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Bifurcation to an entire function

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Abstract. In this paper we will consider a bifurcation that occurs in the topology of the Julia set of meromorphic maps as they become entire. For some meromorphic maps the Julia set will be a Cantor set. We will investigate how the Julia set changes as these meromorphic maps approach the entire function whose Julia set is a Cantor bouquet. Other meromorphic maps have a Julia set that is a Jordan curve. Again we will study how this curve changes as these meromorphic maps approach the entire function whose Julia set is a Cantor bouquet.

1. *Introduction*

Our goal here is to study bifurcations that occur in the topology of the Julia set ([**9, 11**]) of meromorphic functions as they become entire. We will characterize the Julia sets for special kinds of meromorphic maps and use well-known results ([**3, 4**]) about the topology of the Julia set of the entire function $E_\lambda(z) = \lambda e^z$.

The entire functions $E_\lambda(z) = \lambda e^z$ have been studied extensively, mostly by Devaney [**3–7**], but also by Eremenko [**8**], Goldberg [**5**], Keen [**6**], Lyubich [**8**] and McMullen [**13**]. In the parameter plane, hyperbolic components have been identified and their ordering has yielded results about the dynamical plane. In the dynamical plane of the stable exponentials the Julia set consists of closed Jordan arcs, called 'hairs', which have the property that all points on the hair share the same itinerary, given some suitable partitioning of the plane. In the case of

$$E_\lambda(z) = \lambda e^z \quad \text{where } \lambda \in (0, 1/e),$$

Aarts and Oversteegen [**1**] managed to completely parameterize all hairs in the Julia set, showing that in this case the Julia set is a Cantor bouquet. In this set the set of endpoints of these hairs form a special subset. Mayer [**12**] proved that the set of endpoints in this Cantor bouquet is in fact a connected set if we include the point ∞; without the point ∞ it becomes a totally disconnected set. The special properties of this set were further highlighted by Devaney and Goldberg [**5**], who showed that, of all the Julia set, only the endpoints of the

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hairs are accessible from the basin of attraction. This was done by uniformizing the basin of attraction and taking the limit along a ray from the origin to some point of the unit disk.

The exponential function itself is part of a larger class of functions, functions with a constant Schwarzian derivative. These functions have been studied by Devaney and Keen [**6**] and have the advantage that a result by Nevanlinna [**14**] characterizes their asymptotic behavior. Among the maps that have been studied extensively is the tangent family, $T_\lambda(z) = \lambda \tan(z)$, by Keen and Kotus [**10**].

As can be shown by elementary means, the family of all maps with constant Schwarzian derivative is a two-parameter family, which includes the exponential and the tangent family. We will characterize the Julia set of some maps that have exactly one attracting fixed point, two attracting fixed points and an attracting 2-cycle. For the case where the map is not entire and has exactly one attracting fixed point, we will show that the Julia set is a Cantor set. The Julia set for the map that is not entire and that has two attracting fixed points is a Jordan curve going through ∞. The Julia set for an entire map with exactly one attracting fixed point is a Cantor bouquet.

With these two sets of functions we limit onto the entire function and we witness the Julia set changing from a Cantor set to a Cantor bouquet as we approach the entire function with exactly one attracting fixed point from the meromorphic maps that have exactly one attracting fixed point. At the other limit as we approach the same exponential map from the meromorphic map with two attracting fixed points we witness the Julia set changing from a Jordan curve to the Cantor bouquet.

2. *Parameter plane*

Using results from Nevanlinna [**14**] and elementary arguments, one can show that the family of meromorphic functions that have constant Schwarzian derivative can be characterized by two parameters. Every meromorphic map with constant Schwarzian derivative is conjugate to some map of the form

$$F_{\alpha,\beta}(z) = \frac{\alpha e^z + 1}{e^z + \beta} \quad \text{where } \alpha, \beta \in \mathbb{C}.$$

The exponential and tangent families are all conjugate to some elements of the family $F_{\alpha,\beta}$. The exponential functions $E_\lambda(z) = \lambda e^z$, where $\lambda \in \mathbb{C}$, are conjugate to the family $F_{\alpha,0}(z) = \alpha + e^{-z}$, where $\lambda = -e^{-\alpha}$. We focus on the subfamily of $F_{\alpha,\beta}$ by choosing the parameters real:

$$F_{\alpha,\beta}(z) = \frac{\alpha e^z + 1}{e^z + \beta} \quad \text{where } \alpha, \beta \in \mathbb{R}.$$

This subfamily provides analytic tractability, since the real line is invariant and we may use these results to characterize the Julia sets.

Figure 1 shows a portion of the stylized parameter plane for this subfamily.

In this picture the vertical axis denotes β and the horizontal axis denotes α. Hence meromorphic maps corresponding to the α-axis are entire and one can see that there is a period doubling bifurcation at some $\bar{\alpha}$. However, for α large enough these entire functions have exactly one attracting fixed point.

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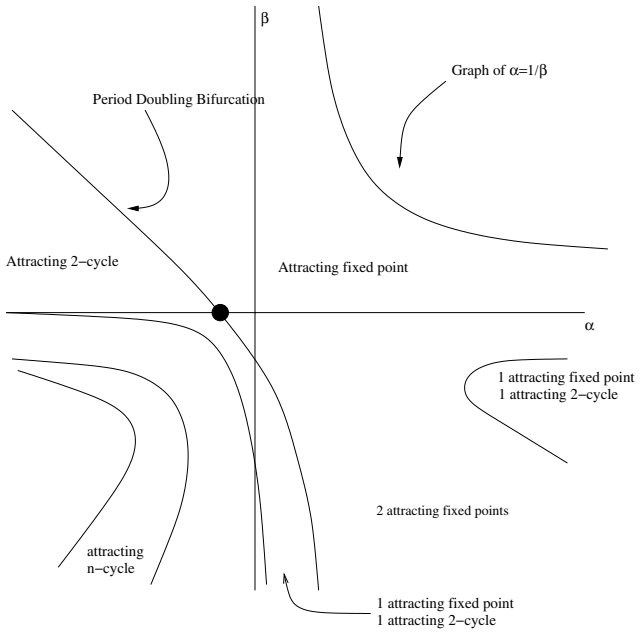


FIGURE 1. The stylized parameter plane for α, β real.

For α large enough and $\beta > 0$ small enough, $F_{\alpha,\beta}$ has exactly one attracting fixed point and the function is decreasing on the real line.

For α large enough and $\beta < 0$ large enough, $F_{\alpha,\beta}$ has a pole on the real line and has exactly two attracting fixed points.

3. *Dynamical plane*

We continue our study of the subfamily

$$F_{\alpha,\beta}(z) = \frac{\alpha e^z + 1}{e^z + \beta}, \quad \alpha, \beta \in \mathbb{R}.$$

Now we will investigate the topology of the Julia sets of functions in our family. We start by reviewing the topology for the Julia set of $F_{\alpha,0}(z) = \alpha + e^{-z}$, which has been shown by Aarts and Oversteegen [**1**] to be a Cantor bouquet if $F_{\alpha,0}$ has exactly one attracting fixed point. Then we characterize the Julia set for functions where both the asymptotic values are in the immediate basin of attraction of one attracting fixed point. After that we will examine functions where the two asymptotic values are each in the immediate basin of attraction of two different attracting fixed points. The case where the two asymptotic values are each in the immediate basin of attraction of an attracting two cycle will be analogous.

3.1. $\beta = 0$.

If $\beta = 0$ then the map $F_{\alpha,0}$ is an exponential function. If $\alpha \in \mathbb{R}$ then $F_{\alpha,0} = E_\alpha$ has either one attracting fixed point or one attracting 2-cycle. The Julia sets in both these cases are well known and have been studied extensively by Devaney and

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Goldberg [**5**]. We will describe in some detail the Julia set of an exponential map that has one attracting cycle. In this case the Julia set will be a 'Cantor bouquet', a set that consists of infinitely many Jordan curves, all limiting onto ∞ and each limiting onto some endpoint in the plane.

To describe this Cantor bouquet we first need to describe the dynamics of points in the Julia set. Because the real line is invariant and completely contained in the Fatou set, we may use it to define a partition

$$R(k) := \{z \mid 2\pi k < \mathrm{Im}(z) < 2\pi(k+1)\}.$$

Now we can assign to each $z \in J(F_{\alpha,0})$ an itinerary

$$S_{\alpha,0}(z) = (s_0, s_1, s_2, \dots)$$

such that

$$s_i = j \quad \text{if and only if } F^i_{\alpha,0} \in R(j).$$

The question arises whether all sequences $(s_0, s_1, \dots)$ are realized by some point $z \in J(F_{\alpha,0})$. Using the definition in Devaney and Krych [**7**], an itinerary is allowable if there exists a real number x such that

$$2\pi|s_j| \le F^j_{\alpha,0}(x) \quad \text{for all } j.$$

Devaney and Krych also showed that not all itineraries are allowable. We denote the set Σ to be a sequence space, where all sequences are allowable under $F_{\alpha,0}$.

Aarts and Oversteegen [**1**] prove the following theorem.

THEOREM 1. *If E_α has one attracting fixed point then $J(E_\alpha)$ is a Cantor bouquet, i.e. $J(E_\alpha)$ is homeomorphic to a straight brush in $[0, \infty) \times \Sigma$.*

See [**1**] for the definition of a straight brush. Hence the Cantor bouquet consists of infinitely many closed Jordan arcs. Each arc has a different endpoint in the plane and none of the arcs intersect one another. Furthermore, all the points that have the same itinerary lie on the same arc, called a hair. It also follows from the properties of the straight brush that the endpoints are dense in the Cantor bouquet and in fact Mayer [**12**] has shown that if $G(F_{\alpha,0})$ is the set of all endpoints of $J(F_{\alpha,0})$, then the set $G(F_{\alpha,0}) \cup \{\infty\}$ is a connected set. However, the set $G(F_{\alpha,0})$ is totally disconnected. Figure 2 shows a picture of a Cantor bouquet in the complex plane.

3.2. $\beta > 0$. We consider functions $F_{\alpha,\beta}$ where $\beta > 0$. Since these functions do not have a pole on the real line, the image of the real line is bounded from above by the larger asymptotic value and from below by the smaller asymptotic value. We will consider only those functions that are decreasing and have exactly one attracting fixed point.

Let Σ' denote the set of one-sided sequences whose entries are either integers or the symbol ∞. If ∞ is an entry in the sequence, then it will terminate the sequence. Σ' consists of all infinite sequences of the form $(s_0, s_1, \dots)$ where $s_j \in \mathbb{Z}$ and all finite sequences of the form $(s_0, s_1, \dots, s_j, \infty)$ where $s_i \in \mathbb{Z}$. We define the topology of Γ and the shift

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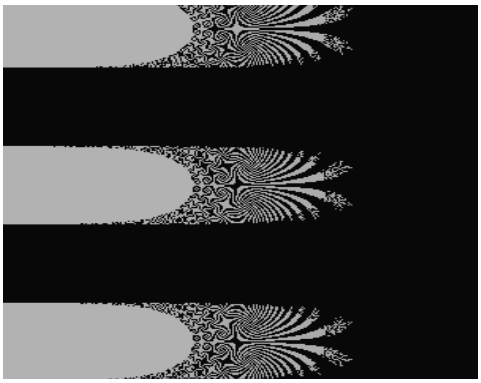


FIGURE 2. The stable region is black, the colored region shows the Cantor bouquet.

automorphism $\sigma : \Sigma' \longrightarrow \Sigma'$ as in Devaney and Keen [**6**]. Will we show for a special case that $\sigma_{|\Sigma'}$ is conjugate to $F_{|J(F)}$.

To do this we will have to define itineraries for points in the Julia set. Let $F_{\alpha,\beta}$ have $\alpha, \beta \in \mathbb{R}$ and $\beta > 0$; furthermore, let us assume that $F_{\alpha,\beta}$ has exactly one attracting fixed point z^* on the real line. For simplicity we will omit the subscripts where no misunderstanding can occur.

Using the fact that the whole real line is in the Fatou set, we can conclude that a neighborhood around the real line as well as all its $2\pi i$ translates is in the Fatou set. To each asymptotic value there exists an unbounded half-plane whose image is a neighborhood of the asymptotic value. For α it is a half-plane on the right and for $1/\beta$ it is a half-plane on the left. Since both asymptotic values are real it follows that both these half-planes are in the Fatou set as well. This yields a ladder-shaped Fatou set, similar to the one in Devaney and Keen [**6**]. The Julia set consists of two half-planes and ladder rungs connecting them; we denote the ladder holes as C_k, where C_0 is immediately above the origin.

Since all points in the Julia set will remain in these C_i we arrive at the following.

Definition 1. For $z \in J(F_{\alpha,\beta})$, the itinerary of z under $F_{\alpha,\beta}$ is the sequence of integers $S_{\alpha,\beta}(z) = (s_0, s_1, \ldots)$, $s_i = k$ iff $F^i_{\alpha,\beta}(z) \in C_k$. If z eventually lands on a pole, i.e. $F^m_{\alpha,\beta}(z) = \infty$ then $S_{\alpha,\beta}(z) = (s_0, \ldots, s_{m-1}, \infty)$, $s_i = k$ iff $F^i_{\alpha,\beta}(z) \in C_k$.

Next we will define the conjugacy between the sequences and points on the complex plane. First consider the inverse map to $F_{\alpha,\beta}$. This map has infinitely many branches. We will define the branches

$$G_k : \mathbb{C} - \mathbb{R} \longrightarrow \{z \mid 2\pi k < \mathrm{Im}(z) < 2\pi(k+1)\}.$$

Since the image of H under $F_{\alpha,\beta}$ is the band D, it follows that $F_{\alpha,\beta}(C_i) \longrightarrow \hat{\mathbb{C}} - D$. Hence the inverse image into C_i is a contraction in the Poincaré metric and it follows that $\lim_{n\to\infty} G_{s_0} \circ \cdots \circ G_{s_n}(C_{n+1})$ exists in C_{s_0} and is unique. For finite sequences s, the preimage is also unique as there is exactly one pole in each C_i. This proves the following.

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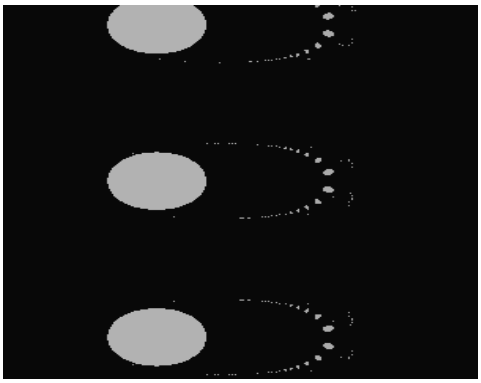


FIGURE 3. The stable region is black. The shaded region shows the neighborhood of points in the Cantor set.

THEOREM 2. *Let $\alpha, \beta \in \mathbb{R}$ and $\beta > 0$. If $F_{\alpha,\beta}$ has an attracting fixed point, then $J(F_{\alpha,\beta})$ is a Cantor set in $\hat{\mathbb{C}}$ and $F_{\alpha,\beta}|_J$ is topologically conjugate to $\sigma|_{\Sigma'}$.*

As an example let us consider the map $F_{1,0.0001}$, which has one attracting fixed point on the real line. Hence the Julia set is a Cantor set in the complex plane. Cantor sets are hard for computers to generate, but Figure 3 shows the neighborhoods of the points in the Cantor set.

This structure in the Julia set will be common among a whole class of functions, namely those that have both asymptotic values in the immediate basin of attraction for the only fixed point.

3.3. $\beta < 0$. In this part of the parameter plane we have to consider the case where $F_{\alpha,\beta}$ has two attracting fixed points and a pole on the real line.

Using some classical results by Baker [**2**] we state without proof the following.

THEOREM 3. *If $F_{\alpha,\beta}$ has two attracting fixed points $\{z_0, z_1\}$ and a repelling fixed point z^*, then the Julia set $J(F_{\alpha,\beta})$ is the boundary between two completely invariant components of the Fatou set.*

The Julia set includes the repelling fixed point and the pole $\ln|\beta| + i\pi$ and all its 2π translates.

THEOREM 4. *Let $F_{\alpha,\beta}$ with $\alpha, \beta \in \mathbb{R}$ and $\beta < 0$ have exactly two attracting fixed points, where each attracting fixed point has an asymptotic value in its immediate basin of attraction. Then $J(F_{\alpha,\beta})$ is a closed Jordan curve in $\hat{\mathbb{C}}$.*

Let Σ'' be a space of infinite and finite sequences, whose entries consist of infinitely many symbols $s_i \in \mathbb{Z} \cup \{p_i \mid i \in \mathbb{Z}\}$:

$$\Sigma'' = \{(s_i)_{i=1}^{\infty} \mid s_i \in \mathbb{Z}\} \cup \bigcup_{k=1}^{\infty} \{(s_i)_{i=1}^{\infty} \mid s_i \in \mathbb{Z} \text{ for all } i < k,\ s_k \in \{p_j \mid j \in \mathbb{Z}\}\}.$$

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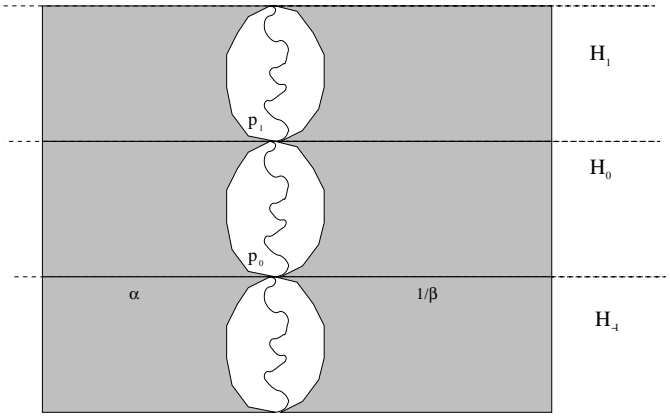


FIGURE 4. The Fatou set is shaded.

For an element $s \in \Sigma''$ in the sequence space we will now define a neighborhood basis as follows.

(i) If s is an infinite sequence then $U_k(s) = \{t \in \Sigma'' \mid t_i = s_i \text{ for all } i \leq k\}$.

(ii) If s is a finite sequence $(s_0, \ldots, s_j, p_m)$ then

$$U_k(s) = \{t \in \Sigma'' \mid t_i = s_i \text{ for all } i \leq k\}$$

for $k < j$ and

$$\begin{aligned} U_k(s) = &\{t \in \Sigma'' \mid t_i = s_i \text{ for all } i \leq k,\ t_{j+1} = m,\ t_{j+2} \leq -k\} \\ &\cup \{t \in \Sigma'' | t_i = s_i \text{ for all } i \leq k,\ t_{j+1} = m-1,\ t_{j+2} \geq k\} \end{aligned}$$

for $k \geq j$.

Proof of Theorem 4. Let $F_{\alpha,\beta}(z) = (\alpha e^z + 1)/(e^z + \beta)$, where $\alpha, \beta \in \mathbb{R}$ and the map has exactly two attracting fixed points. It is clear that $J(F_{\alpha,\beta})$ is closed, periodic, forward and backwards invariant. This set includes the poles of the function $F_{\alpha,\beta}$ at a period of $2\pi i$. Since one of the poles is on the real line, we partition the plane into horizontal strips of height 2π starting from the real line. It follows that $F_{\alpha,\beta}(J(F_{\alpha,\beta}) \cap H_i) = J(F_{\alpha,\beta})$ and in fact $F_{\alpha,\beta}$ is expanding everywhere on $J(F_{\alpha,\beta})$ in the Poincaré metric on a neighborhood of $J(F_{\alpha,\beta})$ in the strip H_i. This is clear if we consider a subset of the Fatou set which is bounded by the Julia set. The gray area in Figure 4 is contained in the Fatou set. However, the image of any component of the white region covers itself, hence we can define a Poincaré metric there. Since the Julia set is not in the shaded area, this implies expansion.

Hence we can define for all $z \in J(F_{\alpha,\beta})$ an itinerary $S_{\alpha,\beta}(z) \in \Sigma''$ with

$$\begin{aligned} s_i = k &\Longleftrightarrow F^i_{\alpha,\beta}(z) \in I_k, \\ s_i = p_k &\Longleftrightarrow F^i_{\alpha,\beta}(z) = (2\pi i k). \end{aligned}$$

We can conclude that $S_{\alpha,\beta}(z) : J(F_{\alpha,\beta}) \longrightarrow \Sigma''$ is a homeomorphism. Since Σ'' is homeomorphic with $\mathbb{R}$ (see [**6**]) it follows that $J(F_{\alpha,\beta})$ is a Jordan curve. □

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FIGURE 5. The Jordan arc is the boundary between the black and the shaded region.

We consider another example: $F_{1,-0.0001}$. This map has two attracting fixed points, each containing an asymptotic value in its immediate basin of attraction. Figure 5 shows this arc as the boundary between the two basins of attraction, one black, the other shaded.

4. *Bifurcation*

We have seen in the preceding section three different types of Julia sets and we wish to investigate the bifurcation of meromorphic maps to the entire map, $F_{\alpha,0}(z) = (\alpha e^z + 1)/(e^z + 0) = \alpha + e^{-z}$, obtained by letting β go to zero in our family of functions $F_{\alpha,\beta}(z) = (\alpha e^z + 1)/(e^z + \beta)$. Given α, β we have defined an itinerary for all $z \in J(F_{\alpha,\beta})$, denoted $S_{\alpha,\beta}(z)$. The inverse map will be denoted $S^{-1}_{\alpha,\beta}(s)$. For $\beta = 0$, $S^{-1}_{\alpha,0}(s)$ yields the whole hair, but we will take $S^{-1}_{\alpha,0}(s)$ to mean the endpoint of the hair with itinerary s. We begin by characterizing the point $z^0_s = S^{-1}_{\alpha,0}(s)$. It is well known that this point can be obtained using the construction of Aarts and Oversteegen [**1**], but we will use the construction in Devaney and Goldberg [**5**], because it will serve our purpose better in later proofs.

For E_α there exists a half-plane $\{z \mid \mathrm{Re}(z) \geq \nu\}$ that maps inside itself and the function E_α is expanding on the complement. For any $z \in J(E_\alpha)$ with an itinerary $s = (s_0, s_1, s_2, \ldots) \in \Sigma$ we will construct a sequence of squares with side length 2π. These will be denoted ${}^0B^\infty_j$ and their right-hand sides will be denoted ${}^0r^\infty_j$.

These boxes will be the limit of the following inductive construction.

For all $k \geq 0$ define the infinite sequence of boxes ${}^0B^k_j$, each with side length 2π and the right boundaries at ${}^0r^k_j$.

Step 1. For $k = 0$ let ${}^0r^k_j = \nu$ for all j.

Step 2. Assume ${}^0B^l_j$ have been defined for $l \leq k$ for all j.

Step 3. For all j select ${}^0B^{k+1}_j$ such that $E_\alpha({}^0B^{k+1}_j)$ just covers ${}^0B^k_{j+1}$. This will be achieved by requiring that the inner radius of the annulus $E_\alpha({}^0B^{k+1}_j)$ intersects ${}^0B^k_{j+1}$ at exactly one point.

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A picture of this construction can be found in Devaney and Goldberg [**5**]. It is clear that ${}^0r_j^{k+1} < {}^0r_j^k$, so as $k \to \infty$ the right-hand sides of the boxes are moving monotonically to the left. Now we define ${}^0r_j^\infty = \lim_{k\to\infty} {}^0r_j^k$ and from that we obtain the boxes ${}^0B_j^\infty$.

It follows that in the limit:

(i) $E_\alpha({}^0B_j^\infty) \supset {}^0B_{j+1}^\infty$;

(ii) $\bigcap_{j\geq 0} E_\alpha^{-j}({}^0B_j^\infty) = S_{\alpha,0}^{-1}(s) = {}^0z_s$;

(iii) as $k \to \infty$, ${}^0r_j^k \searrow {}^0r_j^\infty$.

From these properties we also obtain the following.

COROLLARY. *For all ε there exists a n such that* $\operatorname{diam}\left(\bigcap_{j\leq n} E_\alpha^{-j}({}^0B_j^\infty)\right) < \varepsilon$.

4.1. $\beta > 0$. We can choose $\tilde{\alpha}$ and $\tilde{\beta} > 0$, such that for all $\beta < \tilde{\beta}$ the map $F_{\tilde{\alpha},\beta} = (\tilde{\alpha}e^z + 1)/(e^z + \beta)$ has one attracting fixed point. Now we wish to investigate what happens as $\beta \to 0$. Again we will concentrate on α, β real.

This will be interesting for the following reason. As we have shown earlier, for all $\tilde{\beta} > \beta > 0$, the Julia set of $F_{\tilde{\alpha},\beta}$ is a Cantor set that is bounded to the right and left by the asymptotic tracts. However, for $\beta = 0$, the Julia set of $F_{\tilde{\alpha},0} = \tilde{\alpha} + e^{-z}$ has been shown to be a Cantor bouquet [**1**]. Hence as $\beta \to 0$ we will witness the Julia set of our maps change from a Cantor set to a Cantor bouquet.

If $F_{\alpha,\beta}$ has exactly one attracting fixed point, then by previous results $J(F_{\alpha,\beta})$ is a Cantor set and is homeomorphic to Σ', the set of one-sided sequences that may also be finite. The inverses to $F_{\alpha,\beta}$ are $G_{\alpha,\beta,k}(y) = \ln[(1 - y\beta)/(y - \alpha)] + 2\pi ik$.

Let $\tilde{\alpha}$ and $\tilde{\beta}$ be such that, for all $0 \leq \beta \leq \tilde{\beta}$, $F_{\tilde{\alpha},\beta}$ has exactly one fixed point. As previously shown we can define itineraries for all $z \in J(F_{\tilde{\alpha},\beta})$, where $\beta > 0$. For the case where $\beta = 0$ we will define the itineraries of $z \in J(F_{\tilde{\alpha},0})$ as shown in Aarts and Oversteegen [**1**]. So for all $z \in J(F_{\tilde{\alpha},\beta})$, $\beta > 0$, we will define $S_{\tilde{\alpha},\beta}(z)$ to be the map into Σ'.

We will now begin our investigation of what happens to the Julia set as we meet the exponential. To understand how the bifurcation of the Cantor set to the Cantor bouquet occurs we will partition the Cantor set into several cases. We will distinguish between itineraries that have different properties. So starting with an itinerary we will look at the unique point that has that itinerary and then let $\beta \to 0$. What happens to this point? The first notion that needs to be examined is that of an allowable sequence. Devaney and Krych [**7**] have shown that not all sequences in Σ are actually realized by a point z in the Julia set of the exponential. We recall that under $F_{\alpha,\beta}$ all itineraries in Σ' are allowable.

This, of course, raises the question of what happens to those points that have itineraries that are not allowable under the exponential.

PROPOSITION 1. *Suppose $t \in \Sigma'$ is not allowable under the exponential. Then* $\lim_{\beta\to 0} S_{\tilde{\alpha},\beta}^{-1}(t) = \infty$.

This proposition follows from the observation that if $\lim_{\beta\to 0} S_{\tilde{\alpha},\beta}^{-1}(t)$ had an accumulation point $z^* \in \mathbb{C}$, then by continuity z^* would have an itinerary that is not allowable, and Devaney and Krych [**7**] have shown that these do not exist.

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So we have seen that certain points escape to infinity as we approach the exponential, among which must also be the poles and prepoles of our meromorphic functions. Those are the points that have a finite itinerary. We state the following proposition without proof.

PROPOSITION 2. *Let $t \in \Sigma'$ be a finite sequence. Then for all β with $0 < \beta \leq \tilde{\beta}$, $S^{-1}_{\tilde{\alpha},\beta}(t)$ is unique and $\lim_{\beta\to 0} S^{-1}_{\tilde{\alpha},\beta}(t) = \infty$.*

Now we wish to understand which points in the Cantor set actually stay in the plane and what becomes of them as the Cantor set becomes a Cantor bouquet. The first intuition is that they will become the endpoints of the hairs and this is certainly so if the itinerary is bounded. The proof of this will use the continuity of $F_{\alpha,\beta}$ with respect to β.

PROPOSITION 3. *Let $t \in \Sigma'$ be a bounded sequence. Then for all β with $0 \leq \beta \leq \tilde{\beta}$, $S^{-1}_{\tilde{\alpha},\beta}(t)$ is unique and $\lim_{\beta\to 0} S^{-1}_{\tilde{\alpha},\beta}(t) = S^{-1}_{\tilde{\alpha},0}(t)$.*

Proof. Starting from the exponential $F_{\tilde{\alpha},0}(z) = \tilde{\alpha} + e^{-z}$, we first wish to observe that any point z^* in the Julia set with bounded itinerary has the property that the real part of all points on the forward orbit, $O^+(z^*)$, is bounded.

So if we construct the minimal boxes introduced by Aarts and Oversteegen, with the property that the boxes are in the strips corresponding to the itinerary and cover each other, then the point that stays in the boxes for all time is the endpoint of a hair. The minimal boxes are the ones with this property that have the smallest real parts. These boxes will be denoted $B_0, B_1, \ldots$.

Together with the earlier observation, this implies that the minimal boxes constructed for an itinerary that is bounded are indeed also bounded in the real part, i.e. all the boxes are to the left of some real part which is sufficiently large.

Now we will use the same boxes $B_0, \ldots$ for our meromorphic maps. In general, the main property that the boxes cover themselves may not hold.

As $\beta \to 0$, the big circle goes to the right as the radius gets larger. Since all boxes are bounded from the right, by continuity there is a β^* such that for all $\beta < \beta^*$, $F_{\tilde{\alpha},\beta}(B_i) \supset B_{i+1}$. Hence there is a point that stays in the boxes for all time under the iteration of the meromorphic map. By continuity these points converge to the endpoint of the hairs. □

PROPOSITION 4. *Let $t \in \Sigma'$ be unbounded, but allowable under the exponential. Then $\lim_{\beta\to 0} S^{-1}_{\tilde{\alpha},\beta}(t) = S^{-1}_{\tilde{\alpha},0}(t)$.*

Proof. It suffices to show that for all ε there exists a β small enough such that $|S^{-1}_{\alpha,\beta}(t) - S^{-1}_{\alpha,0}(t)| < \varepsilon$. Let us denote ${}^{\beta}z_t := S^{-1}_{\alpha,\beta}(t)$ and for each ${}^{\beta}z_t$ we will construct a sequence of boxes ${}^{\beta}B^{\infty}_j$. The right-hand side will be denoted as ${}^{\beta}r^{\infty}_j$ and the side lengths will be 2π. We will define these boxes as in the previous section.

By continuity it follows that, for β small enough, the half-plane $\{z|\,\mathrm{Re}(z) \geq \nu\}$ maps inside itself and the map $F_{\alpha,\beta}$ is expanding on the complement. Again the construction will be inductive, but because of the previous remark it follows that

$$ {}^{\beta}r^0_j = {}^0r^0_j \quad \text{for all } j $$

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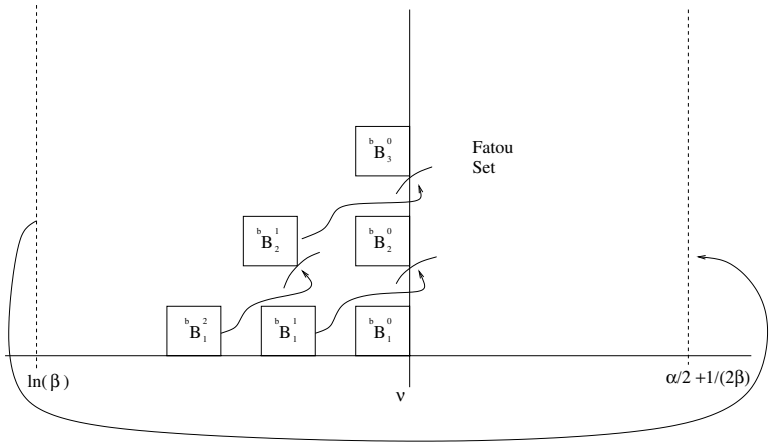


FIGURE 6. The box construction.

and hence

$$ {}^{\beta}B_j^0 = {}^{0}B_j^0 \quad \text{for all } j. $$

Now we construct ${}^{0}B_j^k$.

Step 1. For $k = 0$ let ${}^{\beta}r_j^k = \nu$ for all j.

Step 2. Assume ${}^{\beta}B_j^l$ have been defined for $l \le k$ for all j.

Step 3. For all j select ${}^{\beta}B_j^{k+1}$ such that $F_{\alpha,\beta}({}^{\beta}B_j^{k+1})$ just covers ${}^{0}B_{j+1}^k$. This will be achieved by requiring that the inner radius of the annulus $E_\alpha({}^{0}B_j^{k+1})$ intersects ${}^{0}B_{j+1}^k$ at exactly one point.

In order for Step 3 to work we must choose β small enough such that the image of a box with side length 2π is an annulus of thickness at least 10. The box whose image is the thinnest annulus is the box with right-hand side $\ln(\beta) + 2\pi$. The left boundary of this box maps to a vertical line with real part $(\alpha/2) + (1/2\beta)$ and the right boundary maps to a circle that intersects the real line on the right at approximately $(500\alpha/501) + (1/501\beta)$. Hence the thinnest this annulus can be is $\alpha/2$, but since this occurs far in the right-hand half-plane, it follows that in the complement of $\{z \mid \mathrm{Re}(z) \ge \nu\}$ the thickness is at least 10.

Now by monotonicity it follows that ${}^{\beta}r_j^{k+1} < {}^{\beta}r_j^k$ and for β small enough ${}^{\beta}r_j^k > \ln(\beta)$, because the image of the vertical line at $\ln(\beta)$ is the vertical line at $\alpha/2 + 1/(2\beta)$ as can be seen in Figure 6. Hence any box with right-hand side at $\ln(\beta)$ will not cover the Julia set since its image is completely contained in the Fatou set.

Now we can let ${}^{\beta}r_j^\infty = \lim_{k\to\infty} {}^{\beta}r_j^k$. This limit exists because ${}^{\beta}r_j^k$ is bounded. Hence we have defined the boxes ${}^{\beta}B_j^\infty$, so in the limit:

(i) $F_{\alpha,\beta}({}^{\beta}B_j^\infty) \supset {}^{\beta}B_{j+1}^\infty$;

(ii) $\bigcap_{j\ge 0} F_{\alpha,\beta}^{-j}({}^{\beta}B_j^\infty) = S_{\alpha,\beta}^{-1}(t) = {}^{\beta}z_t$;

(iii) as $k \to \infty$, ${}^{\beta}r_j^k \searrow {}^{\beta}r_j^\infty$.

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Now we consider the finite sequence of boxes

$$ {}^{\beta}B^1_{k-1}, {}^{\beta}B^2_{k-2}, \ldots, {}^{\beta}B^k_0 = {}^0B^k_0. $$

It follows that we can choose β so small that $|{}^0r_j^{k-j} - {}^{\beta}r_j^{k-j}| < \varepsilon/4$. Next we choose n such that $\operatorname{diam}(E_\alpha^{-n}({}^0B^0_n)) < \varepsilon/2$. If this condition holds, then it will also hold for any box ${}^0B^k_n$ since ${}^0r^k_n < {}^0r^0_n$ and the $E'_\alpha(z)$ increases with decreasing real part. For any k let $\bar{\varepsilon}$ be such that

$$ |{}^0r^\infty_n - {}^0r^k_n| < \bar{\varepsilon} \quad \text{and} \quad |{}^{\beta}r^\infty_n - {}^{\beta}r^k_n| < \bar{\varepsilon}. $$

Now we consider the two finite sequence of boxes

$$ {}^{\beta}B^1_{k-1}, {}^{\beta}B^2_{k-2}, \ldots, {}^{\beta}B^k_0 = {}^0B^k_0 \quad \text{and} \quad {}^0B^1_{k-1}, {}^0B^2_{k-2}, \ldots, {}^0B^k_0. $$

The last boxes are identical, but if we pull this box back through one set of boxes with $F^{-1}_{\alpha,\beta}$ and through the other set of boxes with E^{-1}_α then the total cumulative error that any point moves apart is $\sum_{j=1}^n j|{}^0r^k_j - {}^{\beta}r^k_j|$. This implies that the total difference between ${}^0B^\infty_n$ and ${}^{\beta}B^\infty_n$ is

$$ \sum_{j=1}^{n} j|{}^0r^\infty_j - {}^{\beta}r^\infty_j| = \sum_{j=1}^{n} j(|{}^0r^k_j - {}^{\beta}r^k_j| + 2\bar{\varepsilon}). $$

Since this is a finite sum we can choose β so small that $\sum_{j=1}^n j|{}^0r^\infty_j - {}^{\beta}r^\infty_j| < \varepsilon/2$. Hence we have shown that for all $\varepsilon > 0$ there exist a n and β such that:

(i) $\operatorname{diam}(E_\alpha^{-n}({}^0B^\infty_n)) < \varepsilon/2$;

(ii) $\sum_{j=1}^n j|{}^0r^\infty_j - {}^{\beta}r^\infty_j| < \varepsilon/2$.

Since ${}^0z_t \in E_\alpha^{-n}({}^0B^\infty_n)$ and ${}^{\beta}z_t \in F^{-n}_{\alpha,\beta}({}^{\beta}B^\infty_n)$ it follows that for all $\varepsilon > 0$ there exists a $\beta > 0$ such that $|{}^{\beta}z_t - {}^0z_t| < \varepsilon$ and this proves the statement. □

The preceding propositions are summarized in the following theorem.

THEOREM 5. *If $t \in \Gamma$ is allowable under the exponential, then as $\beta \to 0$, $S^{-1}_{\tilde{\alpha},\beta}(t)$ converges to the endpoint of a hair in the Julia set of $F_{\tilde{\alpha},0}$ that has the itinerary t.*

4.2. $\beta < 0$. Now we will consider the other bifurcation, letting β go to zero from the negative side. We choose α such that the functions $F_{\alpha,\beta}$ have exactly two attracting fixed points and each has an asymptotic value in its immediate basin of attraction. Earlier results show that the Julia set is a Jordan curve. However, as $\beta \to 0$ this curve limits onto the Cantor bouquet. We recall that we have also defined for all $z \in J(F_{\alpha,\beta})$ an itinerary $S_{\alpha,\beta} \in \Sigma''$. The sequence space considered now differs from the previous section only by the fact that the poles are indexed. Hence we can show analogously the following.

PROPOSITION 5. *Suppose $t \in \Sigma''$ is not allowable under the exponential or finite. Then $\lim_{\beta\to 0} S^{-1}_{\alpha,\beta}(t) = \infty$.*

Now we can use the box construction from the previous section to show the following analogous result.

THEOREM 6. *If $t \in \Sigma''$ is allowable under the exponential, then as $\beta \to 0$, $S^{-1}_{\tilde{\alpha},\beta}(t)$ converges to the endpoint of a hair in the Julia set of $F_{\tilde{\alpha},0}$ that has the itinerary t.*

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