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Workspace analytic determination of two similar translational parallel manipulators

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SUMMARY

The design of a parallel manipulator for a given workspace would be greatly facilitated, if the analytic expressions of the hypersurfaces bounding the workspace were available. In translational parallel manipulators (TPMs), the hypersurfaces bounding the workspace are actually three-dimensional surfaces, which are the geometric locus of all the positions assumed by an end-effector point when the TPM reaches the workspace borders. These surfaces can be represented in a Cartesian reference system fixed to the frame. This paper studies the workspace of two TPMs, that have the same closure equations and workspace when a few geometric conditions are satisfied. The two TPMs are the translational 3-RUU and the DELTA robot. The kinematic analyses of these two TPMs are different from one another only when velocity and accelerations are considered. The result of this study is that the analytic expression of the surfaces bounding their workspace is a fourth degree polynomial equation, in the coordinates of a plattorm point, which contains all the manipulator geometric parameters. This analytic expression is given in an explicit form. The use of this expression is illustrated through a numerical example.

KEYWORDS: Manipulators; Kinematics; Position analysis; Workspace.

INTRODUCTION

The interest for parallel manipulators (PMs) arises from the fact that they exhibit high stiffness in nearly all configurations and that they often allow the actuators to be put on the frame, which reduces the moving masses to advantage of the dynamic performances.

Parallel architectures have the end-effector (platform) connected to the frame (base) through a number of kinematic chains (legs). Their kinematic analysis is often difficult to address. Nevertheless, many theoretical problems related to it have been solved.[1–6]

Translational parallel manipulators (TPMs) are a relevant family of PMs. TPMs are PMs with three degrees of freedom (dof) that make the platform translate with respect to the base, i.e. they keep the platform orientation constant during motion. As a consequence, the reachable workspace of a TPM is a constant orientation workspace and it can be represented on a three-dimensional Cartesian space (platform's Cartesian space) whose coordinates are the coordinates of a platform point, measured in a reference system fixed to the base. A TPM drawback is the reduced size of the workspace.

The determination of the manipulator workspace is an important stage of the manipulator design. The design of a parallel manipulator for a given workspace would be greatly facilitated, if the analytic expressions of the hypersurfaces bounding the workspace were available. Some researchers[7,8] showed that these analytic expressions can be obtained from the inverse position analysis (IPA) of the manipulator. In fact, such hypersurfaces are the geometric locus of the points of the end-effector space (six-dimensional space whose coordinates are three parameters locating the position of an end-effector point and three parameter locating the end-effector orientation with respect to the frame) corresponding to the end-effector poses in which the number of distinct real solutions of the IPA is different from zero and less than the maximum number of real solutions admitted by the manipulator IPA.[8–10]

In the TPMs, the hypersurfaces bounding the workspace are actually three-dimensional surfaces which are the locus of all the positions assumed by a platform point when the TPM reaches the workspace borders. These surfaces can be represented in a Cartesian reference system fixed to the base.

This paper studies the workspace of two TPMs, that have the same closure equations and workspace when a few geometric conditions are satisfied. The two TPMs are the translational 3-RUU[11] and the DELTA robot.[12–14] The kinematic analyses of these two TPMs are different from one another only when velocity and accelerations are considered.[11,13,14]

The result of this study is that the analytic expression of the surfaces bounding their workspace is a fourth degree polynomial equation, in the coordinates of a platform point, which contains all the manipulator geometric parameters. This analytic expression is given in explicit form. The use of this expression is illustrated through a numerical example.

THE TRANSLATIONAL 3-RUU AND THE DELTA ROBOT

Manipulators of type 3-RUU (Figure 1) are parallel manipulators with three dof in which platform and base are connected by three legs of type RUU (R and U stand for revolute pair and universal joint, respectively). The transla-

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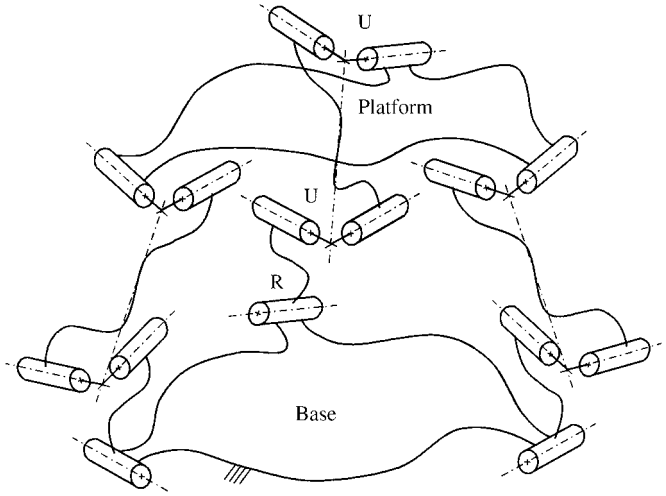


Fig. 1. 3-RUU manipulator.

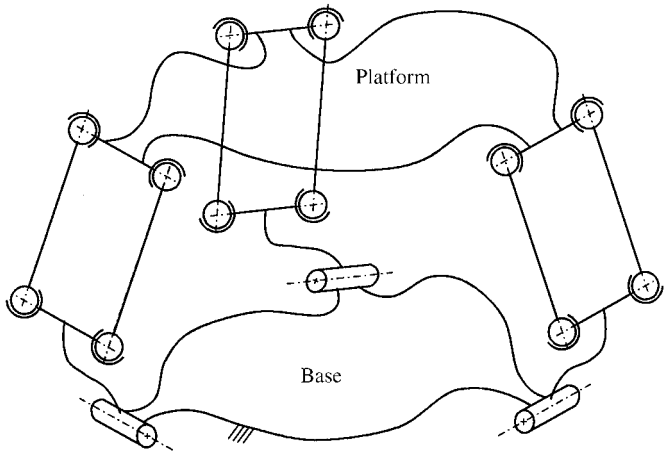


Fig. 3. DELTA robot.

tional 3-RUU (Figure 2) is a 3-RUU manipulator satisfying the following geometric conditions:[11]

(i) in each leg, the axes of the two revolute pairs of the link adjacent to the base are parallel to one another and to the axis of the revolute pair joining the leg to the platform;
(ii) in each leg, the axes of the two revolute pairs of the link connecting the two universal joint are parallel to one another and perpendicular to the axes of all the other revolute pairs of the leg.

When the translational 3-RUU does not assume some singular configurations, named rotation singularities,[11] and its revolute pairs adjacent to the base are actuated, it makes the platform translate with respect to the base.

The DELTA robot (Figure 3) is a TPM in which the platform is connected to the base by means of three legs, each one constituted of three links: the first link connected to the base by a revolute pair, and the other two connected to the first link and to the platform by means of spherical pairs. The centers of the four spherical pairs, present in each leg, are located at the vertices of a parallelogram, which has the side fixed to the first link and the side fixed to the platform parallel to one another, and to the axis of the revolute pair. When the three revolute pairs are actuated the platform translates with respect to the base provided that the DELTA robot does not assume particular singular configurations (rotation singularities).[14]

Figure 4 shows the i-th leg of the DELTA robot and the notation that will be used. With reference to Figure 4, points D_{ji}, $j=1, 2$, are the centers of the spherical pairs adjacent to the link 1_i and points E_{ji}, $j=1, 2$, are the centers of the spherical pairs adjacent to the platform; points B_i and C_i are the middle point of the segments $D_{1i}D_{2i}$ and $E_{1i}E_{2i}$ respectively; point A_i is the foot of the perpendicular through B_i to the axis of the revolute pair.

Moreover, r_i is the length of the segment A_iB_i, q_i is the length of the two rods 2_i and 3_i and h_i is the length of the segments $D_{1i}D_{2i}$ and $E_{1i}E_{2i}$. The vector $\mathbf{w}_{0i}$ is the unit vector of the revolute pair axis. The vectors $\mathbf{w}_{1i}$, $\mathbf{w}_{3i}$ and $\mathbf{w}_{4i}$ are the unit vectors $(\mathbf{D}_{2i}-\mathbf{D}_{1i})/h_i$, $(\mathbf{E}_{2i}-\mathbf{D}_{1i})/|\mathbf{E}_{2i}-\mathbf{D}_{1i}|$ and $(\mathbf{E}_{2i}-\mathbf{E}_{1i})/h_i$, respectively. $\mathbf{w}_{2i}$ is the unit vector perpendicular to the plane of the triangle $D_{1i}D_{2i}E_{2i}$ and $\mathbf{w}_{5i}$ is the unit vector perpendicular to the plane of the triangle $D_{1i}E_{1i}E_{2i}$.

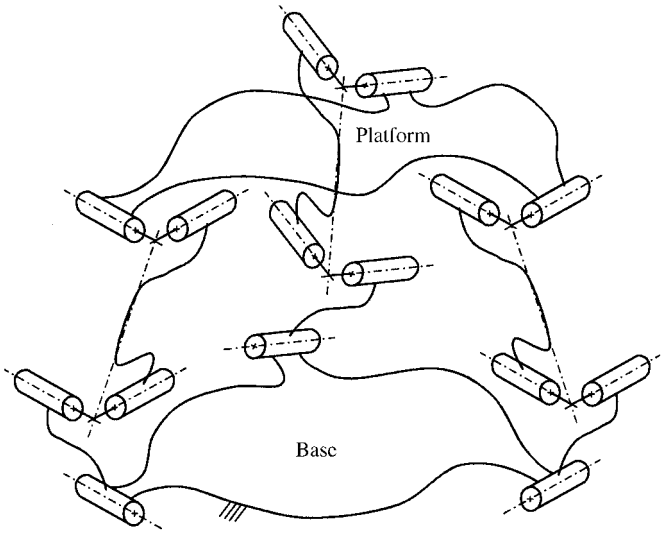


Fig. 2. Translational 3-RUU.

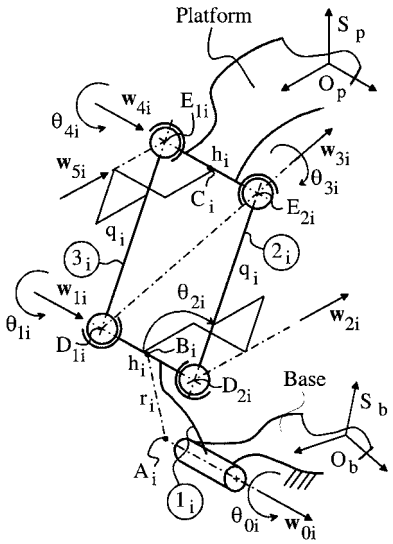


Fig. 4. The i-th leg of the DELTA robot.

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The configuration of the i-th leg of the DELTA robot can be assigned by giving the values of five angles (Figure 4): the joint variable θ_{0i} of the revolute pair, the three angles θ_{1i}, θ_{2i} and θ_{3i} defined as follows:

$$\sin\theta_{1i}=\frac{(\mathbf{A}_i-\mathbf{B}_i)\times\mathbf{w}_{2i}\cdot\mathbf{w}_{1i}}{r_i},\qquad \cos\theta_{1i}=\frac{(\mathbf{A}_i-\mathbf{B}_i)\cdot\mathbf{w}_{2i}}{r_i} \tag{1.1}$$

$$\sin\theta_{2i}=\frac{(\mathbf{D}_{1i}-\mathbf{D}_{2i})\times(\mathbf{E}_{2i}-\mathbf{D}_{2i})\cdot\mathbf{w}_{2i}}{q_ih_i},\quad \cos\theta_{2i}=\frac{(\mathbf{D}_{1i}-\mathbf{D}_{2i})\cdot(\mathbf{E}_{2i}-\mathbf{D}_{2i})}{q_ih_i} \tag{1.2}$$

$$\sin\theta_{3i}=\mathbf{w}_{2i}\times\mathbf{w}_{5i}\cdot\mathbf{w}_{3i},\qquad \cos\theta_{3i}=\mathbf{w}_{2i}\cdot\mathbf{w}_{5i} \tag{1.3}$$

and the angle θ_{4i} which defines the position of the triangle $D_{1i}E_{1i}E_{2i}$ with respect to the platform.

With these notations, the geometric conditions that assure the translation of the platform with respect to the base in the DELTA robot can be written as follows:

$$\mathbf{w}_{0i}=\mathbf{w}_{1i}=\mathbf{w}_{4i} \tag{2.1}$$

$$\theta_{3i}=0 \tag{2.2}$$

When conditions (2) are satisfied the kinematic chain shown in Figure 4 behaves like the kinematic chain of type RUU represented in Figure 5 which satisfies the following geometric conditions:

$$\mathbf{w}_{0i}=\mathbf{w}_{1i}=\mathbf{w}_{4i} \tag{3.1}$$

$$\mathbf{w}_{2i}=\mathbf{w}_{5i} \tag{3.2}$$

$$\mathbf{w}_{0i}\perp\mathbf{w}_{2i} \tag{3.3}$$

$$A_iB_i=r_i \tag{3.4}$$

$$B_iC_i=q_i \tag{3.5}$$

Conditions (3.1) ÷ (3.3) are the geometric conditions (i) and (ii) of the translational 3-RUU, whereas conditions (3.4) and (3.5) are conditions on the sizes of the links. Therefore, provided that a DELTA robot satisfies the conditions for pure translation (conditions (2)), it is equivalent to a translational 3-RUU whose links satisfy conditions (3.4) and (3.5) and vice versa.

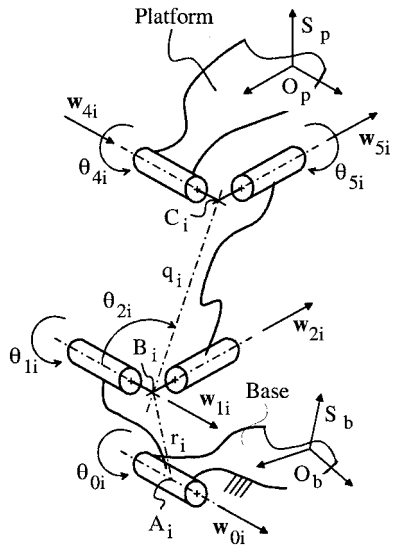


Fig. 5. The i-th leg of the translational 3-RUU corresponding to the DELTA leg of Figure 4.

This equivalence between DELTA robot and translational 3-RUU does not hold any longer when the two manipulators assume configurations that do not satisfy their conditions for the pure translation. In fact, when conditions (2) are not satisfied in the DELTA robot, there is at least one leg in which the unit vector $\mathbf{w}_{2i}$ is not parallel to the unit vector $\mathbf{w}_{5i}$ (see Figure 4), whereas, in the 3-RUU, the unit vectors $\mathbf{w}_{2i}$, i=1, 2, 3, are always parallel to the corresponding unit vectors $\mathbf{w}_{5i}$, i=1, 2, 3, (the two unit vectors belong to the same link (see Figure 5)).

When the position analyses of a DELTA robot and of a translational 3-RUU which satisfy conditions (3.4) and (3.5) take into consideration only the pure translation configurations, the two manipulators can be dealt with as they were the same manipulator. In particular, the reachable workspaces, collecting only the pure translation configurations, are the same for both the manipulators. Hereafter, all the analyses to obtain the analytic expression of the workspace both of the DELTA robot and of the translational 3-RUU will be referred to the translational 3-RUU.

ANALYTIC DETERMINATION OF THE WORKSPACE

Figure 5 shows the i-th leg of a translational 3-RUU. With reference to Figure 5, S_b is a Cartesian reference system, fixed to the base, whose origin is the base point O_b, and S_p is a Cartesian reference system, fixed to the platform, whose origin is the platform point O_p. The coordinates of the point C_i (see Figure 5), measured in S_b, are given by the following vector relationship:

$$\mathbf{C}_i=\mathbf{O}_p+\mathbf{R}_{bp}{}^p(\mathbf{C}_i-\mathbf{O}_p)\quad i=1,2,3 \tag{4}$$

where $\mathbf{R}_{bp}$ is the rotation matrix transforming the components of a vector measured in S_p into the components of the same vector measured in S_b. The left-hand superscript p on the symbol of a vector indicates that the vector is measured in S_p, whereas the vectors without left-hand superscript are measured in S_b.

Since the platform translate with respect to the base, the term $\mathbf{R}_{bp}{}^p(\mathbf{C}_i-\mathbf{O}_p)$ appearing in relationship (4) is a constant vector in S_b. Therefore, relationships (4) show that, in S_b, the coordinates of C_i depend only on the coordinates of the platform point O_p. The coordinates of the platform point O_p measured in S_b will be assumed as the generalized coordinates of the manipulator.

The notations of Figure 5 allow one to write the manipulator constraint equations as follows:

$$(\mathbf{B}_i-\mathbf{A}_i)\cdot\mathbf{w}_{0i}=0\quad i=1,2,3 \tag{5.1}$$

$$(\mathbf{B}_i-\mathbf{A}_i)^2=r_i^2\quad i=1,2,3 \tag{5.2}$$

$$(\mathbf{C}_i-\mathbf{B}_i)^2=q_i^2\quad i=1,2,3 \tag{5.3}$$

Since the coordinates of point C_i can be expressed by means of relationship (4), Eqs. (5) are a system of nine equations in twelve unknowns: the coordinates of the three points B_i,

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$i=1,2,3$, and the coordinates of the point O_p. The i-th triplet of equations (5) is the analytic expression of the constraint imposed on the platform by the i-th leg.

The inverse position analysis (IPA) of the translational 3-RUU consists in the determination of the joint variables of the actuated joints once the coordinates of the platform point O_p are given. The actuated joints of the translational 3-RUU are the three revolute pairs adjacent to the base whose joint variables are the angles θ_{0i}, $i=1,2,3$ (see Figure 5). Since, in the i-th leg, a one-to-one correspondence exists between the value of the angle θ_{0i} and the position of the point B_i, the IPA of the translational 3-RUU can be solved by computing the coordinates of point B_i for $i=1,2,3$. Such a computation can be implemented by solving the i-th triplet of Eqs. (5) for $i=1,2,3$, where the coordinates of point C_i are calculated by using the i-th relationship (4).

Equations (5) can be written in a different way as follows:

$$[(\mathbf{B}_i-\mathbf{C}_i)+(\mathbf{C}_i-\mathbf{A}_i)]\cdot\mathbf{w}_{0i}=0 \quad i=1,2,3 \qquad (6.1)$$

$$[(\mathbf{B}_i-\mathbf{C}_i)+(\mathbf{C}_i-\mathbf{A}_i)]^2=r_i^2 \quad i=1,2,3 \qquad (6.2)$$

$$(\mathbf{B}_i-\mathbf{C}_i)^2=q_i^2 \quad i=1,2,3 \qquad (6.3)$$

Without losing generality, the vector $(\mathbf{B}_i-\mathbf{C}_i)$ can be expressed as follows:

$$(\mathbf{B}_i-\mathbf{C}_i)=k_i\,\mathbf{w}_{0i}+m_i\frac{\mathbf{w}_{0i}\times(\mathbf{C}_i-\mathbf{A}_i)}{|\,\mathbf{w}_{0i}\times(\mathbf{C}_i-\mathbf{A}_i)\,|}+n_i(\mathbf{C}_i-\mathbf{A}_i)\ i=1,2,3 \qquad (7)$$

where k_i, m_i and n_i, $i=1,2,3$, are scalar coefficients, whereas $\mathbf{w}_{0i}$ and $(\mathbf{C}_i-\mathbf{A}_i)$ are vectors whose components are known in S_b (note that, in the IPA, the C_i coordinates are known in S_b).

The introduction of the i-th expression (7) in the i-th triplet of Eqs. (6) yields, after some rearrangements, the following equations:

$$k_i+(n_i+1)[\mathbf{w}_{0i}\cdot(\mathbf{C}_i-\mathbf{A}_i)]=0 \quad i=1,2,3 \qquad (8.1)$$

$$q_i^2+(\mathbf{C}_i-\mathbf{A}_i)^2+2\{n_i(\mathbf{C}_i-\mathbf{A}_i)^2+k_i[\mathbf{w}_{0i}\cdot(\mathbf{C}_i-\mathbf{A}_i)]\}=r_i^2 \quad i=1,2,3 \qquad (8.2)$$

$$k_i^2+m_i^2+n_i^2(\mathbf{C}_i-\mathbf{A}_i)^2+2\,k_i\,n_i[\mathbf{w}_{0i}\cdot(\mathbf{C}_i-\mathbf{A}_i)]=q_i^2 \quad i=1,2,3 \qquad (8.3)$$

where k_i, m_i and n_i, $i=1,2,3$, replace the B_i coordinates in the role of unknowns.

The i-th triplet of Eqs. (8) is a system which comprises two linear equations (Eqs. (8.1) and (8.2)), in the two unknowns k_i and n_i, and one quadratic equation (Eq. (8.3)), in the three unknowns k_i, m_i and n_i. By solving Eqs. (8), the following relationships come out

$$k_i=\frac{-\mathbf{w}_{0i}\cdot(\mathbf{C}-\mathbf{A}_i)[r_i^2-q_i^2+(\mathbf{C}_i-\mathbf{A}_i)^2]}{2\{(\mathbf{C}_i-\mathbf{A}_i)^2-[\mathbf{w}_{0i}\cdot(\mathbf{C}_i-\mathbf{A}_i)]^2\}} \quad i=1,2,3 \qquad (9.1)$$

$$n_i=\frac{r_i^2-q_i^2-(\mathbf{C}_i-\mathbf{A}_i)^2+2[\mathbf{w}_{0i}\cdot(\mathbf{C}_i-\mathbf{A}_i)]^2}{2\{(\mathbf{C}_i-\mathbf{A}_i)^2-[\mathbf{w}_{0i}\cdot(\mathbf{C}_i-\mathbf{A}_i)]^2\}} \quad i=1,2,3 \qquad (9.2)$$

$$m_i=\pm\sqrt{q_i^2-k_i^2-n_i^2(\mathbf{C}_i-\mathbf{A}_i)^2-2\,k_i\,n_i\,\mathbf{w}_{0i}\cdot(\mathbf{C}_i-\mathbf{A}_i)} \quad i=1,2,3 \qquad (9.3)$$

Relationships (9) show that the inverse position analysis of the translational 3-RUU admits real solutions when all the three coefficients m_i, $i=1,2,3$, given by relationships (9.3), are real. This condition is verified when the radicands of the square roots present in the right-hand sides of expressions (9.3) are greater than or equal to zero, i.e. when the following inequality system is satisfied

$$q_i^2-k_i^2-n_i^2(\mathbf{C}_i-\mathbf{A}_i)^2-2\,k_i\,n_i[\mathbf{w}_{0i}\cdot(\mathbf{C}_i-\mathbf{A}_i)]\geq 0 \quad i=1,2,3 \qquad (10)$$

where the scalar coefficients k_i and n_i, $i=1,2,3$, are given by relationships (9.1) and (9.2), respectively, and the position vectors $\mathbf{C}_i$, $i=1,2,3$, are given by relationships (4).

The only unknowns which appear in system (10) are the coordinates of the platform point O_p (see expressions (4) of $\mathbf{C}_i$) measured in S_b; all the other parameters which appear in system (10) are geometric constants of the manipulator. System (10) solutions constitute the geometric locus of all the positions the platform point O_p can reach, and are represented by a fixed volume in S_b. As a consequence, system (10) is the analytic expression of the workspace of the translational 3-RUU. The analytic expressions of the surfaces that are the workspace borders are the following equations

$$q_i^2-k_i^2-n_i^2(\mathbf{C}_i-\mathbf{A}_i)^2-2\,k_i\,n_i[\mathbf{w}_{0i}\cdot(\mathbf{C}_i-\mathbf{A}_i)]=0 \quad i=1,2,3 \qquad (11)$$

The introduction of expressions (9.1), (9.3) and (4) into Eqs. (11) gives, after some rearrangements, Eqs. (A.6) (see the Appendix), which are polynomial equations of the fourth degree in the coordinates (x, y, z) of point O_p, and explicitly contain all the constant geometric parameters of the manipulator. Since any fourth-degree polynomial equation in one unknown can be solved in closed form, the fourth-degree polynomial equations in three unknowns of type $f(x,y,z)=0$ can always be put in the explicit form $z=g(x,y)$. As a consequence, the geometric representation of the solutions of Eqs. (11) in S_b is not difficult to obtain starting from Eqs. (A.6).

In the next section an example of application of the found relationships will be given with reference to a real manipulator.

EXAMPLE OF APPLICATION

The translational 3-RUU that will be considered is geometrically defined as follows (see Figures 5 and 6): (a) the revolute pair axes fixed to the base (platform) are coplanar and intersect one another in three points located at the vertices of an equilateral triangle whose sides have the length s (see Figure 6); (b) $q_i=r_i=1$ [l.u.] (l.u. stands for arbitrary length unit); (c) reference system S_b (S_p) has the origin in the centroid of the base (platform) triangle, x axis

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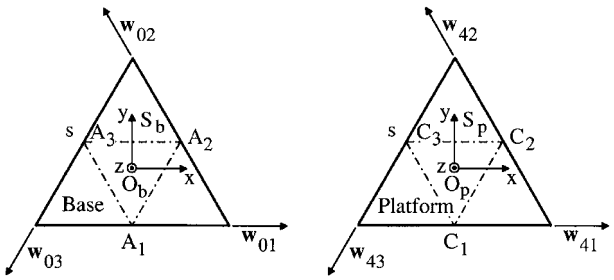


Fig. 6. Geometry of the base and of the platform.

parallel to and oriented according to $\mathbf{w}_{01}$ ($\mathbf{w}_{41}$), y axis lying on the base (platform) triangle and oriented as shown in Figure 6 and z axis chosen according to the right-hand rule.

With this geometry, the vectors $\mathbf{w}_{0i}$, $\mathbf{A}_i$ and $\mathbf{C}_i$, $i=1,2,3$, measured in S_b, have the following expressions:

$$\mathbf{w}_{01}=\{1,0,0\}^T,\ \mathbf{w}_{02}=\left\{-\frac{1}{2},\frac{\sqrt{3}}{2},0\right\}^T,$$

$$\mathbf{w}_{03}=\left\{-\frac{1}{2},\frac{\sqrt{3}}{2},0\right\}^T; \tag{12.1}$$

$$\mathbf{A}_1=\left\{0,\ -\frac{\sqrt{3}}{2}s,0\right\}^T,\ \mathbf{A}_2=\left\{\frac{s}{4},\ -\frac{\sqrt{3}}{12}s,0\right\}^T,$$

$$\mathbf{A}_3=\left\{-\frac{s}{4},\frac{\sqrt{3}}{12}s,0\right\}^T; \tag{12.2}$$

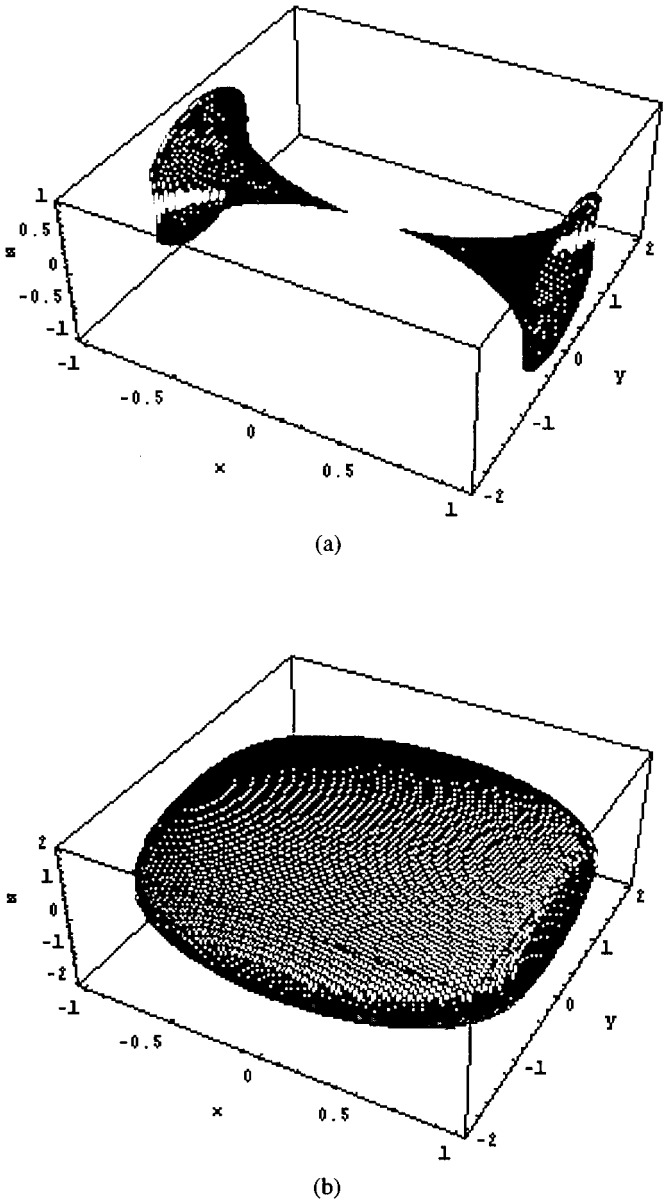


Fig. 7. Surfaces which are the geometric locus of the solutions of Equations (15): (a) solutions of Equation (15.1); (b) solutions of Equation (15.2).

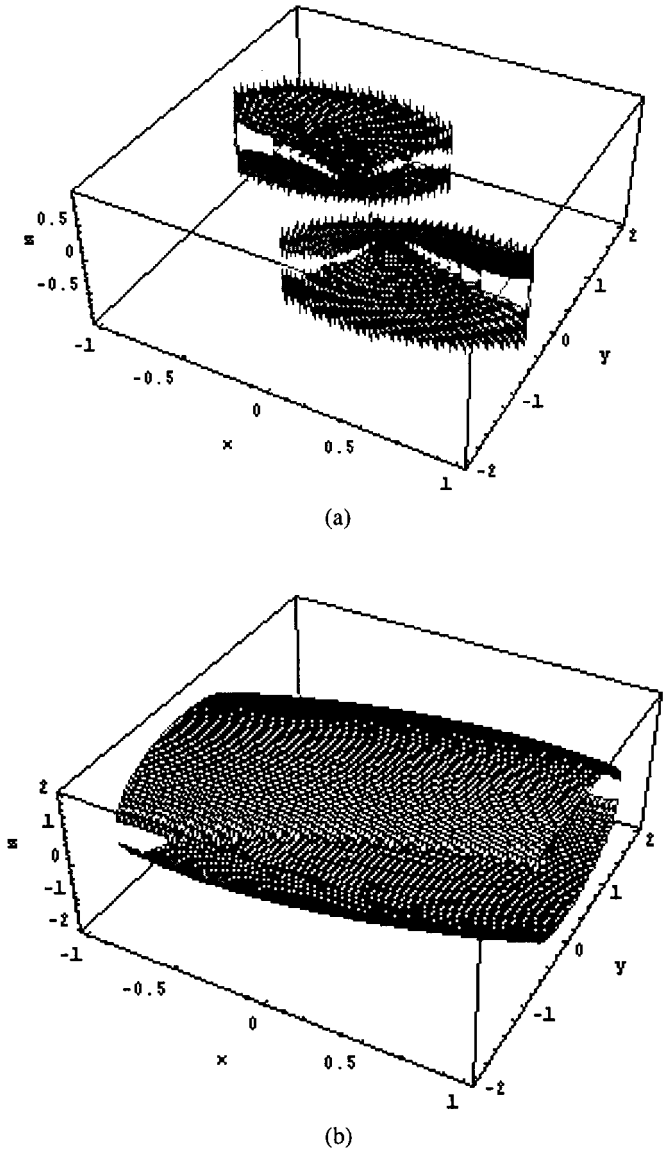


Fig. 8. Surfaces which are the geometric locus of the solutions of Equations (16): (a) solutions of Equation (16.1); (b) solutions of Equation (16.2).

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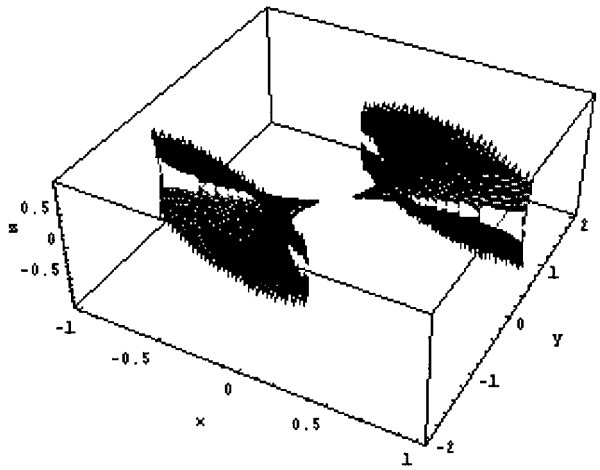
$$\mathbf{C}_1=\left\{x,\ y-\frac{\sqrt{3}}{6}s,\ z\right\}^T,\ \mathbf{C}_2=\left\{x+\frac{s}{4},\ y+\frac{\sqrt{3}}{12}s,\ z\right\}^T,$$

$$\mathbf{C}_3=\left\{x-\frac{s}{4},\ y+\frac{\sqrt{3}}{12}s,\ z\right\}^T; \tag{12.3}$$

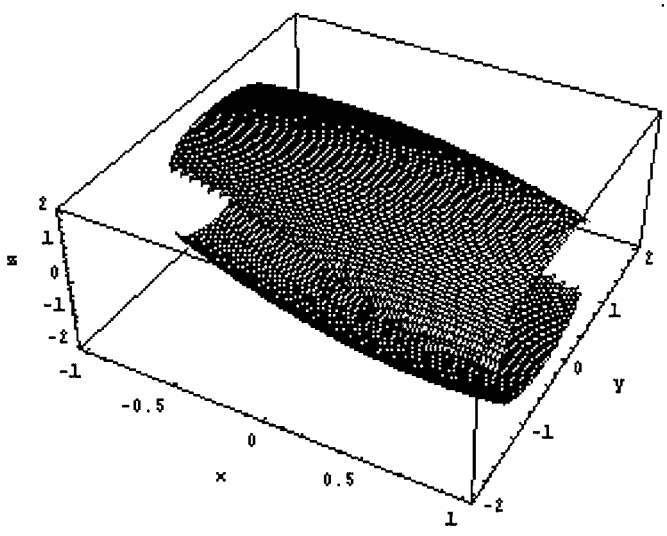
where x, y and z are the coordinates of the origin O_p of S_p, measured in S_b. The introduction of expressions (12) into relationships (10) yields

$$-x^4+4y^2-2x^2y^2-y^4+4z^2-2x^2z^2 -2y^2z^2-z^4\geq 0 \tag{13.1}$$

$$3x^2-x^4+2\sqrt{3}xy+y^2-2x^2y^2-y^4+4z^2 -2x^2z^2-2y^2z^2-z^4\geq 0 \tag{13.2}$$



(a)



(b)

Fig. 9. Surfaces which are the geometric locus of the solutions of Equations (17): (a) solutions of Equation (17.1); (b) solutions of Equation (17.2).

$$3x^2-x^4-2\sqrt{3}xy+y^2-2x^2y^2-y^4+4z^2 -2x^2z^2-2y^2z^2-z^4\geq 0 \tag{13.3}$$

Therefore, the analytic expressions of the surfaces which delimit the manipulator workspace (Equations (11)) are the following ones

$$-x^4+4y^2-2x^2y^2-y^4+4z^2-2x^2z^2 -2y^2z^2-z^4=0 \tag{14.1}$$

$$3x^2-x^4+2\sqrt{3}xy+y^2-2x^2y^2-y^4+4z^2 -2x^2z^2-2y^2z^2-z^4=0 \tag{14.2}$$

$$3x^2-x^4-2\sqrt{3}xy+y^2-2x^2y^2-y^4+4z^2 -2x^2z^2-2y^2z^2-z^4=0 \tag{14.3}$$

By solving Equation (14.1) with respect to z, Equation (14.1) splits into the following four equations in explicit form

$$z=\pm\sqrt{2-x^2-2\sqrt{1-x^2-y^2}} \tag{15.1}$$

$$z=\pm\sqrt{2-x^2+2\sqrt{1-x^2-y^2}} \tag{15.2}$$

The surfaces corresponding to Equations (15) are reported in Figure 7.

By solving Equation (14.2) with respect to z, Equation (14.2) splits into the following four equations in explicit form

$$z=\pm\sqrt{2-x^2-y^2-\sqrt{4-x^2+2\sqrt{3}xy-3y^2}} \tag{16.1}$$

$$z=\pm\sqrt{2-x^2-y^2+\sqrt{4-x^2+2\sqrt{3}xy-3y^2}} \tag{16.2}$$

The surfaces corresponding to Equations (16) are reported in Figure 8.

By solving Equation (14.3) with respect to z, Equation (14.3) splits into the following four equations in explicit form

$$z=\pm\sqrt{2-x^2-y^2-\sqrt{4-x^2-2\sqrt{3}xy-3y^2}} \tag{17.1}$$

$$z=\pm\sqrt{2-x^2-y^2+\sqrt{4-x^2-2\sqrt{3}xy-3y^2}} \tag{17.2}$$

The surfaces corresponding to Equations (17) are reported in Fig. 9.

By combining the surfaces reported in Figures 7, 8 and 9, the platform's Cartesian space is divided into many regions in which the sign of the left-hand sides of inequalities (13) do not change. Therefore, the selection of the regions where

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all the inequalities (13) are satisfied can be easily done by simply calculating the values assumed by the left-hand sides of inequalities (13) in one point per region. The manipulator workspace is constituted by the union of those Cartesian space regions where inequality system (13) is satisfied.

Figures 10, 11 and 12 show the workspace sections at constant x, constant y and constant z respectively. The lines, reported in those figures, are the intersections of the workspace boundaries (surfaces shown in Figures 7, 8 and 9) with the constant coordinate plane and the gray areas are the regions belonging to the manipulator workspace.

CONCLUSIONS

The workspace of two translational parallel manipulators (TPMs) has been studied. The two TPMs have the same

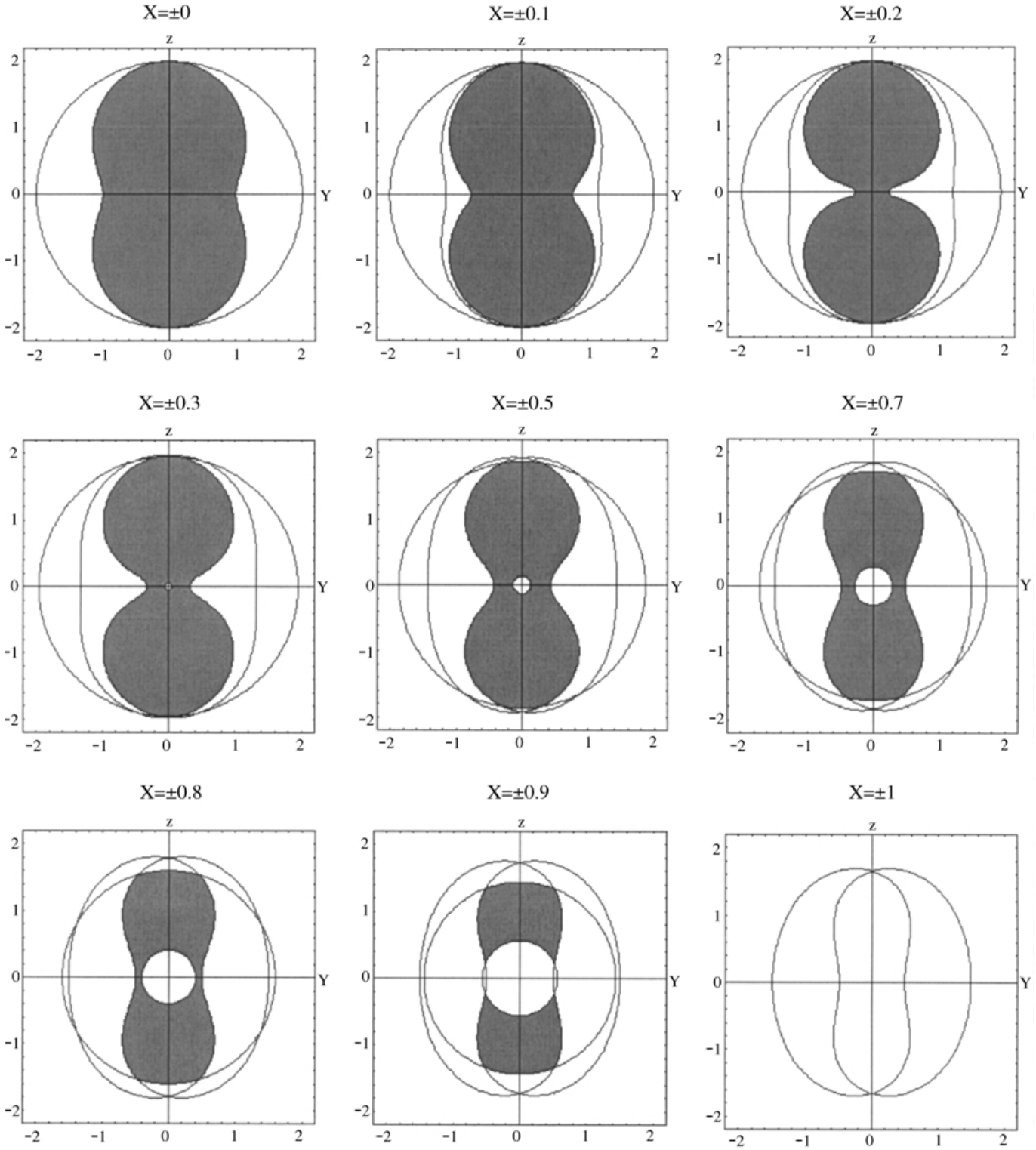


Fig. 10. Workspace sections at constant x. The lines, reported in these figures, are the intersections of the workspace boundaries (surfaces shown in Figures 7, 8 and 9) with the constant coordinate plane, and the gray areas are the regions belonging to the manipulator workspace.

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closure equations and workspace when a few geometric conditions are satisfied. The two TPMs are the translational 3-RUU[11] and the DELTA robot.[12–14] The kinematic analyses of these two TPMs are different from one another only when velocity and accelerations are considered.[11,13,14]

The result of this study is that the analytic expressions of the surfaces that bound their workspace are polynomial equations of the fourth degree in the coordinates of a platform point, which contain all the manipulator's geometric parameters. The explicit forms of these polynomial equations have been found. These analytic expressions facilitate the design of the two TPMs. The use of these expressions has been illustrated through a numerical example.

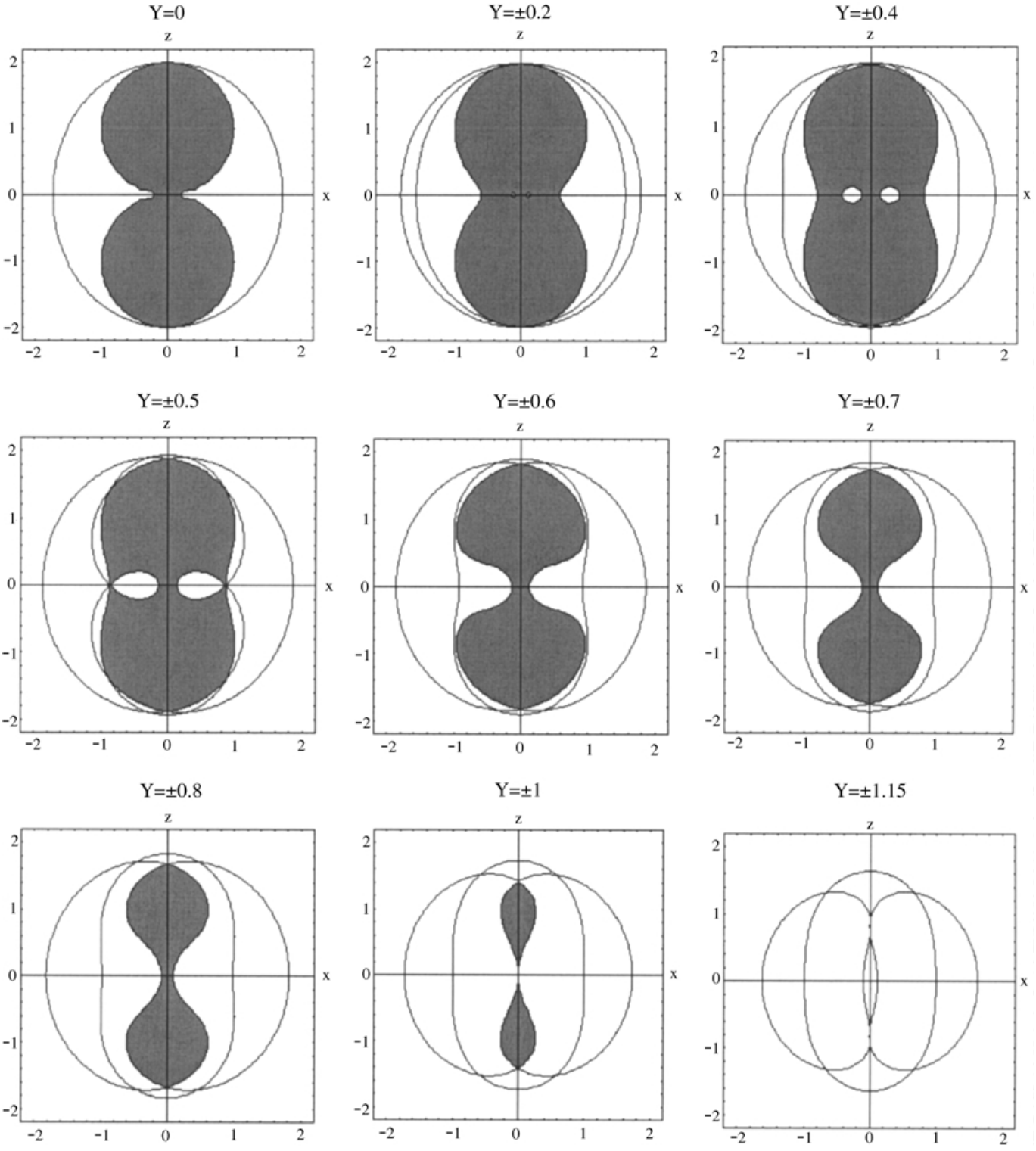


Fig. 11. Workspace sections at constant y. The lines, reported in these figures, are the intersections of the workspace boundaries (surfaces shown in Figures 7, 8 and 9) with the constant coordinate plane, and the gray areas are the regions belonging to the manipulator workspace.

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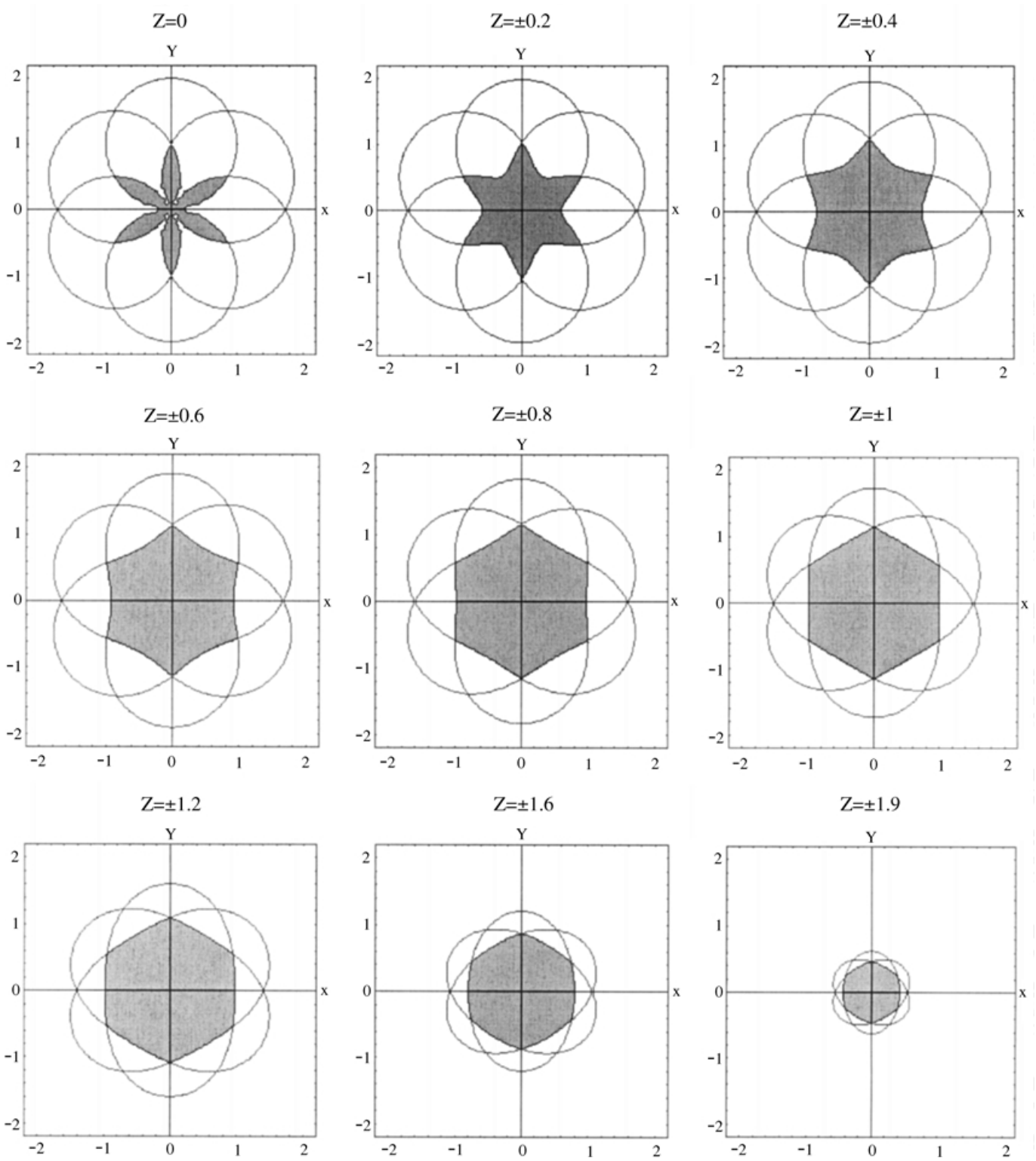


Fig. 12. Workspace sections at constant z. The lines, reported in these figures, are the intersections of the workspace boundaries (surfaces shown in Figures 7, 8 and 9) with the constant coordinate plane, and the gray areas are the regions belonging to the manipulator workspace.

Acknowledgments

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APPENDIX

The substitution of expressions (9.1) and (9.2) for k_i and n_i, respectively, into inequalities (10) yields

$$\frac{4q_i^2\{(\mathbf{C}_i-\mathbf{A}_i)^2-[\mathbf{w}_{0i}\cdot(\mathbf{C}_i-\mathbf{A}_i)]^2\}^2+[\mathbf{w}_{0i}\cdot(\mathbf{C}_i-\mathbf{A}_i)]^2[r_i^2-q_i^2+(\mathbf{C}_i-\mathbf{A}_i)^2]\{r_i^2-q_i^2-3(\mathbf{C}_i-\mathbf{A}_i)^2+4[\mathbf{w}_{0i}\cdot(\mathbf{C}_i-\mathbf{A}_i)]^2\}}{4\{(\mathbf{C}_i-\mathbf{A}_i)^2-[\mathbf{w}_{0i}\cdot(\mathbf{C}_i-\mathbf{A}_i)]^2\}^2}$$
$$-\frac{(\mathbf{C}_i-\mathbf{A}_i)^2[r_i^2-q_i^2-(\mathbf{C}_i-\mathbf{A}_i)^2+2[\mathbf{w}_{0i}\cdot(\mathbf{C}_i-\mathbf{A}_i)]^2\}^2}{4\{(\mathbf{C}_i-\mathbf{A}_i)^2-[\mathbf{w}_{0i}\cdot(\mathbf{C}_i-\mathbf{A}_i)]^2\}^2}\geq 0 \quad i=1,2,3 \tag{A.1}$$

The denominators which appear on the left-hand sides of inequalities (A.1) are always positive; moreover, they never become infinite because they depend on the terms $(\mathbf{C}_i-\mathbf{A}_i)^2$, $i=1,2,3$, which are always finite. Therefore, the sign and the zeros of the left-hand sides of inequalities (A.1) are the same as the ones of their numerators, which involves that the inequality system (A.1) is equivalent to the following one

$$4q_i^2\{(\mathbf{C}_i-\mathbf{A}_i)^2-[\mathbf{w}_{0i}\cdot(\mathbf{C}_i-\mathbf{A}_i)]^2\}^2+[\mathbf{w}_{0i}\cdot(\mathbf{C}_i-\mathbf{A}_i)]^2[r_i^2-q_i^2+(\mathbf{C}_i-\mathbf{A}_i)^2]\{r_i^2-q_i^2-3(\mathbf{C}_i-\mathbf{A}_i)^2+4[\mathbf{w}_{0i}\cdot(\mathbf{C}_i-\mathbf{A}_i)]^2\}$$
$$-(\mathbf{C}_i-\mathbf{A}_i)^2\{r_i^2-q_i^2-(\mathbf{C}_i-\mathbf{A}_i)^2+2[\mathbf{w}_{0i}\cdot(\mathbf{C}_i-\mathbf{A}_i)]^2\}^2\geq 0 \quad i=1,2,3 \tag{A.2}$$

Analogously, Equations (11) become

$$4q_i^2\{(\mathbf{C}_i-\mathbf{A}_i)^2-[\mathbf{w}_{0i}\cdot(\mathbf{C}_i-\mathbf{A}_i)]^2\}^2+[\mathbf{w}_{0i}\cdot(\mathbf{C}_i-\mathbf{A}_i)]^2[r_i^2-q_i^2+(\mathbf{C}_i-\mathbf{A}_i)^2]\{r_i^2-q_i^2-3(\mathbf{C}_i-\mathbf{A}_i)^2+4[\mathbf{w}_{0i}\cdot(\mathbf{C}_i-\mathbf{A}_i)]^2\}$$
$$-(\mathbf{C}_i-\mathbf{A}_i)^2\{r_i^2-q_i^2-(\mathbf{C}_i-\mathbf{A}_i)^2+2[\mathbf{w}_{0i}\cdot(\mathbf{C}_i-\mathbf{A}_i)]^2\}^2=0 \quad i=1,2,3 \tag{A.3}$$

Without losing generality, the vectors $\mathbf{w}_{0i}$ and $\mathbf{A}_i$, $i=1,2,3$, can be written as follows

$$\mathbf{w}_{0i}=\{w_{0ix},\ w_{0iy},\ w_{0iz}\}^T,\quad \mathbf{A}_i=\{a_{ix},\ a_{iy},\ a_{iz}\}^T \quad i=1,2,3 \tag{A.4}$$

where w_{0ix}, w_{0iy}, w_{0iz}, a_{ix}, a_{iy} and a_{iz} are geometric constants of the manipulator; moreover, expressions (4) of vectors $\mathbf{C}_i$ can be written as follows

$$\mathbf{C}_i=\{x,\ y,\ z\}^T+\{c_{ix},\ c_{iy},\ c_{iz}\}^T \quad i=1,2,3 \tag{A.5}$$

where x, y and z are the coordinates of point O_p measured in S_b, while c_{ix}, c_{iy} and c_{iz} are the components, measured in S_b, of the constant vector $\mathbf{R}_{bp}{}^p(\mathbf{C}_i-\mathbf{O}_p)$.

The introduction of expressions (A.4) and (A.5) into Equations (A.3) gives, after some rearrangements the following equations:

$$x^4+2x^2y^2+y^4+2x^2z^2+2y^2z^2+z^4-4x^3a_{ix}-4xy^2a_{ix}-4xz^2a_{ix}+6x^2a_{ix}^2+2y^2a_{ix}^2+2z^2a_{ix}^2-4xa_{ix}^3+a_{ix}^4-4x^2ya_{iy}-4y^3a_{iy}-4yz^2a_{iy}+8xya_{ix}a_{iy}-4ya_{ix}^2a_{iy}+2x^2a_{iy}^2+6y^2a_{iy}^2+2z^2a_{iy}^2-4xa_{ix}a_{iy}^2+2a_{ix}^2a_{iy}^2-4ya_{iy}^3+a_{iy}^4-4x^2za_{iz}-4y^2za_{iz}-4z^3a_{iz}+8xza_{ix}a_{iz}-4za_{ix}^2a_{iz}+8yza_{iy}a_{iz}-4za_{iy}^2a_{iz}+2x^2a_{iz}^2+2y^2a_{iz}^2+6z^2a_{iz}^2-4xa_{ix}a_{iz}^2+2a_{ix}^2a_{iz}^2-4ya_{iy}a_{iz}^2+2a_{iy}^2a_{iz}^2-4za_{iz}^3+a_{iz}^4+4x^3c_{ix}+4xy^2c_{ix}+4xz^2c_{ix}-12x^2a_{ix}c_{ix}-4y^2a_{ix}c_{ix}-4z^2a_{ix}c_{ix}+12xa_{ix}^2c_{ix}-4a_{ix}^3c_{ix}-8xya_{iy}c_{ix}+8ya_{ix}a_{iy}c_{ix}+4xa_{iy}^2c_{ix}-4a_{ix}a_{iy}^2c_{ix}-8xza_{iz}c_{ix}+8za_{ix}a_{iz}c_{ix}+4xa_{iz}^2c_{ix}-4a_{ix}a_{iz}^2c_{ix}+6x^2c_{ix}^2+2y^2c_{ix}^2+2z^2c_{ix}^2-12xa_{ix}c_{ix}^2+6a_{ix}^2c_{ix}^2-4ya_{iy}c_{ix}^2+2a_{iy}^2c_{ix}^2-4za_{iz}c_{ix}^2+2a_{iz}^2c_{ix}^2+4xc_{ix}^3-4a_{ix}c_{ix}^3+c_{ix}^4+4x^2yc_{iy}+4y^3c_{iy}+4yz^2c_{iy}-8xya_{ix}c_{iy}+4ya_{ix}^2c_{iy}-4x^2a_{iy}c_{iy}-12y^2a_{iy}c_{iy}-4z^2a_{iy}c_{iy}+8xa_{ix}a_{iy}c_{iy}-4a_{ix}^2a_{iy}c_{iy}+12ya_{iy}^2c_{iy}-4a_{iy}^3c_{iy}-8yzaiz_{iy}c_{iy}+8za_{iy}a_{iz}c_{iy}+4ya_{iz}^2c_{iy}-4a_{iy}a_{iz}^2c_{iy}+8xyc_{ix}c_{iy}-8ya_{ix}c_{ix}c_{iy}-8xa_{iy}c_{ix}c_{iy}+8a_{ix}a_{iy}c_{ix}c_{iy}+4yc_{ix}^2c_{iy}-4a_{iy}c_{ix}^2c_{iy}+2x^2c_{iy}^2+6y^2c_{iy}^2+2z^2c_{iy}^2-4xa_{ix}c_{iy}^2+2a_{ix}^2c_{iy}^2-12ya_{iy}c_{iy}^2+6a_{iy}^2c_{iy}^2-4za_{iz}c_{iy}^2+2a_{iz}^2c_{iy}^2+4xc_{ix}c_{iy}^2-4a_{ix}c_{ix}c_{iy}^2+2c_{ix}^2c_{iy}^2+4yc_{iy}^3-4a_{iy}c_{iy}^3+c_{iy}^4+4x^2zc_{iz}+4y^2zc_{iz}+4z^3c_{iz}-8xza_{ix}c_{iz}+4za_{ix}^2c_{iz}-8yza_{iy}c_{iz}+4za_{iy}^2c_{iz}-4x^2a_{iz}c_{iz}-4y^2a_{iz}c_{iz}-12z^2a_{iz}c_{iz}+8xa_{ix}a_{iz}c_{iz}-4a_{ix}^2a_{iz}c_{iz}+8ya_{iy}a_{iz}c_{iz}-4a_{iy}^2a_{iz}c_{iz}+12za_{iz}^2c_{iz}-4a_{iz}^3c_{iz}+8xzc_{ix}c_{iz}-8za_{ix}c_{ix}c_{iz}-8xa_{iz}c_{ix}c_{iz}+8a_{ix}a_{iz}c_{ix}c_{iz}+4zc_{ix}^2c_{iz}-4a_{iz}c_{ix}^2c_{iz}+8yzc_{iy}c_{iz}-8za_{iy}c_{iy}c_{iz}-8ya_{iz}c_{iy}c_{iz}+8a_{iy}a_{iz}c_{iy}c_{iz}+4zc_{iy}^2c_{iz}-4a_{iz}c_{iy}^2c_{iz}+2x^2c_{iz}^2+2y^2c_{iz}^2+6z^2c_{iz}^2-4xa_{ix}c_{iz}^2+2a_{ix}^2c_{iz}^2-4ya_{iy}c_{iz}^2+2a_{iy}^2c_{iz}^2-12za_{iz}c_{iz}^2+6a_{iz}^2c_{iz}^2+4xc_{ix}c_{iz}^2-4a_{ix}c_{ix}c_{iz}^2+2c_{ix}^2c_{iz}^2+4yc_{iy}c_{iz}^2-4a_{iy}c_{iy}c_{iz}^2+2c_{iy}^2c_{iz}^2+4zc_{iz}^3-4a_{iz}c_{iz}^3+c_{iz}^4-2x^2q_i^2-2y^2q_i^2-2z^2q_i^2+4xa_{ix}q_i^2-2a_{ix}^2q_i^2+4ya_{iy}q_i^2-2a_{iy}^2q_i^2+4za_{iz}q_i^2-2a_{iz}^2q_i^2-4xc_{ix}q_i^2+4a_{ix}c_{ix}q_i^2-2c_{ix}^2q_i^2-4yc_{iy}q_i^2+4a_{iy}c_{iy}q_i^2-2c_{iy}^2q_i^2-4zc_{iz}q_i^2+4a_{iz}c_{iz}q_i^2-2c_{iz}^2q_i^2+q_i^4-2x^2r_i^2-2y^2r_i^2-2z^2r_i^2+4xa_{ix}r_i^2-2a_{ix}^2r_i^2+4ya_{iy}r_i^2-2a_{iy}^2r_i^2+4za_{iz}r_i^2-2a_{iz}^2r_i^2-4xc_{ix}r_i^2+4a_{ix}c_{ix}r_i^2-2c_{ix}^2r_i^2-4yc_{iy}r_i^2+4a_{iy}c_{iy}r_i^2-2c_{iy}^2r_i^2-4zc_{iz}r_i^2+4a_{iz}c_{iz}r_i^2-2c_{iz}^2r_i^2-2q_i^2r_i^2+r_i^4+4x^2r_i^2w_{0ix}^2-8xa_{ix}r_i^2w_{0ix}^2+4a_{ix}^2r_i^2w_{0ix}^2+8xc_{ix}r_i^2w_{0ix}^2-8a_{ix}c_{ix}r_i^2w_{0ix}^2+4c_{ix}^2r_i^2w_{0ix}^2+8xyr_i^2w_{0ix}w_{0iy}-8ya_{ix}r_i^2w_{0ix}$$

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$$w_{0iy} - 8\,x\,a_{iy}\,r_i^2\,w_{0ix}\,w_{0iy} + 8\,a_{ix}\,a_{iy}\,r_i^2\,w_{0ix}\,w_{0iy} + 8\,y\,c_{ix}\,r_i^2\,w_{0ix}\,w_{0iy} - 8\,a_{iy}\,c_{ix}\,r_i^2\,w_{0ix}\,w_{0iy} + 8\,x\,c_{iy}\,r_i^2\,w_{0ix}\,w_{0iy} - 8\,a_{ix}\,c_{iy}\,r_i^2\,w_{0ix}\,w_{0iy} + 8\,c_{ix}\,c_{iy}\,r_i^2\,w_{0ix}\,w_{0iy} + 4\,y^2\,r_i^2\,w_{0iy}^2 - 8\,y\,a_{iy}\,r_i^2\,w_{0iy}^2 + 4\,a_{iy}^2\,r_i^2\,w_{0iy}^2 + 8\,y\,c_{iy}\,r_i^2\,w_{0iy}^2 - 8\,a_{iy}\,c_{iy}\,r_i^2\,w_{0iy}^2 + 4\,c_{iy}^2\,r_i^2\,w_{0iy}^2 + 8\,x\,z\,r_i^2\,w_{0ix}\,w_{0iz} - 8\,z\,a_{ix}\,r_i^2\,w_{0ix}\,w_{0iz} - 8\,x\,a_{iz}\,r_i^2\,w_{0ix}\,w_{0iz} + 8\,a_{ix}\,a_{iz}\,r_i^2\,w_{0ix}\,w_{0iz} + 8\,z\,c_{ix}\,r_i^2\,w_{0ix}\,w_{0iz} - 8\,a_{iz}\,c_{ix}\,r_i^2\,w_{0ix}\,w_{0iz} + 8\,x\,c_{iz}\,r_i^2\,w_{0ix}\,w_{0iz} - 8\,a_{ix}\,c_{iz}\,r_i^2\,w_{0ix}\,w_{0iz} + 8\,c_{ix}\,c_{iz}\,r_i^2\,w_{0ix}\,w_{0iz} + 8\,y\,z\,r_i^2\,w_{0iy}\,w_{0iz} - 8\,z\,a_{iy}\,r_i^2\,w_{0iy}\,w_{0iz} - 8\,y\,a_{iz}\,r_i^2\,w_{0iy}\,w_{0iz} + 8\,a_{iy}\,a_{iz}\,r_i^2\,w_{0iy}\,w_{0iz} + 8\,z\,c_{iy}\,r_i^2\,w_{0iy}\,w_{0iz} - 8\,a_{iz}\,c_{iy}\,r_i^2\,w_{0iy}\,w_{0iz} + 8\,y\,c_{iz}\,r_i^2\,w_{0iy}\,w_{0iz} - 8\,a_{iy}\,c_{iz}\,r_i^2\,w_{0iy}\,w_{0iz} + 8\,c_{iy}\,c_{iz}\,r_i^2\,w_{0iy}\,w_{0iz} + 4\,z^2\,r_i^2\,w_{0iz}^2 - 8\,z\,a_{iz}\,r_i^2\,w_{0iz}^2 + 4\,a_{iz}^2\,r_i^2\,w_{0iz}^2 + 8\,z\,c_{iz}\,r_i^2\,w_{0iz}^2 - 8\,a_{iz}\,c_{iz}\,r_i^2\,w_{0iz}^2 + 4\,c_{iz}^2\,r_i^2\,w_{0iz}^2 = 0$$

$$i = 1, 2, 3 \qquad \text{(A.6)}$$

Equations (A.6) are polynomial equations of the fourth degree in x, y and z.

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