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# Linear operators with closed range

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This paper continues the investigation begun in (6), and uses the definitions and notation of that paper.

If $T$ is a densely defined closed linear operator from $X$ to $Y$, where $X$ and $Y$ are Banach spaces, then the following statements are well known to be equivalent:

(1) $R(T)$ is closed in $Y$;
(2) $\gamma(T) > 0$;
(3) $R(T^*)$ is closed in $X^*$;
(4) $R(T^*)$ is weak*-closed in $X^*$;
(5) $\gamma(T^*) > 0$.

It is the object of this paper to show how these statements are connected when $T$ is a general densely defined linear operator (not necessarily closed) and $X$ and $Y$ are normed (not necessarily complete) spaces.

In this setting we obviously have (4) $\Rightarrow$ (3). Since $T^*$ is closed and $Y^*$ and $X^*$ are Banach spaces, we know that (3) $\Leftrightarrow$ (5). In (6) we showed that (2) $\Rightarrow$ (5), and that (5) $\not\Rightarrow$ (2). In this paper we show that if $Y$ is complete then (1) $\Rightarrow$ (5), and if $X$ is complete then (3) $\Rightarrow$ (4). We also show that, with $X$ and $Y$ Banach spaces, the hypothesis '$T$ closed' may be weakened, with statements (2) to (5) remaining equivalent. (Note that (1) and (2) together imply that $T$ is closed.)

Proposition 1. *If $R(T)$ is a barrelled subspace of $Y$, then $\gamma(T^*) > 0$.*

*Proof.* We may suppose $R(T) = Y$. For otherwise, $T$ may be 'factorized' as follows:

$$X \xrightarrow{T_0} R(T) \xrightarrow{j} Y$$

(where $j$ is the injection map). Then $T^*$ has the factorization

$$X^* \xleftarrow{T_0^*} Y^*/N(T^*) \xleftarrow{j^*} Y^*$$

(where $j^*$ is the quotient map), and we now need only note that $\gamma(T_0^*) = \gamma(T^*)$.

Let $f_n = T^*g_n \in R(T^*)$ such that $f_n \to f \in X^*$. Then for all $x \in D(T)$

$$\langle x, T^*g_n\rangle \to \langle x, f\rangle.$$

Hence
$$\langle Tx, g_n\rangle \to \langle x, f\rangle.$$

Since $R(T) = Y$, we deduce that $\{g_n\}$ is a weak*-Cauchy sequence in $Y^*$, hence, since $Y$ is barrelled, $\{g_n\}$ is weak*-convergent to an element $g \in Y^*$, i.e.

$$\langle Tx, g_n\rangle \to \langle Tx, g\rangle, \quad \text{all} \quad x \in D(T).$$

Hence
$$\langle x, f\rangle = \langle Tx, g\rangle, \quad \text{all} \quad x \in D(T),$$

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which shows that $g \in D(T^*)$ and $f = T^*g$. So $R(T^*)$ is closed (in fact we have shown that $R(T^*)$ is weak*-sequentially closed), hence $\gamma(T^*) > 0$.

COROLLARY 2. *If $Y$ is complete and $R(T)$ is closed in $Y$, then $\gamma(T^*) > 0$.*

Proposition 1 may be used to furnish a new proof of the following important result due to Kato ((3) Lemma 332):

COROLLARY 3. *If $X$ and $Y$ are Banach spaces and $T: X \to Y$ is a densely defined closed operator such that codim $R(T) < \infty$, then $R(T)$ is closed.*

*Proof.* By (4) Lemma 1·1, $R(T)$ is a barrelled subspace of $Y$, so $\gamma(T^*) > 0$ by Proposition 1. Hence $R(T)$ is closed.

PROPOSITION 4. *If $X$ is complete, then $R(T^*)$ closed in $X^* \Rightarrow R(T^*)$ is weak*-closed in $X^*$.*

*Proof.* Let $B$ be the unit ball in $X^*$. By the Banach–Dieudonné Theorem ((5), p. 275) it is sufficient to prove that $R(T^*) \cap B$ is weak*-closed in $X^*$.

The set $S = \{g \in D(T^*): \|g\| + \|T^*g\| \leqslant 1\}$ is weak*-compact in $Y^*$. This is proved by Joichi (1) when $X$ and $Y$ are complete and $T$ is closed, but his proof is clearly still valid without these assumptions. Since $T^*$ is a closed operator for the weak* topologies on $Y^*$ and $X^*$ it follows that $T^*(S)$ is weak*-closed in $X^*$. Let $B_n = B \cap nT^*(S)$. Then $B_n$ is weak*-closed (hence also closed) in $X^*$ for each $n$, and

$$R(T^*) = \bigcup_{n=1}^{\infty} nB_n.$$

Now by hypothesis $R(T^*)$ is a complete space, so by the Baire Category Theorem some $B_n$ is a neighbourhood of 0. It follows that $B_m = B \cap R(T^*)$ for some $m$, which shows that $B \cap R(T^*)$ is weak*-closed in $X^*$ as required.

PROPOSITION 5. *If $X$ and $Y$ are Banach spaces, and $T: X \to Y$ is a densely defined linear operator such that $N(T) = [R(T^*)]^{\perp}$ ($\perp$ in $X$), then the statements* (2) *to* (5) *are equivalent.*

*Proof.* All that must be proved is (4) $\Rightarrow$ (2). Since $N(T) = [R(T^*)]^{\perp}$ we have $[N(T)]^{\perp} = \overline{R(T^*)}$ (the weak*-closure of $R(T^*)$ in $X^*$). By hypothesis therefore $[N(T)]^{\perp} = R(T^*)$. Hence by (6) Proposition 1·2(3), $\gamma(T) = \gamma(T^*) > 0$.

*Remark.* If $T$ is a closed operator, then $N(T) = [R(T^*)]^{\perp}$ (see (2), Theorem 3·1) while it is easy to give an example of a non-closed operator $T$ for which $N(T) = [R(T^*)]^{\perp}$.

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