Math. Proc. Camb. Phil. Soc. (1985), 98, 247
Printed in Great Britain

Deformations of hyperbolic structures

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(*Received* 17 *October* 1984; *revised* 13 *December* 1984)

1. *Introduction*

In this paper I shall define a deformation of hyperbolic structures on n-manifolds, called '*bending*', by constructing a family of quasiconformal homeomorphisms of hyperbolic $(n+1)$-space. The deformation can be applied to manifolds which contain totally geodesic submanifolds of codimension 1.

A hyperbolic manifold is a $K(\pi, 1)$ manifold with a complete Riemannian metric of constant curvature -1. Two such metrics define the same structure if there is an isometry between them. We consider a stronger equivalence relation by requiring that the isometry be homotopic to the identity. The resulting equivalence classes are called *marked hyperbolic structures*.

The universal covering space of a hyperbolic manifold X^n, with the lifted metric, is isometric to the Poincaré ball, B^n with the metric $g_x(u, v) = 4(u, v)/(1-|x|^2)^2$. A marked hyperbolic structure on X determines, via holonomy, an injective homomorphism of $\pi_1(X)$ onto a discrete subgroup Γ of the group of isometries of B^n, unique up to inner automorphisms of G_n. We can identify X with B^n/Γ. In what follows we shall work with the discrete group Γ. This allows us to treat the more general case where Γ may contain torsion and B^n/Γ is an orbifold, [**10**].

For $n = 3$, hyperbolic structures with infinite volume are related to Kleinian groups, which are discrete subgroups of $G_3 = PSL(2, \mathbb{C})$ acting discontinuously on part of S^2. The deformations of Kleinian groups have been studied by analysing the conformal structures on the components of the boundary of the quotient space. One particularly interesting case is that of quasi-Fuchsian groups: subgroups of G_3 which are quasiconformally conjugate to subgroups of G_2 (see [**7**]).

The same situation occurs in higher dimensions. A subgroup of G_n which is conjugate to a subgroup of G_{n-1} has quotient with infinite volume and may admit deformations even if the corresponding subgroup of G_{n-1} has finite volume quotient and is rigid [**1**, **2**].

The deformation by 'bending' is suggested by Thurston's 'Mickey Mouse' example, [**10**]. If we have a hyperbolic structure on a surface of genus two, we may consider the structure arising when we bend the surface along a simple closed geodesic by an angle $\frac{1}{2}\pi$. Thurston remarks that if the geodesic is short enough, this will give rise to a quasi-Fuchsian group. To extend this idea to higher dimensions we require that the manifold contain a totally geodesic submanifold of codimension one, along which we can bend to define a deformation. Millson has shown that in every dimension there are compact hyperbolic manifolds with arbitrarily many such submanifolds [**8**]. My construction of the bending homeomorphism is similar to Wolpert's construction of a homeomorphism giving rise to the Fenchel–Nielsen deformation [**13**].

If a hyperbolic manifold contains k disjoint totally geodesic submanifolds of

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codimension 1, then bending along these submanifolds defines a k-parameter deformation.

Apart from providing examples in higher dimensions, this deformation may improve our understanding of quasi-Fuchsian groups and their limits. In this case, bending seems to be closely related to the Fenchel–Nielsen deformation [**13**, **6**]. In dimension 3 the definition of the deformation can be extended to the case when the bending locus is a general geodesic lamination. This will be described in a subsequent paper.

The construction is carried out in Sections 3 and 4. In Section 3 we prove Theorem 3·1 which gives conditions for a sequence of mappings to give a quasiconformal deformation. In Section 4 we construct the sequence of bending mappings and show that they satisfy the conditions of Theorem 3·1. In Section 5 we list some properties of the deformation and give an alternative definition of bending which is useful for computations. Section 6 contains examples of bending. In Section 7, we calculate the infinitesimal deformation corresponding to bending.

2. *Preliminaries*

If G is a group and $g \in G$, then we denote the corresponding inner automorphism of G by $g_{\#}: h \mapsto ghg^{-1}$.

Definition. Let Γ be a group, G a Lie group and $k_0 : \Gamma \to G$ an injective homomorphism with discrete image. A *deformation* of k_0 is a mapping $k : W \to \mathrm{Hom}(\Gamma, G)$ from a neighbourhood of the origin in $\mathbb{R}^m$ to the space of homomorphisms of Γ into G, such that $k(0) = k_0$, $k(t)$ is injective with discrete image for every t in W and $t \mapsto k(t)(\gamma)$ is differentiable in W for all γ in Γ.

Two deformations $k : W \to \mathrm{Hom}(\Gamma, G)$ and $k' : W' \to \mathrm{Hom}(\Gamma, G)$ are *equivalent* if there is a diffeomorphism $\psi : W \to W'$, $\psi(0) = 0$, and a differentiable mapping $\Psi : W \to G$ such that $k'(\psi(t)) = \Psi(t)_{\#} k(t)$.

A deformation is *trivial* if it is equivalent to the constant deformation.

A deformation $k : W \to \mathrm{Hom}(\Gamma, G)$ is *degenerate* if there is $W' \subset \mathbb{R}^{m'}$, $m' < m$, a differentiable mapping $\phi : W \to W'$ and a deformation $k' : W' \to \mathrm{Hom}(\Gamma, G)$ such that k is equivalent to $k'\phi$.

When Γ is a discrete subgroup of the group G_n of isometries of B^n we want to consider deformations of $\Gamma \curvearrowright G_n$ which preserve the geometric type of elements of Γ. For this reason we introduce quasiconformal deformations.

Let X and Y be metric spaces and $f : X \to Y$ a homeomorphism. Associate with f the following functions:

$$L_f(x, r) = \sup_{d_X(x,y)=r} d_Y(f(x), f(y)),$$

$$l_f(x, r) = \inf_{d_X(x,y)=r} d_Y(f(x), f(y)),$$

$$H_f(x) = \limsup_{r \to 0} L_f(x, r)/l_f(x, r).$$

Definition. H_f is the *linear dilatation* of f and the homeomorphism f is called *quasiconformal* if H_f is bounded on X.

In what follows we shall only consider quasiconformal mappings between domains of $\mathbb{R}^n$.

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A quasiconformal homeomorphism $f: X \to Y$ is (Fréchet) differentiable at almost every point in X. For a proof of this and other properties of quasiconformal mappings we refer to [11]. We say that f is *K-quasiconformal* if at almost every point x in X we have

$$K^{-1}(L_{Df(x)}(0,1))^n \leqslant \det(Df(x)) \leqslant K(l_{Df(x)}(0,1))^n.$$

A quasiconformal homeomorphism is 1-quasiconformal if and only if $H_f(x) = 1$ for almost every x in X.

PROPOSITION 2·1. [11] *If $f_j: X \to Y_j$ is a sequence of K-quasiconformal homeomorphisms which converge pointwise to a homeomorphism $f: X \to Y$ then f is K-quasiconformal.* ∎

THEOREM 2·2. [3, 9] *If X and Y are domains in $\mathbb{R}^n$, $n \geqslant 3$, and $f: X \to Y$ is a 1-quasiconformal homeomorphism, then f is the restriction of a conformal mapping of $\mathbb{R}^n$.* ∎

Definition. A quasiconformal homeomorphism f of B^n is *compatible* with the subgroup Γ of G_n if $f\gamma f^{-1}$ is in G_n for every γ in Γ.

A family f_t of quasiconformal homeomorphisms of B^n compatible with the group Γ gives rise to a quasiconformal deformation of the inclusion $\Gamma \hookrightarrow G_n$ by defining $k(t) = (f_t)_\#$.

If v is a vector tangent to $x \in B^n$, $\|v\|_x$ denotes the length of v with respect to the hyperbolic metric.

Compatibility can be expressed in terms of distortion of tangent vectors in the following way.

PROPOSITION 2·3. *A quasiconformal homeomorphism $f: B^n \to B^n$ is compatible with Γ if and only if*

$$\|D(f\gamma)(x).u\|_{f\gamma(x)} = \|Df(x).u\|_{f(x)} \tag{1}$$

for all γ in Γ, all u in $\mathbb{R}^n$ and almost every x in B^n.

Proof. If f is compatible with Γ, then $f\gamma f^{-1}$ is an isometry of B^n and this clearly implies (1).

Conversely, if (1) holds then we have $\|D(f\gamma f^{-1}).v\|_{f\gamma(x)} = \|v\|_{f(x)}$ for all v in $\mathbb{R}^n$ and almost every x in B^n. But then $H_{D(f\gamma f^{-1})(x)}(0) = 1$ for almost every x and therefore $f\gamma f^{-1}$ is 1-quasiconformal. By Theorem 2·2 it is then conformal and hence an isometry of B^n. ∎

We shall also require the following result, whose proof is an application of the chain rule.

LEMMA 2·4. *If $f: Y \to W$ is a quasiconformal homeomorphism and $h: X \to Y$ is conformal, then $H_{fh}(x) = H_f(h(x))$ for almost every x in X.* ∎

3. *Criteria for convergence and compatibility*

The central result of this paper is Theorem 3·1. The idea is to construct a quasiconformal homeomorphism compatible with the group Γ step by step, as the infinite product of a sequence of homeomorphisms. We require that the product converge and that it be compatible with the group. The first requirement is satisfied by controlling the support of the homeomorphisms and the second by ensuring that the regions where the homeomorphisms are not conformal are disjoint.

Definition. If f is a quasiconformal homeomorphism of X, then the *support* of f is the

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set $\operatorname{supp}(f) = \operatorname{closure}\{x \in X; f(x) \neq x\}$. The *quasiconformal support* of f is the set $qc\operatorname{supp}(f) = \operatorname{closure}\{x \in X; H_f(x) \neq 1\}$. If U is a subset of $\mathbb{R}^n$, define

$$\operatorname{diam}(U) = \sup\{|x-y|; x, y \in U\}.$$

THEOREM 3·1. *Let Γ be a discrete group of isometries of B^n, and $\{\psi_j\}$ a sequence of K-quasiconformal homeomorphisms of B^n such that:*

(i) *if $j \neq k$ then $qc\operatorname{supp}(\psi_j) \cap qc\operatorname{supp}(\psi_k) = \varnothing$.*

(ii) *if $j \neq k$ then $\operatorname{supp}(\psi_j)$ and $\operatorname{supp}(\psi_k)$ are either disjoint or the one is contained in the other, and if $\operatorname{supp}(\psi_j) \subset \operatorname{supp}(\psi_k)$ then $j > k$.*

(iii) *if $j > k$ then $qc\operatorname{supp}(\psi_k) \cap \operatorname{supp}(\psi_j) = \varnothing$.*

(iv) $\lim_{j\to\infty} \operatorname{diam}(\operatorname{supp}(\psi_j)) = 0$.

(v) *for every $\gamma \in \Gamma$ there is a bijection $\mathbb{N} \to \mathbb{N}$, taking j to k such that for every u in $\mathbb{R}^n$ and every x in B^n at which ψ_j and ψ_k are both differentiable we have*

$$\|D\psi_j(x).u\|_{\psi_j(x)} = \|D(\psi_k\gamma).u\|_{\psi_k\gamma(x)}.$$

Then the sequence b_j of quasiconformal homeomorphisms defined by $b_j = \psi_1\psi_2 \dots \psi_j$ converges uniformly on B^n and the limit $b = \lim_{j\to\infty} b_j$ is a K-quasiconformal homeomorphism compatible with the group Γ.

Proof. Let κ be such that if $i > \kappa$ then $\operatorname{diam}(\operatorname{supp}(\psi_i)) < \delta$. From the definition of b_j we have, for $l > m > \kappa$,

$$b_m^{-1}b_l = \psi_{m+1} \dots \psi_l \quad \text{and} \quad \operatorname{supp}(b_m^{-1}b_l) = \bigcup_{i=m+1}^{l} \operatorname{supp}(\psi_i).$$

If $x \in \operatorname{supp}(b_m^{-1}b_l)$, let $j = \min\{i \in \mathbb{N}; i \geqslant m+1, x \in \operatorname{supp}(\psi_i)\}$. Then for any i such that $m+1 \leqslant i \leqslant l$, if $x \in \operatorname{supp}(\psi_i)$ then $\operatorname{supp}(\psi_i)$ is not disjoint from $\operatorname{supp}(\psi_j)$ and (ii) implies that $\operatorname{supp}(\psi_i) \subset \operatorname{supp}(\psi_j)$. Since ψ_i is a homeomorphism, $\psi_i(\operatorname{supp}(\psi_i)) = \operatorname{supp}(\psi_i)$. It follows that $b_m^{-1}b_l(x) \in \operatorname{supp}(\psi_j)$. But $j > \kappa$ and therefore $|b_m^{-1}b_l(x) - x| < \delta$ for all x. The sequence $\{b_i\}$ converges uniformly to a limit b: $B^n \to B^n$. Proposition 2·1 implies that the limit is quasiconformal.

To prove that b is compatible with the group Γ we shall show that it satisfies the criterion of Proposition 2·3: if $\gamma \in \Gamma$, then for all $u \in \mathbb{R}^n$ and almost every x in B^n we have to show

$$\|D(b\gamma)(x).u\|_{b\gamma(x)} = \|Db(x).u\|_{b(x)}. \tag{2}$$

Let x be a point where b and $b\gamma$ are differentiable.

First assume that $x \notin qc\operatorname{supp}(\psi_i)$ for all $i \in \mathbb{N}$. From (v) we have that for every i there is some k such that

$$\|D\psi_i(x).u\|_{\psi_i(x)} = \|D\psi_k(\gamma(x)).D\gamma(x).u\|_{\psi_k\gamma(x)}$$

is satisfied for all u in $\mathbb{R}^n$. This implies that $\gamma(x) \notin qc\operatorname{supp}(\psi_k)$. As we let i run through $\mathbb{N}$, k also takes all values in $\mathbb{N}$, and we conclude that $\gamma(x) \notin qc\operatorname{supp}(\psi_k)$ for all k in $\mathbb{N}$. Then Db is conformal at x and $\gamma(x)$, and hence satisfies (2).

Now assume that $x \in qc\operatorname{supp}(\psi_j)$. For $i > j$, we have from (iii) that

$$qc\operatorname{supp}(\psi_j) \cap \operatorname{supp}(\psi_i) = \varnothing$$

and hence $b(x) = b_j(x) = \psi_1 \dots \psi_j(x)$. Condition (ii) implies that if $l < k < j$ and $x \in \operatorname{supp}(\psi_j)$ then $\psi_k \dots \psi_j(x) \in \operatorname{supp}(\psi_l)$ if and only if $\psi_{k-1} \dots \psi_j(x) \in \operatorname{supp}(\psi_l)$. By induction we have that if $l < j$ then $\psi_l \dots \psi_j(x) \in \operatorname{supp}(\psi_l)$ if and only if $x \in \operatorname{supp}(\psi_l)$.

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For any $i < j$, let $i' = \min\{k \in \mathbb{N};\ i+1 \leqslant k \leqslant j, \operatorname{supp}(\psi_j) \subset \operatorname{supp}(\psi_k)\}$. If $i+1 \leqslant l < i'$, then $\psi_l \dots \psi_j(x) \notin \operatorname{supp}(\psi_l)$ and hence $\psi_{i+1} \dots \psi_j(x) = \psi_{i'} \dots \psi_j(x)$. Therefore,

$$\psi_{i+1} \dots \psi_j(x) \in \operatorname{supp}(\psi_{i'})$$

and condition (iii) implies that $\psi_{i+1} \dots \psi_j(x) \notin qc \operatorname{supp}(\psi_i)$. This is true for any $i < j$ and therefore

$$\|Db(x).u\|_{b(x)} = \|D\psi_j(x).u\|_{\psi_j(x)}. \tag{3}$$

If $\gamma \in \Gamma$, there is k such that

$$\|D\psi_j(x).u\|_{\psi_j(x)} = \|D(\psi_k\gamma)(x).u\|_{\psi_k\gamma(x)} \tag{4}$$

for every u in $\mathbb{R}^n$. Then $H_{\psi_j}(x) = H_{\psi_k}(\gamma(x))$ and hence $\gamma(x) \in qc \operatorname{supp}(\psi_k)$. By the above arguments, $b\gamma(x) = \psi_1 \dots \psi_k\gamma(x)$ and

$$\|D(b\gamma)(x).u\|_{b\gamma(x)} = \|D(\psi_k\gamma)(x).u\|_{\psi_k\gamma(x)}.$$

But (3), (4) and (5) imply that b satisfies (2). Therefore, b is compatible with the group Γ. ∎

4. *The bending homeomorphism*

Let Γ be a discrete subgroup of G_{n-1} and P a hyperplane of B^{n-1} with the property

$$\inf\{d(\gamma(P), P);\ \gamma \in \Gamma, \gamma(P) \neq P\} = a > 0. \tag{6}$$

Without loss of generality we can assume that 0 is in one of the components of $B^{n-1} - \Gamma(P)$ which lie adjacent to P. Let Γ_P be the subgroup of Γ which leaves P invariant and assume that $\Gamma_P \neq \Gamma$. A hyperplane satisfying the above conditions will be called a *bending hyperplane* for Γ.

Let $\mathscr{P}$ be the set of distinct translates of P.

PROPOSITION 4·1. *If Γ is an infinite discrete subgroup of G_{n-1} and P a bending hyperplane for Γ, then $\mathscr{P}$ is infinite.*

Proof. Assume that $\mathscr{P}$ is finite. By (6), P and its translates are not asymptotic at infinity and there is an element $R \in \mathscr{P}$ and a point $x \in P$ such that $d(x, R) = a$. Any point of $\Gamma_P(x)$ is the unique point on P nearest to an element of $\mathscr{P}$. So if $\mathscr{P}$ is finite, $\Gamma_P(x)$ is also finite. Since Γ is discrete the isotropy group of x is finite and then Γ_P must be finite. There is a bijection $\mathscr{P} \to \Gamma/\Gamma_P$ and hence, if both Γ_P and $\mathscr{P}$ are finite, Γ is also finite, a contradiction. ∎

For $R \in \mathscr{P}$, let Y_R denote the hyperplane in B^n orthogonal to B^{n-1} which intersects B^{n-1} along R, and let Y_R^+ be the closure in $\bar{B}^n$ of the half space bounded by Y_R and disjoint from 0. We define an order relation in $\mathscr{P}$ by $R \gg Q$ if $Y_R^+ \subset Y_Q^+$. It is easy to check that this relation is a partial ordering of $\mathscr{P}$. If Γ and P satisfy (6), then for any R, Q in $\mathscr{P}$, Y_R^+ and Y_Q^+ are either disjoint or the one is contained in the other.

Since Γ is discrete, any closed ball in B^n intersects only a finite number of hyperplanes Y_R. Consider the sequence of balls $\{B_i\}$, $i \geqslant 2$, with centre 0 and euclidean radius $1 - (1/i)$. We can use the finite subsets $\{R; Y_R \cap B_i \neq \varnothing\}$, $i \geqslant 2$, to define a bijection $q\colon \mathbb{N} \to \mathscr{P}$ such that q^{-1} is order-preserving.

We shall define a quasiconformal homeomorphism $\phi\colon \mathbb{C} \to \mathbb{C}$ which bends the real line at 0 by an angle α, in such a way that it is conformal 'near' the real line.

Let α and ζ be real numbers such that

$$0 < \zeta < \tfrac{1}{2}\pi, \quad 0 \leqslant \alpha < \pi - 2\zeta. \tag{7}$$

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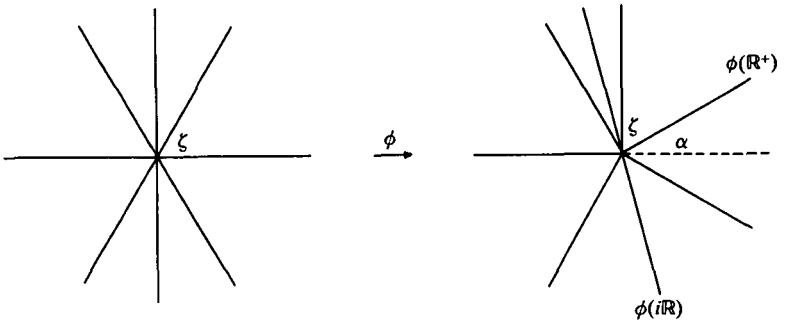


Fig 1.

By $\arg(z)$ we denote the principal value of the argument of a complex number z; $-\pi < \arg(z) \leqslant \pi$. For α and ζ as above, we define $\phi_{\alpha,\zeta}: \mathbb{C} \to \mathbb{C}$ by

$$\phi_{\alpha,\zeta}(z) = \begin{cases} z & (|\arg(z)| \geqslant \pi - \zeta) \\ z\exp(i\alpha) & (|\arg(z)| \leqslant \zeta) \\ z\exp(i\alpha(1-(\arg(z)-\zeta)/(\pi-2\zeta))) & (\zeta < \arg(z) < \pi-\zeta) \\ z\exp(i\alpha(1+(\arg(z)+\zeta)/(\pi-2\zeta))) & (\zeta-\pi < \arg(z) < -\zeta). \end{cases}$$

The subscripts α and ζ will be omitted when there is no ambiguity. For $0 \leqslant \alpha < \pi - 2\zeta$, $\arg(\phi(z))$ is a strictly increasing function of $\arg(z)$ and ϕ is a homeomorphism.

The linear dilatation of ϕ is easy to calculate:

$$H_\phi(z) = \begin{cases} 1 & (|\arg(z)| \geqslant \pi - \zeta) \\ 1 & (|\arg(z)| \leqslant \zeta) \\ (\pi-2\zeta)/(\pi-2\zeta-\alpha) & (\zeta < \arg(z) < \pi-\zeta) \\ (\pi-2\zeta+\alpha)/(\pi-2\zeta) & (\zeta-\pi < \arg(z) < -\zeta). \end{cases}$$

For α and ζ satisfying (7), ϕ is quasiconformal.

We extend the definition of ϕ to negative values of α as follows: if $2\zeta - \pi < \alpha < 0$ then $\phi_{\alpha,\zeta}(z) = \overline{\phi_{-\alpha,\zeta}(\bar{z})}$.

In order to define bending in $\mathbb{R}^n$ we identify $\mathbb{R}e_{n-1} + \mathbb{R}e_n$ with $\mathbb{C}$ by mapping e_{n-1} to 1 and e_n to i. We define

$$\psi^+_{\alpha,\zeta}: \mathbb{R}^n \to \mathbb{R}^n : (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-2}, \phi_{\alpha,\zeta}(x_{n-1} + ix_n))$$

and use the involution $\iota: \mathbb{C} \to \mathbb{C} : z \mapsto -\bar{z}$ to define

$$\psi^-_{\alpha,\zeta}: \mathbb{R}^n \to \mathbb{R}^n : (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-2}, \iota\phi_{\alpha,\zeta}\iota(x_{n-1} + ix_n)).$$

The mappings ψ^+ and ψ^- leave the unit ball and the *upper half-space*

$$\mathscr{H}^n = \{x \in \mathbb{R}^n;\ x_1 > 0\}$$

invariant. Their restrictions to B^n and to $\mathscr{H}^n$ will be denoted by ψ^+ and ψ^- if the domain of the map is clear from the context.

The following Lemma is a consequence of elementary facts about conformal mappings.

LEMMA 4·2. *If $\gamma \in G_{n-1} \subset G_n$ and γ leaves B^{n-2} invariant, then $\gamma\psi^\pm = \psi^\pm\gamma$ if γ preserves the orientation of B^{n-2} and $\gamma\psi^\pm = \psi^\mp\gamma$ if γ reverses the orientation of B^{n-2}.* ∎

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Note that if ψ^+ and ψ^- are differentiable at $x \in B^n$, and $u \in \mathbb{R}^n$, then

$$\|D\psi^-(x).u\|_{\psi^-(x)} = \|D\psi^+(x).u\|_{\psi^+(x)}. \tag{8}$$

For every $R \in \mathscr{P}$, let $\gamma_R \in G_{n-1} \subset G_n$ be an isometry of B^n which is an orientation preserving isometry of B^{n-1} and such that

$$\gamma_R(B^{n-2}) = R \quad \text{and} \quad \gamma_R(\{x \in \bar{B}^n;\, x_{n-1} \geqslant 0\}) = Y_R^+.$$

If $R = q(j)$, define $\psi_j : B^n \to B^n$ by $\psi_j = \gamma_R \psi^+ \gamma_R^{-1}$. Then ψ_j bends the hyperbolic space B^{n-1} inside B^n along R and fixes the half-space of $B^{n-1} - R$ containing 0. Lemma 4·2 implies that ψ_j does not depend on the choice of γ_R.

In the next four Lemmas we shall show that $\{\psi_j\}$ satisfy the conditions of Theorem 3·1.

First we make more precise the sense in which the homeomorphisms ψ_j are conformal 'near' B^{n-1}. It is more convenient to work in the upper half space $\mathscr{H}^n$.

LEMMA 4·3. *Let $\mathscr{S}$ be the set of hyperplanes Y in $\mathscr{H}^n$ orthogonal to $\mathscr{H}^{n-1}$ and distant more than $t = \operatorname{arc\,cosh}(1/\sin\zeta)$ from $\mathscr{H}^{n-2}$. Then*

$$\bigcup_{Y \in \mathscr{S}} Y \cap qc\,\mathrm{supp}\,(\psi^\pm) = \varnothing.$$

Proof. It is clear from the definition of $\psi^\pm$ that

$$qc\,\mathrm{supp}\,(\psi^\pm) = \{x \in \mathscr{H}^n;\, |x_n/x_{n-1}| > \tan\zeta\}.$$

The distance $t = \operatorname{arc\,cosh}(1/\sin\zeta)$ is the distance corresponding to the angle of parallelism ζ. If $x \in \mathscr{H}^{n-1}$ and $d(x, \mathscr{H}^{n-2}) > t$ then $x_1/x_{n-1} < \tan\zeta$. Since $Y \in \mathscr{S}$ is orthogonal to $\mathscr{H}^{n-1}$, its intersection with $\mathscr{H}^{n-1}$ is a great $(n-2)$-hemisphere. It follows that if $x \in Y$ then

$$(x_1^2 + x_n^2)^{\frac{1}{2}}/|x_{n-1}| < \tan\zeta \quad \text{and} \quad \bigcup_{Y \in \mathscr{S}} Y = \{x \in \mathscr{H}^n;\, (x_1^2 + x_n^2)^{\frac{1}{2}}/|x_{n+1}| < \tan\zeta\}. \quad \blacksquare$$

In Lemma 4·4 we show how condition (6) allows us to control the interaction between the homeomorphisms ψ_j.

LEMMA 4·4. *Let $i \neq j$ and $R = q(i)$, $Q = q(j)$. If $\cosh\frac{1}{2}a > 1/\sin\zeta$, then*

(i) $qc\,\mathrm{supp}\,(\psi_i) \cap qc\,\mathrm{supp}\,(\psi_j) = \varnothing$;

(ii) *if $R \gg Q$ then $\mathrm{supp}\,(\psi_i) \subset \mathrm{supp}\,(\psi_j)$, and if R, Q are not related then*

$$\mathrm{supp}\,(\psi_i) \cap \mathrm{supp}\,(\psi_j) = \varnothing;$$

(iii) *if $R \gg Q$ then $qc\,\mathrm{supp}\,(\psi_j) \cap \mathrm{supp}\,(\psi_i) = \varnothing$.*

Proof. Since γ_R is conformal, Lemma 2·4 implies that $qc\,\mathrm{supp}\,(\psi_i) = \gamma_R(qc\,\mathrm{supp}\,(\psi^+))$. Lemma 4·3, translated to the Poincaré ball, implies that $qc\,\mathrm{supp}\,(\psi_i)$ does not intersect any of the hyperplanes orthogonal to B^{n-1} and distant more than t from R. Similarly $qc\,\mathrm{supp}\,(\psi_j)$ does not intersect any of the hyperplanes orthogonal to B^{n-1} and distant more than t from Q. Since $d(R, Q) \leqslant a$, if $a > 2t$ then there is a hyperplane Y orthogonal to B^{n-1} such that $qc\,\mathrm{supp}\,(\psi_i)$ and $qc\,\mathrm{supp}\,(\psi_j)$ do not intersect Y and lie on opposite sides. This proves (i). The proof of (ii) and (iii) follows similarly once we note that $\mathrm{supp}\,(\psi_i) = Y_R^+ \cup qc\,\mathrm{supp}\,(\psi_i)$. $\blacksquare$

The following Lemma is the basic step in the proof of the convergence of $\psi_1\psi_2 \ldots \psi_k \ldots$.

LEMMA 4·5. *If $0 < \zeta < \frac{1}{2}\pi$ then $\lim_{j\to\infty} \mathrm{diam}\,(\mathrm{supp}\,(\psi_j)) = 0$.*

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Proof. If $R = q(i)$, then $\operatorname{supp}(\psi_i)$ is disjoint from any hyperplane orthogonal to B^{n-1} and distant more than $t = \operatorname{arccosh}(1/\sin\zeta)$ from R which does not intersect Y_R^+.

Since Γ is discrete, the number of j such that $Y_{q(j)}$ intersects the hyperbolic ball $N(0,t)$ is finite. Let us choose i large enough so that $d(0,Y_R) > t$ and put $s = d(0,Y_R)$. Let Y be the hyperplane orthogonal to the perpendicular from 0 to Y_R at a distance t from Y_R, and let r be the euclidean radius of the sphere supporting Y. Then we have $r^2+1 = (r+\tanh\frac{1}{2}(s-t))^2$ and hence $r = 1/\sinh(s-t)$. Therefore,

$$\operatorname{diam}(\operatorname{supp}(\psi_i)) \leqslant 2/\sinh(s-t).$$

For any $\delta > 0$ choose s_δ such that $\delta > 2/\sinh(s_\delta - t)$. Then there is K such that if $j > K$ then $\operatorname{diam}(\operatorname{supp}(\psi_j)) < \delta$. |

The final step is a straight forward application of the chain rule, together with (8) and Lemma 4·2.

LEMMA 4·6. *Let $R, Q \in \mathscr{P}$ and $\gamma \in \Gamma$ such that $\gamma(R) = Q$ and $R = q(i)$, $Q = q(j)$. Then* $\|D\psi_i(x).u\|_{\psi_i(x)} = \|D(\psi_j\gamma)(x).u\|_{\psi_j\gamma(x)}$. |

Lemmas 4·3, 4·4, 4·5 and 4·6 show that the sequence $\{\psi_j\}$ satisfies the conditions of Theorem 3·1.

THEOREM 4·7. *Let Γ be a discrete subgroup of G_{n-1} and P a bending hyperplane for Γ. If α and ζ satisfy* (7) *and*

$$\cosh\tfrac{1}{2}a > 1/\sin\zeta, \tag{9}$$

then the mapping $b_\zeta(\alpha): B^n \to B^n$ defined by

$$b_\zeta(\alpha) = \lim_{j\to\infty} \psi_1\psi_2\dots\psi_j$$

is a quasiconformal homeomorphism of B^n compatible with the group Γ. |

The homeomorphism $b_\zeta(\alpha)$ does not depend on the choice of the bijection $q: \mathbb{N} \to \mathscr{P}$, provided that q^{-1} is order preserving. The homomorphism

$$b(\alpha)_\#: \Gamma \to G_n: \gamma \mapsto b_\zeta(\alpha)\,\gamma\, b_\zeta(\alpha)^{-1}$$

does not depend on the choice of ζ, provided that ζ satisfies (7) and (9).

Let η satisfy $\cos\{\frac{1}{2}\eta) = 1/\cosh(\frac{1}{2}a)$, $0 < \eta < \pi$. Then for any θ such that $0 < \theta \leqslant \eta$ we have a family of quasiconformal homeomorphisms of B^n, compatible with the group Γ which define a quasiconformal deformation

$$\beta: (-\theta,\theta) \to \operatorname{Hom}(\Gamma, G_n): \alpha \mapsto (b(\alpha))_\#|\Gamma. \tag{10}$$

The fact that $\alpha \mapsto \beta(\alpha)(\gamma)$ is differentiable for every $\gamma \in \Gamma$ will follow from Theorem 5·5.

5. *Properties of bending*

A. If X is a set in B^n, then $\overline{X}$ will denote the closure of X in the closed unit ball.

Let Γ be a discrete subgroup of G_n and $x \in B^n$. Then the set $L(\Gamma) = \overline{\Gamma(x)} \cap S^{n-1}$ is independent of the choice of x and is called the *limit set* of Γ. The convex hull of $L(\Gamma)$ is denoted by $H(\Gamma)$. It is the smallest closed, convex, Γ-invariant subset of $\overline{B}^n$.

Let $* \in \{+,-\}$ and define $B^n_* = \{x \in B^n;\ \operatorname{sign}(x_n) = *\}$.

LEMMA 5·1. *If $\operatorname{sign}\alpha = *$, then $b(\alpha)(B^n_*)$ is convex.*

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Proof. Let b_0 be the identity on B^n and $b_j = \psi_1 \dots \psi_j$. Then $b_j(B^n_*)$ is the intersection of $b_{j-1}(B^n_*)$ and the half space containing the origin and bound by the $(n-1)$-plane containing $b_j(B^{n-1} \cap Y^+_{q(j)})$. Hence, if $b_{j-1}(B^n_*)$ is convex, so is $b_j(B^n_*)$. Therefore,

$$b(\alpha)(B^n_*) = \bigcap_{j=0}^{\infty} b_j(B^n_*)$$

is convex. |

We say that *the fixed points of the loxodromic elements* of a subgroup Γ of G_n *are dense in* $L(\Gamma) \times L(\Gamma)$ if for any pair of points x, x' in S^{n-1} and neighbourhoods U and U' of x, x' in S^{n-1}, there is a loxodromic element in Γ which has one fixed point in U and the other in U'.

PROPOSITION 5·2. [4]. *If Γ contains loxodromic elements, and the elements of Γ do not have a common fixed point in S^{n-1}, then the fixed points of the loxodromic elements of Γ are dense in $L(\Gamma) \times L(\Gamma)$.* |

THEOREM 5·3. *Let Γ be a discrete subgroup of G_{n-1} and P a bending hyperplane for Γ. Assume that Γ contains hyperbolic elements. Then the deformation defined by bending along P is trivial if and only if P does not intersect the interior of the convex hull $H(\Gamma)$.*

Proof. If $L(\Gamma) \subset \partial P$ then $\Gamma = \Gamma_P$. Therefore, assume that $L(\Gamma)$ is not contained in ∂P. We show that there is a loxodromic element whose axis intersects P transversely.

If the elements of Γ have a common fixed point, then all loxodromic elements have the same axis A, and $H(\Gamma) = A$. If A intersects P is must intersect it transversely.

If the elements of Γ do not have a common fixed point, then the fixed points of loxodromic elements are dense in $L(\Gamma) \times L(\Gamma)$. Since P intersects the interior of $H(\Gamma)$ but $H(\Gamma)$ is not contained in $\overline{P}$, there are points of $L(\Gamma)$ on both components of $S^{n-2} - \overline{P}$ and there is a loxodromic element $\gamma \in \Gamma$ whose fixed points lie on opposite sides of P and whose axis A therefore intersects P transversely.

Now, in either case, if $x \in A$, the distance $d(x, \gamma(x))$ is the minimum distance between any point in B^n and its translate by γ, and it is an invariant of the conjugacy class of γ in G_n. Let $x \in A - P$. Then $d(x, b(\alpha)_\#(\gamma)(x)) < d(x, \gamma(x))$ and therefore $(b(\alpha))_\#(\gamma)$ is not conjugate to γ.

To prove the converse, observe that if the bending locus $\Gamma(P)$ does not intersect the interior of $H(\Gamma)$ then $H(\Gamma)$ is contained in the closure of one component of $B^{n-1} - \Gamma(P)$. This component is then invariant by the group. But since the restriction of $b(\alpha)$ to any component of $B^{n-1} - \Gamma(P)$ is an isometry into B^n, the deformation defined by conjugation by $b(\alpha)$ is trivial. |

Definition. If $K \subset S^n$ and card $(K) \geqslant 2$, we say that K *spans* an m-sphere $S \subset S^n$ if S is the sphere of lowest dimension which contains K.

THEOREM 5·4. *Let Γ be a discrete subgroup of G_{n-1} and P_1, P_2 bending hyperplanes for Γ such that the deformations β_1, β_2 defined by bending along P_1, P_2 are not trivial. Assume that $L(\Gamma) \cap \partial P_i$ spans ∂P_i, $i = 1, 2$. If there are $\alpha_1 \neq 0$, $\alpha_2 \neq 0$ in the domain of definition of β_1, β_2 and $g \in G_n$ such that for all $\gamma \in \Gamma$*

$$\beta_1(\alpha_1)(\gamma) = g_\# \beta_2(\alpha_2)(\gamma) \tag{11}$$

then $\alpha_1 = \alpha_2$ and $P_2 \subset \Gamma(P_1)$.

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Proof. (11) implies that $(b_1(\alpha_1))^{-1} g b_2(\alpha_2)$ commutes with elements of Γ, and hence it fixes the points of $L(\Gamma)$. So if $x \in L(\Gamma)$,

$$g b_2(\alpha_2)(x) = b_1(\alpha_1)(x). \tag{12}$$

Since $\Gamma_{P_1} \neq \Gamma$, $L(\Gamma)$ contains points not in P_1 and therefore $L(\Gamma)$ spans S^{n-2}. Since $b_1(\alpha_1)|\partial(\gamma P_1)$, for $\gamma \in \Gamma$, is conformal, $b_1(\alpha_1)|L(\Gamma)$ determines $b_1(\alpha_1)|\Gamma(\partial P_1)$. But $b_1(\alpha_1)|B^{n-2}$ is determined by its restriction to $\Gamma(\partial P_1)$: if U is a component of $B^{n-1} - \Gamma(P_1)$ and R, Q are translates of P_1 adjacent to U, then U is mapped by $b_1(\alpha_1)$ conformally into the sphere spanned by $b_1(\alpha_1)(R)$ and $b_1(\alpha_1)(Q)$. Similarly $b_2(\alpha_2)|B^{n-1}$ is determined by $b_2(\alpha_2)|L(\Gamma)$. Therefore (12) holds for every $x \in B^{n-1}$.

In particular, g maps totally geodesic pieces of $b_2(\alpha_2)(B^{n-1})$ to totally geodesic pieces of $b_1(\alpha_1)(B^{n-1})$ and therefore $\alpha_1 = \alpha_2$. The mapping $(b_1(\alpha_1))^{-1} g b_2(\alpha_2)$ maps the set $\Gamma(P_2)$ to the set $\Gamma(P_1)$, and by (12), $\Gamma(P_2) = \Gamma(P_1)$. ∎

B. We shall give an alternative definition of the deformation β, using a decomposition of the group Γ into a generalized free product. This definition gives no way to determine the bounds of the domain of the deformation, but it is useful in computations.

If R is a $(n-2)$-plane in B^n we define the isometry $\sigma_R(\alpha): B^n \to B^n$ to be the unique isometry which is the identity on R and is conjugate in G_n to multiplication by the matrix

$$\begin{pmatrix} I & & 0 \\ & \cos\alpha & -\sin\alpha \\ 0 & & \\ & \sin\alpha & \cos\alpha \end{pmatrix}.$$

THEOREM 5·5. *Let Γ be a discrete subgroup of G_{n-1} and P a bending hyperplane for Γ. Let U_1 and U_2 be the components of $B^{n-1} - \Gamma(P)$ adjacent to P and assume that $U_2 \subset Y_P^+$. Let Γ_i be the subgroup of Γ leaving U_i invariant. Then we can express the deformation $\beta(\alpha)$ as follows:* (i) *If there is no element of Γ mapping U_1 to U_2, then Γ is an amalgamated product, $\Gamma = \Gamma_{1*_{\Gamma_P}} \Gamma_2$ and*

$$\beta(\alpha)(\gamma) = \gamma \qquad (\gamma \in \Gamma_1),$$
$$\beta(\alpha)(\gamma) = \sigma_P(\alpha)\, \gamma\, \sigma_P^{-1}(\alpha) \qquad (\gamma \in \Gamma_2).$$

(ii) *If there is an element $\gamma_0 \in \Gamma$ mapping U_1 to U_2 then Γ is generated by Γ_1, and γ_0, γ_0^{-1} conjugates $\Gamma_1 \cap \Gamma_P$ to a subgroup of Γ_1 and*

$$\beta(\alpha)(\gamma) = \gamma \qquad (\gamma \in \Gamma_1),$$
$$\beta(\alpha)(\gamma_0) = \sigma_P(\alpha)\, \gamma_0.$$

Proof. Since no element of Γ reverses the orientation of B^{n-1}, any element is uniquely determined by its restriction to U_1 or to U_2. Note that

$$b(\alpha)|U_1 = 1 \quad \text{and} \quad b(\alpha)|U_2 = \sigma_p(\alpha)|U_2. \quad \blacksquare$$

C. We shall extend the definition of the deformation to the case where we bend along several disjoint totally geodesic submanifolds of a hyperbolic manifold.

Let $\mathscr{P}$ be a countable Γ-invariant set of disjoint hyperbolic $(n-2)$-planes in B^{n-1}. Without loss of generality we can assume that $0 \notin \mathscr{P}$, and let Y_P^+ be the half space of P not containing 0. We order the elements of $\mathscr{P}$ with the relation $R \gg Q$ if $Y_R^+ \subset Y_Q^+$ and define a bijection $q: \mathbb{N} \to \mathscr{P}$ as in Section 4, such that q^{-1} is order preserving. We define

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a function $a: \mathscr{P} \to \mathbb{R}$ by $a(P) = \inf\{d(P, Q); Q \in \mathscr{P}, Q \neq P\}$ and assume that $a(P) > 0$ for all $P \in \mathscr{P}$. Finally we assume that $\mathscr{P}/\Gamma$ is finite and let $k = \text{card}(\mathscr{P}/\Gamma)$.

Let $\{e_i; i = 1, \ldots, k\}$ be the standard basis of $\mathbb{R}^k$ and choose a bijection between $\{e_i\}$ and the elements of $\mathscr{P}/\Gamma$. Let $\zeta: \mathscr{P} \to (0, \frac{1}{2}\pi)$ and $\alpha: \mathscr{P} \to (-\pi, \pi)$ be invariant functions. The functions a, ζ and α are constant on Γ-orbits. We shall write a_i to denote the value of a on the orbit corresponding to e_i, and similarly for ζ_i and α_i.

Define $U = \{u \in \mathbb{R}^k; \cos(\frac{1}{2}u_i) > 1/\cosh(\frac{1}{2}a_i), -\pi < u_i < \pi\}$. Let W be a neighbourhood of the origin in $\mathbb{R}^k$ such that $\overline{W} \in U$. Then there is a point y in U such that $y_i > 0$ for $i = 1, \ldots, k$ and the convex hull of the 2^k points $(\pm y_1, \ldots, \pm y_k)$ contains $\overline{W}$ in its interior. Let $\zeta: \mathscr{P} \to (0, \frac{1}{2}\pi)$ be the invariant function defined by $y_i = \pi - 2\zeta_i$. Then $\sin(\zeta_i) = \cos(\frac{1}{2}y_i) > 1/\cosh(\frac{1}{2}a_i)$. If $\alpha: \mathscr{P} \to (-\pi, \pi)$ is such that $(\alpha_1, \ldots, \alpha_k) \in W$ then $|\alpha(R)| < \pi - 2\zeta(R)$ for $R \in \mathscr{P}$. For each $R \in \mathscr{P}$ choose γ_R in G_{n-1} such that $\gamma_R(B^{n-2}) = R$ and $\gamma_R(\{x \in \overline{B}^n; x_{n-1} \geqslant 0\}) = Y_R^+$. Define $\psi_R = \gamma_R \psi^+_{\alpha(R), \zeta(R)} \gamma_R^{-1}$.

Using the results of Section 4 we can show that for every $w \in W$, the sequence $\{\psi_{q(i)}\}$ satisfies the conditions of Theorem 3·1. So it defines a family of quasiconformal homeomorphisms $b(w): B^n \to B^n$ compatible with Γ and hence a deformation

$$\beta: W \to \text{Hom}(\Gamma, G_n): w \mapsto b(w)_\#.$$

The proof of the following result is almost identical to that of Theorem 5·4.

PROPOSITION 5·6. *Let $p: \text{Hom}(\Gamma, G_n) \to \text{Hom}(\Gamma, G_n)/G_n$ be the quotient map for the action of G_n on the left by inner automorphisms. If $L(\Gamma) \cap \partial P$ spans ∂P for each $P \in \mathscr{P}$ and the deformation defined by bending along each $\Gamma(P)$ is not trivial, then*

$$p\beta: W \to \text{Hom}(\Gamma, G_n)/G_n$$

is injective. |

COROLLARY 5·7. *If Γ is a discrete torsion free subgroup of G_{n-1} such that the manifold B^{n-1}/Γ is compact and contains k disjoint totally geodesic connected submanifolds, then there is a k-parameter non-degenerate deformation of Γ in G_n.* |

A similar result has been proved by Johnson and Millson [5].

6. *Examples of bending*

I use the algebraic definition of bending to describe the deformation in the simplest non-trivial case, when Γ is an infinite cyclic group generated by a (strictly) hyperbolic element γ, and P is a hyperplane intersecting transversely the axis of γ. In this case there is a totally geodesic subspace of B^n of dimension 3 which is invariant by Γ and the bending homeomorphisms $b(\alpha)$. Therefore, we can restrict our attention to 3-dimensional hyperbolic space.

We shall work in $\mathscr{H}^3$. We can assume, without loss of generality, that γ is the transformation $x \mapsto e^s x$, with minimal displacement s, and that P is a geodesic in $\mathscr{H}^2$ intersecting the axis A of γ at the point $(1, 0, 0)$ in $\mathscr{H}^3$. Choose the orientation of A and the normal to P to be positive towards ∞. We define η to be the angle from the axis A to the normal to P, so that $-\frac{1}{2}\pi < \eta \leqslant \frac{1}{2}\pi$.

The rotation $\sigma_P(\alpha)$ is given by the matrix

$$\begin{pmatrix} \cos\frac{1}{2}\alpha + i\sin\eta\,\sin\frac{1}{2}\alpha & i\cos\eta\,\sin\frac{1}{2}\alpha \\ i\cos\eta\,\sin\frac{1}{2}\alpha & \cos\frac{1}{2}\alpha - i\sin\eta\,\sin\frac{1}{2}\alpha \end{pmatrix}.$$

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Then $\beta(\alpha)(\gamma) = \sigma_P(\alpha)\gamma$ is loxodromic and corresponds to the matrix in $SL(2,\mathbb{C})$

$$B(\alpha,\eta) = \begin{pmatrix} e^{\frac{1}{2}s}(\cos\frac{1}{2}\alpha + i\sin\eta\,\sin\frac{1}{2}\alpha) & i\,e^{-\frac{1}{2}s}\cos\eta\,\sin\frac{1}{2}\alpha \\ i\,e^{\frac{1}{2}s}\cos\eta\,\sin\frac{1}{2}\alpha & e^{-\frac{1}{2}s}(\cos\frac{1}{2}\alpha - i\sin\eta\,\sin\frac{1}{2}\alpha) \end{pmatrix}.$$

By differentiating the trace of this function we find that the minimal displacement $s(\alpha)$ has a maximum at $\alpha = 0$.

When A and P intersect orthogonally, $\eta = 0$ and $B(\alpha, 0)$ represents a hyperbolic element for values of α such that $\cos\frac{1}{2}\alpha > 1/\cosh\frac{1}{2}s$. For $\alpha = 2\,\mathrm{arccosh}\,(1/\cosh\frac{1}{2}s)$ the matrix $B(\alpha, 0)$ becomes parabolic. So in this case the domain of the deformation β determined in Section 4 is the best possible.

In contrast, when $\eta \neq 0$ the matrix $B(\alpha,\eta)$ represents a loxodromic element for $-\pi < \alpha \leqslant \pi$. The deformation can be obtained by the quasiconformal deformation $b(\alpha)$ for $|\alpha|$ in the range of values specified in Section 4. But it is possible to define the deformation by a different family of quasiconformal deformations for a larger range of $|\alpha|$.

The class of discrete subgroups of G_n with a bending hyperplane contains the examples described by Apanasov in [**1**]. I give here a description of such a group, correcting certain errors in [**1**] and [**6**].

Let C be a cube in $\mathbb{R}^3$ with side of length 1 and vertices $\pm\frac{1}{2}e_1 \pm \frac{1}{2}e_2 \pm \frac{1}{2}e_3$. Let $S_1, \ldots, S_8$ be eight spheres in $\mathbb{R}^3$ with centres at the vertices of C and radius $\sqrt{3}/3$; and $S_9, \ldots, S_{14}$ six spheres with centre $(5/6)e_1$, $-(5/6)e_1$, $(5/6)e_2$, $-(5/6)e_2$, $(5/6)e_3$, $-(5/6)e_3$ respectively and radius $\sqrt{10}/6$. If $j = 9, \ldots, 14$ then S_j is orthogonal to all the spheres S_i, $i = 1, \ldots, 8$, that it intersects. For $i = 1, \ldots, 8$, let L_i be the plane through the origin, the point e_3 and the centre of S_i. Define also $L_9 = L_{10} = \{x \in \mathbb{R}^3;\ x_3 = 0\}$ and for $j = 11, \ldots, 14$, $L_j = \{x \in \mathbb{R}^3;\ x_1 = 0\}$.

Let γ_i be the conformal mapping of $\mathbb{R}^3$ obtained by inversion in S_i and reflection in L_i. Let Γ be the group generated by $\gamma_1, \ldots, \gamma_{14}$. It leaves invariant the sphere S_0 with centre at the origin and radius $\sqrt{15}/6$. Γ is a subgroup of index four of the group generated by reflections in $S_1, S_9, S_{13}, L_1, L_9, L_{13}$.

Regard $\mathbb{R}^3$ as $\partial\mathscr{H}^4$ and denote by S_j^+, L_i^+ the totally geodesic 3-space in $\mathscr{H}^4$ bounded by S_j, L_i. The group Γ acts as a group of isometries of the 3-space S_0^+. Let $P_i = S_i^+ \cap S_0^+$. Then P_9 and P_{13} are two bending hyperplanes for Γ giving rise to distinct non trivial deformations, by bending along non-orientable surfaces. The distance between P_9 and one of its nearest translates, say P_{11}, is arc cosh $(3/2)$, while the distance between P_{13} and one of its nearest translates, say $\gamma_9(P_{11})$, is 2 arc cosh $(3/2)$. The easiest way to calculate the distance between them is to use the hyperboloid model for hyperbolic space [**10**].

I shall describe bending along the translates of P_{13}. By Theorem 5·5 the group $\beta(\alpha)(\Gamma)$ is generated by elements $\gamma_j(\alpha)$ where $\gamma_j(\alpha) = \gamma_j$ for $1 \leqslant j \leqslant 12$, and

$$\gamma_j(\alpha) = \sigma_{P_j}(\alpha)\gamma_j \quad \text{for} \quad j = 13, 14.$$

The transformation $\gamma_{13}(\alpha)$ is defined by reflection in L_{13} and inversion in the sphere $S_{13,\alpha}$ intersecting S_{13} along $S_{13} \cap S_0$ and making an angle $\frac{1}{2}\alpha$ with S_{13}. $S_{13,\alpha}$ has centre te_3 and radius r where

$$t = 5/(6 - 2\sqrt{6}\tan\tfrac{1}{2}\alpha) \quad \text{and} \quad r^2 = t^2 - t + (5/12).$$

It is easy to check that $S_{13,\alpha}$ is orthogonal to S_j, $j = 1, \ldots, 4$. The domain of the

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deformation given in Section 4 is in this case $|\alpha| < \alpha_\infty = 2\arccos(2/3)$. The values of t corresponding to $\pm\alpha_\infty$ are

$$t_+ = 5(1+\sqrt{(5/6)}) \quad t_- = 5(1-\sqrt{(5/6)}).$$

It is clear from Apanasov's description of the deformation that at these values the sphere $S_{13,\alpha}$ becomes tangent to S_9, S_{10}, S_{11}, S_{12} and hence also to its translate $\gamma_9(S_{14,\alpha})$.

7. *Infinitesimal deformations*

If J is an open interval containing 0, Γ is a finitely generated group and

$$k : J \to \mathrm{Hom}(\Gamma, G)$$

is a deformation, then Weil[**12**] shows that the mapping $c_k : \gamma \mapsto d/dt(k(t)(\gamma))$ is a crossed homomorphism for Γ with values in $\mathfrak{g}$, the Lie algebra of G regarded as a Γ-module under the adjoint action of $k(0)(\Gamma)$.

If k and k' are conjugate deformations then $c_k - c_{k'}$ is a principal crossed homomorphism (a 1-coboundary).

Definition. An *infinitesimal deformation* of the injective homomorphism $k(0) : \Gamma \to G$ is a cohomology class in $H^1(\Gamma, \mathfrak{g})$.

When $G = G_n$ and $k(0)$ is the inclusion of a discrete finitely generated subgroup Γ of G_n we want to consider the case of a quasiconformal deformation, obtained by conjugation with a family of quasiconformal homeomorphisms f_t compatible with the group.

Let G_n act on the set of continuous vector fields on B^n by

$$\gamma.\eta(x) = D\gamma(\gamma^{-1}(x)).\eta(\gamma^{-1}(x)).$$

Define $\Xi(x) = d/dt(f_t(x))$. Then the compatibility of f_t implies that, if $\gamma \in \Gamma$, $\Xi(x) - \gamma.\Xi(x)$ is an infinitesimal isometry of B^n [**6**]. The infinitesimal deformation corresponding to the deformation (f_t) is then the cohomology class of the crossed homomorphism $c(\gamma) : x \mapsto \Xi(x) - \gamma.\Xi(x)$.

Consider now the infinitesimal deformation corresponding to bending. Start by calculating the vector field corresponding to the family of homeomorphisms $b(\alpha)$.

Recall that $\iota(z) = -\bar{z}$. Then from the definition of ϕ we find that

$$d/d\alpha(\phi_\alpha(z)) - d/d\alpha(\iota\phi_\alpha\iota(z)) = iz$$

for $z \in \mathbb{C}$, and hence

$$\frac{d}{d\alpha}(\psi_\alpha^+(x)) - \frac{d}{d\alpha}(\psi_\alpha^-(x)) = (0, \ldots, 0, -x_n, x_{n-1}). \tag{13}$$

Define two vector fields, $\xi : x \mapsto d/d\alpha(\psi_\alpha^+(x))$ and $\chi : x \mapsto (0, \ldots, 0, -x_n, x_{n-1})$. For $R \in \mathscr{P}$ and γ_R as defined in Section 4, let $\psi_{R,\alpha} = \gamma_R \psi_\alpha^+ \gamma_R^{-1}$. Then

$$\frac{d}{d\alpha}(\psi_{R,\alpha}(x)) = \gamma_R.\xi(x) \quad \text{and therefore} \quad \Xi(x) = \frac{d}{d\alpha}(b(\alpha)(x)) = \sum_{R\in\mathscr{P}} \gamma_R.\xi(x).$$

The infinitesimal deformation of bending is the class of c: $\Gamma \to \mathfrak{g}$ defined by

$$c(\delta)(x) = \frac{d}{d\alpha}(\beta_\alpha(\delta)(x)) = \Xi(x) - \delta.\Xi(x)$$

$$= \sum_{R\in\mathscr{P}} \gamma_R.\xi(x) - (\delta\gamma_R).\xi(x).$$

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Taking into consideration the action of Γ on $\mathscr{P}$, we have

$$c(\delta)(x) = \sum_{R \in \mathscr{P}} \gamma_R . \xi(x) - (\delta \gamma_{\delta^{-1}(R)}) . \xi(x).$$

If $\delta(Y^+_{\delta^{-1}(R)}) = Y^+_R$, then $\gamma_R^{-1} \delta \gamma_{\delta^{-1}(R)}$ preserves the orientation of B^{n-2} and

$$(\delta \gamma_{\delta^{-1}(R)}) . \xi(x) = \gamma_R . \xi(x).$$

Otherwise $\gamma_R^{-1} \delta \gamma_{\delta^{-1}(R)}$ reverses the orientation of B^{n-2} and, from Lemma 4·2,

$$(\delta \gamma_{\delta^{-1}(R)}) . \xi(x) = \gamma_R . \left(\frac{d}{d\alpha} (\psi_\alpha^-) \right)(x).$$

So in this case (13) implies that $\gamma_R . \xi(x) - (\delta \gamma_{\delta^{-1}(R)}) . \xi(x) = \gamma_R . \chi(x)$.

PROPOSITION 7·1. *Let Γ be a finitely generated discrete subgroup of G_n and P a bending hyperplane for Γ. The infinitesimal bending deformation along P is represented by the crossed homomorphism*

$$c(\delta)(x) = \sum_{\substack{R \in \mathscr{P} \\ \delta(Y^+_{\delta^{-1}(R)}) \neq Y^+_R}} \gamma_R . \chi(x). \quad \blacksquare$$

Johnson and Millson prove in [5] that when Γ is torsion free, cocompact and $n \geqslant 5$, if P and Q are bending hyperplanes for Γ such that $P \cap Q \neq \varnothing$, and c_P, c_Q are the corresponding infinitesimal deformations then $c_P + c_Q$ is not necessarily an integrable infinitesimal deformation of Γ in G_n.

This work is part of a Ph.D. thesis [6] and was carried out at King's College London, under the supervision of Luke Hodgkin. While I was at King's College I was financially supported for two years by the University of London with a University Postgraduate Scholarship.

I am grateful to Bill Harvey, who suggested the problem and helped to clarify many ideas in countless conversations.

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