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Dynamics of induced systems

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Abstract. In this paper we study the dynamical properties of actions on the space of compact subsets of the phase space. More precisely, if X is a metric space, let 2^X denote the space of non-empty compact subsets of X provided with the Hausdorff topology. If f is a continuous self-map on X, there is a naturally induced continuous self-map f_* on 2^X. Our main theme is the interrelation between the dynamics of f and f_*. For such a study, it is useful to consider the space $\mathcal{C}(K, X)$ of continuous maps from a Cantor set K to X provided with the topology of uniform convergence, and f_* induced on $\mathcal{C}(K, X)$ by composition of maps. We mainly study the properties of transitive points of the induced system $(2^X, f_*)$ both topologically and dynamically, and give some examples. We also look into some more properties of the system $(2^X, f_*)$.

1. *Introduction*

For us a dynamical system (X, f) consists of a continuous map f on a non-empty metric space X, i.e. $f : X \to X$. We examine here the dynamics of the system $(2^X, f_*)$ where 2^X is the space of non-empty compact subsets of X.

Such a study was first undertaken by Bauer and Sigmund [**7**]. The dynamics on 2^X is richer than the dynamics on X, which can be easily assessed from the results and examples in Bauer and Sigmund [**7**], Glasner and Weiss [**11**], Banks [**6**] and Sharma and Nagar [**22**, **23**]. This provides us a strong motivation for the study of dynamics of $(2^X, f_*)$.

In this article we look into some known results—improving them, some new concepts arising from them—and answer some natural questions with a view to further developing this study. A recent interesting paper which overlaps ours to some extent is [**17**].

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2. *Spaces of continuous maps and spaces of compact subsets*

We first review some useful results and set up notation. All our basic definitions and notations are as in Akin [**1**], though we make full attempt to explain each of these as we proceed. All our spaces are assumed to be non-empty.

Recall that any two perfect, compact, zero-dimensional metric spaces are homeomorphic and we will refer to any such as a Cantor set. A Polish space is a separable space which admits a complete metric. As a compact metric space is complete and separable, it is Polish. A Polish space with no isolated points is called perfect. Any perfect Polish space contains a Cantor set. A G_δ subset of a Polish space is Polish. It follows that any non-empty open subset of a perfect Polish space contains a Cantor set. For an exposition of this material, see, for example, Akin [**2**].

Every compact metric space is a continuous image of a Cantor set. We include the brief proof.

PROPOSITION 2.1. *If X is a compact, metrizable space and K is a Cantor set then there is a continuous map from K onto X.*

Proof. Let $\mathcal{B} = \{U_1, U_2, \ldots\}$ be a countable base for X and let Q be the closure in $X \times \{0, 1\}^{\mathbb{N}}$ of $\{(x, a) : a_i = 1 \Leftrightarrow x \in U_i\}$. Clearly, Q is contained in the closed set $\{(x, a) : a_i = 1 \text{ and } x \in \bar{U}_i \text{ or } a_i = 0 \text{ and } x \in X \backslash U_i\}$. It follows that if $(x_1, a_1), (x_2, a_2) \in Q$ and $U_i \in \mathcal{B}$ with $x_1 \in U_i$, $x_2 \in X \backslash U_i$ then $(a_1)_i = 1$, $(a_2)_i = 0$. Hence, the projection map from Q to $\{0, 1\}^{\mathbb{N}}$ is injective and so Q is a zero-dimensional compact metric space. Clearly, $\pi_1 : Q \to X$ is surjective. $Q \times K$ is perfect and so there is a homeomorphism $h : K \to Q \times K$. $\pi_1 \circ h : K \to X$ is surjective. □

For a metric space X, we follow Illanes and Nadler [**13**] in letting 2^X denote the space of non-empty compact subsets of X. Notice that we are excluding $\emptyset \subset X$ which is sometimes regarded as an isolated point of 2^X. If K is a compact metrizable space we let $\mathcal{C}(K, X)$ denote the space of continuous maps from K to X. Each of these has a natural metric induced from d on X. (We will use d for all the metrics which occur, allowing context to make the referent clear.), On 2^X we use the Hausdorff metric. For $A, B \in 2^X$,

$$d(A, B) = \max\{d(a, B) : a \in A\} \cup \{d(b, A) : b \in B\}, \tag{2.1}$$

where $d(a, B) = \min\{d(a, b) : b \in B\}$. Thus, $d(A, B) < \epsilon$ if and only if each set is in the open ϵ neighborhood of the other, or, equivalently, each point of A is within ϵ of a point in B and *vice versa*.

When X is compact, we occasionally use an equivalent topology on 2^X. Define for any collection $\{U_i : 1 \le i \le n\}$ of opene (= open and non-empty) sets,

$$\langle U_1, U_2, \ldots, U_n \rangle = \left\{ E \in 2^X : E \subseteq \bigcup_{i=1}^{n} U_i,\ E \bigcap U_i \neq \phi,\ 1 \le i \le n \right\}. \tag{2.2}$$

The topology on 2^X, generated by such collection as basis, is known as the *Vietoris topology*.

As a topological object, 2^X has a very rich structure. It is known that if X contains a *Peano continuum* then 2^X contains the Hilbert cube. We refer the reader to Illanes and Nadler [**13**] or Schori and West [**20**] for more details.

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On $\mathcal{C}(K, X)$ we use the sup metric. For $u, v \in \mathcal{C}(K, X)$,

$$d(u, v) = \max\{d(u(x), v(x)) : x \in K\}, \tag{2.3}$$

with the topology of uniform convergence.

If $\{A_n\}$ is a sequence of closed sets in a topological space then

$$\overline{\bigcup_n \{A_n\}} = \bigcup_n \{A_n\} \cup \limsup_n \{A_n\} \quad \text{where } \limsup_n \{A_n\} = \bigcap_k \overline{\bigcup_{n \geq k} \{A_n\}}. \tag{2.4}$$

The results stated below are standard. One can refer to [**1**, **13**, **18**] for more details. For the sake of completeness we briefly discuss the proofs of the results which we shall use and in the process establish some of our notation.

LEMMA 2.2. *For a metric space X, and $\{A_n\}$ a sequence in 2^X:*

(a) *if $\{A_n\}$ converges to A in 2^X then $\overline{\bigcup_n \{A_n\}}$ is compact;*

(b) *if $\overline{\bigcup_n \{A_n\}}$ is compact and $\{A_n\}$ is Cauchy then A_n converges to* $\limsup\{A_n\}$*;*

(c) *if X is complete then 2^X is complete;*

(d) *if X is compact then 2^X is compact;*

(e) *if K is compact and X is complete, then $\mathcal{C}(K, X)$ is complete.*

Proof. We consider the sequence $\{A_n\}$ in 2^X for a metric space X.

(a) Let $\{x_n\}$ be a sequence in $\overline{\bigcup_n \{A_n\}}$. If there is an $N \in \mathbb{N}$ such that infinitely many x_i are in the compact set $\bigcup_{n \leq N} A_n$ then clearly $\{x_n\}$ has a convergent subsequence. Otherwise, since $A_n \to A$, it follows from the definition of the Hausdorff metric that there are y_i in A with $d(x_i, y_i) \to 0$. Since A is compact, $\{y_i\}$ has a convergent subsequence, and the corresponding subsequence of $\{x_i\}$ converges to the same point.

(b) Let $\epsilon > 0$. Since $\{A_n\}$ is a Cauchy sequence there is an $N > 0$ such that all the A_n for $n \geq N$ are within the $\epsilon/3$ neighborhoods of each other, and so are within $2\epsilon/3$ of $\overline{\bigcup_{n \geq N} \{A_n\}}$. Now $\overline{\bigcup_{n \geq N} A_n} \to \limsup A_n$ as $N \to \infty$, so A_n is within the ϵ neighborhood of $\limsup A_n$ when $n \geq N$. It follows that $\lim A_n = \limsup A_n$. In particular, if $A_n \to A$ then $A = \limsup A_n$.

(c) Let $\{A_n\}$ be a Cauchy sequence in 2^X. We show that $\overline{\bigcup A_n}$ is compact. Since a compact metric space is complete, this will imply that $\{A_n\}$ converges. Now since $\overline{\bigcup A_n}$ is closed, it is sufficient to show that it is totally bounded. Let $\epsilon > 0$. Again there is an $N \in \mathbb{N}$ such that all A_n for $n \geq N$ are within the $\epsilon/3$ neighborhood of A_N. Therefore, a finite subset of A_N which is $\epsilon/3$ dense in A_N is ϵ dense in $\overline{\bigcup_{n \geq N} A_n}$. Since $\bigcup_{n < N} A_n$ is compact, there is a finite ϵ dense set in $\overline{\bigcup A_n}$.

(d) If X is compact and $\epsilon > 0$ then there exists D a finite subset of X which is ϵ-dense, i.e. for every $x \in X$ there exists $p \in D$ such that $d(x, p) < \epsilon$. If $A \in 2^X$ then the set of points of D which are less than ϵ from a point of A forms an element of 2^X which has distance less than ϵ from A. So the finite set 2^D is ϵ-dense in 2^X. Thus, 2^X is totally bounded as well as complete, and so is compact.

(e) This is well known. A pointwise Cauchy sequence of functions converges to a function, and if the sequence is uniformly Cauchy then convergence is uniform. Finally, the uniform limit of continuous functions is continuous. □

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We will let (n) denote $\{1, \ldots, n\}$ and write $X^{(n)}$ for the n-fold product of copies of X. Hence, $f^{(n)}$ denotes the n-fold product of copies of f, as opposed to f^n which is the n-fold iterate of a map f on X.

PROPOSITION 2.3. *Let X be a metric space and K be a compact metrizable space.*

(a) *Define $i_X : X \to 2^X$ by $x \mapsto \{x\}$ and $i_X : X \to \mathcal{C}(K, X)$ by $x \mapsto c_x$ where c_x is the function which takes the constant value x. Both maps are isometric inclusions.*

(b) *Define $i_{X,n} : X^{(n)} \to 2^X$ by $(x_1, \ldots, x_n) \mapsto \{x_1, \ldots, x_n\}$ and $\vee : 2^{2^X} \to 2^X$ by $F \mapsto \bigcup F$. Each of these has Lipschitz constant 1, where on the product space $X^{(n)}$ we use the* max *of the distances between corresponding coordinates. Hence, the map $(2^X)^{(n)} \to 2^X$ given by $(A_1, \ldots, A_n) \mapsto A_1 \cup \cdots \cup A_n$ has Lipschitz constant 1. The image of $i_{X,n}$ is closed and $\vee$ is surjective.*

(c) *For $D \subset X$ let $\mathrm{FIN}(D) \subset 2^X$ be the collection of all finite subsets of D. If D is dense in X then $\mathrm{FIN}(D)$ is dense in 2^X. For $\mathcal{A}$ a clopen partition of K we let $\mathcal{C}(\mathcal{A}, D) \subset \mathcal{C}(K, X)$ denote the functions with values in D which are constant on each element of $\mathcal{A}$. If K is zero-dimensional (e.g. a Cantor set), D is dense and $\{\mathcal{A}_n\}$ is a sequence of clopen partitions of K whose mesh tends to zero then $\bigcup_n \{\mathcal{C}(\mathcal{A}_n, D)\}$ is dense in $\mathcal{C}(K, X)$.*

(d) *If X is separable then so are $\mathcal{C}(K, X)$ and 2^X.*

(e) *The map $\mathrm{Im} : \mathcal{C}(K, X) \to 2^X$ defined by $\mathrm{Im}(u) = u(K)$ has Lipschitz constant 1. If K is a Cantor set then* Im *is surjective.*

(f) If X has no isolated points then neither does 2^X, and if K is zero-dimensional then $\mathcal{C}(K, X)$ has no isolated points either. If X is connected then 2^X is connected, and if K is a Cantor set then $\mathcal{C}(K, X)$ is connected if X is connected.

(g) *For $h : K_1 \to K_2$ a continuous map of compact metrizable spaces, the map $h^* : \mathcal{C}(K_2, X) \to \mathcal{C}(K_1, X)$ defined by $h^*(u) = u \circ h$ has Lipschitz constant at most 1. If h is surjective then h^* is an isometric inclusion and $\mathrm{Im}(h^*(u)) = \mathrm{Im}(u)$.*

(h) *For $f : X_1 \to X_2$ a continuous map of metric spaces, the maps $f_* : \mathcal{C}(K, X_1) \to \mathcal{C}(K, X_2)$ and $f_* : 2^{X_1} \to 2^{X_2}$ are defined by $u \mapsto f \circ u$ and $A \mapsto f(A)$, respectively. If f is uniformly continuous or continuous, then the maps f_* are uniformly continuous or continuous respectively, and $f_*(\mathrm{Im}(u)) = \mathrm{Im}(f_*(u))$.*

(i) *The following subsets are closed:*

$$\begin{aligned} INT &= \{(A, B) \in 2^X \times 2^X : A \cap B \neq \emptyset\}, \\ INC &= \{(A, B) \in 2^X \times 2^X : A \subset B\}, \\ EPS &= \{(x, A) \in X \times 2^X : x \in A\}. \end{aligned} \tag{2.5}$$

If Y is a closed subset of X then 2^Y is closed, regarded as a subset of 2^X. If U is an open subset of X then 2^U is open, regarded as a subset of 2^X, and $2^X \backslash 2^{X \backslash U} = \{A : A \cap U \neq \emptyset\}$ is open.

Proof. The isometry and Lipschitz constant results in (a), (b), (e) and (g) are easy to check.

(b) We check that $F \in 2^{2^X}$ implies $\bigcup F$ is compact and so the map $\vee$ is well defined.

If $\{x_n\}$ is a sequence in $\bigcup F$, then there exists a sequence $\{A_n\} \in F$ such that $x_n \in A_n$. Since F is compact we can assume by going to a subsequence that $\{A_n\}$

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converges to $A \in F$. Clearly, $\bigcup F \supset \bigcup_n \{A_n\} \cup A = \overline{\bigcup_n \{A_n\}}$ and the latter is compact by Lemma 2.2(a). It follows that $\{x_n\}$ has a subsequence which converges to a point of $\bigcup F$. Thus, $\bigcup F$ satisfies the Bolzano–Weierstrass property and so is compact. Since $\vee \circ i_{2^X} = 1_{2^X}$, $\vee$ is surjective.

When K is a Cantor set, Im is surjective by Proposition 2.1. If A contains $n+1$ distinct points then for some $\epsilon > 0$ they are at least 2ϵ apart. It follows that A has distance at least ϵ from every set of cardinality at most n, i.e. from the image of $i_{X,n}$. Hence, the latter image is closed.

The map $(2^X)^{(n)} \to 2^X$ is $\vee \circ i_{2^X,n}$.

(c) It is easy to see that if D is dense in X then FIN(D) is dense in 2^X; see the proof of Lemma 2.2(d). If $\delta > 0$ is an ϵ modulus of uniform continuity for $u \in \mathcal{C}(K, X)$ and $\mathcal{A}$ is a clopen partition of mesh less than δ then there exists $v \in \mathcal{C}(\mathcal{A}, D)$ which is ϵ close to u in $\mathcal{C}(K, X)$.

(d) Let D be a countable dense subset of X. By (c), FIN(D) is a countable dense set in 2^X.

For $\mathcal{C}(K, X)$, first assume that K is a Cantor set. By (c), $\bigcup_n \{\mathcal{C}(\mathcal{A}_n, D)\}$ is a countable dense subset of $\mathcal{C}(K, X)$ when K when $\{\mathcal{A}_n\}$ is a sequence of clopen partitions with mesh tending to zero.

When K is not a Cantor set, we apply Proposition 2.1 to get $h : K_1 \to K$ a continuous surjection with K_1 a Cantor set. Then h^* is an isometric inclusion of $\mathcal{C}(K, X)$ onto a subset of $\mathcal{C}(K_1, X)$ which is separable by the previous Cantor set case. Any subset of a separable metric space is separable.

(f) If X has no isolated points then neither does FIN(X) nor $\mathcal{C}(\mathcal{A}, X)$ for any clopen partition $\mathcal{A}$. Since 2^X has a dense subset with no isolated points, there are no isolated points in 2^X. When K is a Cantor set we similarly apply (c).

If X is connected then each $\mathcal{C}(\mathcal{A}, X)$ is homeomorphic to $X^{(n)}$ where n is the cardinality of $\mathcal{A}$ and so is connected. By choosing a sequence $\{\mathcal{A}_n\}$ of clopen partitions where each refines its predecessor and with the mesh tending to zero we obtain a dense subset of $\mathcal{C}(K, X)$ as an increasing union of connected subsets. It follows that $\mathcal{C}(K, X)$ is connected and so 2^X, its image under Im, is connected as well.

If X is a Cantor set then we can choose as metric on X an *ultrametric* which satisfies the strengthening of the triangle inequality:

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}.$$

This is equivalent to saying that $V_\epsilon = \{(x, y) : d(x, y) < \epsilon\}$ is an open equivalence relation for all $\epsilon > 0$. This implies that the open balls are clopen. The ultrametric condition then holds for the induced metrics on 2^X and $\mathcal{C}(K, X)$ and so each of these is totally disconnected. Since 2^X is compact and perfect it is a Cantor set.

(h) The results are easy when f is uniformly continuous. In general, if $\{A_n\}$ is a sequence in 2^{X_1} converging to A then Lemma 2.2(a) implies that $A \subset \overline{\bigcup_n \{A_n\}}$ and the latter is compact. So we can use uniform continuity of the restriction of f to this set to show that $\{f_*(A_n)\}$ converges to $f_*(A)$. Similarly, if $\{u_n\}$ converges to u in $\mathcal{C}(K, X_1)$ then

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$\{\mathrm{Im}(u_n)\}$ converges to $\mathrm{Im}(u)$ in 2^{X_1} and so $u(K) \subset \overline{\bigcup_n \{u_n(K)\}}$ and the latter is compact. Again apply uniform continuity of f on the subset to show that $\{f_*(u_n)\}$ converges to $f_*(u)$.

(i) It is easiest to proceed by checking that $A \cap B = \emptyset$, $A \cap (X \backslash B) \neq \emptyset$ and $x \notin A$ are open conditions. We leave the details to the reader. □

3. *Induced system dynamics*

A dynamical system (X, f) consists of a continuous map f on a metric space X. The *orbit* of a point $x \in X$ is the set $\{x, f(x), f^2(x), \ldots\}$. The point x is *periodic* if there exists $n \in \mathbb{N}$ such that $f^n(x) = x$. The orbit of a periodic point, which is a finite set, is called a *periodic orbit*.

An *action map* $\pi : (X_1, f_1) \to (X_2, f_2)$ is a continuous map $\pi : X_1 \to X_2$ such that $f_2 \circ \pi = \pi \circ f_1$. When π is surjective we call it a *factor map* and say that (X_2, f_2) is a factor of (X_1, f_1).

For a system (X, f) we call $(2^X, f_*)$ and $(\mathcal{C}(K, X), f_*)$ the *induced systems*. The map $\mathrm{Im} : \mathcal{C}(K, X) \to 2^X$ defines an action map between the induced systems. When K is a Cantor set, Im is surjective and so a factor map.

PROPOSITION 3.1. *Let K be a Cantor set. If in a dynamical system (X, f) the periodic points are dense then the periodic points are dense in the induced systems $(2^X, f_*)$ and $(\mathcal{C}(K, X), f_*)$.*

Proof. Let PER be the dense set of periodic points in X. Any finite subset of PER is a periodic point of 2^X and if $u \in \mathcal{C}(\mathcal{A}, \mathrm{PER})$ with $\mathcal{A}$ a clopen partition of K, then u is a periodic point of $\mathcal{C}(K, X)$. By Proposition 2.3(c) above, FIN(PER) is dense in 2^X and $\bigcup_{\mathcal{A}} \{\mathcal{C}(\mathcal{A}, \mathrm{PER})\}$ is dense in $\mathcal{C}(K, X)$. □

Remark. Each periodic orbit is a fixed point in 2^X. So if X is infinite and the periodic points are dense, or, more generally, if there are infinitely many periodic points in X, then there are infinitely many fixed points in 2^X. It follows that if X is compact and contains infinitely many periodic points then the set of fixed points for the induced system on 2^X has an accumulation point. Such a system cannot, for example, be expansive.

For $A, B \subset X$ we define by $N(A, B)$ the *hitting time set* with respect to f on X:

$$N(A, B) = \{n \in \mathbb{N} : A \cap f^{-n}(B) \neq \emptyset\} = \{n \in \mathbb{N} : f^n(A) \cap B \neq \emptyset\}.$$

Clearly, if f_1 and f_2 are continuous functions on X_1 and X_2 respectively and $A_1, B_1 \subset X_1$, $A_2, B_2 \subset X_2$, then the hitting time set $N(A_1 \times A_2, B_1 \times B_2)$ with respect to the product map $f_1 \times f_2$ on $X_1 \times X_2$, is exactly $N(A_1, B_1) \cap N(A_2, B_2)$.

The system is called *topologically transitive* when $N(U, V) \neq \emptyset$ for every pair of opene sets $U, V \subset X$. While f need not be surjective, its image is clearly dense and so U opene implies $f^{-n}(U)$ is opene for every $n \in \mathbb{N}$.

The system is called *weakly mixing* when the product system $(X \times X, f \times f)$ is topologically transitive. It then follows that the n-fold product system $(X^{(n)}, f^{(n)})$ is weakly mixing for $n = 1, 2, \ldots$. This follows by induction using the beautiful *Furstenberg intersection lemma* [**8**]. We prove a strengthening due to Petersen [**19**].

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THEOREM 3.2. *For a dynamical system (X, f), assume that $N(U, V) \cap N(U, U) \neq \emptyset$ for every pair of opene sets $U, V \subset X$ and so that the system is topologically transitive. For all opene sets $U_1, V_1, U_2, V_2 \subset X$ there exist opene sets $U_3, V_3 \subset X$ such that*

$$N(U_3, V_3) \subset N(U_1, V_1) \cap N(U_2, V_2). \tag{3.1}$$

In particular, the system is weakly mixing.

Proof. $N(U_1, V_1) \neq \emptyset$ implies there exists $n_1 \in \mathbb{N}$ such that $U_0 = U_1 \cap f^{-n_1}(V_1)$ is opene. $N(U_0, U_2) \neq \emptyset$ implies there exists $n_2 \in \mathbb{N}$ such that $U = U_1 \cap f^{-n_1}(V_1) \cap f^{-n_2}(U_2)$ is opene. Since f is transitive, $f^{-n_1-n_2}(V_2)$ is opene. We have

$$\begin{aligned} &N(U, U) \cap N(U, f^{-n_1-n_2}(V_2)) \\ &\quad \subset N(U_1, f^{-n_2}(U_2)) \cap N(f^{-n_1}(V_1), f^{-n_1}(f^{-n_2}(V_2))) \\ &\quad = N(U_1, f^{-n_2}(U_2)) \cap N(f^{n_1}(f^{-n_1}(V_1)), f^{-n_2}(V_2)) \\ &\quad \subset N(U_1, f^{-n_2}(U_2)) \cap N(V_1, f^{-n_2}(V_2)). \end{aligned} \tag{3.2}$$

Fix $n_0 \in N(U_1, f^{-n_2}(U_2)) \cap N(V_1, f^{-n_2}(V_2))$. With $n = n_0 + n_2$ the sets $U_3 = U_1 \cap f^{-n}(U_2)$, $V_3 = V_1 \cap f^{-n}(V_2)$ are opene.

Let $k \in N(U_3, V_3)$. Then $f^{-k}(V_3) \cap U_3 \neq \emptyset$. That is, $f^{-k}(V_1) \cap f^{-n-k}(V_2) \cap U_1 \cap f^{-n}(U_2) \neq \emptyset$. Hence $k \in N(U_1, V_1) \cap N(f^{-n}(U_2), f^{-n}(V_2)) = N(U_1, V_1) \cap N(U_2, V_2)$.

As before, $N(f^{-n}(U_2), f^{-n}(V_2)) \subset N(U_2, V_2)$ and so (3.1) follows. □

As a consequence, we obtain the following lemma.

LEMMA 3.3. *If f_i is a continuous map on a metric space X_i for $i = 1, \ldots, k$ and the product system $(2^{X_1} \times \cdots \times 2^{X_k}, (f_1)_* \times \cdots \times (f_k)_*)$ is transitive then the product system $(X_1 \times \cdots \times X_k, f_1 \times \cdots \times f_k)$ is weakly mixing.*

Proof. Given opene $U, V \subset X_1 \times \cdots \times X_k$, we show that $N(U, U) \cap N(U, V) \neq \emptyset$ and then apply Theorem 3.2.

Choose opene $U_i, V_i \subset X_i$ such that $U_1 \times \cdots \times U_k \subset U$ and $V_1 \times \cdots \times V_k \subset V$. Let $\bar{U}_i = 2^{U_i}$ and $\bar{V}_i = \{A \in 2^{X_i} : A \cap U_i \neq \emptyset\} \cap \{A \in 2^{X_i} : A \cap V_i \neq \emptyset\}$. Since $(f_1)_* \times \cdots \times (f_k)_*$ is transitive there exist $A_i \in \bar{U}_i$ and $n \in \mathbb{N}$ such that $((f_i)_*)^n(A_i) \in \bar{V}_i$ for $i = 1, \ldots, k$. So there exist $x_i, y_i \in A_i$ with $(f_i)^n(x_i) \in U_i$, $(f_i)^n(y_i) \in V_i$. Since $A_i \in 2^{U_i}$, $x_i, y_i \in U_i$. Hence,

$$n \in N(U_1 \times \cdots \times U_k, U_1 \times \cdots \times U_k) \cap N(U_1 \times \cdots \times U_k, V_1 \times \cdots \times V_k)$$

and so $n \in N(U, U) \cap N(U, V)$. □

It is easy to see that a factor of a topologically transitive system is topologically transitive and so a factor of a weakly mixing system is weakly mixing.

We now strengthen the result in [**6**, **23**].

THEOREM 3.4. *Let K be a Cantor set. If f is a continuous map on a metric space X then the following are equivalent.*

(a) *(X, f) is weakly mixing.*

(b) *$(\mathcal{C}(K, X), f_*)$ is topologically transitive.*

(c) *$(2^X, f_*)$ is topologically transitive.*

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When these conditions hold, then the product system on $(X \times \mathcal{C}(K, X) \times 2^X, f \times f_* \times f_*)$ *is weakly mixing and so each of the induced systems is weakly mixing.*

Proof. (a) $\Rightarrow$ (b). We prove that when (X, f) is weakly mixing, $(X \times \mathcal{C}(K, X), f \times f_*)$ is topologically transitive. Notice that if $\mathcal{A}$ is a clopen partition of K then $\mathcal{C}(\mathcal{A}, X)$ is an invariant set and the subsystem $(\mathcal{C}(\mathcal{A}, X), f_*)$ is clearly isomorphic to the product system $(X^{(n)}, f^{(n)})$ with n the cardinality of $\mathcal{A}$. Hence, $(X \times \mathcal{C}(\mathcal{A}, X), f \times f_*)$ is topologically transitive for any clopen partition $\mathcal{A}$. Now let $U_1, V_1 \subset X$, $U_2, V_2 \subset \mathcal{C}(K, X)$ be opene. By choosing the mesh of $\mathcal{A}$ fine enough we obtain a clopen partition such that $U_2 \cap \mathcal{C}(\mathcal{A}, X)$, $V_2 \cap \mathcal{C}(\mathcal{A}, X)$ are opene relative to $\mathcal{C}(\mathcal{A}, X)$. Hence, $N(U_1 \times (U_2 \cap \mathcal{C}(\mathcal{A}, X)), V_1 \times (V_2 \cap \mathcal{C}(\mathcal{A}, X))) \subset N(U_1 \times U_2, V_1 \times V_2)$ is non-empty as required. It then follows that the factor $(\mathcal{C}(K, X), f_*)$ is topologically transitive, proving (b).

(b) $\Rightarrow$ (c). Because K is a Cantor set, Im is a factor map and so transitivity on 2^X follows from transitivity on $\mathcal{C}(K, X)$.

(c) $\Rightarrow$ (a). This is Lemma 3.3 with $k = 1$.

Above we showed that (X, f) weakly mixing implies $(X \times \mathcal{C}(K, X), f \times f_*)$ is topologically transitive. Since the product system $(X \times X, f \times f)$ is weakly mixing, we see that the product system on $X \times X \times \mathcal{C}(K, X \times X)$ is transitive. As $(\mathcal{C}(K, X \times X), (f \times f)_*)$ is isomorphic to $(\mathcal{C}(K, X) \times \mathcal{C}(K, X), f_* \times f_*)$ it follows that the system on $X \times \mathcal{C}(K, X)$ is weakly mixing and so the system on $X \times \mathcal{C}(K, X) \times \mathcal{C}(K, X)$ is weakly mixing. Applying Im on the third factor, we see that $(X \times \mathcal{C}(K, X) \times 2^X, f \times f_* \times f_*)$ is a factor of the latter and so is weakly mixing as well. □

COROLLARY 3.5. *If* (X, f) *is weakly mixing then the restriction of the product system to each of the closed, invariant subsets* $INT, INC \subset 2^X \times 2^X$, $EPS \subset X \times 2^X$ *are weakly mixing systems.*

Proof. Let K be a Cantor set. We will show that each of these systems is a factor of the weakly mixing system $(\mathcal{C}(K, X), f_*)$.

Let $\{K_0, K_1, K_2\}$ be a partition of K by three clopen sets. The map $\mathcal{C}(K, X) \to INT$ by $u \mapsto (u(K_0 \cup K_1), u(K_0 \cup K_2))$ is clearly an action map. If $(A, B) \in INT$ we can choose continuous surjections $u_0 : K_0 \to A \cap B$, $u_1 : K_1 \to A$, $u_2 : K_2 \to B$ and concatenate to obtain $u \in \mathcal{C}(K, X)$ mapping to (A, B). The map $\mathcal{C}(K, X) \to INC$ by $u \mapsto (u(K_0), u(K))$ is an action map. If $(A, B) \in INC$ then choose surjections $u_0 : K_0 \to A$, $u_{12} : K_1 \cup K_2 \to B$ and concatenate.

Finally, fix $e \in K$ and map $\mathcal{C}(K, X)$ to EPS by $u \mapsto (u(e), u(K))$. Again this is an action map. If $(x, A) \in EPS$ then we can choose a continuous surjection $u_0 : K \to A$. Let $y \in K$ with $u_0(y) = x$. Since K is homogeneous (e.g. there are Cantor sets which are topological groups) we can choose a homeomorphism h on K such that $h(e) = y$. Then $u = u_0 \circ h$ maps to (x, A). □

4. *Transitive points for induced systems*

For the system (X, f) and $x \in X$, the *omega limit set of* x, denoted $\omega f(x)$, is the set of limit points of the orbit sequence. Thus, $\omega f(x) = \limsup\{\{f^n(x)\} : n \in \mathbb{N}\}$. We call x a *transitive point* when $\omega f(x) = X$. If $\mathcal{B}$ is a base for the topology then the set of

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transitive points, Trans_f, is $\bigcap\{\bigcup_{k\geq n} f^{-k}(U) : k \in \mathbb{N}, U \in \mathcal{B}\}$. Clearly, if X contains a transitive point then the system is topologically transitive. In fact, every $N(U, V)$ is infinite. Furthermore, the space X is separable and, unless X consists of a single periodic orbit, the space has no isolated points. Conversely, if X is complete as well as separable then topological transitivity implies that Trans_f is a dense G_δ set. We apply the Baire category theorem to the above description of Trans_f with $\mathcal{B}$ a countable base.

Two points $x_1, x_2 \in X$ are called *asymptotic* when

$$\lim_{n\to\infty} d(f^n(x_1), f^n(x_2)) = 0.$$

In that case, $\omega f(x_1) = \omega f(x_2)$. In particular, if $x_1 \in \mathrm{Trans}_f$ then $x_2 \in \mathrm{Trans}_f$.

If X is a complete, separable, metric space with no isolated points, and so is a perfect Polish space, we will say that X is a *CSP metric space*. If a metric space X is CSP then by Lemma 2.2 and Proposition 2.3 the same is true of 2^X and $\mathcal{C}(K, X)$. Because X is perfect, a point $x \in X$ is a transitive point if and only if the orbit sequence $\{f^n(x) : n \in \mathbb{N}\}$ is dense in X.

If X consists of a single periodic orbit with more than one point, then (X, f) is not weakly mixing. If (X, f) is weakly mixing then by Theorem 3.4 the induced systems are weakly mixing as well and so admit transitive points.

The goal of this section is to examine the characteristics of these transitive points.

Definition 4.1. Let (X, f) be a dynamical system. A *Kronecker subset* L is a Cantor set contained in X such that $\{f^n|L : n \in \mathbb{N}\}$ is dense in $\mathcal{C}(L, X)$.

This view of Kronecker subsets comes from Katznelson [**15**].

PROPOSITION 4.2. *If L is a Kronecker set for (X, f) and L_0 is a non-empty clopen subset of L then L_0 is a Kronecker set for (X, f).*

Proof. If u_0 is an arbitrary element of $\mathcal{C}(L_0, X)$ and $\epsilon > 0$, then extend u_0 arbitrarily on the clopen set $L\backslash L_0$ to obtain $u \in \mathcal{C}(L, X)$. There exists $n \in \mathbb{N}$ such that $d(u(x), f^n(x)) < \epsilon$ for all $x \in L$ and so, a fortiori, $d(u_0(x), f^n(x)) < \epsilon$ for all $x \in L_0$. □

THEOREM 4.3. *L is a Kronecker set for a system (X, f) if and only if the inclusion map $u_L : L \to X$ is a transitive point for the induced system $(\mathcal{C}(L, X), f_*)$. Equivalently, u is a transitive point for $(\mathcal{C}(K, X), f_*)$, with K a Cantor set, if and only if u is injective with $\mathrm{Im}(u) = u(K)$ a Kronecker set for (X, f).*

Proof. u is a transitive point for $(\mathcal{C}(K, X), f_*)$ if and only if $\{f^n \circ u : n \in \mathbb{N}\}$ is dense in $\mathcal{C}(K, X)$. Since $\mathcal{C}(K, X)$ distinguishes points of K, it follows that u must be an injective map in that case and so, by compactness, it is a homeomorphism onto its image $L = u(K)$. So $u^* : \mathcal{C}(L, X) \to \mathcal{C}(K, X)$ is an isomorphism between the induced systems. So u is a transitive point of $\mathcal{C}(K, X)$ if and only if u_L is a transitive point of $\mathcal{C}(L, X)$. The f_* orbit of u_L is $\{f^n|L : n \in \mathbb{N}\}$ and so u_L is a transitive point if and only if L is a Kronecker subset. □

COROLLARY 4.4. *If (X, f) is a dynamical system with X a CSP metric space, then (X, f) admits a Kronecker set if and only if the system is weakly mixing.*

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Proof. By Theorem 4.3, (X, f) admits a Kronecker subset if and only if the induced system on $\mathcal{C}(K, X)$ is topologically transitive. By Theorem 3.4 this occurs if and only if (X, f) is weakly mixing. □

In extending these results to the n-fold product, we recall the natural set isomorphisms $(A^B)^C \cong A^{B\times C} \cong (A^C)^B$. We thus obtain natural isometric isomorphisms:

$$\mathcal{C}(K, X^{(n)}) \cong \mathcal{C}(K \times \{1, \ldots, n\}, X) \cong \mathcal{C}(K, X)^{(n)}. \tag{4.1}$$

Note that if K is a Cantor set then $K \times \{1, \ldots, n\}$ is a Cantor set with clopen partition $\{K \times \{i\} : i = 1, \ldots, n\}$. We will write $(u_1, \ldots, u_n)$ for an element of any of these.

COROLLARY 4.5. *Let $L \subset X^{(n)}$ with coordinate factors $L_i = \pi_i(L)$. L is a Kronecker set for $(X^{(n)}, f^{(n)})$ if and only if the following three conditions hold.*

- *The restriction of the coordinate projection $\pi_i : L \to L_i$ is injective and so is a homeomorphism for $i = 1, \ldots, n$.*
- *The subsets $\{L_1, \ldots, L_n\}$ of X are pairwise disjoint.*
- *The union $L_1 \cup \cdots \cup L_n$ is a Kronecker set for (X, f).*

Conversely, if $\{L_1, \ldots, L_n\}$ is a partition of a Kronecker set for (X, f) by non-empty clopen subsets then, for any choice of homeomorphisms $u_i : K \to L_i$ of a fixed Cantor set K, the image $(u_1, \ldots, u_n)(K) \subset X^{(n)}$ is a Kronecker set for $(X^{(n)}, f^{(n)})$.

Proof. L is a Kronecker set for $(X^{(n)}, f^{(n)})$ if and only if it is the image of some transitive point $(u_1, \ldots, u_n) \in \mathcal{C}(K, X^{(n)})$ for some Cantor set K. Via the isomorphisms of (4.1) this is true if and only if $(u_1, \ldots, u_n)$ is a transitive point of $\mathcal{C}(K \times \{1, \ldots, n\}, X)$ which has image $L_1 \cup \cdots \cup L_n$ and is partitioned by $\{L_1, \ldots, L_n\}$. By Theorem 4.3 a transitive point for either induced system is an injective map and so is a homeomorphism onto its image. □

A subset A of X is called *wandering* for (X, f) if $\{A, f(A), f^2(A), \ldots\}$ is a sequence of pairwise disjoint sets. It then follows that the bi-infinite sequence $\{f^n(A) : n \in \mathbb{Z}\}$ consists of pairwise disjoint sets. For if $x \in f^{-n}(A) \cap f^m(A)$ with $-n < m$ then $f^n(x) \in A \cap f^{n+m}(A)$.

A pair of points (x_1, x_2) is called *proximal* for (X, f) if 0 is a limit point of the sequence $\{d(f^n(x_1), f^n(x_2))\}$. The pair is *asymptotic* if $0 = \lim_n \{d(f^n(x_1), f^n(x_2))\}$. A subset A of X is called *uniformly proximal* for (X, f) if 0 is a limit point of the sequence of diameters $\{\text{diam}\,(f^n(A))\}$.

We now consider $(2^X, f_*)$.

THEOREM 4.6. *Let (X, f) be a weakly mixing system with X a CSP metric space. Assume that C is a transitive point for the induced system $(2^X, f_*)$.*

(a) *If g_1, g_2 are a continuous functions on X which commute with f and $g_1(C) \cap g_2(C) \neq \emptyset$ then $g_1 = g_2$. In fact, if there exist $c_1, c_2 \in C$ with $g_1(c_1)$ asymptotic to $g_2(c_2)$ then $g_1 = g_2$.*

(b) *The set C is wandering for (X, f). In particular, $\{C, f(C), f^2(C), \ldots\}$ is a pairwise disjoint sequence of transitive points for $(2^X, f_*)$.*

(c) *If μ is an invariant probability measure for f, then $\mu(C) = 0$.*

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(d) *The set C is nowhere dense in X.*
(e) *The set C has infinitely many components. In particular, C is infinite.*
(f) *Every point of C is a transitive point for f.*
(g) *If $x \in X$ then there is a sequence of iterates $f^{n_i}(C)$ which converges to $\{x\}$. In particular, C is a uniformly proximal set for (X, f).*

Proof. (a) Suppose that $c_1, c_2 \in C$ with $g_1(c_1)$ asymptotic to $g_2(c_2)$. If $x \in X$ then there exists a sequence j_n such that $f_*^{j_n}(C)$ converges to $\{x\}$. Hence, $\{g_a(f^{j_n}(c_a))\}$ converges to $g_a(x)$ for $a = 1, 2$. But $g_a(f^{j_n}(c_a)) = f^{j_n}(g_a(c_a))$ for $a = 1, 2$ and so the two limits are the same because $g_1(c_1)$ is asymptotic to $g_2(c_2)$. Hence, $g_1(x) = g_2(x)$. Since x was arbitrary, $g_1 = g_2$.

(b) If $f^n(C) \cap f^m(C) \neq \emptyset$ for some integers $n, m \geq 0$ then from (a) it follows that $f^n = f^m$. If $n > m$ then for every x the orbit of x would be $\{x, f(x), \ldots, f^n(x)\}$ which is finite. As (X, f) admits transitive points and X is perfect, this cannot be true and so $n = m$. Thus, C is wandering and so the bi-infinite sequence $\{f^n(C) : n \in \mathbb{Z}\}$ is pairwise disjoint.

Since $\omega f_*(f^n(C)) = \omega f_*(C) = 2^X$ it follows that $f^n(C)$ is a transitive point for every $n \in \mathbb{N}$.

(c) Since the measure is invariant the sequence of disjoint sets $\{f^{-k}(C) : k \in \mathbb{N}\}$ all have measure $\mu(C)$. Since the sum is finite, $\mu(C) = 0$.

(d) If C had a non-empty interior then since X is perfect there would exist $U, V \subset C$ disjoint opene subsets of C. If $n \in N(U, V)$ then n would be a positive integer such that $C \cap f^n(C) \neq \emptyset$ and this would contradict (b).

(e) Let $\{x_1, \ldots, x_n\}$ be a set of n distinct points and $2\epsilon < \min\{d(x_i, x_k) : \text{for } i \neq k = 1, \ldots, n\}$. There exists $j > 0$ such that $f^j(C)$ is ϵ close to $\{x_1, \ldots, x_n\}$. With $B_\epsilon(x)$ the ϵ ball centered at x, we see that $\{f^{-j}(B_\epsilon(x_i)) : i = 1, \ldots, n\}$ is a partition of C into n opene sets and hence the number of components is at least n.

(f), (g) For every $x \in X$ the set $\{x\}$ is the limit of some subsequence $\{f^{n_j}(C)\}$ in 2^X. □

LEMMA 4.7. *Assume $C \in 2^X$.*
(a) *If $A \in 2^X$ such that*

$$\lim_{j \to \infty} \max\{d(f^j(a), f^j(C)) : a \in A\} = 0, \tag{4.2}$$

then $C \cup A$ is asymptotic to C with respect to f_*. In particular, if C is a transitive point for f_* then so is $C \cup A$.
(b) *Let F be a finite subset of X with $F \cap C = \emptyset$. If $B \in 2^X$ is perfect and $(f_*)^{j_i}(C \cup F) \to B$ then $(f_*)^{j_i}(C) \to B$. If $C \cup F$ is a transitive point for f_* then so is C.*

Proof. (a) $\max\{d(f^j(a), f^j(C)) : a \in A\}$ is the Hausdorff distance from $(f_*)^j(C \cup A)$ to $(f_*)^j(C)$.

(b) It suffices to show that every convergent subsequence of $f_*^{j_i}(C)$ has limit B. Assume that subsequence of $f_*^{j_i}(C)$ converges to B_1. By going to a further subsequence we can assume that $f_*^{j_i}(F)$ converges to F_1. Since the finite sets with cardinality at most that of F form a closed subset of 2^X, F_1 is finite. Continuity of the map $\cup$ implies that $B_1 \cup F_1 = B$.

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Since B is perfect and B_1 is closed, $B_1 = B$. If $C \cup F$ is a transitive point and B is a Cantor set transitive point then B is in $\omega f_*(C \cup F)$ and so is in $\omega f_*(C)$. Hence, the latter contains $\omega f_*(B) = 2^X$. □

THEOREM 4.8. *Let $(C_1, \ldots, C_n)$ be a transitive point for $(f_*)^{(n)}$ on $(2^X)^{(n)}$.*

(a) *The C_i are pairwise disjoint and so form a partition of $C = \bigcup_i C_i$. If $c_i \in C_i$ for $i = 1, \ldots, n$ then $(c_1, \ldots, c_n)$ is a transitive point for $f^{(n)}$.*

(b) *The union $C = \bigcup_i C_i$ is a transitive point for f_* and it contains $c_1, \ldots, c_n$ such that $(c_1, \ldots, c_n)$ is a transitive point for $f^{(n)}$. The set of transitive points C for which such a decomposition exists is dense in the set of transitive points for f_*.*

Proof. (a) Let $\{x_1, \ldots, x_n\}$ be distinct points with $2\epsilon < \min\{d(x_i, x_k) : \text{for } i \neq k = 1, \ldots, n\}$. There exists j such that $(f_*)^j(C_i)$ is ϵ close to $\{x_i\}$ for $i = 1, \ldots, n$ and so $\{f^j(C_i) : i = 1, \ldots, n\}$ are pairwise disjoint and hence $\{C_i : i = 1, \ldots, n\}$ are pairwise disjoint. If $c_i \in C_i$ then $f^j(c_i)$ is ϵ close to x_i for $i = 1, \ldots, n$. This proves that $(c_1, \ldots, c_n)$ is a transitive point of $f^{(n)}$.

(b) The map $(2^X)^{(n)} \to 2^X$ by $(A_1, \ldots, A_n) \mapsto \bigcup_i A_i$ is surjective and continuous and so is a factor map from $(f_*)^{(n)}$ to f_*. A surjective continuous map takes dense sets to dense sets, and a factor map takes transitive points to transitive points. □

Call C an *n-decomposable transitive point* when there exists $(C_1, \ldots, C_n)$ a transitive point for $(f_*)^{(n)}$ on $(2^X)^{(n)}$ such that $C = \bigcup_i C_i$.

COROLLARY 4.9. *If C is an n-decomposable, transitive point then it contains at least n distinct accumulation points.*

Proof. If C is n-decomposable then it admits a clopen partition $\{C_1, \ldots, C_n\}$ with each C_i infinite and so with each containing an accumulation point. □

We will need the following routine result.

LEMMA 4.10. *Let L be a totally disconnected compact metric space. If $\{C_1, \ldots, C_n\}$ are pairwise disjoint, closed non-empty subsets of L then there exists a clopen partition $\{L_1, \ldots, L_n\}$ of L such that $C_i \subset L_i$ for $i = 1, \ldots, n$.*

Proof. Inductively we can for $i = 1, \ldots, n-1$ choose L_i a clopen subset of L which contains C_i and which is disjoint from L_j for $j < i$ and from C_j for $j > i$. Then let $L_n = L \backslash (L_1 \cup \cdots \cup L_{n-1})$. Thus, $\{L_1, \ldots, L_n\}$ is the required clopen partition of L. □

A subset E of X is called *independent* for (X, f) if any finite sequence $(e_1, e_2, \ldots, e_n)$ of points in E is a transitive point in the product system $(X^{(n)}, f^{(n)})$. Independent sets were studied by Iwanik [**14**], and he proved the existence of large independent sets for weakly mixing systems. A subset of X is called *scrambled* when every pair of points in it is proximal and no pair of distinct points is asymptotic. An independent set is clearly scrambled and is also *wandering*. Observe that if $e_2 = f^n(e_1)$ then (e_1, e_2) is not a transitive point for $(X \times X, f \times f)$.

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THEOREM 4.11. *Let (X, f) be weakly mixing with X a CSP metric space. Assume that L is a Kronecker subset of X.*

(a) *L is a transitive point for f_* which is n-decomposable for every n. In fact, if $\{L_1, \ldots, L_n\}$ is a partition of L by non-empty clopen subsets then $(L_1, \ldots, L_n)$ is a transitive point for $(f_*)^{(n)}$. If $c_1, \ldots, c_n$ are distinct points in L then $(c_1, \ldots, c_n)$ is a transitive point for $f^{(n)}$, i.e. L is an independent set.*

(b) *If C is any closed infinite subset of L of X then C is a transitive point for f_*. If $\{C_1, \ldots, C_n\}$ are pairwise disjoint, closed infinite subsets of L then $(C_1, \ldots, C_n)$ is a transitive point for $(f_*)^{(n)}$.*

(c) *If C is a closed subset of L which contains n accumulation points then it is an n-decomposable transitive point. If it contains exactly n accumulation points then it is not $(n+1)$-decomposable.*

Proof. (a) L is a Kronecker subset if and only if the inclusion map is a transitive point for f_* on $\mathcal{C}(L, X)$. Since the map $\mathrm{Im} : \mathcal{C}(L, X) \to 2^X$ is a factor map, it takes transitive points to transitive points. Applied to the inclusion map of L itself, this implies that $L \in 2^X$ is a transitive point.

If $\{L_1, \ldots, L_n\}$ is a partition of L by non-empty clopen subsets then by Proposition 4.2 each is a Kronecker set. Let $(B_1, \ldots, B_n) \in (2^X)^{(n)}$. Choose $g_i \in \mathcal{C}(L_i, X)$ with $g_i(L_i) = B_i$. Concatenate to define $g \in \mathcal{C}(L, X)$. There exists $j > 0$ such that $f^j|L$ is close to g and so $f^j(L_i)$ is close to B_i for $i = 1, \ldots, n$. Hence, $(L_1, \ldots, L_n)$ is a transitive point for $(f_*)^{(n)}$. Thus, L is n-decomposable. Alternatively, one can apply Corollary 4.5.

If $c_1, \ldots, c_n$ are distinct points of L then there exists a partition $L_1, \ldots, L_n$ of L with $c_i \in L_i$ for $i = 1, \ldots, n$. Since $(L_1, \ldots, L_n)$ is a transitive point for $(f_*)^{(n)}$, $(c_1, \ldots, c_n)$ is a transitive point for $f^{(n)}$ by Theorem 4.8(a).

(b) If $\{C_1, \ldots, C_n\}$ are pairwise disjoint, closed infinite subsets of L, then by Lemma 4.10 there is a clopen partition $\{L_1, \ldots, L_n\}$ with $C_i \subset L_i$ for $i = 1, \ldots, n$.

Let $(B_1, \ldots, B_n) \in (2^X)^{(n)}$ and $\epsilon > 0$. There exist $M > 0$ and (not necessarily distinct) points $x_{ij} \in B_i$, $i = 1, \ldots, n$, $j = 1, \ldots, M$, so that $\{x_{ij} : j = 1, \ldots, M\}$ is $\epsilon/2$ dense in B_i for $i = 1, \ldots, n$.

Now choose distinct points $c_{ij} \in C_i$, $i = 1, \ldots, n$, $j = 1, \ldots, M$. This is where we need that each C_i is infinite. By Lemma 4.10 again, there exist clopen partitions $\{L_{ij} : j = 1, \ldots, M\}$ of L_i for $i = 1, \ldots, n$ such that $c_{ij} \in L_{ij}$ for all i, j.

Define $g \in \mathcal{C}(L, X)$ by $g(x) = x_{ij}$ for $x \in L_{ij}$. There exists $k > 0$ such that $f^k|L$ is $\epsilon/2$ close to g and so $f^k(C_i)$ is $\epsilon/2$ close to $\{x_{i1}, \ldots, x_{iM}\}$ and so is ϵ close to B_i for $i = 1, \ldots, n$. This shows that $(C_1, \ldots, C_n)$ is a transitive point for $(f_*)^{(n)}$.

(c): Let $c_1, \ldots, c_n$ be distinct accumulation points of C. By Lemma 4.10 there is a clopen partition $\{C_1, \ldots, C_n\}$ of C with $c_i \in C_i$. Since C_i is a neighborhood in C of the accumulation point c_i, it is infinite. By (b), $(C_1, \ldots, C_n)$ is an $(f_*)^{(n)}$ transitive point and so C is n-decomposable. If there are only n accumulation points then C is not $(n+1)$-decomposable by Corollary 4.9. □

By combining Corollary 3.5 with Theorem 4.3 we sharpen the former.

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COROLLARY 4.12. *If L is a Kronecker subset of X and $\{L_0, L_1, L_2\}$ is a clopen partition of L, then (L, L_0) is a transitive point for INC and $(L_0 \cup L_1, L_0 \cup L_2)$ is a transitive point for INT. If $e \in L$ then (e, L) is a transitive point for EPS.*

COROLLARY 4.13.

(a) *There exists a transitive point C for f_* which contains a single accumulation point $c^* \in C$, and so $C\backslash\{c^*\}$ is a set of isolated points in X. In particular, C is countable. Such a set C is never 2-decomposable. However, it can be chosen so that if $c_1, \ldots, c_n$ are distinct points of C then $(c_1, \ldots, c_n)$ is a transitive point for $f^{(n)}$.*

(b) *If A is any infinite, compact, totally disconnected metric space there exists a transitive point C for f_* which is homeomorphic to A.*

Proof. (a) Any set of isolated points in a separable metric space is countable. A set with a single accumulation point is not 2-decomposable by Corollary 4.9.

Let L be a Kronecker subset and let $\{c_k : k = 1, 2, \ldots\}$ be a sequence of distinct points in L converging to a point c^* in L. Let C be the points of this sequence together with the limit point. Since the sequence is convergent, the points of $C\backslash\{c^*\}$ are isolated. As it is an infinite subset of L, it is a transitive point for f_* by Theorem 4.11(b).

(b) The product $A \times L$ is perfect as well as totally disconnected and compact metric. Hence, $A \times L$ is homeomorphic to L. Hence, L contains a homeomorphic copy of A. Such a set is an infinite closed subset of L and so is a transitive point by Theorem 4.11(b) again. □

A subset of X is called *strongly scrambled* when every pair of points in it is proximal and recurrent. Any Kronecker set is strongly scrambled. The map f is called *uniformly rigid* when some sequence of iterates f^{j_i} converges uniformly to the identity on X. In that case, every pair of points is recurrent and so no pair of distinct points is asymptotic. Glasner and Maon [**10**] have constructed uniformly rigid, weakly mixing, minimal homeomorphisms on the torus. If f is uniformly rigid then every transitive point C for f_* is strongly scrambled.

We note that Hernandez, King and Mendez-Lango [**12**] have constructed Cantor sets with dense orbit in $(2^X, f_*)$.

5. *Examples*

We first construct an explicit example of a Kronecker set.

Let $X = \{0, 1\}^{\mathbb{N}}$ along with the shift map σ be the one-sided shift space.

Consider the induced system $(2^X, \sigma_*)$.

Let $\mathcal{L}(X)$ denote the language of X, and define W_n to be the set of all words of length n in $\mathcal{L}(X)$. So we have

$$W_1 = \{w_1^{(1)}, w_2^{(1)}\},\ W_2 = \{w_1^{(2)}, w_2^{(2)}, w_3^{(2)}, w_4^{(2)}\}, \ldots,\ W_n = \{w_1^{(n)}, \ldots, w_{2^n}^{(n)}\}, \ldots$$

where $w_i^{(j)}$ can be considered lexicographically in each W_j, i.e. in each W_j, $w_1^{(j)} < w_2^{(j)} < \cdots < w_{2^j-1}^{(j)} < w_{2^j}^{(j)}$.

Also we consider S_n, the group of permutations on n elements.

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Let $\mathcal{U}^{n,k} \subset \mathcal{P}(W_n)$, for $1 < k \leq 2^n$, be the collection of all sets in the power set of W_n with k elements. We note that there will be ${}_{2^n}C_k$ of them, which we can enumerate as $U_1^{n,k}, U_2^{n,k}, \ldots, U_{{}_{2^n}C_k}^{n,k}$. We note that $|U_j^{n,k}| = k$, and for each $u_j^{n,k} \in U_j^{n,k}$, we have $|u_j^{n,k}| = n$.

Define $C \in 2^X$ as

$$\begin{aligned} C = \{ & w^{(1)}_{\rho_{1_1}(1)} w^{(1)}_{\rho_{1_1}(2)} w^{(1)}_{\rho_{1_2}(1)} w^{(1)}_{\rho_{1_2}(2)} w^{(2)}_1 w^{(2)}_2 \\ & \cdots w^{(2)}_4 u_1^{2,2} u_2^{2,2} \cdots u_6^{2,2} u_1^{2,3} u_2^{2,3} u_3^{2,3} u_4^{2,3} u_1^{2,4} w^{(3)}_{\rho_{3_1}(1)} w^{(3)}_{\rho_{3_1}(2)} \\ & \cdots w^{(3)}_{\rho_{3_1}(2^3)} \cdots w^{(3)}_{\rho_{3_{2^3!}}(1)} w^{(3)}_{\rho_{3_{2^3!}}(2)} \cdots w^{(3)}_{\rho_{3_{2^3!}}(2^3)} w^{(4)}_1 w^{(4)}_2 w^{(4)}_3 w^{(4)}_4 \\ & \cdots w^{(4)}_{2^4} u_1^{4,2} \cdots u_{{}_{2^4}C_2}^{4,2} \cdots u_1^{n,2^n} w^{(5)}_{\rho_{5_1}(1)} w^{(5)}_{\rho_{5_1}(2)} \cdots w^{(5)}_{\rho_{5_1}(2^5)} \cdots w^{(5)}_{\rho_{5_{2^5!}}(1)} w^{(5)}_{\rho_{5_{2^5!}}(2)} \\ & \cdots w^{(5)}_{\rho_{5_{2^5!}}(2^5)} w^{(6)}_1 \cdots u_1^{2n,2} \cdots u_{{}_{2^{2n}}C_2}^{2n,2} \cdots u_{{}_{2^{2n}}C_{2^{2n}-1}}^{2n,2^{2n}-1} u_1^{2n,2^{2n}} w^{(2n+1)}_{\rho_{2n+1_1}(1)} \\ & \cdots w^{(2n+1)}_{\rho_{2n+1_1}(2^{2n+1})} \cdots / u_j^{n,k} \in U_j^{n,k},\ \rho_{k_j} \in S_{2^k} \text{ for each } n, k, j \in \mathbb{N}\}, \end{aligned}$$

i.e. C consists of all sequences with consecutive blocks of all words of length 1 arranged according to all permutations in S_2, followed by consecutive blocks of all words of length 2 arranged lexicographically, followed by words of length 2 from each set containing 2, 3, 4 of these words of length 2, followed by consecutive blocks of all words of length 3 arranged according to all permutations in $S_8, \ldots$, followed by consecutive blocks of all words of length $2n$ arranged lexicographically, followed by words of length $2n$ from each set containing $2, 3, \ldots, 2^{2n}$ of these words of length $2n$, followed by consecutive blocks of all words of length $2n+1$ arranged according to all permutations in $S_{2^{2n+1}}, \ldots$.

We claim that C is compact. It is enough to see that C is closed.

Let $x = x_1 \cdots x_p \cdots \notin C$. This means that x cannot be realized as a sequence with a consecutive arrangement of blocks of all words of length n for $n \in \mathbb{N}$ as done in C. Let t be the smallest index where this difference can be realized, and let j be largest even integer such that

$$\begin{aligned} r = t - \Bigg(& 2 \cdot 1 \cdot 2^1! + 2^2 \cdot 2 + 2 \cdot \sum_{k=2}^{4} {}_4C_k + 2^3 \cdot 3 \cdot 2^3! \\ & + \cdots + 2^j \cdot j + j \cdot \sum_{k=2}^{2^j} {}_{2^j}C_k \Bigg) > 0, \end{aligned}$$

and $t - r < 2^{j+1} \cdot j$.

Then $x \in [x_1 \cdots x_h]$, but $C \cap [x_1 \cdots x_h] = \phi$, when $h = 2 \cdot 1 \cdot 2^1! + 2^2 \cdot 2 + 2 \cdot \sum_{k=2}^{4} {}_4C_k + 2^3 \cdot 3 \cdot 2^3! + \cdots + 2^j \cdot j + j \cdot \sum_{k=2}^{2^j} {}_{2^j}C_k + 2^{j+1} \cdot (j+1)$.

We see that C is transitive in $(2^X, \sigma_*)$.

Take any basic open set $\langle A_1, \ldots, A_q \rangle$ in the Vietoris topology on 2^X. We can always have a $p \in 2\mathbb{N}$ large enough so that the cylinders $[a_1^i \cdots a_p^i] \subset A_i$, $1 \leq i \leq q$. For some $j \in 1, \ldots, 2^p$, we have $U_j^{p,q} = \{a_1^1 \cdots a_p^1, \ldots, a_1^q \cdots a_p^q\}$. Then we have $\sigma_*^t(C) \subset \langle [a_1^1 \cdots a_p^1], \ldots, [a_1^q \cdots a_p^q] \rangle \subset \langle A_1, \ldots, A_q \rangle$, where $t = 2 \cdot 1 \cdot 2^1! + 2^2 \cdot 2 + 2 \cdot \sum_{k=2}^{4} {}_4C_k + 2^3 \cdot 3 \cdot 2^3! + \cdots + 2^p \cdot p + p \cdot \sum_{k=2}^{j-1} {}_{2^p}C_k$.

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We note that each $c \in C$ is a transitive point for (X, σ), and for every $n \in \mathbb{N}$, $C = C_1 \cup \cdots \cup C_n$ such that $(C_1, \ldots, C_n)$ is a $(\sigma_*)^{(n)}$ transitive point and so C is n-decomposable.

We now specialize to $X = \{0, 1\}^{\mathbb{Z}}$ with σ the shift homeomorphism.

THEOREM 5.1. *There exists a transitive point C for σ_* which is a countable set with a single accumulation point $c^* \in C$. In addition, any pair of points $c_1, c_2 \in C$ is asymptotic. In particular, (c_1, c_2) is never a transitive point for $\sigma \times \sigma$.*

Proof. Call a set of finite words an EL set (equal length) if they all have the same length. Let $\{A_1, A_2, \ldots\}$ be a sequence which counts all those EL sets on the alphabet $\{0, 1\}$ whose common length is odd. Let $2\ell_k + 1$ be the common length of the words in A_k. In each A_k choose a particular word a_k^* which we call a *special word*. For example, using lexicographic ordering we could choose the first word in the set A_k. Let $N_k = \sum_{j=1}^{k}(2\ell_j + 1)$.

We now construct inductively a sequence of EL sets $\{B_1, \ldots\}$ for which N_k is the common length of the words in B_k and there is a distinguished element $b_k^* \in B_k$. Each member of B_k is a concatenation of blocks of length $2\ell_1 + 1, \ldots, 2\ell_k + 1$. We refer to these as the first block, second block, …, kth block.

Let $B_1 = A_1$ and $b_1^* = a_1^*$.

Inductively we concatenate, defining B_{k+1} to be the union of the two sets $\{b_k^*\} \cdot A_{k+1}$ and $B_k \cdot \{a_{k+1}^*\}$. The intersection of these two sets is the single word $b_k^* \cdot a_{k+1}^*$ which we define to be b_{k+1}^*.

Thus, $b_{k+1}^* = a_1^* \cdot a_2^* \cdot \ldots \cdot a_k^* \cdot a_{k+1}^*$.

Inductively it is clear that for each word in B_{k+1} at most one block is not a special word.

Define $c \in C$ when $c_i = 0$ for all $i \leq 0$ and when $c_{[1,\ldots,N_k]} \in B_k$ for all k. Let c^* be the special element of C with $c_i^* = 0$ for all $i \leq 0$ and $c_{[1,\ldots,N_k]}^* = b_k^*$.

If $a_k \in A_k \backslash \{a_k^*\}$ and the kth block of $c \in C$ is a_k then every other block on (the positive side) is a special word. Thus, c agrees with c^* except in finitely many places and it is clear that c is an isolated point of C. Thus, the points of any finite subset of C all agree with c^* after some finite coordinate level. It follows that any two points are asymptotic. This in turn implies that no pair in C defines a transitive point for $\sigma \times \sigma$.

Finally, we observe that the words from the $-\ell_{k+1}$ to the $+\ell_{k+1}$ coordinate occurring in the set $\sigma^{N_k+\ell_{k+1}+1}(C)$ are exactly the words of A_{k+1} centered in the middle. Since the sequence $\{A_k\}$ lists all EL sets of words of odd length it follows that the iterates $\sigma_*^j(C)$ approach every element of 2^X arbitrarily closely. That is, C is a transitive point for σ_*. □

We can obtain Cantor set examples by using the *asymptotic extension technique*. For $B \in 2^X$ with $X = \{0, 1\}^{\mathbb{Z}}$ let $B_N = \pi_N^{-1}(\pi_N(B))$ where $\pi_N : \{0, 1\}^{\mathbb{Z}} \to \{0, 1\}^{[N,\infty)}$ is the coordinate projection. It is clear that B and B_N are asymptotic points of 2^X, with respect to σ_*, since they agree beyond the Nth coordinate. Furthermore, every point of B_N is asymptotic to a point of B. In particular, if B is a transitive point for σ_* then each B_N is as well.

COROLLARY 5.2. *There exists a transitive point C for σ_* which is a Cantor set such that any pair of points $c_1, c_2 \in C$ is asymptotic. In particular, (c_1, c_2) is never a transitive point for $\sigma \times \sigma$.*

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Proof. Let B be the countable set constructed for Theorem 5.1 so that every point of B is asymptotic to c^*. Clearly, every point of $C = B_N$ is asymptotic to c^*. Since the coordinates below N are arbitrary, B_N is a Cantor set. □

Remark. By extending the idea of Theorem 5.1 one can construct B to be a disjoint union of sets $B_1, \ldots, B_n$, each an infinite countable set contained in a single asymptotic class and so that $(B_1, \ldots, B_n)$ is a transitive point for $(\sigma)^{(n)}$. Then B_N is a Cantor set n-decomposable transitive point which meets only n asymptotic classes and so is not $(n+1)$-decomposable.

Using this asymptotic extension idea in a different context, we can obtain examples of transitive points in 2^X with non-trivial components.

We specialize to $X = \mathbb{R}^2/\mathbb{Z}^2$ with $\pi : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ the canonical projection. We define the metric d on $\mathbb{R}^2/\mathbb{Z}^2$ by $d(a+\mathbb{Z}^2, b+\mathbb{Z}^2) = \min\{|a-b+z| : z \in \mathbb{Z}^2\}$. Hence, π has Lipschitz constant 1. Any closed $C \subset \mathbb{R}^2/\mathbb{Z}^2$ of diameter less than 1 is contained in a ball in $\mathbb{R}^2/\mathbb{Z}^2$, and by lifting the ball we obtain a closed $C^+ \subset \mathbb{R}^2$ such that $\pi : C^+ \to C$ is a homeomorphism. If $C \subset \mathbb{R}^2/\mathbb{Z}^2$ is a Cantor set of any diameter, we can choose a clopen partition of C by sets of diameter less than 1 and lift each separately to obtain a Cantor set $C^+ \subset \mathbb{R}^2$ such that $\pi : C^+ \to C$ is a homeomorphism.

Now let t be a Thom torus map on $\mathbb{R}^2/\mathbb{Z}^2$, e.g. the diffeomorphism induced by the linear map T with matrix $\left(\begin{smallmatrix} 3 & 1 \\ 5 & 2 \end{smallmatrix}\right)$. This is an Anosov diffeomorphism and so is a factor of the shift map. In particular, it is weakly mixing. The eigenvalues $\lambda_\pm = (5 \pm \sqrt{21})/2$ satisfy $4 < \lambda_+ < 5$ and $0 < \lambda_- < \frac{1}{2}$. Let v_+ and v_- be corresponding eigenvectors of unit length. The lines parallel to v_- and to v_+ project to the stable and unstable foliations. In particular, for any $a \in \mathbb{R}^2$ and any real number s,

$$|T^j(a) - T^j(a + sv_-)| = |s|(\lambda_-)^j < |s|2^{-j}. \tag{5.1}$$

In particular, all points on the stable manifold of $\pi(a)$, i.e. $\pi(a + \mathbb{R}v_-)$, are asymptotic with respect to t.

Let L be a Kronecker subset of $\mathbb{R}^2/\mathbb{Z}^2$ for t. There exists a sequence $j_k \to \infty$ such that $t^{j_k}|L$ converges to the inclusion map of L into $\mathbb{R}^2/\mathbb{Z}^2$. Hence, no two distinct points of L are asymptotic. Hence, the stable manifolds of any two distinct points in L are disjoint. Let L^+ be a Cantor set in $\mathbb{R}^2$ such that $\pi|L^+ \to L$ is a homeomorphism. Define the map $Q : L^+ \times \mathbb{R} \to \mathbb{R}^2/\mathbb{Z}^2$ by $Q(a, s) = \pi(a + sv_-)$. The map Q is clearly continuous. It is injective because the v_- lines through distinct points of L^+ do not intersect. For $M > 0$ let $C_M = Q(L^+ \times [-M, M])$. By compactness, the restriction of Q is a homeomorphism from $L^+ \times [-M, M]$ onto C_M. From equation (5.1) it follows that the Hausdorff distance between $T^j(L^+)$ and $T^j(L^+ + [-M, M]v_-)$ converges to zero. Hence, the distance between $t^j_*(L)$ and $t^j_*(C_M)$ converges to zero. That is, L and C_M are asymptotic points of 2^X. Since L is a transitive point, C_M is as well. Thus, we have proved the following theorem.

THEOREM 5.3. *There exists a transitive point C for t_* which is homeomorphic to $L \times [0, 1]$ where L is a Cantor set. In particular, every component of C is an interval.*

Remark. Note that $\{C_M : M = 1, 2, \ldots\}$ is an increasing sequence of transitive points for t_* whose union is dense in $\mathbb{R}^2/\mathbb{Z}^2$.

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In examples of this sort any pair of points which lie in the same component of the transitive point C are asymptotic. The question arises whether this is always true. We obtain a partial result suggesting that it is true.

For a compact metric space X define the *component diameter* to be

$$C\ \mathrm{diam}(X) = \sup\{d(x, y) : x \text{ and } y \text{ lie in the same component of } X\}. \tag{5.2}$$

Equivalently, $C\ \mathrm{diam}(X)$ is the supremum of the diameters of the components of X.

LEMMA 5.4. *If $\{B_i\}$ is a sequence in 2^X converging to a totally disconnected $C \in 2^X$ then*

$$\lim_{i\to\infty} C\ \mathrm{diam}(B_i) = 0. \tag{5.3}$$

Proof. Given $\epsilon > 0$, choose a clopen partition of C by elements of diameter less than ϵ and enlarge each to an open subset of X. That is, we can obtain disjoint opene sets $U_1, \ldots, U_n$ whose union U contains C. There exists N so that $i \geq N$ implies $B_i \subset U$ and so each component is entirely contained in one of the U_i. Hence, $i \geq N$ implies that $C\ \mathrm{diam}(B_i) < \epsilon$. □

We conjecture that if C is a transitive point for f_* on 2^X, then

$$\lim_{i\to\infty} C\ \mathrm{diam}((f_*)^i(C)) = 0.$$

The closest we can get is the following result.

THEOREM 5.5. *Assume that f is a homeomorphism. C is a transitive point for f_* on 2^X and $B \in 2^X$ then there exists a sequence $\{j_i\}$ of integers tending to infinity such that*

$$\lim_{i\to\infty} \{(f_*)^{j_i}(C)\} = B \quad \textit{and} \quad \lim_{i\to\infty} C\ \mathrm{diam}((f_*)^{j_i}(C)) = 0. \tag{5.4}$$

Proof. Let L be a Cantor set transitive point for f_* and choose sequences n_p, m_q so that $(f_*)^{n_p}(C) \to L$ and $(f_*)^{m_q}(L) \to B$ as $p, q \to \infty$. Diagonalize. That is, given $i > 0$, choose q_i so that $d(B, (f_*)^{m(q_i)}(L)) < 1/(2i)$ and then p_i so that $d((f_*)^{m(q_i)}(L), (f_*)^{m(q_i)+n(p_i)}(C)) < 1/(2i)$ and $C\ \mathrm{diam}((f_*)^{m(q_i)+n(p_i)}(C)) < 1/i$. Because f is a homeomorphism, $(f_*)^{m(q_i)}(L)$ is a Cantor set, and so Lemma 5.4 implies that p_i exists. Let $j_i = m_{(q_i)} + n_{(p_i)}$. □

If $H(K)$ is the homeomorphism group of the Cantor set K then $H(K)$ acts on $\mathcal{C}(K, X)$ by $tg = g \circ t^{-1}$. This action commutes with f_*. The map $\mathrm{Im} : \mathcal{C}(K, X) \to 2^X$ is constant on the $H(K)$ orbits. Let $\mathcal{C}_{in}(K, X)$ be the set of injective continuous maps. This is a dense G_δ subset of $\mathcal{C}(K, X)$ and, if f is a homeomorphism, then $\mathcal{C}_{in}(K, X)$ is invariant under f_*. It maps to $CANTOR \subset 2^X$ the dense G_δ subset of Cantor sets in X. Furthermore, the fibers of the map $\mathcal{C}_{in}(K, X) \to CANTOR$ are exactly the $H(K)$ orbits.

6. *Minimality and distality for the induced dynamics of compact systems*

The system (X, f) is *minimal* if every $x \in X$ is a transitive point. We note that X is always a fixed point for $(2^X, f_*)$, and so the system $(2^X, f_*)$ is never minimal when (X, f) is non-trivial. Here we look at a related notion.

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In this section X will always be a compact metric space.

Definition 6.1. The system (X, f) is said to be *strongly transitive* if the negative semiorbit of x, $O^-(x) = \{y \in X : f^n(y) = x \text{ for some } n \in \mathbb{N}\}$, is dense in X for every $x \in X$. Equivalently, (X, f) is strongly transitive if for every opene U, $X = \bigcup_{n=1}^{\infty} f^n(U)$.

A system (X, f) is called *locally eventually onto* if, for every opene U in X, there exists an $n \in \mathbb{N}$ such that $f^n(U) = X$.

Remark. We can immediately make the following observations.

(1) A minimal system is always strongly transitive, and if f is a homeomorphism then minimality and strong transitivity are equivalent.
(2) A strongly transitive system is always topologically transitive.
(3) A locally eventually onto system is always strongly transitive and weakly mixing.

A simple example of a strongly transitive system is the one-sided shift space on a finite alphabet. It is easy to see that this shift space will be locally eventually onto but not minimal. Also, the two-sided shift space on a finite alphabet, where the shift map is now a homeomorphism, fails to be strongly transitive. The irrational rotation is a minimal system so is strongly transitive but is not locally eventually onto.

Example 6.2. A strongly transitive system (X, f) need not be weakly mixing. In particular, in this case $(2^X, f_*)$ will not be topologically transitive.

Proof. Let X be the closure in $\{0, 1\}^{\mathbb{N}}$ of the sequences of the form

$$0^k 10^{3^{n_1}} 10^{3^{n_2}} 10^{3^{n_3}} 1 \cdots 10^{3^{n_i}} 1 \cdots$$

where $n_i \in \mathbb{N}$. Consider the shift map σ on X.

For any finite word w with which an element of X begins and any $x \in X$, there exists a finite word u such that $wux \in X$. It follows that the system (X, σ) is strongly transitive. The length of u may depend, in an unbounded way, upon x, because the initial run of 0s in x would have to be extended in u to get to length a power of 3 less 1.

Observe that between any two distinct 1s in a sequence in X the number of shifts is divisible by 2. Hence, there is no $n \in \mathbb{N}$ for which $\sigma^n[001] \cap [1] \neq \emptyset$ and also $\sigma^n[010] \cap [1] \neq \emptyset$ simultaneously. So our system (X, σ) is not weakly mixing. By Theorem 3.4, $(2^X, \sigma_*)$ cannot be topologically transitive. □

THEOREM 6.3. *For a dynamical system (X, f) the following are equivalent.*

(i) *(X, f) is locally eventually onto.*
(ii) *$(2^X, f_*)$ is locally eventually onto.*
(iii) *$(2^X, f_*)$ is strongly transitive.*
(iv) *The negative orbit $O^-(\{X\})$ of the point X is dense in 2^X.*
(v) *$i_X(X) \subset 2^X$ is contained in the closure of $\bigcup_{i \geq 0} (f_*)^{-i}(\{X\})$ in 2^X.*

Proof. It is obvious that (ii) ⇒ (iii) ⇒ (iv) ⇒ (v).

(v) ⇒ (i). We are assuming that the backward orbit of the point $X \in 2^X$ meets every neighborhood of $\{x\} \in 2^X$ for any $x \in X$. Thus, any opene subset of X contains a closed set A which maps onto X under some iterate of f. That is, f is locally eventually onto.

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(i) $\Rightarrow$ (ii). Since f is locally eventually onto, it is surjective and so $f^n(U) = X$ implies $f^i(U) = X$ for $i \geq n$. Now, given $A \in 2^X$, take a Vietoris neighborhood of A, i.e. a list of closed sets $C_1, \ldots, C_k$ covering A and such that each interior meets A. There exists $n \in \mathbb{N}$ so that $f^n(\text{Int}C_i) = X$ for $i = 1, \ldots, k$. For any $B \in 2^X$, let $D_i = (f|C_i)^{-n}(B)$. Each $D_i \subset C_i$ and meets $\text{Int}C_i$. Hence, $A_1 = \bigcup_{i=1}^k D_i$ is in the Vietoris neighborhood of A, while $f^n(A_1) = B$. Hence, f_* is locally eventually onto. □

The *proximal relation* $P \subset X \times X$ for (X, f) is the set of proximal pairs in $X \times X$ so that $(x, y) \in P$ if $\liminf\{d(f^n x, f^n y)\} = 0$, $n \in \mathbb{N}$. Equivalently, for any $\epsilon > 0$ there is an $n \in \mathbb{N}$ such that $d(f^n x, f^n y) < \epsilon$. The relation P is reflexive, symmetric, and f invariant. A point x is called a *distal point* if $(x, y) \in P$ only when $y = x$. The system (X, f) is said to be *distal* when all the points are distal points, i.e. if P is the diagonal $\Delta = \{(x, x) : x \in X\}$. That is, there are no non-trivial proximal pairs. It is obvious that if (X, f) is distal then the map f is injective.

A point is called *almost periodic* if its orbit closure is a compact minimal system. A dynamical system is called *pointwise almost periodic* if every point is almost periodic. If a system is pointwise almost periodic, then the map is surjective. For a compact system (X, f) the compactness of the orbit closures is automatic, but the definition applies to non-compact systems as well.

A classical theorem of Ellis says that (X, f) is distal if and only if the product system $(X \times X, f \times f)$ is pointwise almost periodic. That is, every orbit closure in $X \times X$ is minimal. In particular, if (X, f) is distal then it is pointwise almost periodic and f is a homeomorphism.

Notice that if two almost periodic points are proximal then they have the same orbit closure. That is, they are contained in the same minimal subset. It follows that (X, f) is distal if and only if it is pointwise almost periodic and the restriction of f to each minimal subset is distal.

A dynamical system is *equicontinuous* if the set of the iterates of the map forms a uniformly equicontinuous family. That is, (X, f) is equicontinuous if for every $\epsilon > 0$, there is a $\delta > 0$ such that for all $x, x' \in X$, and $n \in \mathbb{N}$, $d(x, x') < \delta$ implies $d(f^n x, f^n x') < \epsilon$.

The *regionally proximal relation* $RP \subset X \times X$ for (X, f) is an enlargement of the proximal relation P. We say that $(x, y) \in RP$ if there are sequences $\{x_n\}$ and $\{y_n\}$ with $x_n \to x$ and $y_n \to y$ and integers k_n such that $d(f^{k_n}(x_n), f^{k_n}(y_n)) \to 0$. The relation RP is closed as well as reflexive, symmetric, and f invariant. It is known that (X, f) is equicontinuous if and only if $RP = \Delta$. That is, whenever $x \neq y$, $(x, y) \notin RP$. It follows that if (X, f) is equicontinuous then it is distal, and it is well known that the converse fails.

LEMMA 6.4. *Assume that (X, f) is distal and that B is an almost periodic point for $(2^X, f_*)$. If $A \in 2^X$ and (A, B) is a proximal pair for f_* then $A \subset B$. If A is also an almost periodic point then $A = B$.*

Proof. We only sketch the proof as it uses the machinery from [**4**] of the action of $\beta\mathbb{Z}$ on X and its induced action $\circ$ (circle operation) on 2^X; see also [**5**] and [**9**]. We omit a review of the applications (due largely to Ellis) of the idempotents of $\beta\mathbb{Z}$.

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We use the following facts.

- If $(x, y) \in P$ and x is almost periodic, then there exists an idempotent $u \in \beta\mathbb{Z}$ such that $uy = ux = x$.
- If u is an idempotent element of $\beta\mathbb{Z}$ then, for any $x \in X$, $(x, ux) \in P$. In particular, if (X, f) is distal then $ux = x$ for every idempotent u and every $x \in X$.
- If $u \in \beta\mathbb{Z}$, $A \in 2^X$ and $u \circ A \in 2^X$ denotes the element obtained by the induced action on 2^X, then $ux \in u \circ A$ for all $x \in A$.

So if $(A, B) \in 2^X \times 2^X$ is a proximal pair with B almost periodic, there is an idempotent u such that $u \circ A = u \circ B = B$. Since (X, f) is distal, $ux = x$ for all $x \in X$. Hence, for all $x \in A$, $x = ux \in u \circ A = B$.

If A is also almost periodic, then we can reverse the argument to get $B \subset A$. □

We immediately obtain the following corollary.

COROLLARY 6.5. *If (X, f) is distal and M is a minimal subset of 2^X then the subsystem (M, f_*) is distal.*

We recall the following lemma, e.g. from [**3**, Lemma 2.8].

LEMMA 6.6. *If the almost periodic points are dense in the systems (X_1, f_1) and (X_2, f_2) then they are dense in the product system $(X_1 \times X_2, f_1 \times f_2)$.*

Proof. Since the product of dense sets is dense in the product, we may assume that each system is minimal. Let (x_1, x_2) be an almost periodic point of $X_1 \times X_2$. For all $k_1, k_2 \in \mathbb{N}$, $(f^{k_1}(x_1), f^{k_2}(x_2))$ are almost periodic points of $X_1 \times X_2$. Since each factor is minimal, this collection of almost periodic points is dense in $X_1 \times X_2$. □

Compare the following result with Proposition 3.1.

PROPOSITION 6.7. *If in the system (X, f) the almost periodic points are dense then the almost periodic points are dense in the induced systems $(2^X, f_*)$ and $(\mathcal{C}(K, X), f_*)$.*

Proof. By Lemma 6.6, and induction, the almost periodic points are dense in the product systems $(X^{(n)}, f^{(n)})$ for all $n \in \mathbb{N}$.

If $(y_1, \ldots, y_n)$ is an almost periodic point of $X^{(n)}$ then $i_{X,n}(y_1, \ldots, y_n) = \{y_1, \ldots, y_n\}$ is an almost periodic point of 2^X. If $\mathcal{A} = \{A_1, \ldots, A_n\}$ is a clopen partition of K then the function $u : K \to X$ with $u(A_i) = \{y_i\}$ for $i = 1, \ldots, n$ is an almost periodic point of $\mathcal{C}(\mathcal{A}, X)$.

The result follows because $\mathrm{FIN}(X) = \bigcup_n i_{X,n}(X^{(n)})$ is dense in 2^X and $\bigcup_{\mathcal{A}} \mathcal{C}(\mathcal{A}, X)$ is dense in $\mathcal{C}(K, X)$. □

COROLLARY 6.8. *Assume that (X, f) is a distal dynamical system. In the induced system $(2^X, f_*)$ each almost periodic point is distal and these points are dense in 2^X.*

Proof. Every almost periodic point is distal by Corollary 6.5. Since every point of X is almost periodic, the almost periodic points are dense in 2^X by Proposition 6.7. □

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It is not true that (X, f) distal implies $(2^X, f_*)$ is distal.

For the unit disc D, define the map $f : D \to D$ by $f(r, \theta) = (r, r + \theta)$, using polar coordinates. Clearly, (D, f) is distal. Let $r_n \mapsto i$, where r_n are rationals and i is irrational. Consider $A = \{(r_n, 0) : n \in \mathbb{N}\} \cup \{(i, 0)\} \in 2^X$. In the orbit f_* closure of A there exists a set B which contains the full circle at the radius i. Since the radius i circle is f invariant, A is not in the orbit closure of B. Thus, $A \in 2^X$ is not almost periodic.

We now clarify the situation by extending the result from [**23**] which says that the system (X, f) is equicontinuous if and only if the system $(2^X, f_*)$ is equicontinuous. The result below shows that if (X, f) is distal but not equicontinuous then $(2^X, f_*)$ is not distal.

THEOREM 6.9. *Let (X, f) be a compact dynamical system and K be a Cantor set.*

The following are equivalent.

(i) (X, f) *is equicontinuous.*
(ii) $(2^X, f_*)$ *is equicontinuous.*
(iii) $(2^X, f_*)$ *is distal.*
(iv) $(2^X, f_*)$ *is pointwise almost periodic.*
(v) $(\mathcal{C}(K, X), f_*)$ *is equicontinuous.*
(vi) $(\mathcal{C}(K, X), f_*)$ *is pointwise almost periodic.*

Proof. Recall that the definitions of equicontinuity and pointwise almost periodicity apply to non-compact systems like $(\mathcal{C}(K, X), f_*)$.

It is easy to show that (i) $\Rightarrow$ (ii). Since compact equicontinuous systems are distal, (ii) $\Rightarrow$ (iii), and since distal systems are pointwise almost periodic, (iii) $\Rightarrow$ (iv).

(iii) $\Rightarrow$ (i). Suppose that (X, f) is not equicontinuous. Then there are x and y in X with $x \neq y$ and $(x, y) \in RP$. Then we have $x_n \to x$, $y_n \to y$ and $k_n \in \mathbb{Z}$ with $(f^{k_n}(x_n), f^{k_n}(y_n)) \to (z, z)$ for some $z \in X$. Let $C = \{x_1, x_2, \ldots, x\}$ and $D = \{y_1, y_2, \ldots, y\}$. We may suppose that $C \cap D = \emptyset$. We show that the orbit closure of (C, D) in $2^X \times 2^X$ is not minimal. Let (C', D') be the limit of some subsequence of $\{(f_* \times f_*)^{k_n}(C, D)\}$. Since $z \in C' \cap D'$, we have $C' \cap D' \neq \emptyset$. In the terminology of Proposition 2.3(i), $(C', D') \in INT$.

Now if the $f_* \times f_*$ orbit closure of (C, D) were minimal, then (C, D) would be in the $f_* \times f_*$ orbit closure of (C', D'). Since INT is closed and invariant, this would imply $C \cap D \neq \emptyset$, a contradiction.

(iv) $\Rightarrow$ (iii). We first show that if $(2^X, f_*)$ is pointwise almost periodic, then (X, f) is distal.

If $(x, y) \in P$ with $x \neq y$ then there exists $z \in X$ such that (z, z) is in the $f \times f$ orbit closure of (x, y) and hence $\{z\}$ is in the f_* orbit closure of $\{x, y\}$. Since the set of singletons $i_X(X)$ is closed, it follows that $\{x, y\}$ is not in the f_* orbit closure of $\{z\}$. Hence, $\{x, y\}$ is not almost periodic. Contrapositively, $(2^X, f_*)$ pointwise almost periodic implies that (X, f) is distal.

Since (X, f) is distal, Corollary 6.5 implies that the restriction of f_* to every minimal subset is distal. Hence, if $(2^X, f_*)$ is pointwise almost periodic, it is distal.

(i) $\Leftrightarrow$ (v). It is easy to check that f equicontinuous implies f_* is equicontinuous. The converse follows by restricting to the subset of constant functions in $\mathcal{C}(K, X)$.

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(v) $\Rightarrow$ (vi). For $u \in \mathcal{C}(K, X)$, let M be the f_* orbit closure of u. Since (X, f) is equicontinuous and u is uniformly continuous, the set $\{f^n \circ u : n = 0, 1, \ldots\}$ is equicontinuous in $\mathcal{C}(K, X)$ and so has compact closure by the Arzelà–Ascoli theorem. That is, the subsystem $(M, f_*|M)$ is compact and equicontinuous. For compact systems, equicontinuity implies distality implies pointwise almost periodicity. Hence, each point of M is almost periodic. In particular, u is almost periodic.

(vi) $\Rightarrow$ (iv). Because $\mathrm{Im} : \mathcal{C}(K, X) \to 2^X$ is a surjective action map between the induced systems, $(\mathcal{C}(K, X), f_*)$ pointwise almost periodic implies $(2^X, f_*)$ is. □

Remark. The equivalence of (i) and (ii) is [**7**, Proposition 7]. The equivalence of (i)–(iv) has been proved independently as [**16**, Theorem 3.4].

A system (X, f) is *sensitive* if there exist a $\delta > 0$ such that for any $x_1 \in X$ and opene set U containing x_1, there exist $x_2 \in U$ and $m \in \mathbb{N}$ such that $d(f^m(x_1), f^m(x_2)) > \delta$.

It is known that minimal systems are either equicontinuous or sensitive. Various forms of sensitivities for the induced system $(2^X, f_*)$ are studied by Sharma and Nagar in [**22**], and among other things there is an example of a sensitive (X, f) for which $(2^X, f_*)$ is not sensitive.

7. (X, f) *can be isomorphic to its own induced system*

In this section we will again restrict attention to compact systems (X, f).

The dynamical system $(2^X, f_*)$ induces the system $(2^{2^X}, f_{**})$, and in many cases we believe that iterating this will lead to a transfinite process. Hence, it would be nice to characterize those compact dynamical systems (Y, g) which are of the form $(2^X, f_*)$ for some (X, f). In this section, we look into some cases where this process of inducing definitely gives us non-conjugate dynamics or where this process of inducing stabilizes.

A subset $A \subset X$ is called + *invariant* when $f(A) \subset A$ and invariant when $f(A) = A$. Observe that if A is closed then it is invariant if and only if as an element of 2^X it is a fixed point for f_*. Hence, if f is transitive then if A is a fixed point for f_* then either $A = X$ or A has empty interior. If f is *totally transitive*, i.e. every power f^n for $n = 1, 2, \ldots$ is transitive, then if A is a periodic point for f_*, and hence a fixed point for some f_*^n, then either $A = X$ or A has empty interior. In particular, a weakly mixing system is totally transitive and so this then holds.

If $F \subset 2^X$ is a closed invariant set for f_* then as an element of 2^{2^X} it is a fixed point and hence $\vee(F) = \bigcup F$ is a fixed point for f_*, i.e. an invariant set for f. In particular, the union of any periodic orbit of f_* is a fixed point of f_*. Notice that the union of any finite collection of periodic points of f_* is a periodic point (with period dividing the lowest common multiple of the periods of the members of the collection). The intersection, if non-empty, is also a periodic point.

Recall that if the periodic points are dense in X then the same holds for 2^X. We will later see that the converse is not true.

THEOREM 7.1. *Let (X, f) be a compact dynamical system. The periodic points are dense in $(2^X, f_*)$ if their closure contains $i(X) \subset 2^X$.*

If f is totally transitive and the periodic points are dense in 2^X then there are infinitely many distinct fixed points for f_*.

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Proof. Assume that every singleton can be approximated by a periodic point of 2^X.

If $\{x_1, \ldots, x_k\} \subset X$ and $\epsilon > 0$ then there exist periodic points $A_1, \ldots, A_k \in 2^X$ such that $d(A_i, \{x_i\}) < \epsilon$ for $i = 1, \ldots, k$. Then $A = A_1 \cup \cdots \cup A_k$ is a periodic point of 2^X with $d(A, \{x_1, \ldots, x_k\}) < \epsilon$. Hence, the closure of the periodic points contains $\mathrm{FIN}(X)$ which is dense in 2^X.

Let A denote the union of an arbitrary finite collection of periodic orbits for f_*, other than the fixed point X itself. Then A is a closed, f invariant set. So if f is totally transitive, then A is nowhere dense. Each union of a periodic orbit is a fixed point for f_*. There is an opene set $U \subset X$ disjoint from A and it contains a periodic point B for f_*. The union of the associated periodic orbit is disjoint from A and so is distinct from the previous fixed points. □

COROLLARY 7.2. *If (X, f) is a mixing subshift of finite type then it is not isomorphic to $(2^Y, g_*)$ for any compact system (Y, g).*

Proof. A mixing subshift of finite type is totally transitive and has dense periodic points. Because it is also expansive, it cannot admit infinitely many distinct fixed points. □

An *inverse system* is a sequence between compact metrizable spaces: $\{\pi_i : X_{i+1} \to X_i : i = 1, 2, \ldots\}$. The associated *inverse limit* is the compact metrizable space $X_\infty = \{x \in \Pi_i X_i : x_i = \pi_i(x_{i+1}) \text{ for } i = 1, 2, \ldots\}$, which is metrizable when the X_i are. The projection map to the ith coordinate is denoted $\pi^i : X_\infty \to X_i$. If F is a closed subset of X_∞ we let $F_i = \pi^i(F)$. If $x \in X_\infty$ is such that $\pi^i(x) \in F_i$ for all i then $\{F \cap (\pi^i)^{-1}(x_i)\}$ is a decreasing sequence of non-empty compacta and so the intersection, which is $F \cap \{x\}$, is non-empty. That is, $x \in F$. Thus, the restrictions $\pi^i : F \to F_i$ and $\pi_i : F_{i+1} \to F_i$ are all surjective and so F is the inverse limit of the system $\{\pi_i : F_{i+1} \to F_i\}$. It thus follows that the map from $\{(\pi_i)_* : 2^{X_{i+1}} \to 2^{X_i}\}$ is an inverse system. Furthermore, the maps $(\pi^i)_* : 2^{X_\infty} \to 2^{X_i}$ induce a natural homeomorphism between 2^{X_∞} and the inverse limit of the system $\{(\pi_i)_* : 2^{X_{i+1}} \to 2^{X_i}\}$. Refer to [**13**, **21**] for more details.

Now let X be any compact metric space. Define $X_1 = 2^X$, $X_2 = 2^{2^X} = 2^{X_1}$ and $\pi_1 = \vee : X_2 \to X_1$. Inductively, define $X_{i+1} = 2^{X_i}$ and $\pi_i = (\pi_{i-1})_* : X_{i+1} = 2^{X_i} \to 2^{X_{i-1}} = X_i$. Let X_∞ denote the inverse limit of this inverse system. As remarked in the previous paragraph, there is a natural homeomorphism from 2^{X_∞} to the inverse limit of the system $\{(\pi_i)_* : 2^{X_{i+1}} \to 2^{X_i}\}$ which is our original system with the index shifted by 1. That is, there is a natural homeomorphism $q : 2^{X_\infty} \to X_\infty$.

If (X, f) is a compact dynamical system then, letting $f_1 = f_*$ and $f_{i+1} = (f_i)_*$, we obtain an inverse limit of dynamical systems with f_∞ the map induced on X_∞ and q is an isomorphism from $(2^{X_\infty}, (f_\infty)_*)$ to (X_∞, f_∞). Since the inverse limit of weakly mixing systems is weakly mixing, (X_∞, f_∞) is weakly mixing when (X, f) is. Furthermore, the pair of action maps $(\pi_1)_* : 2^{X_\infty} \to 2^{2^X}$ and $\pi_1 : X_\infty \to 2^X$ maps the isomorphism q to $\vee$. Thus, we have proved the following result.

THEOREM 7.3. *If (X, f) is a compact dynamical system then $(2^X, f_*)$ is a factor of a compact dynamical system (X_∞, f_∞) which is isomorphic to $(2^{X_\infty}, (f_\infty)_*)$. Furthermore, if (X, f) is weakly mixing then (X_∞, f_∞) is weakly mixing.*

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A well-known inverse limit construction provides the example, promised above, of a system with no periodic points but whose induced system has dense periodic points.

If $\{n_i\}$ is a strictly increasing sequence of positive integers with $n_i | n_{i+1}$ then the set of inclusions of $n_{i+1}\mathbb{Z} \to n_i\mathbb{Z}$ induces an inverse system $\{\pi_i : \mathbb{Z}/n_{i+1}\mathbb{Z} \to \mathbb{Z}/n_i\mathbb{Z}\}$ of epimorphisms of finite rings. The inverse limit $\mathbb{Z}_{\{n_i\}}$ is a compact ring which is topologically a Cantor set. The associated *odometer* or *adding machine* uses the translation homeomorphism $t(x) = x + 1$. The odometer $(\mathbb{Z}_{\{n_i\}}, t)$ is minimal and equicontinuous (but clearly not totally transitive).

Every clopen subset is a periodic element for $(2^{\mathbb{Z}_{\{n_i\}}}, t_*)$, as it is the preimage of a finite subset of $\mathbb{Z}/\mathbb{Z}_{\{n_i\}}$ for sufficiently large i. Thus, any point of x can be approximated by a periodic element of 2^X with non-empty interior and so with the union of the orbit equal to X. It follows that the periodic points are dense in 2^X. Hence, from Theorem 7.1 it follows that the periodic points are dense in $2^{\mathbb{Z}_{\{n_i\}}}$. On the other hand, $(\mathbb{Z}_{\{n_i\}}, t)$ is minimal and so has no periodic points.

Finally, it would be interesting to see which dynamical systems can be represented as $(2^X, f_*)$ for some system (X, f). This is far from settled, but we have obtained some obstructions. Of course, 2^X can never be minimal. Theorem 6.3 tells us that it cannot be strongly transitive unless it is locally eventually onto, and by Theorem 6.9, it cannot be distal unless it is equicontinuous.

Moreover, $(2^X, f_*)$ has some structure which apparently does not occur in most dynamical systems, namely the relations *INT*, *INC*, *EPS* defined in Proposition 2.3(i). These are closed $f_* \times f_*$ invariant relations, and an interesting problem is to determine their dynamical significance.

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