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CERTIFIED Σ_1-SENTENCES

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Abstract. In this paper, we study the employment of Σ_1-sentences with certificates, i.e., Σ_1-sentences where a number of principles is added to ensure that the witness is sufficiently number-like. We develop certificates in some detail and illustrate their use by reproving some classical results and proving some new ones. An example of such a classical result is Vaught's theorem of the strong effective inseparability of R_0.

We also develop the new idea of a theory being $\mathsf{R}_{0\mathsf{p}}$-*sourced*. Using this notion, we can transfer a number of salient results from R_0 to a variety of other theories.

§1. Introduction. In this paper, we study *certificates*. These are *theories-of-a-number* with a free parameter for the number in question, or, more precisely, for the *number-like object*. In other words, certificates specify a property of a number. This property is roughly that the object specified is sufficiently like a number. A salient property of theories-of-a-number is that they have finite models.

The main focus of this paper is on certificates as a tool to metamathematical results. Thus, the paper can be viewed as a study of certificates as a method. We develop one specific certificate and provide the necessary lemmas for its employment. We apply the certificate to, possibly non-standard, witnesses of Σ^0_1-sentences. This use of the certificate is in constant interaction with the salient theories R_0 and R. Our presentation provides more detail than previous presentations, so that many subtleties of what is going on become clearly visible here for the first time.

We extend the classical results obtained by the use of certificates by defining a wider class of theories, the $\mathsf{R}_{0\mathsf{p}}$-*sourced theories*. These theories behave in some important respects like the salient theory R_0.

The paper presents a number of applications of the use of certificates, which are important in themselves, but also serve to illustrate the use of the method well. These are:

- Certain theorems by Cobham and Vaught, the contents of which are explained in Section 2 below. We introduce these results in our preparatory Section 2. The detailed treatment then will be in Section 7. Our version of the second Vaught theorem is a generalization to the $\mathsf{R}_{0\mathsf{p}}$-sourced case.
- A variety of results concerning degree structures of interpretability. These results are in Section 8.

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Theories-of-a-number have counterparts for various other data-types, like sets, multi-sets, sequences, and strings. The alternative that is developed and used is theories-of-a-finite-set. Here is a list of examples of uses of theories-of-a-number and their kin that we noted. We do not have any pretence of completeness here.

- In [2], Harvey Friedman uses theories-of-a-number to prove the density of the interpretability degrees of finitely axiomatized theories. This result was proved earlier, by another method, in [10]. We present a version of the result in Theorem 8.9 and Corollary 8.11.
- In [20], Albert Visser uses theories-of-a-number to reprove (and improve) Vaught's result [18] that every c.e. Vaught theory is axiomatizable by a scheme.
- In [11], Fedor Pakhomov uses theories-of-a-finite-set to construct an R-like set theory that proves its own consistency.
- In [12], Fedor Pakhomov and Albert Visser show the following. Consider a finitely axiomatised extension A of c.e. theory U in a possibly extended signature. Suppose A is conservative over U. Then, there is a conservative extension B of U in the signature of A, such that $A \vdash B$ and $B \nvdash A$. They use theories-of-a-finite-set to prove this result.

1.1. Plan of the paper. The plan of the paper is as follows. In Section 2, we give a first presentation of both certain results by Cobham and Vaught and a preliminary explanation of the use of certificates. Section 3 fixes some basic definitions and provides pointers to relevant literature. In Section 4, we develop the basic facts about certificates of a Σ^0_1-witness and the theories R_0 and R. In Section 5, we generalize the results of Section 4 by replacing R_0 by theories from a class that has R_0 as its source. We provide examples to illustrate that many salient theories are in that class. Section 6 provides basic facts about witness comparison, which is an important tool that we use in the subsequent sections. The section is needed since the interaction between certificates and witness comparisons is somewhat delicate. In Section 7, we apply the methods developed in the previous sections to prove two theorems due to Vaught. Finally, in Section 8, we apply these methods to prove various results about degrees of interpretability. Sections 7 and 8 can be read independently of each other.

1.2. History of the paper. The present paper succeeds and replaces Taishi Kurahashi's earlier preprint *Incompleteness and undecidability of theories consistent with* R. The materials of the preprint are contained in the present paper.

§2. Theorems of Cobham and Vaught.

In this section, we give a presentation of current status of certain theorems of Cobham and Vaught. After some preparatory work, we take up this thread again in Section 7.

Let $\mathbb{L}_\mathsf{a}$ be the signature $\{0, \mathsf{s}, +, \times, \leqslant\}$ of first-order arithmetic. Let ω denote the set of all natural numbers. For each $n \in \omega$, the numeral $\mathsf{s} \ldots \mathsf{s}0$ (n applications of s) for n is denoted by $\underline{n}$. A central object of study in this paper is the $\mathbb{L}_\mathsf{a}$-theory R introduced by Tarski, Mostowski, and Robinson in [15].

DEFINITION 2.1 (The theory R). The theory R is axiomatized by the following sentences: For $m, n \in \omega$,

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R1. $\underline{m} + \underline{n} = \underline{m+n}$

R2. $\underline{m} \times \underline{n} = \underline{m \times n}$

R3. $\underline{m} \neq \underline{n}$ (if $m \neq n$)

R4. $\forall x \left(x \leqslant \underline{n} \to \bigvee_{i \leqslant n} x = \underline{i}\right)$

R5. $\forall x \, (x \leqslant \underline{n} \vee \underline{n} \leqslant x)$.

It was proved in [15] that the original R is essentially undecidable, that is, every consistent extension of R is undecidable. Here, we comment on the difference between our formulation and the original one of the theory R. The signature adopted in [15] does not contain the symbol $\leqslant$ and the formula $x \leqslant y$ is introduced as the abbreviation for $\exists z \; z + x = y$. Our signature $\mathbb{L}_{\mathsf{a}}$ contains $\leqslant$ as a primitive symbol and our version of R does not prove the equivalence between the formulas $x \leqslant y$ and $\exists z \; z + x = y$. So, our R is strictly weaker than the original. Jones and Shepherdson [6] pointed out that the essential undecidability of R also holds without using the equivalence $x \leqslant y \leftrightarrow \exists z \; z + x = y$.

The $\mathbb{L}_{\mathsf{a}}$-theory R_0 is obtained from R by replacing the axiom R5 with the following R5′:

R5′: $\underline{m} \leqslant \underline{n}$ (if $m \leqslant n$) .

Alternatively, we can axiomatize R_0 by dropping R5 altogether and replacing R4 by:

R4′: $\forall x \left(x \leqslant \underline{n} \leftrightarrow \bigvee_{i \leqslant n} x = \underline{i}\right)$.

We make R1-4 plus R5′ our official axiomatization.

Remark 2.2. Our definitions of R and R_0 correspond to those in Švejdar's paper [14]. Vaught's definition of R_0 in [17] is in the same spirit as ours, but still differs. He presents the theory in a relational format without identity. There is a mistake in Vaught's statement of the axioms. The theory as given in the paper clearly has a decidable extension. As possible repairs, one could add the axioms for the totality of the functions or replace, both in Axiom I and II, the second occurrence of $\to$ by $\leftrightarrow$. Jones and Shepherdson in [6] discuss both the version of R with and without a defined relation $\leqslant$. They use R_0' for our R_0.

It is easy to see that R_0 is a proper subtheory of R. We note that all the axioms of R_0 can be rewritten as Δ_0-formulas. This leads to a nice observation by Vítězslav Švejdar.

Theorem 2.3 (Cf. Švejdar [14]). *An $\mathbb{L}_{\mathsf{a}}$-theory T is Σ_1-complete if and only if $T \vdash \mathsf{R}_0$.*

Cobham observed that R is interpretable in R_0, and that, hence, R_0 is essentially undecidable (see Vaught [17] and Jones and Shepherdson [6]).[1] We give Cobham's interpretation in Section 7.

In the formulation of R_0, if $x \leqslant y$ is defined by $\exists z \; z + x = y$ as in [15], rather than primitive as in the present paper, then the axiom R5′ is redundant since it can be derived from R1. On the other hand, note that, in our signature $\mathbb{L}_{\mathsf{a}}$, the theory obtained from R_0 by removing R5′ has a complete consistent decidable extension (see Jones and Shepherdson [6]).

[1] It is also easy to prove the essential undecidability of R_0 directly.

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We say that a theory T is *essentially hereditarily undecidable* if every $\mathbb{L}_{\mathsf{a}}$-theory consistent with T is undecidable (cf. [26]). It is shown in [15] that every finitely axiomatized essentially undecidable theory is also essentially hereditarily undecidable. Here, since the theory R_0 is not finitely axiomatizable, it is nontrivial whether R_0 is essentially hereditarily undecidable. In fact, there exists a computably axiomatized essentially undecidable theory having a decidable subtheory (cf. Ehrenfeucht [1] and Putnam [13]). Then, Cobham proved the following interesting theorem.

THEOREM 2.4 (Cobham, see Vaught [17, Theorem 1.5]). *The theory* R_0 *is essentially hereditarily undecidable.*

A proof of Cobham's theorem was presented in Vaught [17]. Vaught also showed two strengthenings of Cobham's theorem.

For each $i \in \omega$, let W_i denote the c.e. set with the index i. We say that a pair (X, Y) of disjoint c.e. sets is *effectively inseparable* if for any $i, j \in \omega$, if $X \subseteq \mathsf{W}_i$, $Y \subseteq \mathsf{W}_j$, and $\mathsf{W}_i \cap \mathsf{W}_j = \emptyset$, then we can effectively find an element x such that $x \notin \mathsf{W}_i \cup \mathsf{W}_j$. For each theory T, let $T_{\mathfrak{p}}$ and $T_{\mathfrak{r}}$ be the set of all theorems of T and the set of all sentences refutable in T, respectively. We say that a consistent theory T is *strongly effectively inseparable* if the pair $(T_{\mathfrak{p}}, \emptyset_{\mathfrak{r}})$ is effectively inseparable (cf. [9]). The first strengthening is the following.

THEOREM 2.5 (Vaught [17, Theorem 5.2]). *The theory* R_0 *is strongly effectively inseparable.*

In fact, Cobham's theorem follows easily from Theorem 2.5. The second one is the following theorem that immediately implies Cobham's theorem, but no proof was presented in Vaught's paper.

THEOREM 2.6 (Vaught [17, Theorem 7.1]). *For any c.e.* $\mathbb{L}_{\mathsf{a}}$*-theory* U*, if* $\mathsf{R}_0 + U$ *is consistent, then there exists a finitely axiomatized* $\mathbb{L}_{\mathsf{a}}$*-theory* S *extending* R_0 *such that* $S + U$ *is also consistent.*

Recently, a more comprehensible proof of Cobham's theorem (Theorem 2.4) was also given in Visser [22].

DEFINITION 2.7 (Pure Δ_0- and Σ_1-formulas). Let φ be an $\mathbb{L}_{\mathsf{a}}$-formula.

- We say that φ is a *pure* Δ_0*-formula* if φ is Δ_0 and satisfies the following conditions:
  1. For any atomic formula of the form $t_1 \leqslant t_2$ contained in φ, terms t_1 and t_2 are both variables;
  2. Every atomic formula of the form $t_1 = t_2$ contained in φ is of one of the forms $x_0 = x_1$, $0 = x_0$, $\mathsf{s}x_0 = x_1$, $x_0 + x_1 = x_2$, and $x_0 \times x_1 = x_2$, where x_0, x_1, and x_2 are variables.
- We say that φ is a *pure* Σ_1*-formula* if φ is of the form $\exists \vec{x}\, \varphi_0(\vec{x})$, where $\varphi_0(\vec{x})$ is a pure Δ_0-formula. Here, the block $\vec{x}$ of quantifiers is allowed to be empty.

Our version of predicate logic does not contain the logical constants $\top$ and $\bot$ as primitive symbols. It is then shown that every pure Δ_0-formula contains at least one free variable. An effective procedure to obtain an equivalent pure Σ_1-formula for each Σ_1-formula is presented in [22].

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PROPOSITION 2.8 (Visser [22]). *For any Σ_1-formula $\varphi(\vec{x})$, a pure Σ_1-formula $\varphi^\circ(\vec{x})$ satisfying the following conditions is effectively found*:

1. $\mathbb{N} \models \forall \vec{x}\, (\varphi(\vec{x}) \leftrightarrow \varphi^\circ(\vec{x}))$,
2. $\forall \vec{x}\, (\varphi^\circ(\vec{x}) \rightarrow \varphi(\vec{x}))$ *is logically valid.*

Here, we outline the proof of Cobham's theorem presented in [22]. At first, the finite $\mathbb{L}_\mathsf{a}$-theory TN (the theory of a number) is introduced. Then, for each pure Σ_1-sentence of the form $\exists \vec{x}\, \sigma_0(\vec{x})$, where $\sigma_0(\vec{x})$ is a pure Δ_0-formula, let $[\sigma]$ be the finite $\mathbb{L}_\mathsf{a}$-theory

$$\mathsf{TN} + \exists v\, \exists \vec{x} < v\, \sigma_0(\vec{x}).$$

Let $\sigma^\star$ be an $\mathbb{L}_\mathsf{a}$-sentence saying that there exists the least number n such that the finite $\mathbb{L}_\mathsf{a}$-structure $\{0, 1, \ldots, n\}$ is a model of $[\sigma]$. Then, the following three clauses hold for each pure Σ_1-sentence σ:

1. If $\mathbb{N} \models \sigma$, then $\mathsf{R}_0 \vdash \sigma^\star$.
2. If $\mathbb{N} \models \neg\, \sigma$, then $[\sigma] \vdash \mathsf{R}_0$.
3. $(\mathsf{R}_0 + \sigma^\star) \rhd [\sigma]$.

Here, $T \rhd T'$ means that T' is interpretable in T (see Section 3 for the definition). Let U be any $\mathbb{L}_\mathsf{a}$-theory such that $\mathsf{R}_0 + U$ is consistent. We would like to show that U is undecidable. We may assume that U is a c.e. theory. Then, the set $X := \{\sigma \mid \sigma$ is a pure Σ_1 sentence and $\mathsf{R}_0 + \sigma^\star + U$ is consistent$\}$ is Π_1-definable. Since the set $Y := \{\sigma \mid \sigma$ is a true pure Σ_1-sentence$\}$ is not Π_1-definable, we have $X \neq Y$. By the first clause above, we have $Y \subsetneq X$, and hence $X \nsubseteq Y$. Then, we get a false pure Σ_1-sentence σ such that $\mathsf{R}_0 + \sigma^\star + U$ is consistent. By the second clause, we have $[\sigma] \vdash \mathsf{R}_0$, and thus the theory $[\sigma]$ is essentially undecidable. Since $[\sigma]$ is finite and $(\mathsf{R}_0 + \sigma^\star) \rhd [\sigma]$ by the third clause, there exists a finite subtheory S of $\mathsf{R}_0 + \sigma^\star$ such that $S \rhd [\sigma]$. Then, S is essentially undecidable and $S + U$ is consistent. Since S is finite, we conclude that U is undecidable.

However, it seems that the proof by Visser cannot be used to prove Vaught's theorems (Theorems 2.5 and 2.6) as it is, because the notion of interpretability between theories is used in it.

In this paper, we prove the following theorem using a modification of Visser's strategy.

THEOREM 2.9 (Certified extension theorem). *For each Σ_1-sentence σ, we can effectively find a sentence $[\sigma]$ satisfying the following conditions*:

1. $[\sigma] \vdash \sigma$.
2. *If* $\mathbb{N} \models \sigma$, *then* $\mathsf{R}_0 \vdash [\sigma]$.
3. *If* $\mathbb{N} \models \neg\, \sigma$, *then* $[\sigma] \vdash \mathsf{R}_0$.

We call such sentences $[\sigma]$ certified Σ_1-sentences for R_0.

We note that Visser's $[\sigma]$ does give us the analogs of (1) and (3) of Theorem 2.9. However the analog of (2) fails. We only get: if $\mathbb{N} \models \sigma$, then $\mathsf{R}_0 \rhd [\sigma]$.

§3. Basic definitions. We only present a brief outline of the basic notions. For more detail, we refer the reader to, e.g., [22, Appendix A].

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A *theory* U in this paper is a theory of predicate logic of finite signature Θ. A theory U is given by a finite signature Θ and a set of axioms X of the signature Θ.

DEFINITION 3.1. The conjunction of the finitely many axioms for identity of U is $\mathfrak{id}_\Theta$ or $\mathfrak{id}_U$. The axioms of identity are officially part of the logic but at times we will treat them as if they were part of the axioms of the theory.

Let us fix an infinite sequence of variables $v_0, v_1, \ldots$. Suppose Θ is a relational signature. *A one-dimensional parameter-free translation* $\tau : \Theta \to \Xi$ specifies a domain predicate δ_τ, with at most v_0 free, and, for an n-ary predicate symbol R of Θ a Ξ-formula R_τ, such that the free variables of R_τ are among $v_0, \ldots, v_{n-1}$. We treat identity as if it were a predicate from the signature rather than a logical predicate. We lift the translation to the full Θ-signature as follows:

- $(R(\vec{x}))^\tau := R_\tau[\vec{v} := \vec{x}]$. Here we assume an automatic mechanism of renaming variables in case of clashes.
- $(\cdot)^\tau$ commutes with the propositional connectives.
- $(\forall x \psi)^\tau := \forall x\, (\delta_\tau[v_0 := x] \to \psi^\tau)$.
- $(\exists x \psi)^\tau := \exists x\, (\delta_\tau[v_0 := x] \wedge \psi^\tau)$.

If Γ is a set of Θ-sentences, we write Γ^τ for $\{\phi^\tau \mid \phi \in \Gamma\}$.

We can extend translations to m-dimensional ones by translating a variable from the Θ-signature to a sequence of variables of length m of the Ξ-signature. We can also allow parameters in our interpretations.

We can define the identity translation and composition of translations in the obvious way.

An *interpretation* K of U in V is a triple (U, τ, V), where τ is a translation from the signature Θ_U of U to the signature Θ_V of V. We demand that, for every U-sentence ϕ such that $U \vdash \phi$, we have $V \vdash \phi^\tau$. We write $\mathsf{K} : U \lhd V$ or $\mathsf{K} : V \rhd U$ for K is an interpretation of U in V. We write $U \lhd V$ or $V \rhd U$ for: there is a K such that $\mathsf{K} : U \lhd V$.

We have also the notion of *local interpretability*. The theory V *locally interprets the theory* U, or $V \rhd_{\mathsf{loc}} U$, iff for each finitely axiomatized subtheory U_0 of U, we have $V \rhd U_0$.

We will make use of the following operations on theories. Consider theories U and V.

- The theory $U \mathbin{\mathbb{O}\!\!\!\vee} V$ is defined as follows. The signature of $U \mathbin{\mathbb{O}\!\!\!\vee} V$ is the disjoint sum of the signatures of U and V and, in addition, a fresh 0-ary predicate P. The axioms of $U \mathbin{\mathbb{O}\!\!\!\vee} V$ are all $P \to \phi$, where ϕ is an axiom of U, plus all $\neg P \to \psi$, where ψ is an axiom of V.
- The theory $U \mathbin{\mathbb{O}\!\!\!\wedge} V$ is defined as follows. The signature of $U \mathbin{\mathbb{O}\!\!\!\wedge} V$ is the disjoint sum of the signatures of U and V and, in addition, a fresh 1-ary predicate $\triangle$. The axioms of $U \mathbin{\mathbb{O}\!\!\!\wedge} V$ are the relativizations $\phi^\triangle$ of all axioms of ϕ of U w.r.t. $\triangle$, plus the relativizations $\psi^{\neg\triangle}$ of all axioms of ψ of V with respect to the complement of $\triangle$, plus axioms saying that neither $\triangle$ nor its complement are empty.

We have the following important properties.

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THEOREM 3.2.

a. $(U \curlyvee V) \rhd W$ *iff* $U \rhd W$ *and* $V \rhd W$.
b. $W \rhd (U \curlywedge V)$ *iff* $W \rhd U$ *and* $W \rhd V$.

Thus, $U \curlyvee V$ is (an implementation of) the *infimum* of U and V in the degrees of interpretability and $U \curlywedge V$ is (an implementation of) the *supremum* of U and V in the degrees of interpretability. There is something notationally awkward about representing an infimum by $\curlyvee$ and a supremum by $\curlywedge$. This awkwardness is due to a legacy problem. In the boolean intuition the most informative element is the bottom, where in the interpretation-ordering the most informative element is the top. We follow the boolean intuition here, treating $\curlyvee$ as a kind of disjunction of theories and $\curlywedge$ as a kind of conjunction.

§4. Certified extension. We prove Theorem 2.9. Our proof is based on the ideas from [22], but we exclude from the proof the use of $\sigma^\star$ and interpretability.

Let $x < y$ be an abbreviation of $x \leqslant y \wedge x \neq y$. We define certain special elements as follows.

DEFINITION 4.1 (Certification). An element v is *certified*, or $\mathsf{cert}(v)$, if it satisfies the following formulas. These formulas together form the certificate.

A1. $0 \leqslant v$
A2. $\forall x < v \;\; \mathsf{s}x \leqslant v$
A3. $\forall x \, (x \leqslant 0 \leftrightarrow x = 0)$
A4. $\forall x < v \, \forall y \, (y \leqslant \mathsf{s}x \leftrightarrow (y \leqslant x \vee y = \mathsf{s}x))$
A5. $\forall x, y, z \leqslant v \;\; \mathsf{s}((x \times y) + z) \neq 0$
A6. $\forall x, y, z, w \leqslant v \;\; \mathsf{s}((x \times y) + z) = \mathsf{s}w \to (x \times y) + z = w$
A7. $\forall x, y \leqslant v \, (x \times y) + 0 = x \times y$
A8. $\forall x, y, z \leqslant v \, (x \times y) + \mathsf{s}z = \mathsf{s}((x \times y) + z)$
A9. $\forall x \leqslant v \; x \times 0 = 0$
A10. $\forall x, y \leqslant v \; x \times \mathsf{s}y = (x \times y) + x$

We note that of the properties defining certification, only A3 and A4 are not *prima facie* Δ_0. However, we can rewrite A3 as $\forall x \leqslant 0 \; x = 0 \wedge 0 \leqslant 0$ and we can rewrite A4 as

$$\forall x < v \, (\forall y \leqslant \mathsf{s}x \, (y \leqslant x \vee y = \mathsf{s}x) \wedge \forall y \leqslant x \; y \leqslant \mathsf{s}x \wedge \mathsf{s}x \leqslant \mathsf{s}x).$$

So, modulo equivalence in predicate logic, certification is Δ_0.

The properties A5–A8 look a bit different from the usual axioms in certificates. Their specific form is needed to prove Lemma 4.8, which in its turn is needed to prove the negative atomic cases of Lemma 4.11.

REMARK 4.2. We aimed to keep our definition of certification reasonably simple. This has the advantage that it made clear that we can use a fairly light property. As we will see in Example 5.9, it is possible to add all kinds of convenient properties to certification that preserve our intended application. An example of such a property is linearity of $\leqslant$ below v.

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We say that σ is a *pure 1-Σ_1-sentence* if it is of the form $\exists x\, \sigma_0(x)$, where $\sigma_0(x)$ is a pure Δ_0-formula.

We strengthen Proposition 2.8 as follows.

PROPOSITION 4.3. *For any Σ_1-sentence λ, a pure 1-Σ_1-sentence $\lambda^\bullet$ satisfying the following conditions can be effectively found*:

1. $\mathbb{N} \models \lambda \leftrightarrow \lambda^\bullet$,
2. $\lambda^\bullet \to \lambda$ *is logically valid.*

PROOF. Consider any Σ_1-formula λ. By Proposition 2.8, we can effectively find a pure Σ_1-formula λ° such that $\mathbb{N} \models \lambda \leftrightarrow \lambda^\circ$ and $\vdash \lambda^\circ \to \lambda$. Suppose that λ° is of the form $\exists \vec{v}\, \lambda_0(\vec{v})$ for some pure Δ_0-formula $\lambda_0(\vec{v})$. Define $\lambda^\bullet$ to be the pure 1-Σ_1-sentence $\exists x\, \exists \vec{v} \leqslant x\, \lambda_0(\vec{v})$. Then, $\lambda^\bullet$ satisfies the conditions (1) and (2). ⊣

We are now ready to define certified Σ_1-sentences.

DEFINITION 4.4. Let σ be a pure 1-Σ_1-sentence of the form $\exists x\, \sigma_0(x)$. We define:

$$\sigma^{\mathsf{cert}} := \exists x \left(\mathsf{cert}(x) \wedge \sigma_0(x)\right).$$

The following theorem is the heart of the technical part of our results.

THEOREM 4.5. *Let σ be a pure 1-Σ_1-sentence. Then*:

1. $\sigma^{\mathsf{cert}} \vdash \sigma$.
2. *If* $\mathbb{N} \models \sigma$, *then* $\mathsf{R}_0 \vdash \sigma^{\mathsf{cert}}$.
3. *If* $\mathbb{N} \models \neg\, \sigma$, *then* $\sigma^{\mathsf{cert}} \vdash \mathsf{R}_0$.

For each Σ_1-sentence λ, we may define $[\lambda]$ to be the sentence $(\lambda^\bullet)^{\mathsf{cert}}$. Then, Theorem 2.9 immediately follows from Theorem 4.5 and Proposition 4.3.

Before proving Theorem 4.5, we investigate an $\mathbb{L}_{\mathsf{a}}$-model $\mathcal{M}$. We assume that

(†): $\mathfrak{v}$ is a designated certified element and k is a natural number such that, for all $m < k$, we have $\mathcal{M} \models \underline{m} \neq \mathfrak{v}$.

Our first lemma is concerned with successor.

LEMMA 4.6 (†). *For each $m \leqslant k$, we have $\mathcal{M} \models \underline{m} \leqslant \mathfrak{v}$.*

PROOF. We prove the lemma by induction on $m \leqslant k$. For $m = 0$, we have $\mathcal{M} \models 0 \leqslant \mathfrak{v}$ by A1. Suppose that the lemma holds for m with $m + 1 \leqslant k$. Then, $\mathcal{M} \models \underline{m} \leqslant \mathfrak{v}$ by the induction hypothesis. Since $m < k$ and, hence, by Assumption (†), $\mathcal{M} \models \underline{m} \neq \mathfrak{v}$, we find $\mathcal{M} \models \underline{m} < \mathfrak{v}$. So, by A2, we may conclude that $\mathcal{M} \models \underline{m+1} \leqslant \mathfrak{v}$. ⊣

In the proofs of the following lemmas, we will use Lemma 4.6 without any reference.

LEMMA 4.7 (†). *For any $m \leqslant k$, we have* $\mathcal{M} \models \forall y \left(y \leqslant \underline{m} \leftrightarrow \bigvee_{l \leqslant m} y = \underline{l}\right)$.

PROOF. We prove our lemma by induction on $m \leqslant k$. The case $m = 0$ is precisely A3. Suppose we have our equivalence for m with $m + 1 \leqslant k$. We note that $m < k$,

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and hence $\mathcal{M} \models \underline{m} < \mathfrak{v}$. Thus, we have, by A4:

$$\begin{aligned} \mathcal{M} \models y \leqslant \underline{m+1} &\leftrightarrow y \leqslant \mathsf{s}\underline{m} \\ &\leftrightarrow y \leqslant \underline{m} \vee y = \mathsf{s}\underline{m} \\ &\leftrightarrow \bigvee_{l \leqslant m} y = \underline{l} \vee y = \underline{m+1} \\ &\leftrightarrow \bigvee_{l \leqslant m+1} y = \underline{l}. \end{aligned}$$

⊣

LEMMA 4.8 (†). *For any $m, n, p \leqslant k$, we have $\mathcal{M} \models (\underline{m} \times \underline{n}) + \underline{p} = \underline{(m \times n) + p}$.*

PROOF. We prove the lemma by induction on n. For $n = 0$, we prove that $\mathcal{M} \models (\underline{m} \times 0) + \underline{p} = \underline{p}$ by induction on p. For $p = 0$, we see $\mathcal{M} \models (\underline{m} \times 0) + 0 = \underline{m} \times 0$ holds by A7. Since $\mathcal{M} \models \underline{m} \times 0 = 0$ by A9, we obtain $\mathcal{M} \models (\underline{m} \times 0) + 0 = 0$.

Suppose that the statement holds for p with $p + 1 \leqslant k$. Then, by A8 and the induction hypothesis for p, we get

$$\mathcal{M} \models (\underline{m} \times 0) + \underline{(p+1)} = (\underline{m} \times 0) + \mathsf{s}\underline{p} = \mathsf{s}((\underline{m} \times 0) + \underline{p}) = \mathsf{s}\underline{p} = \underline{p+1}.$$

We have proved that the lemma holds for $n = 0$.

Suppose that the lemma holds for n with $n + 1 \leqslant k$. We prove that $\mathcal{M} \models (\underline{m} \times \underline{(n+1)}) + \underline{p} = \underline{(m \times (n+1)) + p}$ by induction on p. For $p = 0$, by A7, A10, and the induction hypothesis for n, using that $m \leqslant k$,

$$\begin{aligned} \mathcal{M} \models (\underline{m} \times \underline{(n+1)}) + 0 &= \underline{m} \times \mathsf{s}\underline{n} \\ &= (\underline{m} \times \underline{n}) + \underline{m} \\ &= \underline{(m \times n) + m} \\ &= \underline{(m \times (n+1)) + 0}. \end{aligned}$$

Assume that the statement holds for p with $p + 1 \leqslant k$. By A8 and the induction hypothesis for p,

$$\begin{aligned} \mathcal{M} \models (\underline{m} \times \underline{(n+1)}) + \underline{(p+1)} &= (\underline{m} \times \underline{(n+1)}) + \mathsf{s}\underline{p} \\ &= \mathsf{s}((\underline{m} \times \underline{(n+1)}) + \underline{p}) \\ &= \mathsf{s}(\underline{(m \times (n+1)) + p}) \\ &= \underline{(m \times (n+1)) + (p+1)}. \end{aligned}$$

⊣

LEMMA 4.9 (†). *Let $m, n \leqslant k$.*

1. $\mathcal{M} \models \underline{m} \times \underline{n} = \underline{m \times n}$,
2. $\mathcal{M} \models \underline{m} + \underline{n} = \underline{m + n}$.

PROOF. 1. By Lemma 4.8, we get $\mathcal{M} \models (\underline{m} \times \underline{n}) + 0 = \underline{m \times n}$. By A7, we obtain $\mathcal{M} \models \underline{m} \times \underline{n} = \underline{m \times n}$.

2. If $k = 0$, then $m = n = 0$. By A7, $\mathcal{M} \models (0 \times 0) + 0 = 0 \times 0$. Since $\mathcal{M} \models 0 \times 0 = 0$ by A9, we obtain $\mathcal{M} \models 0 + 0 = 0$.

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If $k \geq 1$, then by Lemma 4.8 and Clause 1, we obtain

$$\mathcal{M} \models \underline{m} + \underline{n} = (\underline{m} \times \underline{1}) + \underline{n} = \underline{m + n}.$$

⊣

LEMMA 4.10 (†). *Suppose $m \leqslant k^2 + k$ and $l \leqslant k$ and $m \neq l$. Then, $\mathcal{M} \models \underline{m} \neq \underline{l}$.*

PROOF. We prove our result by induction on $l \leqslant k$. We will use the fact that (‡) every $m \leqslant k^2 + k$, can be written as $(k \times m_0) + m_1$ for some $m_0, m_1 \leqslant k$.

For the base case, suppose $l = 0$, $m \leqslant k^2 + k$, and $m \neq l$. We have $m = m' + 1$, so $\underline{m} = \mathsf{s}\underline{m'}$. So we are done by A5 in combination with (‡) and Lemma 4.8.

We treat the successor case. Suppose $l = l' + 1$ and we have the desired result for l'. Suppose also $m \leqslant k^2 + k$, and $m \neq l$. In case $m = 0$, we are done by A5 in combination with (‡) and Lemma 4.8. Suppose, $m = ((k \times m_0) + m_1) + 1$, where $m_0, m_1 \leqslant k$. Suppose $\mathcal{M} \models \underline{m} = \underline{l}$. Then, by Lemma 4.8, we find $\mathcal{M} \models \mathsf{s}((\underline{k} \times \underline{m_0}) + \underline{m_1}) = \mathsf{s}\underline{l'}$. By A6, we may conclude $\mathcal{M} \models (\underline{k} \times \underline{m_0}) + \underline{m_1} = \underline{l'}$. But this contradicts the induction hypothesis. ⊣

LEMMA 4.11 (†). *For any pure Δ_0-formula $\varphi(x_0, \ldots, x_i)$, and $n_0, \ldots, n_i \leqslant k$, if $\mathbb{N} \models \varphi(\underline{n_0}, \ldots, \underline{n_i})$, then $\mathcal{M} \models \varphi(\underline{n_0}, \ldots, \underline{n_i})$.*

PROOF. For any pure Δ_0-formula $\varphi(x_0, \ldots, x_i)$, and $n_0, \ldots, n_i \leqslant k$, we simultaneously prove the following two clauses by induction on the construction of φ:

1. if $\mathbb{N} \models \varphi(\underline{n_0}, \ldots, \underline{n_i})$, then $\mathcal{M} \models \varphi(\underline{n_0}, \ldots, \underline{n_i})$,
2. if $\mathbb{N} \models \neg\varphi(\underline{n_0}, \ldots, \underline{n_i})$, then $\mathcal{M} \models \neg\varphi(\underline{n_0}, \ldots, \underline{n_i})$.

Firstly, we prove that the statement holds for atomic formulas. We distinguish the following cases.

- φ is of the form $x_0 = x_1$.
  1. If $\mathbb{N} \models \underline{n_0} = \underline{n_1}$, then $n_0 = n_1$, and hence $\mathcal{M} \models \underline{n_0} = \underline{n_1}$.
  2. If $\mathbb{N} \models \underline{n_0} \neq \underline{n_1}$, then $n_0 \neq n_1$. By Lemma 4.10, $\mathcal{M} \models \underline{n_0} \neq \underline{n_1}$ because $n_0, n_1 \leqslant k$.
- φ is of the form $0 = x_0$.
  1. If $\mathbb{N} \models 0 = \underline{n_0}$, then $0 = n_0$, and hence $\mathcal{M} \models 0 = \underline{n_0}$.
  2. If $\mathbb{N} \models 0 \neq \underline{n_0}$, then $0 \neq n_0$. By Lemma 4.10, $\mathcal{M} \models 0 \neq \underline{n_0}$ because $n_0 \leqslant k$.
- φ is of the form $\mathsf{s}x_0 = x_1$.
  1. If $\mathbb{N} \models \mathsf{s}\underline{n_0} = \underline{n_1}$, then $n_0 + 1 = n_1$, and $\mathcal{M} \models \underline{n_0 + 1} = \underline{n_1}$. This means $\mathcal{M} \models \mathsf{s}\underline{n_0} = \underline{n_1}$.
  2. If $\mathbb{N} \models \mathsf{s}\underline{n_0} \neq \underline{n_1}$, then $n_0 + 1 \neq n_1$. If $k = 0$, then $n_0 = n_1 = 0$. By A5, we get $\mathcal{M} \models \mathsf{s}((\overline{0} \times \overline{0}) + 0) \neq 0$. By Lemma 4.8, $\mathcal{M} \models (0 \times 0) + 0 = 0$, and hence $\mathcal{M} \models \underline{1} \neq 0$. This means $\mathcal{M} \models \mathsf{s}\underline{n_0} \neq \underline{n_1}$. If $k \geq 1$, then $n_0 + 1 \leqslant k^2 + k$. By Lemma 4.10, we have $\mathcal{M} \models \mathsf{s}\underline{n_0} \neq \underline{n_1}$.
- φ is of the form $x_0 + x_1 = x_2$.
  1. If $\mathbb{N} \models \underline{n_0} + \underline{n_1} = \underline{n_2}$, then $n_0 + n_1 = n_2$, and $\mathcal{M} \models \underline{n_0 + n_1} = \underline{n_2}$. By Lemma 4.9, $\mathcal{M} \models \underline{n_0} + \underline{n_1} = \underline{n_2}$ because $n_0, n_1 \leqslant k$.
  2. If $\mathbb{N} \models \underline{n_0} + \underline{n_1} \neq \underline{n_2}$, then $n_0 + n_1 \neq n_2$. Since $n_0 + n_1 \leqslant k^2 + k$ and $n_2 \leqslant k$, by Lemma 4.10, we have $\mathcal{M} \models \underline{n_0 + n_1} \neq \underline{n_2}$. By Lemma 4.9, we have $\mathcal{M} \models \underline{n_0} + \underline{n_1} \neq \underline{n_2}$.

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- φ is of the form $x_0 \times x_1 = x_2$.
  1. If $\mathbb{N} \models \underline{n_0} \times \underline{n_1} = \underline{n_2}$, then $n_0 \times n_1 = n_2$, and $\mathcal{M} \models \underline{n_0 \times n_1} = \underline{n_2}$. By Lemma 4.9, $\mathcal{M} \models \underline{n_0} \times \underline{n_1} = \underline{n_2}$ because $n_0, n_1 \leqslant k$.
  2. If $\mathbb{N} \models \underline{n_0} \times \underline{n_1} \neq \underline{n_2}$, then $n_0 \times n_1 \neq n_2$. By Lemma 4.10, $\mathcal{M} \models \underline{n_0 \times n_1} \neq \underline{n_2}$ because $n_0 \times n_1 \leqslant k^2 + k$ and $n_2 \leqslant k$. By Lemma 4.9, $\mathcal{M} \models \underline{n_0} \times \underline{n_1} \neq \underline{n_2}$.
- φ is of the form $x_0 \leqslant x_1$.
  1. If $\mathbb{N} \models \underline{n_0} \leqslant \underline{n_1}$, then $n_0 \leqslant n_1$. Since $\mathcal{M} \models \bigvee_{l \leqslant n_1} \underline{n_0} = \underline{l}$, we have, by Lemma 4.7, $\mathcal{M} \models \underline{n_0} \leqslant \underline{n_1}$ because $n_1 \leqslant k$.
  2. If $\mathbb{N} \models \underline{n_0} \not\leqslant \underline{n_1}$, then $n_1 < n_0$. For each $l \leqslant n_1$, we have $n_0 \neq l$. We have $M \models n_0 \neq \overline{l}$ by Lemma 4.10 because $n_0, l \leq k$. Then, $\mathcal{M} \models \bigwedge_{l \leqslant n_1} \underline{n_0} \neq \underline{l}$. Since $n_1 \leqslant k$, by Lemma 4.7, we have $\mathcal{M} \models \underline{n_0} \not\leqslant \underline{n_1}$.

Secondly, we prove the induction steps. The case that φ is one of the forms $\neg \varphi_0$ and $\varphi_0 * \varphi_1$ for $* \in \{\wedge, \vee, \to\}$ is easily shown by the induction hypothesis. It suffices to show the case that φ is of the form $\exists y \leqslant x_j \, \varphi_0(x_0, \dots, x_i, y)$, where the claim holds for φ_0. The case that φ is of the form $\forall y \leqslant x_j \, \varphi_0(x_0, \dots, x_i, y)$ is proved similarly.

1. Suppose $\mathbb{N} \models \exists y \leqslant \underline{n_j} \, \varphi_0(\underline{n_0}, \dots, \underline{n_i}, y)$. Then, there exists an n_{i+1} such that $n_{i+1} \leq n_j \leq k$ and $\mathbb{N} \models \varphi_0(\underline{n_0}, \dots, \underline{n_i}, \underline{n_{i+1}})$. By the induction hypothesis, $\mathcal{M} \models \varphi_0(\underline{n_0}, \dots, \underline{n_i}, \underline{n_{i+1}})$. Also, we have already proved $\mathcal{M} \models \underline{n_{i+1}} \leqslant \underline{n_j}$. Therefore, we obtain $\mathcal{M} \models \exists y \leqslant \underline{n_j} \, \varphi_0(\underline{n_0}, \dots, \underline{n_i}, y)$.

2. Suppose we have $\mathbb{N} \models \neg \exists y \leqslant \underline{n_j} \, \varphi_0(\underline{n_0}, \dots, \underline{n_i}, y)$, or, equivalently, we have $\mathbb{N} \models \forall y \leqslant \underline{n_j} \, \neg \varphi_0(\underline{n_0}, \dots, \underline{n_i}, y)$. Then, we have $\mathbb{N} \models \neg \varphi_0(\underline{n_0}, \dots, \underline{n_i}, \underline{l})$, for each $l \leqslant n_j$. By the induction hypothesis, we have $\mathcal{M} \models \neg \varphi_0(\underline{n_0}, \dots, \underline{n_i}, \underline{l})$ for each $l \leqslant n_j$. Thus,

$$\mathcal{M} \models \forall y \Big(\bigvee_{l \leqslant n_j} y = \underline{l} \to \neg \varphi_0(\underline{n_0}, \dots, \underline{n_i}, y) \Big).$$

By Lemma 4.7, we obtain $\mathcal{M} \models \forall y \leqslant \underline{n_j} \, \neg \varphi_0(\underline{n_0}, \dots, \underline{n_i}, y)$, and, hence, $\mathcal{M} \models \neg \exists y \leqslant \underline{n_j} \, \varphi_0(\underline{n_0}, \dots, \underline{n_i}, y)$. ⊣

We have finished our investigation of the model $\mathcal{M}$. We prove one further lemma as a bridge between the conditions on $\mathcal{M}$ of the preceding lemmas and a false Σ_1-sentence and, then, we are ready for the proof of Theorem 4.5.

LEMMA 4.12. *Let σ be a pure 1-Σ_1-sentence of the form $\exists x \, \sigma_0(x)$, where $\sigma_0(x)$ is a pure Δ_0-formula. Suppose $\mathbb{N} \models \neg \sigma$. Let $\mathcal{N}$ be an $\mathbb{L}_{\mathsf{a}}$-model and $\mathfrak{v} \in \mathcal{N}$ be a witness of σ^{cert} in $\mathcal{N}$, that is, $\mathcal{N} \models \mathsf{cert}(\mathfrak{v}) \wedge \sigma_0(\mathfrak{v})$. Then $\mathcal{N} \models \underline{m} < \mathfrak{v}$, for every $m \in \omega$.*

PROOF. Suppose $\mathbb{N} \models \neg \sigma$ and $\mathcal{N} \models \mathsf{cert}(\mathfrak{v}) \wedge \sigma_0(\mathfrak{v})$.

We first show that, for every $m \in \omega$, we have $\mathcal{N} \models \underline{m} \neq \mathfrak{v}$. Suppose $\mathcal{N} \models \underline{m} = \mathfrak{v}$, for some $m \in \omega$. Let $m^\star$ be the least such m. Then, the condition (†) holds for $\mathcal{N}$, $\mathfrak{v}$, and $m^\star$. Since $\mathbb{N} \models \neg \sigma_0(\underline{m^\star})$, we have, by Lemma 4.11, $\mathcal{N} \models \neg \sigma_0(\underline{m^\star})$. But this is impossible.

Since, for every m, we have $\mathcal{M} \models \underline{m} \neq \mathfrak{v}$, Lemma 4.6 gives us that, for every m, we have $\mathcal{M} \models \underline{m} < \mathfrak{v}$. ⊣

We now prove Theorem 4.5.

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PROOF OF THEOREM 4.5. *Ad 1:* The implication from σ^{cert} to σ is immediate.

Ad 2: It is obvious that $\mathbb{N} \models \forall v\, \mathsf{cert}(v)$ holds. Suppose $\mathbb{N} \models \sigma$. It follows that $\mathbb{N} \models \sigma^{\mathsf{cert}}$. Since σ^{cert} is Σ_1, we have, by Σ_1-completeness (Theorem 2.3), that $\mathsf{R}_0 \vdash \sigma^{\mathsf{cert}}$.

Ad 3: Suppose $\mathbb{N} \models \neg\sigma$. We prove that the theory σ^{cert} is Σ_1-complete.

Let $\mathcal{N}$ be any $\mathbb{L}_{\mathsf{a}}$-model of σ^{cert}. So, for some $\mathfrak{v}$, we have $\mathcal{N} \models \mathsf{cert}(\mathfrak{v}) \wedge \sigma_0(\mathfrak{v})$. By Lemma 4.12, we find that $\mathcal{N} \models \underline{m} \neq \mathfrak{v}$ for every $m \in \omega$. Hence, the condition (†) holds for $\mathcal{N}$, $\mathfrak{v}$, and all $k \in \omega$.

Let ψ be any Σ_1-sentence such that $\mathbb{N} \models \psi$. By Proposition 4.3, there exists a pure Δ_0-formula $\delta(x)$ satisfying the following conditions:

1. $\mathbb{N} \models \psi \leftrightarrow \exists x\, \delta(x)$,
2. $\exists x\, \delta(x) \rightarrow \psi$ is logically valid.

Then, $\mathbb{N} \models \exists x\, \delta(x)$, and, hence, $\mathbb{N} \models \delta(\underline{n})$ for some n. By Lemma 4.11, we have $\mathcal{N} \models \delta(\underline{n})$. By the completeness theorem, we obtain $\sigma^{\mathsf{cert}} \vdash \delta(\underline{n})$.[2] Thus, $\sigma^{\mathsf{cert}} \vdash \exists x\, \delta(x)$. Therefore, we obtain $\sigma^{\mathsf{cert}} \vdash \psi$.

Finally, by Theorem 2.3, we conclude that $\sigma^{\mathsf{cert}} \vdash \mathsf{R}_0$. ⊣

§5. Certified extension generalized. In this section, we generalize Theorem 2.9 to a wide class of further base theories.

5.1. The theory $\mathsf{R}_{0\mathsf{p}}$. We start with reproving Theorem 2.9 for a slightly different base theory. We define:

- $\mathbb{L}_{\mathsf{ap}}$ is the arithmetical signature $\mathbb{L}_{\mathsf{a}}$ extended by a unary predicate symbol P.
- $\mathfrak{id}_{\mathsf{ap}} := \mathfrak{id}_{\mathbb{L}_{\mathsf{ap}}}$. (The notion $\mathfrak{id}$ is explained in Definition 3.1.)
- $\mathsf{R}_{0\mathsf{p}}$ is the the $\mathbb{L}_{\mathsf{ap}}$-theory obtained by extending R_0 with the axioms $\mathsf{P}(\underline{n})$, for all $n \in \omega$.

We have:

THEOREM 5.1 (Second Certified Extension Theorem). *For each Σ_1-sentence σ, we can effectively find a sentence $[\![\sigma]\!]$ satisfying the following conditions*:

1. $[\![\sigma]\!] \vdash \sigma$.
2. *If* $\mathbb{N} \models \sigma$, *then* $\mathsf{R}_{0\mathsf{p}} \vdash [\![\sigma]\!]$.
3. *If* $\mathbb{N} \models \neg\sigma$, *then* $[\![\sigma]\!] \vdash \mathsf{R}_{0\mathsf{p}}$.

PROOF. For each pure 1-Σ_1-sentence σ, let $\sigma^{\mathsf{cert}_{\mathsf{p}}}$ be the $\mathbb{L}_{\mathsf{a}}$-sentence:

$$\bigwedge \mathfrak{id}_{\mathsf{ap}} \wedge \exists v \left(\mathsf{cert}(v) \wedge \sigma_0(v) \wedge \forall x \leqslant v\, \mathsf{P}(x)\right).$$

The proof of Theorem 4.5 can be repeated for $\mathsf{R}_{0\mathsf{p}}$ by using the sentence $\sigma^{\mathsf{cert}_{\mathsf{p}}}$. Thus, for each Σ_1-sentence σ, it is shown that $[\![\sigma]\!] = (\sigma^{*})^{\mathsf{cert}_{\mathsf{p}}}$ satisfies the required conditions. ⊣

The addition of $\bigwedge \mathfrak{id}_{\mathsf{ap}}$ in the definition of $[\![\sigma]\!]$ is superfluous in the context of the proof of Theorem 5.1. It is added since it also delivers the following simple insight.

THEOREM 5.2. *Suppose τ is a translation from $\mathbb{L}_{\mathsf{ap}}$ to a signature of some theory. Let σ be a Σ_1-sentence. Then, $\mathsf{K}_\tau : [\![\sigma]\!]^\tau \rhd [\![\sigma]\!]$, where K_τ is the interpretation based on τ.*

[2] In fact, the witnessing proof can be directly read off from the proofs of the lemmas.

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5.2. $\mathsf{R}_{0\mathsf{p}}$**-sourced theories.** We define:

- Let T and U be c.e. theories and let Θ be the signature of T. Let τ be a translation from Θ to the U-signature. The theory U is τ-T-*sourced* iff U is deductively equivalent to $(T + \mathfrak{id}_\Theta)^\tau$. The theory U is T-sourced if it is τ-T-sourced, for some τ.

In this paper we just focus on $\mathsf{R}_{0\mathsf{p}}$-sourced theories. We have:

THEOREM 5.3 (Generalized certified extension theorem). *Suppose W is τ-$\mathsf{R}_{0\mathsf{p}}$-sourced. Then, for each Σ_1-sentence σ, we can effectively find a sentence $[\![\sigma]\!]$ satisfying the following conditions*:

1. $[\![\sigma]\!]^\tau \vdash \sigma^\tau$.
2. *If* $\mathbb{N} \models \sigma$, *then* $W \vdash [\![\sigma]\!]^\tau$.
3. *If* $\mathbb{N} \models \neg\sigma$, *then* $[\![\sigma]\!]^\tau \vdash W$.

PROOF. This is immediate from Theorem 5.1. Note that, for (1), we use Theorem 5.2. ⊣

Here is a first simple example of a $\mathsf{R}_{0\mathsf{p}}$-sourced theory.

EXAMPLE 5.4. Any finitely axiomatized theory A that interprets R_0 is $\mathsf{R}_{0\mathsf{p}}$-sourced. We note that this example is not very useful, since we already know that the applications we want from $\mathsf{R}_{0\mathsf{p}}$-sourced theories hold for finitely axiomatized theories.

Here is a second example. We remind the reader of Vaught's set theory VS. It is a theory in the signature with the single binary relation symbol $\in$, axiomatized by the following axioms.

VSn: $\forall x_0 \ldots \forall x_{n-1}\, \exists z\, \forall y\, (y \in z \leftrightarrow \bigvee_{i<n} y = x_i)$

We note that in the case that $n = 0$, we have an axiom that guarantees the existence of some empty sets. We have:

THEOREM 5.5. VS *interprets* R_0.

PROOF. In [25, Appendix A], it was proven that VS interprets R. So, *a fortiori*, VS interprets R_0. ⊣

THEOREM 5.6. *The theory* VS *is* $\mathsf{R}_{0\mathsf{p}}$*-sourced.*

The idea of the proof is to represent VS as Adjunctive Set Theory AS with local size restrictions on the sets to which one can apply adjunction.

PROOF. Let v be the translation on which an interpretation of R_0 in VS is based. We assume that v is one-dimensional—as the translation provided by [25] is. The many-dimensional case only requires minor adaptations.

We extend v to $v^\star$ by providing a translation of P. The statement $\mathsf{P}_{v^\star}(x)$ will roughly say that VS0 and VS2 and that, whenever we have a set y of cardinality $\leqslant x$, we may adjoin any z to y. Here VS0 and VS2 are needed to provide the necessary coding machinery. Here are the ingredients for formulation of our statement.

- f is an injection from objects to numbers or $\mathsf{in}(f)$ iff f is a set of pairs, f is functional w.r.t. $=$, the objects in the range of f are in δ_v, and f is injective in the sense that, if $f(u) =_v f(v)$, then $u = v$.

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- $\mathsf{dom}(f, y)$ iff $\forall w\,(w \in y \leftrightarrow \exists v\, f(w) = v)$.
- $\mathsf{card}_{\leqslant}(y, x)$ iff there is an f with $\mathsf{in}(f)$ and $\mathsf{dom}(f, y)$ and $\forall v \in y\, f(v) <_v x$.
- $\mathsf{P}_{v^\star}(x)$ iff VS0 and VS2 and

$$\forall y\, \forall z\, (\mathsf{card}_{\leqslant}(y, x) \to \exists w\, \forall u\, (u \in w \leftrightarrow (u \in x \vee u = z))).$$

The deductive equivalence between $\mathsf{R}_{0\mathsf{p}}^{v^\star}$ and VS is now easy to verify. ⊣

We provide two larger classes of examples of $\mathsf{R}_{0\mathsf{p}}$-sourced theories. Consider any signature $\mathcal{L}$. Let τ_0 be a translation of $\mathcal{L}$ in $\mathbb{L}_{\mathsf{a}}$.

Let $\chi_0(x_0, \dots, x_{k_0-1}), \dots, \chi_m(x_0, \dots, x_{k_m-1})$ be any $\mathcal{L}$-formulas and let $X_0, \dots, X_m$ be any computable relations on ω such that $X_i \subseteq \omega^{k_i}$ for each $i \leqslant m$. We write $\underline{n}$ for the τ_0-numerals in the context of $\mathcal{L}$. We define the theory $\mathsf{R}_0[\tau_0; \chi_0, \dots, \chi_m; X_0, \dots, X_m]$ as follows:

$$\mathsf{R}_0^{\tau_0} + \bigcup_{i \leqslant m} \{\chi_i(\underline{n_0}, \dots, \underline{n_{k_i-1}}) \mid (n_0, \dots, n_{k_i-1}) \in X_i\}.$$

We use $\mathsf{R}_0[\tau_0; \vec{\chi}; \vec{X}]$ as an abbreviation of $\mathsf{R}_0[\tau_0; \chi_0, \dots, \chi_m; X_0, \dots, X_m]$. If $\mathcal{L}$ is $\mathbb{L}_{\mathsf{a}}$ and if τ_0 is the identity translation on $\mathbb{L}_{\mathsf{a}}$, we simply omit τ_0.

THEOREM 5.7. *Let $\mathcal{L}$ be a signature, let τ_0 be a translation of $\mathbb{L}_{\mathsf{a}}$ into $\mathcal{L}$. Let $\vec{\chi}$ be any $\mathcal{L}$-formulas and let $\vec{X}$ be computable relations on ω matching $\vec{\chi}$. Then,* $\mathsf{R}_0[\tau_0; \vec{\chi}; \vec{X}]$ *is* $\mathsf{R}_{0\mathsf{p}}$*-sourced.*

PROOF. To simplify inessentially we assume that τ_0 is 1-dimensional. For each $i \leqslant m$, let $\xi_i(x_0, \dots, x_{k_i-1})$ be an $\mathbb{L}_{\mathsf{a}}$-formula representing X_i in R_0. That is, for any $n_0, \dots, n_{k_i-1} \in \omega$, if $(n_0, \dots, n_{k_i-1}) \in X_i$, then $\mathsf{R}_0 \vdash \xi_i(\underline{n_0}, \dots, \underline{n_{k_i-1}})$; and if $(n_0, \dots, n_{k_i-1}) \notin X_i$, then $\mathsf{R}_0 \vdash \neg\, \xi_i(\underline{n_0}, \dots, \underline{n_{k_i-1}})$. Let $\chi^*(x)$ be the $\mathcal{L}$-formula

$$\bigwedge_{i \leqslant m} \forall x_0 \leqslant_{\tau_0} x \cdots \forall x_{k_i-1} \leqslant_{\tau_0} x\, (\xi_i^{\tau_0}(x_0, \dots, x_{k_i-1}) \to \chi_i(x_0, \dots, x_{k_i-1})).$$

We extend τ_0 to τ by setting $\mathsf{P}_\tau(x) := \chi^*(x)$. Then, clearly, $\mathsf{R}_0[\tau_0; \vec{\chi}; \vec{X}]$ is deductively equivalent to $\mathsf{R}_{0\mathsf{p}}^{\tau}$. ⊣

EXAMPLE 5.8. The theory R_0 is $\mathsf{R}_{0\mathsf{p}}$-sourced since it is deductively equivalent to $\mathsf{R}_0[x = x; \emptyset]$ and the theory R is $\mathsf{R}_{0\mathsf{p}}$-sourced since it is deductively equivalent to $\mathsf{R}_0[\forall y\, (y \leqslant x \vee x \leqslant y); \omega]$.

EXAMPLE 5.9. Suppose $\forall \vec{x}\, P_0(\vec{x})$ is a true pure Π_1-sentence. Consider $W := \mathsf{R}_0[P_0; \omega^n]$, where n is the length of $\vec{x}$. Then, by Σ_1-completeness, W is deductively equivalent to R_0. However, modulo deductive equivalence, the corresponding $(\sigma^{\mathsf{cert}_{\mathsf{p}}})^\tau$ has the form $\exists z\, (\mathsf{cert}(z) \wedge \forall \vec{x} \leqslant z\, P_0(\vec{x}) \wedge \exists z\, \sigma_0(z))$. So, we may think of $\mathsf{cert}(z) \wedge \forall \vec{x} \leqslant z\, P_0(\vec{x})$ as replacing $\mathsf{cert}(z)$. Thus, we may add all kind of desirable properties to cert like the linearity of $\leqslant$ below the certified element. This may be useful if we want to use certification as the basis of an interpretation of a stronger theory in R_0.

In the development below, we use the notion of *depth-of-quantifier-alternations complexity*, henceforth, simply *complexity*. For a careful exposition of this notion, see [24].

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THEOREM 5.10. *Consider a finitely axiomatized theory A and a number n and a computable set X of A-sentences of complexity $\leqslant n$. Suppose that*:

- *A interprets R_0 via an interpretation based on translation τ_0.*
- *There is an A formula Φ such that $A \vdash \Phi(\ulcorner \phi \urcorner) \leftrightarrow \phi$, for all A-sentences ϕ of complexity $\leqslant n$. Here the numerals are the τ_0-numerals. (Note that these could be sequences modulo a definable equivalence relation.)*

Then $A + X$ is an $\mathsf{R}_{0\mathsf{p}}$-sourced c.e. theory.

PROOF. We write $\top$ for the non-empty zero-ary relation[3] and $\bigwedge A$ for the sentence that is the conjunction of the axioms of A. We note that $A + X$ is deductively equivalent to $\mathsf{R}_0[\tau_0; \bigwedge A, \Phi; \top, X]$. ⊣

A theory is *restricted*, iff it can be axiomatized by axioms of complexity $\leqslant n$, for some fixed n. In [24] it has been verified that sequential restricted c.e. theories can be written as $A + X$, where A and X satisfy the conditions of Theorem 5.10. So, we have:

COROLLARY 5.11. *Any restricted sequential c.e. theory is $\mathsf{R}_{0\mathsf{p}}$-sourced.*

Examples of restricted sequential c.e. theories are:

- $\mathcal{U}_{\mathsf{PA}} := \mathsf{S}^1_2 + \mathsf{Con}_1(\mathsf{PA}) + \mathsf{Con}_2(\mathsf{PA}) + \dots$, where the $\mathsf{Con}_i(\mathsf{PA})$ are consistency statements where we restrict the PA-axioms to those with Gödel number $\leqslant i$ and where we restrict the proofs to those in which only formulas of depth of quantifier alternations $\leqslant i$ occur.
- $\mathrm{I}\Delta_0 + \Omega_1 + \Omega_2 + \dots$,
- $\mathsf{EA} + \mathsf{Con}(\mathsf{EA}) + \mathsf{Con}(\mathsf{EA} + \mathsf{Con}(\mathsf{EA})) + \dots$,
- PRA (in a suitable version in finite signature).

We end with a closure condition.

THEOREM 5.12. *Suppose U and V are $\mathsf{R}_{0\mathsf{p}}$-sourced theories in the same signature $\mathcal{L}$, as witnessed by τ and τ'. Suppose the restriction of τ and τ' to the arithmetical signature is a shared part τ_0. Then $U \cup V$ is $\mathsf{R}_{0\mathsf{p}}$-sourced.*

PROOF. We take as witnessing translation $\tau^\star$ for $U \cup V$, the translation τ_0 on the arithmetical vocabulary and $\mathsf{P}_{\tau^\star}(\vec{x}) := (\mathsf{P}_\tau(\vec{x}) \wedge \mathsf{P}_{\tau'}(\vec{x}))$. Alternatively, we note that $U \cup V$ is deductively equivalent to $\mathsf{R}_0[\tau_0; \mathsf{P}_\tau(\vec{x}), \mathsf{P}_{\tau'}(\vec{x}); \omega, \omega]$. ⊣

§6. Witness comparisons and fixed points. In this section, we give the basic definitions and facts for witness comparison. Moreover, we discuss the Gödel fixed point lemma and its interaction with witness comparison.

6.1. Comparing the witnesses. We define witness comparison. Let $\phi := \exists x\, \phi_0(x)$ and $\psi := \exists y\, \psi_0(y)$. We define:

- $\phi \leqslant \psi :\leftrightarrow \exists x\, (\phi_0(x) \wedge \forall y < x \neg \psi_0(y))$,
- $\phi < \psi :\leftrightarrow \exists x\, (\phi_0(x) \wedge \forall y \leqslant x \neg \psi_0(y))$,
- $(\phi \leqslant \psi)^\perp := (\psi < \phi)$,
- $(\phi < \psi)^\perp := (\psi \leqslant \phi)$.

[3] If we represent relations as sets of tuples, then $\top = \{\varepsilon\}$, where ε is the empty sequence.

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We have to do some preliminary work to compensate for the fact that in R_0 we are lacking the axiom R5 which says, for every numeral n, that $x \leqslant \underline{n} \vee \underline{n} \leqslant x$. We say that x is *well-behaved* or $\mathsf{wb}(x)$ iff it satisfies A1 and A2 of the definition of certification, i.o.w., $\mathsf{wb}(x)$ iff $0 \leqslant x$ and $\forall y < x\ \mathsf{s}y \leqslant x$. We say that a sentence is *well-behaved* if it is of the form $\exists x\,(\mathsf{wb}(x) \wedge \psi(x))$.

REMARK 6.1. The idea of well-behavedness, though not the name, is due to Cobham, see Jones and Shepherdson [6]. In fact, for our purposes, we could also have worked with *certified* instead of *well-behaved*, but we found it attractive to use the lightest possible means to obtain the results.

We have the following lemma.

LEMMA 6.2.

a. $\mathsf{R}_0 \vdash \mathsf{wb}(x) \to (x < \underline{n} \vee \underline{n} \leqslant x)$.
b. $\mathsf{R}_0 \vdash \mathsf{wb}(x) \to (x \leqslant \underline{n} \vee \underline{n} < x)$.

PROOF. (a) can be proved by a simple induction on n and (b) is immediate from (a). ⊣

It follows that:

LEMMA 6.3. *Suppose σ and σ' are 1-Σ_1-sentences and σ' is well-behaved.*

a. *If* $\mathbb{N} \models \sigma \leqslant \sigma'$, *then* $\mathsf{R}_0 \vdash \neg\, \sigma' < \sigma$.
b. *If* $\mathbb{N} \models \sigma < \sigma'$, *then*, $\mathsf{R}_0 \vdash \neg\, \sigma' \leqslant \sigma$.

PROOF. Let $\sigma = \exists x\, \sigma_0(x)$ and $\sigma' = \exists y\, (\mathsf{wb}(y) \wedge \sigma_0'(y))$.

We treat (a). Suppose $\sigma \leqslant \sigma'$ is true in the natural numbers. Then, for some n, we have $\mathsf{R}_0 \vdash \sigma_0(\underline{n})$. Moreover, for all $k < n$, we have (†) $\mathsf{R}_0 \vdash \neg\, \sigma_0'(\underline{k})$. We reason in R_0. Suppose $\sigma' < \sigma$. Then, for some well-behaved y, we have $\sigma_0'(y)$ and (‡) $\forall z \leqslant y\, \neg\, \sigma_0(z)$. By Lemma 6.2, we find that $y < \underline{n}$ or $\underline{n} \leqslant y$. The first disjunct contradicts (†) and the second disjunct contradicts (‡).

The proof of (b) is similar. ⊣

We prove the result that gives us the desired applications.

THEOREM 6.4. *Suppose σ and σ' are 1-Σ_1-sentences, σ' is well-behaved and ρ is a Σ_1-sentence.*

a. *If* $\mathbb{N} \models \sigma \leqslant \sigma'$ *and* $\mathsf{R}_0 \vdash \rho \leftrightarrow \sigma' < \sigma$, *then* $[\rho]$ *is inconsistent*.
b. *If* $\mathbb{N} \models \sigma < \sigma'$ *and* $\mathsf{R}_0 \vdash \rho \leftrightarrow \sigma' \leqslant \sigma$, *then* $[\rho]$ *is inconsistent*.

PROOF. We prove (a). Suppose $\mathbb{N} \models \sigma \leqslant \sigma'$ and $\mathsf{R}_0 \vdash \rho \leftrightarrow \sigma' < \sigma$. By Lemma 6.3, we have $\mathsf{R}_0 \vdash \neg\, \sigma' < \sigma$, and hence $\mathsf{R}_0 \vdash \neg\, \rho$. It follows that $\mathbb{N} \models \neg\, \rho$. So, by Theorem 2.9, that $[\rho] \vdash \mathsf{R}_0$, and, hence, $[\rho] \vdash \neg\, \rho$. On the other hand, again by Theorem 2.9, we have $[\rho] \vdash \rho$. *Ergo*, $[\rho]$ is inconsistent. ⊣

By a trivial adaptation of the above argument, we also have:

THEOREM 6.5. *Let σ and σ' be 1-Σ_1-sentences, where σ' is well behaved. Let ρ be any Σ_1-sentence. We have*:

a. *If* $\mathbb{N} \models \sigma \leqslant \sigma'$ *and* $\mathsf{R}_0 \vdash \rho \leftrightarrow \sigma' < \sigma$, *then* $[\![\rho]\!]$ *is inconsistent*.
b. *If* $\mathbb{N} \models \sigma < \sigma'$ *and* $\mathsf{R}_0 \vdash \rho \leftrightarrow \sigma' \leqslant \sigma$, *then* $[\![\rho]\!]$ *is inconsistent*.

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6.2. The fixed point lemma. In R_0 we can prove the representability of all recursive functions as the following lemma describes.

LEMMA 6.6. *For every recursive function F there is a 1-Σ_1-formula $\sigma(x, y)$ such that, whenever $F(n) = m$, we have $\mathsf{R}_0 \vdash \forall y\, (\sigma(\underline{n}, y) \leftrightarrow \underline{m} = y)$.*

PROOF. Consider any recursive function F and let $\sigma^\star(x, y) = \exists z\, \sigma_0^\star(x, y, z)$ be any 1-Σ_1-formula representing the graph of F. We take

$$\sigma(x, y) :\leftrightarrow \exists z \big(\mathsf{wb}(z) \wedge y \leqslant z \wedge \exists u \leqslant z\, \sigma_0^\star(x, y, u) \wedge \forall a \leqslant z\, \forall b \leqslant z\, (\sigma_0^\star(x, a, b) \to a = y)\big).$$

We now use Lemma 6.2, to mimic the well-known proof of the analog of the Lemma for the case of R. ⊣

We can prove the usual fixed point lemma using a representation of the substitution function provided by Lemma 6.6. However, we need a bit more.

THEOREM 6.7.

i. *Suppose $\sigma(x)$ is Σ_1. Then, we can find a Σ_1-formula η such that $\mathsf{R}_0 \vdash \eta \leftrightarrow \sigma(\ulcorner \eta \urcorner)$.*
ii. *Suppose $\sigma(x, y)$ and $\sigma'(x, y)$ are Σ_1-formulas. We can find Σ_1-formulas η and η' such that $\mathsf{R}_0 \vdash \eta \leftrightarrow \sigma(\ulcorner \eta \urcorner, \ulcorner \eta' \urcorner)$ and $\mathsf{R}_0 \vdash \eta' \leftrightarrow \sigma'(\ulcorner \eta \urcorner, \ulcorner \eta' \urcorner)$.*

PROOF. *We treat (i).*

We can obtain the desired result by a careful modification of the usual proof of the fixed point lemma. Alternatively, we can proceed as follows. Let $\Sigma_1^\dagger$ be the class given by $\chi ::= \sigma \mid (\chi \wedge \chi) \mid \exists v\, \chi$. Here σ ranges over Σ_1-sentences. We can easily rewrite a $\Sigma_1^\dagger$-sentence to a Σ_1-sentence by moving the relevant existential quantifiers out. The usual fixed point calculation delivers a $\Sigma_1^\dagger$-sentence using the wide scope elimination for the substitution function. Suppose we arithmetize this normalization function say as norm. We can represent this function in R_0 by Lemma 6.6. We can construct a Σ_1-formula $\sigma'(x)$ that functions as $\sigma(\mathsf{norm}(x))$. Let η' be the ordinary Gödel fixed point of σ' and let η be the normalized form of η'. Then, we have:

$$\begin{aligned} \mathsf{R}_0 \vdash \eta &\leftrightarrow \eta' \\ &\leftrightarrow \sigma'(\ulcorner \eta' \urcorner) \\ &\leftrightarrow \sigma(\ulcorner \eta \urcorner). \end{aligned}$$

The alternative proof is easily adapted to deliver also the desired double fixed point promised in (ii). ⊣

We note that the trick of the alternative proof will work for all kinds of normalizations.

A disadvantage of the modified Gödel-style fixed point construction is that it does not preserve witness comparison form. For the results in the next sections, this is not really needed, but it may sometimes lead to less elegant formulations. How nice it would be if we could preserve almost all forms of the original formula. An elegant way to do this is to employ a self-referential Gödel-numbering, which has self-reference built in.

The idea of a self-referential Gödel numbering was introduced by Saul Kripke in [7, Footnote 6]. It was worked out in some detail in [19]. Recently, two papers

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appeared exploring the notion further, to wit [8] and [4]. As far as we know the only place where the idea is truly applied is [3].

We follow the realization of [19]. The idea is simple. We extend $\mathbb{L}_\mathsf{a}$ with a fresh constant c and employ a standard Gödel numbering for the extended language. We then take the standard Gödel number $\ulcorner \phi(\mathsf{c}) \urcorner$ for the extended language to be the self-referential Gödel number of $\phi(\ulcorner \phi(\mathsf{c}) \urcorner)$ for $\mathbb{L}_\mathsf{a}$. Without further measures, the resulting Gödel numbering is not functional from sentences to numbers. The functionality of a Gödel number is not strictly needed, but we opt for making the numbering functional by stipulating that we take the smallest number assigned to a sentence by our non-functional version. In [19] it is carefully verified that, for a decent choice of the input standard Gödel numbering, the numbering so obtained fulfills all the desiderata of a self-referential Gödel numbering. We write $\lceil \phi \rceil$ for the self-referential Gödel number of ϕ. The crucial property is that for any $\phi(x)$ with at most x free, we can effectively find a ψ with $\psi = \phi(\lceil \psi \rceil)$.

In this paper we mostly opt for the ordinary numbering, accepting the use of Theorem 6.7 as the way to go. We will prove Theorem 8.17 twice, once with an ordinary numbering and once with a self-referential one in order to illustrate the use of a self-referential numbering.

§7. Vaught's theorems revisited. We give two proofs of Theorem 2.5 and prove a generalization of Theorem 2.6.

THEOREM 2.5 (Vaught [17, 5.2]). *The theory* R_0 *is strongly effectively inseparable.*

FIRST PROOF. Let $i, j \in \omega$ be such that $\mathsf{R}_{0\mathfrak{p}} \subseteq \mathsf{W}_i$, $\emptyset_\mathfrak{r} \subseteq \mathsf{W}_j$, and $\mathsf{W}_i \cap \mathsf{W}_j = \emptyset$. Let (X, Y) be any effectively inseparable pair of c.e. sets. We can clearly find 1-Σ_1-formulas $\xi(x)$ and $\eta(x)$ that represent X respectively Y. We can clearly arrange that ξ and η are well-behaved.

For each natural number n, let σ_n be the 1-Σ_1-sentence $\xi(\underline{n}) \leqslant \eta(\underline{n})$. We can effectively find natural numbers i' and j' such that $\mathsf{W}_{i'} = \{n \in \omega \mid [\sigma_n] \in \mathsf{W}_i\}$ and $\mathsf{W}_{j'} = \{n \in \omega \mid [\sigma_n] \in \mathsf{W}_j\}$.[4] Obviously, $\mathsf{W}_{i'} \cap \mathsf{W}_{j'} = \emptyset$.

Suppose $n \in X$. Since X and Y are disjoint, we find $\mathbb{N} \models \sigma_n$. By Theorem 2.9, we have $\mathsf{R}_0 \vdash [\sigma_n]$, that is, $[\sigma_n] \in \mathsf{R}_{0\mathfrak{p}}$. Then, $[\sigma_n] \in \mathsf{W}_i$, and hence $n \in \mathsf{W}_{i'}$.

Suppose $n \in Y$. We find $\mathbb{N} \models \sigma_n^\perp$. By Theorem 6.4, we obtain $[\sigma_n] \vdash \perp$, that is $[\sigma_n] \in \emptyset_\mathfrak{r}$. Hence, $[\sigma_n] \in \mathsf{W}_j$, and, so, $n \in \mathsf{W}_{j'}$.

We have shown that $\mathsf{W}_{i'}$ and $\mathsf{W}_{j'}$ separate (X, Y). By the effective inseparability of (X, Y), we can effectively find a number $m^\star$ such that $m^\star \notin \mathsf{W}_{i'} \cup \mathsf{W}_{j'}$. Then, $[\sigma_{m^\star}] \notin \mathsf{W}_i \cup \mathsf{W}_j$. Thus, we have shown that $(\mathsf{R}_{0\mathfrak{p}}, \emptyset_\mathfrak{r})$ is effectively inseparable. ⊣

SECOND PROOF. Let X and Y be any c.e. sets separating $\mathsf{R}_{0\mathfrak{p}}$ and $\emptyset_\mathfrak{r}$. We assume that $x \in X$ and $x \in Y$ are represented by well-behaved 1-Σ_1-formulas ξ and η. By Theorem 6.7, we can effectively find a Σ_1-sentence ρ satisfying the following equivalence: $\mathsf{R}_0 \vdash \rho \leftrightarrow \eta([\rho]) \leqslant \xi([\rho])$.

Suppose $[\rho] \in X$. Since X and Y are disjoint, we have $[\rho] \notin Y$. It follows that $\mathbb{N} \models \xi([\rho]) < \eta([\rho])$ and, thus, by Theorem 6.4, that $[\rho]$ is inconsistent, i.e., $[\rho] \in \emptyset_\mathfrak{r}$. We may conclude that $X \cap \emptyset_\mathfrak{r}$ is non-empty. But this is a contradiction.

[4] In order to avoid notational overload, we omit the Gödel numbering brackets around σ_n.

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Suppose $[\rho] \in Y$. Since X and Y are disjoint, we have $\mathbb{N} \models \rho$. By Theorem 2.9, we find $\mathsf{R}_0 \vdash [\rho]$, i.e., $[\rho] \in \mathsf{R}_{0\mathfrak{p}}$. Thus, $Y \cap \mathsf{R}_{0\mathfrak{p}}$ is non-empty. This is a contradiction.

Therefore, $[\rho] \notin X \cup Y$. Since we can find $[\rho]$ effectively, we have shown that $(\mathsf{R}_{0\mathfrak{p}}, \emptyset_{\mathfrak{r}})$ is effectively inseparable. ⊣

Let $\mathcal{F}$ be the set of all $\mathbb{L}_{\mathfrak{a}}$-sentences having a finite model. Since $\mathsf{R}_{0\mathfrak{p}} \subseteq \mathcal{F}$ and $\mathcal{F} \cap \emptyset_{\mathfrak{r}} = \emptyset$, we obtain the following version of Trakhtenbrot's theorem as a corollary.

COROLLARY 7.1 (Trakhtenbrot [16]). *The pair* $(\mathcal{F}, \emptyset_{\mathfrak{r}})$ *is effectively inseparable.*

We note that our version of Trakhtenbrot's Theorem is formulated for the signature $\mathbb{L}_{\mathfrak{a}}$. We can generalize it to other signatures (with at least one relation symbol with arity ≥ 2) by the usual tricks of translating a finite signature into the signature with one binary relation symbol; see e.g., [5, Chapter 5.5]. Alternatively, we can replace the use of theories-of-a-number by the use of very weak set theories as developed, e.g., in [11].

We then turn to Theorem 2.6. Before generalizing Theorem 2.6, we introduce the following notions.

DEFINITION 7.2. Let T be an $\mathcal{L}$-theory and $\mathcal{X}$ be a set of $\mathcal{L}$-sentences.

1. We say that T is *effectively half-essentially $\mathcal{X}$-incomplete* iff there exists a partial computable function Φ such that for any natural number i, if W_i is a c.e. $\mathcal{L}$-theory such that $T + \mathsf{W}_i$ is consistent, then $\Phi(i)$ converges, $\Phi(i) \in \mathcal{X}$, $T \nvdash \Phi(i)$ and $\mathsf{W}_i \nvdash \neg\Phi(i)$.
2. We say that T is *$\mathcal{X}$-creative* iff there exists a partial computable function Ψ such that for any natural number i, if $T_{\mathfrak{p}} \cap \mathsf{W}_i = \emptyset$, then $\Psi(i)$ converges, $\Psi(i) \in \mathcal{X}$, and $\Psi(i) \notin T_{\mathfrak{p}} \cup \mathsf{W}_i$.

Actually, we prove that these two notions are equivalent.

PROPOSITION 7.3. *For any $\mathcal{L}$-theory T and any set $\mathcal{X}$ of $\mathcal{L}$-sentences, the following are equivalent*:

a. *T is effectively half-essentially $\mathcal{X}$-incomplete.*
b. *T is $\mathcal{X}$-creative.*

PROOF. "(a) to (b)". Let Φ be a partial computable function witnessing the effective half-essential $\mathcal{X}$-incompleteness of T. Let W_i be any c.e. set such that $T_{\mathfrak{p}} \cap \mathsf{W}_i = \emptyset$. By using the recursion theorem, we can effectively find a natural number k from i such that

$$\mathsf{W}_k = \begin{cases} \{\neg\Phi(k)\} & \text{if } \Phi(k)\downarrow \text{ and } \Phi(k) \in \mathsf{W}_i, \\ \emptyset & \text{otherwise.} \end{cases}$$

If $T + \mathsf{W}_k$ were inconsistent, then there would be a sentence φ such that $T \vdash \varphi$ and $\mathsf{W}_k \vdash \neg\varphi$. Since T is consistent, we would have $\Phi(k)\downarrow \in \mathsf{W}_i$ and $\mathsf{W}_k = \{\neg\Phi(k)\}$. In this case, we would obtain $T \vdash \Phi(k)$. Hence $\Phi(k) \in T_{\mathfrak{p}} \cap \mathsf{W}_i$, a contradiction.

Thus, $T + \mathsf{W}_k$ is consistent. By the effective half-essential $\mathcal{X}$-incompleteness of T, we have $\Phi(k)\downarrow \in \mathcal{X}$, $T \nvdash \Phi(k)$, and $\mathsf{W}_k \nvdash \neg\Phi(k)$. In particular, $\mathsf{W}_k \nvdash \neg\Phi(k)$ implies $\Phi(k) \notin \mathsf{W}_i$. Therefore the partial computable function $\Psi(i) := \Phi(k)$ witnesses the $\mathcal{X}$-creativity of T.

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"(b) to (a)". Let Ψ be a partial computable function witnessing the $\mathcal{X}$-creativity of T. Let W_i be any c.e. $\mathcal{L}$-theory such that $T + \mathsf{W}_i$ is consistent. We can effectively find a number k from i such that $\mathsf{W}_k = \mathsf{W}_{i\mathfrak{r}}$. Then, $T_\mathfrak{p} \cap \mathsf{W}_k = \emptyset$. By the $\mathcal{X}$-creativity of T, we have $\Psi(k)\downarrow \in \mathcal{X}$ and $\Psi(k) \notin T_\mathfrak{p} \cup \mathsf{W}_{i\mathfrak{r}}$. Therefore, the partial computable function $\Phi(i) := \Psi(k)$ witnesses the effective half-essential $\mathcal{X}$-incompleteness of T. ⊣

We proceed with a generalization of Theorem 2.6. For each $\mathcal{L}$-theory T, let $\mathsf{coTh}_T := \{\varphi \mid \varphi \text{ is an } \mathcal{L}\text{-sentence and } \varphi \vdash T\}$.

THEOREM 7.4. *Every* $\mathsf{R}_{0\mathfrak{p}}$*-sourced c.e. theory* T *is* coTh_T*-creative.*

PROOF. Let T be a τ-$\mathsf{R}_{0\mathfrak{p}}$-sourced c.e. $\mathcal{L}$-theory and let W_i be any c.e. set such that $T_\mathfrak{p} \cap \mathsf{W}_i = \emptyset$. Let α_i be a well-behaved 1-Σ_1-formula that represents W_i and let Pr_T be a well-behaved 1-Σ_1-formula that represents provability in T. By Theorem 6.7, we can effectively find a Σ_1-sentence $\jmath$ from i satisfying $\mathsf{R}_0 \vdash \jmath \leftrightarrow \alpha_i(\llbracket \jmath \rrbracket^\tau) \leqslant \mathsf{Pr}_T(\llbracket \jmath \rrbracket^\tau)$.[5]

Suppose $T \vdash \llbracket \jmath \rrbracket^\tau$. Then, we have $\mathbb{N} \models \mathsf{Pr}_T(\llbracket \jmath \rrbracket^\tau) < \alpha_i(\llbracket \jmath \rrbracket^\tau)$ because $T_\mathfrak{p} \cap \mathsf{W}_i = \emptyset$. By Theorem 6.5, $\llbracket \jmath \rrbracket$ is inconsistent. Then, we have $T \vdash \neg \llbracket \jmath \rrbracket^\tau$. This is a contradiction.

Suppose $\llbracket \jmath \rrbracket^\tau \in \mathsf{W}_i$. Since $T_\mathfrak{p} \cap \mathsf{W}_i = \emptyset$, we have $\mathbb{N} \models \jmath$. By Theorem 5.3, we find $T \vdash \llbracket \jmath \rrbracket^\tau$. Thus, $T_\mathfrak{p} \cap \mathsf{W}_i$ is non-empty. This is a contradiction.

Therefore, we obtain $\llbracket \jmath \rrbracket^\tau \notin T_\mathfrak{p} \cup \mathsf{W}_i$. This implies that $\jmath$ is false. By Theorem 5.3 again, we obtain that $\llbracket \jmath \rrbracket^\tau \vdash T$, i.e., $\llbracket \jmath \rrbracket^\tau \in \mathsf{coTh}_T$. Then the partial computable function $\Psi(i) := \llbracket \jmath \rrbracket^\tau$ witnesses the coTh_T-creativity of T. ⊣

COROLLARY 7.5. *Every* $\mathsf{R}_{0\mathfrak{p}}$*-sourced c.e. theory* T *is effectively half-essentially* coTh_T*-incomplete.*

Consider a $\mathsf{R}_{0\mathfrak{p}}$-sourced c.e. $\mathcal{L}$-theory T. If U is a c.e. $\mathcal{L}$-theory such that $T + U$ is consistent, then by Corollary 7.5, we can effectively find a sentence φ such that $\varphi \vdash T$, $T \nvdash \varphi$, and $U \nvdash \neg\varphi$. Then the $\mathcal{L}$-theory A axiomatized by φ is a proper extension of T such that $A + U$ is consistent. This shows that Corollary 7.5 is in fact a generalization of Theorem 2.6.

COROLLARY 7.6. *Every* $\mathsf{R}_{0\mathfrak{p}}$*-sourced c.e. theory is deductively equivalent to the intersection of the deductive closures of its finite same-signature extensions.*

We note that not every c.e. theory is deductively equivalent to the intersection of the deductive closures of its finite same-signature extensions. For example, PA has no consistent finite same-signature extensions. So, the relevant intersection would be the inconsistent $\mathbb{L}_\mathfrak{a}$-theory.

We can define the notions "effective $\mathcal{X}$-inseparability" and "strong effective $\mathcal{X}$-inseparability" in the forms that witnesses can be found from the set $\mathcal{X}$. The following theorem is a special case of a theorem proved in our paper [9].

THEOREM 7.7 (Kurahashi and Visser [9, Theorem 5.5]). *If* T *is* coTh_T*-creative and effectively inseparable, then* T *is strongly effectively* coTh_T*-inseparable.*

[5] As before, we omit Gödel numbering brackets. Note that, as an intermediate step, we have to find a 1-Σ_1-formula $\sigma(x)$ such that $\mathsf{R}_0 \vdash \sigma(\ulcorner \beta \urcorner) \leftrightarrow \mathsf{Pr}_T(\ulcorner \llbracket \beta \rrbracket^\tau \urcorner)$.

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Since R_0 is effectively inseparable, every $\mathsf{R}_{0\mathsf{p}}$-sourced theory is also effectively inseparable. Thus, Theorems 7.4 and 7.7 establish the following theorem which is a generalization of Theorem 2.5 and a strengthening of Theorem 7.4.

THEOREM 7.8. *For any* $\mathsf{R}_{0\mathsf{p}}$*-sourced c.e. theory* T*, we have that* T *is strongly effectively* coTh_T*-inseparable.*

Theorem 7.8 is the strongest form of the first incompleteness theorem given in the present paper. Of course, one can prove Theorem 7.8 directly in a similar way as described in the proof of Theorem 2.5 above.

We close this section with the following application of Theorem 7.8.

DEFINITION 7.9. Let T be a c.e. $\mathcal{L}$-theory and $\mathcal{X}$ be a set of $\mathcal{L}$-sentences. We say that T is *effectively uniformly essentially* $\mathcal{X}$*-incomplete* iff there exists a partial computable function Φ such that for every computable sequence of consistent c.e. extensions U_i of T with index j, we have that $\Phi(j)$ converges, $\Phi(j) \in \mathcal{X}$, and for all i, $U_i \nvdash \Phi(j)$ and $U_i \nvdash \neg\, \Phi(j)$.

We proved in [9] the following theorem.

THEOREM 7.10 (Kurahashi and Visser [9, Theorem 2.9]). *Let* T *be any c.e.* $\mathcal{L}$*-theory and* $\mathcal{X}$ *be any set of* $\mathcal{L}$*-sentences. The following are equivalent*:

a. T *is effectively* $\mathcal{X}$*-inseparable.*
b. T *is effectively uniformly essentially* $\mathcal{X}$*-incomplete.*

By combining this theorem with Theorem 7.8, we obtain the following strengthening of Corollary 7.5.

COROLLARY 7.11. *Every* $\mathsf{R}_{0\mathsf{p}}$*-sourced c.e. theory* T *is effectively uniformly essentially* coTh_T*-incomplete.*

§8. Various facts about degree structures. In this section, we provide various applications of our framework to degrees of interpretability.

8.1. Useful insights. We first remind the reader of a special property of R.

THEOREM 8.1 (Visser [21, Theorem 6]). *For any c.e. theory* T*, we have that* $\mathsf{R} \rhd T$ *if and only if every finite subtheory of* T *has a finite model.*

We note that this property is inherited by every c.e. theory that is mutually interpretable with R.

We have the following definition.

- The theory T is a *globalizer* iff, for every c.e. theory W, whenever $T \rhd_{\mathsf{loc}} W$, then $T \rhd W$.

Examples of globalizers are PRA, PA, and ZF. Theorem 8.1 has the following useful consequence, which also appears in [21].

THEOREM 8.2. R *is a globalizer.*

PROOF. Suppose $\mathsf{R} \rhd_{\mathsf{loc}} U$. Then, for every finitely axiomatized subtheory U_0 of U, we have $\mathsf{R} \rhd U_0$. So, for every finitely axiomatized subtheory U_0 of U, there is a finitely axiomatized subtheory A of R, such that $A \rhd U_0$. Since A has a finite model,

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so has U_0. We may conclude that every finitely axiomatized subtheory U_0 of U has a finite model. *Ergo*, $\mathsf{R} \rhd U$. ⊣

Cobham has shown that R_0 is mutually interpretable with R; see [6]. It follows that the insights contained in Theorems 8.1 and 8.2 are inherited by R_0.

We give two useful results. We assume that we have a Σ_1-representation of interpretability $\rhd$ for the case that the interpreted theory is finitely axiomatized and the interpreting theory is computably enumerable. For later use we also assume that this representation is well-behaved. *Par abus de langage*, we write $\rhd$ both for the meta-notion and for its theory-internal representation.

THEOREM 8.3. *Let W be a τ-$\mathsf{R}_{0\mathsf{p}}$-sourced c.e. theory and let A be finitely axiomatized. We can effectively find a Σ_1-sentence λ from an index of W, such that $\mathbb{N} \models \lambda$, $[\![\lambda]\!]^\tau \rhd A$, and $W \rhd A$ are equivalent.*

PROOF. Let W be τ-$\mathsf{R}_{0\mathsf{p}}$-sourced and let A be finitely axiomatized. By Theorem 6.7, we obtain a Σ_1-sentence λ satisfying the following equivalence:

$$\mathsf{R}_0 \vdash \lambda \leftrightarrow [\![\lambda]\!]^\tau \rhd A.$$

Suppose $[\![\lambda]\!]^\tau \rhd A$. Then, we have $\mathbb{N} \models \lambda$ and, thus, $W \vdash [\![\lambda]\!]^\tau$ by Theorem 5.3. Hence, $W \rhd A$.

Conversely, suppose that $W \rhd A$. If $\mathbb{N} \models \neg\lambda$, then we have $[\![\lambda]\!]^\tau \vdash W$ by Theorem 5.3, and, hence, $[\![\lambda]\!]^\tau \rhd A$. By the fixed point equation, we find $\mathbb{N} \models \lambda$, contradicting our assumption that $\mathbb{N} \models \neg\lambda$. So, we may conclude that $\mathbb{N} \models \lambda$, and, thus, $[\![\lambda]\!]^\tau \rhd A$. ⊣

REMARK 8.4. The proof of Theorem 8.3 is strongly reminiscent of the proof of Löb's Theorem. Regrettably, it does not seem that we can take the further step to obtain an analog of Löb's theorem, to wit:

$$W \rhd A \text{ iff } (W + [\![W \rhd A]\!]^\tau) \rhd A.$$

The left-to-right direction is trivial, but we do not know about the right-to-left direction at the moment.

EXAMPLE 8.5. Juvenal Murwanashyaka asked whether there is a finitely axiomatized theory B that interprets VS but does not interpret AS. Theorem 8.3 provides an example that, additionally, is a same-signature extension of VS.

We can see this as follows. Since VS is $\mathsf{R}_{0\mathsf{p}}$-sourced, we have, by Theorem 8.3, a finite same-signature-theory B $(= [\![\lambda]\!]^\tau)$, such that $B \rhd \mathsf{AS}$ iff $\mathsf{VS} \rhd \mathsf{AS}$. However, $\mathsf{VS} \not\rhd \mathsf{AS}$, since, otherwise, a finite subtheory of VS would interpret AS. Such finite subtheories are interpretable in the theory of non-surjective pairing, i.e., $\mathsf{VS0} + \mathsf{VS2}$, and, as is well-known this theory has a decidable extension. On the other hand, AS is essentially undecidable. Since $\mathbb{N} \models \neg\lambda$, we have that B is an extension of VS by Theorem 5.3.

Similarly, we can specify a finitely axiomatized same-signature extension of PRA that does not interpret $\mathrm{I}\Sigma_1$.

COROLLARY 8.6. *Let W be a τ-$\mathsf{R}_{0\mathsf{p}}$-sourced c.e. theory, and let T be any c.e. theory.*

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i. *Suppose* $W \not\rhd_{\mathsf{loc}} T$. *Then, there is a false* Σ_1*-sentence* λ *such that* $[\![\lambda]\!]^\tau \not\rhd_{\mathsf{loc}} T$.
ii. *Suppose* W *is a globalizer and* $W \not\rhd T$. *Then, there is a false* Σ_1*-sentence* λ *such that* $[\![\lambda]\!]^\tau \not\rhd T$.

Proof. We prove (i). Let W be τ-$\mathsf{R}_{0\mathsf{p}}$-sourced and let T be any c.e. theory. Suppose $W \not\rhd_{\mathsf{loc}} T$. It follows that $W \not\rhd A$, for some finitely axiomatized subtheory A of T. We apply Theorem 8.3 to find a false Σ_1-sentence λ such that $[\![\lambda]\!]^\tau \not\rhd A$. It follows that $[\![\lambda]\!]^\tau \not\rhd_{\mathsf{loc}} T$.
(ii) is immediate from (i). ⊣

Here is another result in the same spirit as Theorem 8.3 that uses a Rosser argument. We remind the reader of our representation of $\rhd$ is well-behaved (under the assumption that the interpreted theory is finitely axiomatized and the interpreting theory is c.e.).

Theorem 8.7. *Let* W *be a* τ-$\mathsf{R}_{0\mathsf{p}}$*-sourced theory, let* T *be a c.e. theory such that* $T \rhd_{\mathsf{loc}} W$, *and let* A *be finitely axiomatized. Then, there is a* Σ_1*-sentence* θ, *which is* R_0*-provably equivalent to a witness comparison sentence, such that the following are equivalent*:
a. $((T \mathbin{\owedge} [\![\theta]\!]^\tau) \rhd A)$ *or* $(T \rhd (A \mathbin{\owedge} [\![\theta]\!]^\tau))$,
b. $T \rhd A$.

Proof. Suppose $T \rhd_{\mathsf{loc}} W$. By Theorem 6.7, we obtain a Σ_1-sentence θ satisfying the following equivalence:

$$\mathsf{R}_0 \vdash \theta \leftrightarrow ((T \mathbin{\owedge} [\![\theta]\!]^\tau) \rhd A) \leqslant (T \rhd (A \mathbin{\owedge} [\![\theta]\!]^\tau)).$$

Clearly (b) implies (a). We show that (a) implies (b). Suppose we have (a). Let $\eta := ((T \mathbin{\owedge} [\![\theta]\!]^\tau) \rhd A) \leqslant (T \rhd (A \mathbin{\owedge} [\![\theta]\!]^\tau))$. Clearly, we have $\mathbb{N} \models \eta$ or $N \models \eta^\perp$.
Suppose $\mathbb{N} \models \eta$. It follows that $\mathbb{N} \models \theta$ and $(T \mathbin{\owedge} [\![\theta]\!]^\tau) \rhd A$. It follows that $W \vdash [\![\theta]\!]^\tau$, by Theorem 5.3. Since $T \rhd_{\mathsf{loc}} W$, we find $T \rhd [\![\theta]\!]^\tau$. Hence, $T \rhd (T \mathbin{\owedge} [\![\theta]\!]^\tau) \rhd A$.
Suppose $\mathbb{N} \models \eta^\perp$. It follows that $\mathbb{N} \models (T \rhd (A \mathbin{\owedge} [\![\theta]\!]^\tau)) < ((T \mathbin{\owedge} [\![\theta]\!]^\tau) \rhd A)$. By Theorem 6.5, $[\![\theta]\!]$ is inconsistent, so $[\![\theta]\!]^\tau$ is inconsistent. It also follows from $\mathbb{N} \models \eta^\perp$ that $T \rhd (A \mathbin{\owedge} [\![\theta]\!]^\tau)$. Hence, $T \rhd A$. ⊣

The following theorem, which is in particular the case where T and A are same-signature-theories of W, can be proved in the same way.

Theorem 8.8. *Let* W *be a* τ-$\mathsf{R}_{0\mathsf{p}}$*-sourced theory, let* T *be a c.e. theory in the signature of* W *such that* $T \supseteq W$, *and let* A *be finitely axiomatized theory in the signature of* W. *Then, there is a* Σ_1*-sentence* θ, *which is* R_0*-provably equivalent to a witness comparison sentence, such that the following are equivalent*:
a. $((T + [\![\theta]\!]^\tau) \rhd A)$ *or* $T \rhd B$, *where* $B = \{\phi \vee [\![\theta]\!]^\tau \mid \phi \in A\}$.
b. $T \rhd A$.

8.2. Applications of certified extension. We turn to the consideration of various density results.

Theorem 8.9. *Consider c.e. theories* T *and* S *such that* $S \not\rhd_{\mathsf{loc}} T$. *Then, there exists a c.e. theory* U *such that* $T \rhd U$ *and* $S \not\rhd_{\mathsf{loc}} U$ *and* $U \not\rhd_{\mathsf{loc}} T$. *Moreover, if* $T \rhd S$, *then* $U \rhd S$ *and, if* $T \rhd_{\mathsf{loc}} S$, *then* $U \rhd_{\mathsf{loc}} S$.

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PROOF. Suppose $S \not\rhd_{\mathsf{loc}} T$. Then, we can find a finite subtheory A of T, such that $S \not\rhd A$. Since $S \rhd_{\mathsf{loc}} \mathsf{R}_0$ and R_0 is a τ-$\mathsf{R}_{0\mathsf{p}}$-sourced theory, we can apply Theorem 8.7 to S and A. Let θ be the promised Σ_1-sentence. Since $S \not\rhd A$, we find $(S \mathbin{\textcircled{\scriptsize +}} [\![\theta]\!]^\tau) \not\rhd A$ and $S \not\rhd (A \mathbin{\textcircled{\scriptsize \vee}} [\![\theta]\!]^\tau)$.

Let $U := (S \mathbin{\textcircled{\scriptsize +}} [\![\theta]\!]^\tau) \mathbin{\textcircled{\scriptsize \vee}} T$. It is immediate that $T \rhd U$. Moreover, it is also immediate that, if $T \rhd S$, then $U \rhd S$ and, if $T \rhd_{\mathsf{loc}} S$, then $U \rhd_{\mathsf{loc}} S$.

Suppose $S \rhd_{\mathsf{loc}} U$. Then, $S \rhd (A \mathbin{\textcircled{\scriptsize \vee}} [\![\theta]\!]^\tau)$. *Quod non*. Suppose $U \rhd_{\mathsf{loc}} T$. Then, $(S \mathbin{\textcircled{\scriptsize +}} [\![\theta]\!]) \rhd A$. *Quod non*. ⊣

QUESTION 8.10. *The proof of Theorem* 8.9, *seems to use specific properties of* R_0. *Is there a good way to generalize it*?

We have immediately the following corollaries.

COROLLARY 8.11. *Consider c.e. theories* S *and* T. *Suppose* $T \mathrel{\underset{\sim}{\lhd}}_{\mathsf{loc}} S$. *Then, there exists a c.e. theory* U *such that* $T \mathrel{\underset{\sim}{\lhd}}_{\mathsf{loc}} U \mathrel{\underset{\sim}{\lhd}}_{\mathsf{loc}} S$.

The density of the degrees of local interpretability of c.e. theories was first proved by Jan Mycielski, Pavel Pudlák, and Alan Stern in their classical paper [10, Corollary 6.17].

COROLLARY 8.12. *Consider c.e. theories* S *and* T, *where either* S *is a globalizer or* T *is finitely axiomatized. Suppose* $T \mathrel{\underset{\sim}{\lhd}} S$. *Then, there exists a c.e. theory* U *such that* $T \mathrel{\underset{\sim}{\lhd}} U \mathrel{\underset{\sim}{\lhd}} S$.

PROOF. We note that if either S is a globalizer or T is finitely axiomatized, then $S \rhd_{\mathsf{loc}} T$ iff $S \rhd T$. ⊣

EXAMPLE 8.13. Consider INF the theory in the signature of identity with axioms saying, for each n, that there are at least n elements and TWO the theory in the signature of identity with an axiom saying that there are precisely two elements. Then, we have $\mathsf{TWO} \mathrel{\underset{\sim}{\lhd}} \mathsf{INF}$. Every theory that has a finite model is interpretable in TWO and every theory that has only infinite models proves INF. So, there can be no theory strictly $\lhd$-between TWO and INF. *Ergo*, density fails in general in the degrees of interpretability of c.e. theories.

PROBLEM 8.14. *Example* 8.13 *seems too easy. What if we do have density for all theories with no finite models*? *So, it would be good to have some further classes of examples.*

In one special case, we can constrain the in-between theories a bit more. The following theorem is a generalization of [22, Theorem 2].

THEOREM 8.15. *Let* W *be a* τ-$\mathsf{R}_{0\mathsf{p}}$-*sourced c.e. theory. Consider c.e. theories* S *and* T *such that* $W \subseteq S \subseteq T$ *and* $S \not\rhd_{\mathsf{loc}} T$. *Then, there exists a c.e. theory* U *such that* $S \subseteq U \subseteq T$ *and* $S \not\rhd_{\mathsf{loc}} U$ *and* $U \not\rhd_{\mathsf{loc}} T$.

PROOF. Suppose W is τ-$\mathsf{R}_{0\mathsf{p}}$-sourced, $W \subseteq S \subseteq T$ and $S \not\rhd_{\mathsf{loc}} T$. Then, we can find a finite subtheory A of T, such that $S \not\rhd A$. We apply Theorem 8.8 to $W, S \supseteq W$, and A. Let θ be the promised Σ_1-sentence. Since $S \not\rhd A$, we find $(S + [\![\theta]\!]^\tau) \not\rhd A$ and $S \not\rhd B$, where $B = \{\phi \vee [\![\theta]\!]^\tau \mid \phi \in A\}$.

Let $U := \{\phi \vee \psi \mid \phi \in T \text{ and } \psi \in (S + [\![\theta]\!]^\tau)\}$. It is immediate that $S \subseteq U \subseteq T$.

Since B is a finite subtheory of U and $S \not\rhd B$, we have that $S \not\rhd_{\mathsf{loc}} U$. Also since $U \subseteq (S + [\![\theta]\!]^\tau)$ and $(S + [\![\theta]\!]^\tau) \not\rhd A$, we have $U \not\rhd_{\mathsf{loc}} T$. ⊣

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COROLLARY 8.16. *Suppose that W is a τ-R_{0p}-sourced c.e. theory. Consider c.e. theories S and T such that $W \subseteq S \subseteq T$, where either S is a globalizer or T is finitely axiomatized. Suppose $S \not\rhd T$. Then, there exists a c.e. theory U such that $S \subseteq U \subseteq T$ and $S \not\rhd U$ and $U \not\rhd T$.*

THEOREM 8.17. *Any finite theory is the supremum of the finite theories strictly below it in the lattice of the interpretability degrees of c.e. theories.*

We will give two proofs. The first uses an ordinary Gödel numbering and the second a self-referential one. We remind the reader that we chose the representation of $\rhd$ in such a way that, as long as the interpreted theory is finite and the interpreting one c.e., it is well-behaved and 1-Σ_1.

PROOF WITH ORDINARY GÖDEL NUMBERING. Suppose A is a finitely axiomatized theory. If A is in the minimal degree, we are immediately done. Suppose A is non-minimal. Suppose U interprets all finitely axiomatized theories strictly below A. We have to show that $U \rhd A$.

By Theorem 6.7(ii), we can find Σ_1-sentences ρ and θ such that:

- $\mathsf{R}_0 \vdash \rho \leftrightarrow ((U \rhd (A \mathbin{\owedge} [\theta])) \vee (([\rho] \mathbin{\owedge} [\theta]) \rhd A)) \leqslant (U \rhd (A \mathbin{\owedge} [\rho]))$.
- $\mathsf{R}_0 \vdash \theta \leftrightarrow (U \rhd (A \mathbin{\owedge} [\rho])) < ((U \rhd (A \mathbin{\owedge} [\theta])) \vee (([\rho] \mathbin{\owedge} [\theta]) \rhd A))$.

Let $\eta := (U \rhd (A \mathbin{\owedge} [\theta])) \vee (([\rho] \mathbin{\owedge} [\theta]) \rhd A)) \leqslant (U \rhd (A \mathbin{\owedge} [\rho]))$.

CLAIM 1. *Suppose $U \rhd (A \mathbin{\owedge} [\theta])$. Then, $U \rhd A$.*

PROOF OF CLAIM 1. Suppose $U \rhd (A \mathbin{\owedge} [\theta])$. Then, $\mathbb{N} \models \eta$ or $\mathbb{N} \models \eta^\perp$. In the first case, we have, by Theorem 6.4, that $[\theta] \vdash \bot$. Hence, $U \rhd A$.

In the second case, we have $U \rhd (A \mathbin{\owedge} [\rho])$. By Theorem 6.4, we find $[\rho] \vdash \bot$. Hence, again, $U \rhd A$.

So, in both cases, we may conclude $U \rhd A$. ⊣

CLAIM 2. *Suppose $U \rhd (A \mathbin{\owedge} [\rho])$. Then, $U \rhd A$.*

PROOF OF CLAIM 2. Suppose $U \rhd (A \mathbin{\owedge} [\rho])$. It follows that $\mathbb{N} \models \eta$ or $\mathbb{N} \models \eta^\perp$. In the first case it follows that (a) $U \rhd (A \mathbin{\owedge} [\theta])$ or (b) $([\rho] \mathbin{\owedge} [\theta]) \rhd A$. In subcase (a), we find, by Claim 1, that $U \rhd A$. In subcase (b), we have $\mathbb{N} \models \rho$, and, hence, by Theorem 2.9, that $\mathsf{R}_0 \vdash [\rho]$. So, $[\rho]$ has a finite model, and, thus, A is in the minimal degree, contradicting our assumption on A.

Suppose $\mathbb{N} \models \eta^\perp$. In that case, we have $U \rhd (A \mathbin{\owedge} [\rho])$. So, by Theorem 6.4, we find $U \rhd A$. ⊣

CLAIM 3. *We have either $(A \mathbin{\owedge} [\rho]) \not\rhd A$ or $(A \mathbin{\owedge} [\theta]) \not\rhd A$.*

PROOF OF CLAIM 3. Suppose we have both $(A \mathbin{\owedge} [\rho]) \rhd A$ and $(A \mathbin{\owedge} [\theta]) \rhd A$. Then, $([\rho] \mathbin{\owedge} [\theta]) \rhd A$. It follows that $\mathbb{N} \models \eta$ or $\mathbb{N} \models \eta^\perp$, and, hence, that $\mathbb{N} \models \rho$ or $\mathbb{N} \models \theta$. In both cases we may conclude that $([\rho] \mathbin{\owedge} [\theta])$ has a finite model, so A is in the minimal degree, contradicting our assumption on A. ⊣

We are now ready to prove the theorem. By Claim 3, one of $A \mathbin{\owedge} [\rho]$ and $A \mathbin{\owedge} [\theta]$ is strictly below A, and, hence, below U. If $U \rhd (A \mathbin{\owedge} [\theta])$, it follows by Claim 1 that $U \rhd A$. If $U \rhd (A \mathbin{\owedge} [\rho])$, it follows by Claim 2 that $U \rhd A$. So, A is indeed the supremum of the finite elements strictly below it.

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We now give our proof using a self-referential Gödel numbering. The proof will be largely the same, only we need just one fixed point. We note that e.g., the arithmetized form of $U \rhd A$ in this proof is the form appropriate for the self-referential Gödel numbering and, thus, is different from the case of the ordinary numbering. We opted to keep the same notations for readability's sake, but the reader should keep the point in mind.

PROOF WITH SELF-REFERENTIAL GÖDEL NUMBERING. Suppose A is a finitely axiomatized theory. If A is in the minimal degree, we are immediately done. Suppose A is non-minimal. Suppose U interprets all finitely axiomatized theories strictly below A. We have to show that $U \rhd A$. We find ρ with:

$$\rho = ((U \rhd (A \otimes [\rho^\perp])) \vee (([\rho] \otimes [\rho^\perp]) \rhd A)) \leqslant (U \rhd (A \otimes [\rho])).$$

CLAIM 1. *Suppose* $U \rhd (A \otimes [\rho^\perp])$. *Then*, $U \rhd A$.

PROOF OF CLAIM 1. Suppose $U \rhd (A \otimes [\rho^\perp])$. Then, $\mathbb{N} \models \rho$ or $\mathbb{N} \models \rho^\perp$. In the first case, we have, by Theorem 6.4, that $[\rho^\perp] \vdash \perp$. Hence, $U \rhd A$.

In the second case, we have $U \rhd (A \otimes [\rho])$. By Theorem 6.4, we find $[\rho] \vdash \perp$. Hence, again, $U \rhd A$.

So, in both cases, we may conclude $U \rhd A$. ⊣

CLAIM 2. *Suppose* $U \rhd (A \otimes [\rho])$. *Then*, $U \rhd A$.

PROOF OF CLAIM 2. Suppose $U \rhd (A \otimes [\rho])$. It follows that $\mathbb{N} \models \rho$ or $\mathbb{N} \models \rho^\perp$. In the first case it follows that (a) $U \rhd (A \otimes [\rho^\perp])$ or (b) $([\rho] \otimes [\rho^\perp]) \rhd A$. In subcase (a), we find, by Claim 1, that $U \rhd A$. In subcase (b), we have $\mathbb{N} \models \rho$, and, hence, by Theorem 2.9, that $\mathsf{R}_0 \vdash [\rho]$. So, $[\rho]$ has a finite model, and, thus, A is in the minimal degree, contradicting our assumption on A. ⊣

Suppose $\mathbb{N} \models \rho^\perp$. In that case, we have $U \rhd (A \otimes [\rho])$. So, by Theorem 6.4, we find $U \rhd A$.

CLAIM 3. *We have either* $(A \otimes [\rho]) \not\rhd A$ *or* $(A \otimes [\rho^\perp]) \not\rhd A$.

PROOF OF CLAIM 3. Suppose we have both $(A \otimes [\rho]) \rhd A$ and $(A \otimes [\rho^\perp]) \rhd A$. Then, $([\rho] \otimes [\rho^\perp]) \rhd A$. It follows that $\mathbb{N} \models \rho$ or $\mathbb{N} \models \rho^\perp$. In both cases we may conclude that $([\rho] \otimes [\rho^\perp])$ has a finite model, so A is in the minimal degree, contradicting our assumption on A. ⊣

We are now ready to prove the theorem. By Claim 3, one of $A \otimes [\rho]$ and $A \otimes [\rho^\perp]$ is strictly below A, and, hence, below U. If $U \rhd (A \otimes [\rho^\perp])$, it follows by Claim 1 that $U \rhd A$. If $U \rhd (A \otimes [\rho])$, it follows by Claim 2 that $U \rhd A$. So, A is indeed the supremum of the finite elements strictly below it.

EXAMPLE 8.18. We note that both in the local and in the global degrees of interpretability of c.e. theories, the degree of the theory INF is not the supremum of the degrees of the theories strictly below them, so *a fortiori*, it is not the supremum of the degrees of the finitely axiomatizable ones.

COROLLARY 8.19. *In the lattice of c.e. degrees of interpretability, no theory A can be finitely axiomatized, non-minimal, join-irreducible, and compact.*

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PROOF. Suppose A is finitely axiomatized, non-minimal, join-irreducible, and compact. By Theorem 8.17, A is the supremum of the finitely axiomatized theories strictly below it. Hence, by compactness, it is mutually interpretable with the supremum of a finite number of finite theories strictly below it. By join-irreducibility, it follows that A is mutually interpretable with a finite theory strictly below it. A contradiction. ⊣

THEOREM 8.20. *Consider a c.e. theory W.*

i. *Suppose W is mutually locally interpretable with a $\mathsf{R}_{0\mathsf{p}}$-sourced theory. Then, in the degrees of local interpretability of c.e. theories, W is the infimum of the finitely axiomatized theories above it.*
ii. *Suppose W is mutually interpretable with a $\mathsf{R}_{0\mathsf{p}}$-sourced globalizer. Then, in the degrees of interpretability of c.e. theories, W is the infimum of the finitely axiomatized theories above it.*

PROOF. We first prove (i). It is clearly sufficient to prove the result for the case that W is a τ-$\mathsf{R}_{0\mathsf{p}}$-sourced theory, for some τ. Suppose all finitely axiomatized theories that interpret W locally interpret T. We want to show that $W \rhd_{\mathsf{loc}} T$. Suppose, in order to obtain a contradiction, that $W \not\rhd_{\mathsf{loc}} T$. Let λ be the sentence provided by Corollary 8.6(i) such that $[\![\lambda]\!]^\tau \not\rhd_{\mathsf{loc}} T$. Since λ is false, we have $[\![\lambda]\!]^\tau \rhd W$ and, hence, $[\![\lambda]\!]^\tau \rhd_{\mathsf{loc}} T$. A contradiction.

We turn to (ii). Again is sufficient to prove the result for the case that W is a τ-$\mathsf{R}_{0\mathsf{p}}$-sourced theory, for some τ. Suppose all finitely axiomatized theories that interpret W interpret T. We want to show that $W \rhd T$. Suppose, in order to obtain a contradiction, that $W \not\rhd T$. Let λ be the sentence provided by Corollary 8.6(ii) such that $[\![\lambda]\!]^\tau \not\rhd T$. Since λ is false, we have $[\![\lambda]\!]^\tau \rhd W$ and, hence, $[\![\lambda]\!]^\tau \rhd T$. A contradiction. ⊣

COROLLARY 8.21. *Suppose W is mutually interpretable with a c.e. sequential globalizer. Then, W is the interpretability infimum of all finitely axiomatized theories above it (w.r.t. $\lhd$).*

PROOF. Any sequential globalizer U is mutually interpretable with a restricted sequential theory, to wit $\mho_U$, which is, of course, itself a globalizer; see [23]. By Corollary 5.11, the theory $\mho_U$ is a $\mathsf{R}_{0\mathsf{p}}$-sourced. We now apply Theorem 8.20. ⊣

So, e.g., PA is the infimum in the degrees of local interpretability of c.e. theories of the finitely axiomatized theories that locally interpret it.

§9. Conclusions.

We presented the following two new methods.

- *Certification of Σ_1-witnesses:* We introduced the notion of the certification of an element (Definition 4.1), and explored some consequences of the certification. The certified extension theorem on R_0 (Theorem 2.9) is one of the main results of this study.
- *$\mathsf{R}_{0\mathsf{p}}$-sourced theories:* We developed a generalization of the argument concerning R_0 to $\mathsf{R}_{0\mathsf{p}}$-sourced theories, which allows the generalized certified extension theorem (Theorem 5.3) to be applied, for example, to Vaught's set theory VS (Theorem 5.6).

Our two methods have the following two applications:

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- Certified Σ_1-sentences can be successfully applied to provide proofs of Vaught's two theorems. Furthermore, we proved the strong effective coTh_T-inseparability of $\mathsf{R}_{0\mathsf{p}}$-sourced c.e. theories, which yields Vaught's two theorems (Theorem 7.8).
- Certified Σ_1-sentences can also be applied to the study of the degrees of interpretability of theories. We proved some density results (Corollaries 8.12 and 8.16) and studied sufficient conditions for a theory to be the supremum of the theories below it or the infimum of the theories above it (Theorems 8.17 and 8.20).

In our paper [9], we specifically discussed topics related to the first application. This paper may be read in connection with the present paper.

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