times the first, we get

$$\begin{aligned}(&-4\alpha^2-4\beta^2-12\delta^2+8\alpha x+8\beta y-4y^2-12x^2,\ -2\alpha-4\delta x,\ -2\beta+y-2\delta y,\\&4\delta-8\delta^2-12x^2-2y^2,\ 4x-24\delta x).\end{aligned}$$

Using y and the last four components as new co-ordinates we see that this 2-jet is

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equivalent to $(\theta(u_1, \ldots, u_5), u_2, \ldots, u_5)$. The coefficient of $u_1^2 (= y^2)$ in θ is -1, non-zero, so $\phi \circ j_1^k \tilde{G}$ is of type $\Sigma^{1,0}$ as required.

Finally we come to the case of the umbilics or D_4 singularities. The condition for transversality of $j_1^k G$ to the D_4 orbit is easily seen to be $(6a_1a_3 + 6a_0a_2 - 2a_1^2 - 2a_2^2) \neq 0$ (see (2), (5)). What of the condition that $j_1^k \tilde{G}$ is transverse to $N(D_4)$? Well, the orbit has tangent space equal to $\mathfrak{m}^3/\mathfrak{m}^{k+1}$. The 3-jet of the distance-squared function is $-2\gamma(a_0x^3 + a_1x^2y + a_2xy^2 + a_3y^3) = -2\gamma C$ and from each component of $N(D_4)$ (the one or three $\mathrm{cl}(A_3)$ sets) we get an additional vector L^2, where L is the relevant linear factor of C. A short computation shows that the condition for non-transversality of $j_1^k \tilde{G}$ to $N(D_4)$ is that L^2, $\partial C/\partial x$ and $\partial C/\partial y$ be linear dependent as quadratic forms. Choosing $L = x$, $C = xQ$, it is easy to see that this can only happen if C has a repeated factor. In other words the condition automatically holds.

Now we wish to apply the transversality theorem of § 2. The point we have to check is that the co-dimension of the bad sets to be avoided is > 2. Of course we have been working with the quadratic form diagonalized, but this is no problem. For the A singularities one simply counts the conditions obtained to find the co-dimension, for the D singularities one adds two to the number of conditions (the quadratic form has equal eigenvalues means two conditions are satisfied). The proof is easy. (Of course we have to check that the conditions yield semi-algebraic sets of the correct dimensions but that is easy too.)

Thus in each case we have found that the set of bad Monge–Taylor jets is of the right co-dimension to be avoided, and by the transversality Theorem of § 2 our Theorem 1 is proved.

REFERENCES

(1) Arnold, V. I. Wavefront evolution and equivariant Morse lemma. *Comm. Pure Appl. Math.* **29** (1976), 557–582.

(2) Bruce, J. W., Giblin, P. J. and Gibson, C. G. Source genericity of caustics by reflexion in $\mathbb{R}^3$. *Proc. Royal Soc. London* A 381, 83–116 (1982).

(3) Gibson, C. G. *Singular points of smooth mappings*. Pitman Research Notes in Mathematics, no. 25 (1979).

(4) Looijenga, E. J. N. Structural stability of families of C^∞-functions. (Thesis, University of Amsterdam, 1974.)

(5) Porteous, I. R. The normal singularities of submanifold. *J. Differential Geom.* **5** (1971), 543–564.

(6) Wall, C. T. C. Geometric properties of generic differentiable manifolds. In *Geometry and Topology*, vol. III, Springer Lecture Notes in Mathematics, no. 597 (1976).

https://doi.org/10.1017/S030500410006062X Published online by Cambridge University Press