Euro. Jnl of Applied Mathematics (2005), *vol.* 16, *pp.* 21–36. © 2005 Cambridge University Press
doi:10.1017/S0956792505005863 Printed in the United Kingdom

A nonlocal mean curvature flow for multi-dimensional image processing

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(*Received 17 March 2004; revised 22 December 2004*)

In this paper, we propose a mean curvature flow equation with nonlocal nonlinearities for an image processing model on noisy surfaces, which corresponds to the negative L^2-gradient flow of surface energy functional in (1.4). We investigate the initial value problem mathematically. Furthermore, numerical simulations are shown, which exhibit that the model performs smoothing operation on a complicated level surface of a given image data while preserving its geometric shape feature under suitable parameters chosen.

1 Introduction

In the field of image processing, PDE (partial differential equation)-based methods have been intensively studied during the past two decades [3, 6, 7]. Many important PDE approaches for two-dimensional (2-D) image processing have been presented, such as the nonlinear diffusion equations of the Malik-Perona type [5, 11], the geometric flow of mean curvature type [1, 2], and variational methods such as the Mumford-Shah [10] and TV (total variation) type [13].

On the other hand, three-dimensional (3-D) image processing has also had an attraction from the point of view of not only a mathematical viewpoint, but also in real fields such as medical image analysis [12, 8]. Suppose that a complicated, closed level surface (or curve) is somehow extracted from given noisy image data at high resolution. Then we generally wish to smooth the complicated surface while preserving its geometric shape feature as much as possible. This task is important from the point of the mathematical morphology (see Alvarez *et al.* [2] and Cao [4]).

In this paper, we propose an anisotropic mean curvature flow equation with a nonlocal repulsive term to process a complicated level surface (or curve) of image data obtained for the elimination of geometric tiny features. To describe our image processing model precisely, we begin with the mathematical setting in any space dimensional situation.

Let

$$X : M^n \to \mathbb{R}^{n+1}$$

be a smooth, closed, embedded hypersurface into $\mathbb{R}^{n+1}$ for an n-dimensional differentiable manifold M^n with the induced metric $g = (g_{ij})$. Let σ_X be a characteristic function on

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$\mathbb{R}^{n+1}$ defined by

$$\sigma_X(x) = \begin{cases} 1 & \text{if } x \in \mathrm{Ext}(X), \\ 0 & \text{if } x \in X, \\ -1 & \text{if } x \in \mathrm{Int}(X), \end{cases} \tag{1.1}$$

where $\mathrm{Ext}(X)$ (resp. $\mathrm{Int}(X)$) is the exterior (resp. interior) region of X. Set χ be a binary function on $\{-1, 0, 1\}$ such that

$$\chi(-1) = 1, \quad \chi(0) = \chi(1) = 0. \tag{1.2}$$

We denote by $\{e^{\tau\Delta}\}_{\tau \geqslant 0}$ a heat semigroup generated by the Cauchy problem on X:

$$\frac{\partial u}{\partial \tau} = \Delta u \qquad \text{on } X \times [0, \infty) \tag{1.3}$$

subject to the initial condition $u(0) = u_0$. Here, Δ denotes the Laplace-Beltrami operator on X, i.e. in local coordinates $(\xi^1, \cdots, \xi^n)$ on M^n,

$$\Delta = g^{ij}\big(D_i D_j - \Gamma_{ij}^k D_k\big),$$

where $D_i = \partial/\partial\xi^i$, $(g^{ij}) = (g_{ij})^{-1} = (\langle D_i X, D_j X\rangle)^{-1}$, and Γ_{ij}^k are the Christoffel symbols of g.

Given $X_0 : M^n \to \mathbb{R}^{n+1}$ with the metric g_0, let us introduce the following energy functional E for X associated with X_0:

$$E(X; X_0) = \int_X G\big(|\tilde{\nabla}(e^{\tau\tilde{\Delta}} H_0)|^2\big)\, dA + K\left[\int_{\mathbb{R}^{n+1}} \chi(\sigma_X\, \sigma_{X_0})\, dx\right]^{\frac{n}{n+1}}, \tag{1.4}$$

where

- $\tilde{\nabla}$ is the gradient operator on X_0, and $|\tilde{\nabla} u|^2 = \sum g_0^{ij} D_i u D_j u$;
- $\tilde{\Delta}$ and the Laplace-Beltrami operator on X_0;
- H_0 is the mean curvature on X_0;
- $dA = \sqrt{|g|}\, d\xi^1 \ldots d\xi^n$ is the area element on X, where $|g| = \det(g_{ij})$;
- $G(s)$ is a nonincreasing, positive, smooth function on $[0, \infty)$ with $G(0) = 1$;
- τ and K are positive parameters.

Practically in 3-D (resp. 2-D) image processing, X_0 corresponds to an input complex, noisy level surface (resp. contour) extracted from a raw image data, and X is the output image to be processed.

Roughly speaking, each term of $E(X; X_0)$ performs as follows:

(a) The first term represents a generalized area depending on the weight G. The argument $|\tilde{\nabla}(e^{\tau\tilde{\Delta}} H_0)|$ is approximately a local mean value of the gradient of mean curvature H_0 in a neighborhood of a point X_0.

(b) The quantity $\int_{\mathbb{R}^{n+1}} \chi(\sigma_{X_0}\, \sigma_X)\, dx$ corresponds to the volume of such an enclosing region, only if X is sufficiently close to X_0. Then the integral is considered as a distance-like function between X and X_0.

(c) If $G \equiv 1$, then the first term and second one have the same homogeneous degree under the dilation $\{x \mapsto \lambda x \mid x \in \mathbb{R}^{n+1},\ \lambda > 0\}$ in virtue of the exponent $n/(n+1)$.

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The plan of the paper is as follows. In § 2, as a multi-dimensional image processing model we present some nonlocal mean curvature flow based on functional E. § 3 is concerned with the local existence and uniqueness of classical solutions of the initial value problem. Finally, in § 4 some numerical simulations are exhibited in the case of 2-D processing ($n = 1$), where the experimental results obtained by our model are compared with the curve shortening flow ($K = 0$) for several complicated original curves.

2 The model

In this section, we shall propose a curvature flow equation based on energy (1.4) for applying the discretization to image processing as a multiscale method. For the derivation, we begin to recall the fundamental variation formulae.

For a hypersurface $X : M^n \to \mathbb{R}^{n+1}$ and a smooth function ϕ on X, consider the one-parameter variation

$$X_{t,\phi} = \{x + t\phi(x)N(x) \,;\, x \in X\},$$

where N is the outward unit normal vector field on X. It is well-known that the first variation formula of the area $|X| = \int_X dA$ is

$$\delta_{\phi N}(|X|) = \frac{d}{dt}\bigg|_{t=0} |X_{t,\phi}| = n \int_X \phi H \, dA,$$

where H is the mean curvature on X. On the other hand, the first variation formula for $|\mathrm{Int}(X)| = \int_{\mathbb{R}^{n+1}} \chi(\sigma_X)\, dx$, the volume of interior region of X, is given by

$$\delta_{\phi N}(|\mathrm{Int}(X)|) = \frac{d}{dt}\bigg|_{t=0} |\mathrm{Int}(X_{t,\phi})| = n \int_X \phi \, dA.$$

Thus, by letting

$$\mathrm{dist}(X, X_0) = \int_{\mathbb{R}^{n+1}} \chi(\sigma_X\, \sigma_{X_0})\, dx = \int_{\{\sigma_X\, \sigma_{X_0} = -1\}} dx, \tag{2.1}$$

we obtain the first variation for the second term on the (1.4):

$$\delta_{\phi N}\big(\mathrm{dist}(X, X_0)^{n/(n+1)}\big) = \frac{n}{n+1} \mathrm{dist}(X, X_0)^{-1/(n+1)} \int_X \phi\, \sigma_{X_0}\, dA.$$

We are now in a position to mention our nonlocal mean curvature flow. We propose the following regularized flow equation of the negative L^2-gradient flow of the Euler–Lagrange equation of E:

$$\frac{d}{dt} X(t) = -\Big[\, G\big(|\tilde{\nabla}(e^{\tau\tilde{\Delta}} H_0)|^2\big) H + K\, \mathrm{dist}(X, X_0)^{-1/(n+1)} \sigma^{\epsilon}_{X_0} \Big]\, N, \tag{2.2}$$

where $\sigma^{\epsilon}_{X_0}$ is some regularization of σ_{X_0} by a smooth function β^{ϵ} as follows. For two parameters ϵ, d with $0 < \epsilon < d$, suppose that $\beta^{\epsilon}(r)$ is a nondecreasing C^2-function on $\mathbb{R}$ such that $\beta^{\epsilon} = 0$ ($|r| \leqslant \epsilon$), $\beta^{\epsilon} = -1$ ($r \leqslant -d$), $\beta^{\epsilon} = 1$ ($r \geqslant d$). Thus, the regularization of

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σ_{X_0} is defined by

$$\sigma^{\epsilon}_{X_0}(x) = \begin{cases} \beta^{\epsilon}(\sigma_{X_0}(x)\mathrm{dist}(x, X_0)) & \text{if } \mathrm{dist}(x, X_0) < d, \\ \sigma_{X_0}(x) & \text{otherwise,} \end{cases} \tag{2.3}$$

where d is small enough for the consistency in general.

Our purpose in the sequel is to consider the initial value problem of a family of smooth embedding

$$X : [0, T) \times M^n \to \mathbb{R}^{n+1}$$

such that $X(t) := X(M^n \times \{t\})$ is driven by (2.2) satisfying initial hypersurface $X(0) = X_0$.

Although the regularization procedure through function β^{ϵ} is necessary for solving the Cauchy problem mathematically, the role of the function is still more important in the context of image processing. If we wish to process a noisy original image by applying the discretization of (2.2), the parameter ϵ should be chosen as a minimal scale of the locally tiny geometric features that we desire to neglect. Therefore, mostly details of the image will be forced to smooth as much as possible. On the other hand, if ϵ is smaller (and K is relatively large), and the processed image will remain close to an original image. For instance, see Figure 4 in §4, in which the amplitude of noisy part is about 0.2, and thus it is reasonable to choose $\epsilon = 0.2$.

We finish this section with observing the spherical situation of our model as a simple illustration. If $X_0 = \{|x| = R_0\}$, then $X(t)$ is a sphere of radius $R(t)$, satisfying

$$\frac{d}{dt}R(t) = -\frac{n}{R} - \frac{K\beta^{\epsilon}(R_0 - R)}{(\omega_n|R_0^{n+1} - R^{n+1}|)^{1/(n+1)}} =: f(R), \quad R(0) = R_0, \tag{2.4}$$

where ω_n denotes the measure of unit sphere. If we choose $K > 0$ such that $f(R_0 - d) > 0$, i.e.

$$K > \frac{n(\omega_n(R_0^{n+1} - (R_0 - d)^{n+1}))^{1/(n+1)}}{R_0 - d}, \tag{2.5}$$

then $f(R)$ has at least two zeros $\underline{R}$ and $\{\overline{R}_1, \cdots, \overline{R}_k\}$ in $(0, R_0 - \epsilon)$ such that $0 < \underline{R} < R_0 - d \leqslant \overline{R}_1 < \cdots < \overline{R}_k < R_0 - \epsilon$. Then, $R(t)$ globally exists and converges to $\overline{R}_k$ as $t \to \infty$. If K is small enough, satisfying the inequality converse to (2.5), $X(t) = \{|x| = R(t)\}$ shrinks to a one-point in finite time. Additionally, consider the problem (2.4) with general initial data $R(0) = A > 0$. Then one easily see that if $A \geqslant \underline{R}$, the solution globally exists, and besides if $A \notin \{\underline{R}, \overline{R}_1, \cdots, \overline{R}_k\}$, then the solution converges to the nearest stable point. On the other hand, if $A < \underline{R}$, a finite-time extinction occurs.

3 Mathematical analysis

In this section, we shall prove the short-time existence and uniqueness of a family of smooth hypersurface X, evolved by the motion of (2.2). The proof of local existence given here is two steps: we first give a representation of the moving surface $X(t)$ as a normal graph over a smooth reference manifold Σ. Secondly the proof of the existence part is given within the framework of Hölder space (see the precise definitions below) as in Lunardi's parabolic PDE theory [9].

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Theorem 3.1 *Suppose that X_0 is a closed, embedded hypersurface in $\mathbb{R}^{n+1}$ of class $C^{2+\alpha}$. Then there exists sufficiently small $T > 0$ and a unique family of closed, embedded $X = \{X(t) ; 0 \leqslant t \leqslant T\}$ of (2.2) in $C^{1+\alpha/2,2+\alpha}([0,T] \times M^n)$. Moreover, $X(t)$ is of class C^∞ for $t > 0$.*

For a given initial hypersurface X_0, we can always find a smooth reference hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ such that X_0 is a graph over Σ in the outer unit normal direction $N \in C^\infty(\Sigma, S^n)$ with distance less than d to X_0, precisely as follows: for $\varepsilon_0 > 0$ small enough, then the distance from $s \in \Sigma$ to X_0 is a function $u_0 \in C^{2+\alpha}(\Sigma)$ with $||u_0||_\infty < \varepsilon_0$, i.e.

$$\Phi_\Sigma(s, u_0(s)) := s + u_0(s)N$$

is a $C^{2+\alpha}$-diffeomorphism from Σ to X_0 such that $X_0 = \Phi_\Sigma(\Sigma, u_0(\Sigma))$. Furthermore, as its extension, there is some $d > 0$ such that

$$\Phi_\Sigma : \Sigma \times (-d, d) \to \mathbb{R}^{n+1}, \quad \Phi_\Sigma(s, r) := s + rN(s)$$

is a C^∞-diffeomorphism from $\Sigma \times (-d, d)$ onto $\mathscr{R}$:= its image by Φ_Σ. That is, $\mathscr{R}$ consists exactly of those points in $\mathbb{R}^{n+1}$ with distance less than d to Σ. We may assume that ε_0 is so much less than d without loss of generality.

Since that $X(t)$ is obtained by a smooth, small deformation of Σ in $\mathscr{R}$, the parametrization is represented by a graph u as follows:

$$x = s + u(s,t)N \tag{3.1}$$

where $u : \Sigma \times [0,T)$ is a smooth function with $||u||_\infty < d$. Let $\{e_i ; 1 \leqslant i \leqslant n\}$ be an orthonormal basis for the tangent space $T_s\Sigma$, and denote the derivative in direction by e_i by $x_{,i} = e_i(u) = \nabla_{e_i} u$, etc. Then one obtain

$$x_{,i} = u_{,i}N + (u+1)e_i,$$

and the induced metric $g = (g_{ij})$

$$g_{ij} = \langle x_{,i}, x_{,j} \rangle = u_{,i}u_{,j} + (u+1)^2\delta_{ij},$$

and the inverse is

$$g^{ij} = \frac{1}{(u+1)^2}\left(\delta_{ij} - \frac{u_{,i}u_{,j}}{(u+1)^2 + |\nabla u|^2}\right).$$

The normal vector of $X = X(t)$ is given by

$$N_X = \frac{(u+1)N - u_{,i}e_i}{\sqrt{(u+1)^2 + |\nabla u|^2}},$$

and hence, the second fundamental forms metric (h_{ij}) is

$$h_{ij} = \langle x_{,i}, N_{X,j} \rangle = \frac{2u_{,i}u_{,j} + (u+1)^2\delta_{ij} - u_{,ij}}{\sqrt{(u+1)^2 + |\nabla u|^2}},$$

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and the mean curvature is

$$H = \mathrm{trace}(g^{ij}h_{jk}) = g^{ij}h_{ji}$$
$$= \frac{n}{\sqrt{(u+1)^2+|\nabla u|^2}} - \frac{1}{u+1}\mathrm{div}\left(\frac{\nabla u}{\sqrt{(u+1)^2+|\nabla u|^2}}\right).$$

Furthermore, the square norm of the gradient of a function f on X is

$$|\nabla_X f|^2 = g^{ij}f_{,i}f_{,j} = \frac{1}{(u+1)^2}\left(|\nabla u|^2 - \frac{u_{,i}u_{,j}}{(u+1)^2+|\nabla u|^2}f_{,i}f_{,j}\right),$$

and the Laplacian on X is given by

$$\Delta f = g^{ij}(f_{,ij} - \Gamma^k_{ij}f_{,k})$$
$$= \frac{1}{(u+1)^2}\left(\Delta f - \frac{u_{,i}u_{,j}}{(u+1)^2+|\nabla u|^2}f_{,ij} - \Gamma^k_{ij}f_{,k}\right),$$

where Γ^k_{ij} is the Christoffel symbols of (g_{ij}),

$$\Gamma^k_{ij} = \frac{1}{2}g^{kl}(g_{jl,i} + g_{il,j} - g_{ij,l})$$
$$= \frac{u_{,k}}{(u+1)^2}\left(u_{,ij} - \frac{|\nabla u|^2u_{,ij} + (u+1)(2u_{,i}u_{,j} - |\nabla u|^2\delta_{ij})}{(u+1)^2+|\nabla u|^2}\right)$$
$$+ \frac{1}{u+1}(u_{,i}\delta_{jk} + u_{,j}\delta_{ik} - u_{,k}\delta_{ij}).$$

By taking $\mathbb{R}^{n+1}$-inner product of the flow equation (2.2) and N, we have

$$\langle \partial_t X, N\rangle = -\left(G(|\tilde{\nabla}(e^{\tau\tilde{\Delta}}H_0)|^2)H + K\,\mathrm{dist}(X,X_0)^{-\frac{1}{n+1}}\sigma^\epsilon_{X_0}\right)\langle N_X, N\rangle.$$

Note that $\partial_t X = \langle \partial_t x, N\rangle N_X$, and hence

$$\langle \partial_t x, N\rangle = -G(|\tilde{\nabla}(e^{\tau\tilde{\Delta}})H|^2)H - K\,\mathrm{dist}(X,X_0)^{-\frac{1}{n+1}}\sigma^\epsilon_{X_0}.$$

On the other hand, we know

$$\langle \partial_t x, N\rangle = \frac{(u+1)\partial_t u}{\sqrt{(u+1)^2+|\nabla u|^2}}.$$

Moreover, one sees

$$\mathrm{dist}(X,X_0)^{-\frac{1}{n+1}}\sigma^\epsilon_{X_0} = \left[\int_{\mathscr{R}}\chi(\sigma[u]\sigma[u_0])\,dx\right]^{-\frac{1}{n+1}}\beta^\epsilon(u-u_0),$$

where

$$\sigma[u](x) = \sigma[u](s+rN) = \begin{cases} 1 & \text{if } r > u, \\ 0 & \text{if } r = u, \\ -1 & \text{if } r < u. \end{cases}$$

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Hence, we get the governing equation of u:

$$\frac{\partial u}{\partial t} = G\big(|\tilde{\nabla}(e^{\tau\tilde{\Delta}}H_0)|^2\big)\left[\frac{\sqrt{(u+1)^2+|\nabla u|^2}}{(u+1)^2}\operatorname{div}\left(\frac{\nabla u}{\sqrt{(u+1)^2+|\nabla u|^2}}\right)-\frac{n}{u+1}\right]$$
$$-K\left[\int_{\mathscr{R}}\chi(\sigma[u]\sigma[u_0])\,dx\right]^{-\frac{1}{n+1}}\frac{\sqrt{(u+1)^2+|\nabla u|^2}}{u+1}\beta^{\epsilon}(u-u_0). \tag{3.2}$$

Conversely, if u solves this equation with $\sup_{[0,T]\times\Sigma}|u|<d$, then $X(t)$ having the parametrization $s+uN$, evolves by (2.2).

To show the local existence for the Cauchy problem of (3.2), we now introduce the space of Hölder continuous functions. For an n-dimensional, closed, differential manifold M with a smooth atlas $(\Omega_k,\varphi_k)_{k\in\Lambda}$, let us denote the local coordinate of the point in Ω_k by $\xi=(\xi^1,\cdots,\xi^n)\in\varphi_k(\Omega_k)$ (often, the index k is omitted). and D_if denotes the first order derivative of f w.r.t. ξ^i-coordinate, and also $D=(D_1f,\cdots,D_nf)$, $D_{ij}f=D_j(D_if)$, $D^2=(D_{ij})$. For $a<b$, we set

$$C^{1,2}([a,b]\times M)=\{f\in C([a,b]\times M)\,;\,\partial_t f,\,D_{ij}f\in C([a,b]\times\varphi_k(\Omega_k))\ \forall i,j,\ \forall k\in\Lambda\},$$

where

$$||f||_{C^{1,2}}=||f||_\infty+||\partial_t f||_\infty+\sum_{i=1}^n||D_if||_\infty+\sum_{i=1}^n||D_{ij}f||_\infty.$$

For $0<\alpha<1$, we set

$$C^{\alpha/2,\alpha}([a,b]\times M)=\{f\in C([a,b]\times M)\,;\,f(\cdot,\xi)\in C^{\alpha/2}([a,b])\ \forall\xi\in M,$$
$$f(t,\cdot)\in C^{\alpha}(\overline{\varphi_k(\Omega_k)})\ \forall t\in[a,b],\ \ \forall k\in\Lambda,\ ||f||_{C^{\alpha/2,\alpha}([a,b]\times M)}<\infty\},$$

where

$$||f||_{C^{\alpha/2,\alpha}([a,b]\times M)}=||f||_\infty+[f]_{C^{\alpha/2,0}([a,b]\times M)}+[f]_{C^{0,\alpha}([a,b]\times M)},$$
$$[f]_{C^{\alpha/2,0}([a,b]\times M)}=\max_{\xi\in M}\sup_{t,t'\in[a,b],\,t'<t}\frac{|f(t,\xi)-f(t',\xi)|}{(t-t')^{\alpha/2}},$$
$$[f]_{C^{0,\alpha}([a,b]\times M)}=\max_{t\in[a,b],k\in\Lambda}\sup_{\xi,\,\xi'\in\overline{\varphi_k(\Omega_k)},\,\xi\neq\xi'}\frac{|f(t,\xi)-f(t,\xi')|}{|\xi-\xi'|^{\alpha}}.$$

Now we define the parabolic Hölder space $C^{1+\alpha/2,2+\alpha}([a,b]\times M)$ as follows:

$$C^{1+\alpha/2,2+\alpha}([a,b]\times M)=\{f\in C^{1,2}([a,b]\times M)\,;$$

$$\partial_t f,\,D_{ij}f\in C^{\alpha/2,\alpha}([a,b]\times\overline{\varphi_k(\Omega_k)})\,\forall i,j,\forall k\in\Lambda\},$$

where

$$||f||_{C^{1+\alpha/2,2+\alpha}([a,b]\times M)}=||f||_\infty+||\partial_t f||_{C^{\alpha/2,\alpha}}+\sum_{i=1}^n||D_if||_\infty+\sum_{i,j=1}^n||D_{ij}f||_{C^{\alpha/2,\alpha}}.$$

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We will use the following existence and optimal regularity result in class $C^{\alpha/2,\alpha}$ (see Lunardi [9]). Note that such results still hold on a general, closed manifold M.

Theorem 3.2 *Let M be a sufficiently smooth, closed manifold of dimension n, and let a_{ij}, b_i, $c, f \in C^{\alpha/2,\alpha}([0,T]\times M)$. Suppose that a_{ij} satisfies the uniform ellipticity condition:*

$$\sum_{i,j=1}^{n} a_{ij}(t,\xi)\eta^i\eta^j \geqslant \nu|\eta|^2, \quad 0 \leqslant t \leqslant T,\ \xi \in M,\ \eta \in \mathbb{R}^n,$$

for some $\nu > 0$. Set

$$\mathscr{A}(t,\xi) = \sum_{i,j=1}^{n} a_{ij}(t,\xi)D_{ij} + \sum_{i=1}^{n} b_i(t,\xi)D_i + c(t,\xi),$$

and consider the Cauchy problem:

$$\partial_t u = \mathscr{A}u + f, \quad u(0) = u_0. \tag{3.3}$$

If $u_0 \in C^{2+\alpha}(M)$, then there is a unique solution u of (3.3) in $C^{1+\alpha/2,2+\alpha}([0,T]\times M)$ such that

$$||u||_{C^{1+\alpha/2,2+\alpha}} \leqslant C(||u_0||_{C^\alpha} + ||f||_{C^{\alpha/2,\alpha}}).$$

Especially, if coefficients a_{ij}, b_i, c are time-independent in $C^\alpha(M)$, then for every $u_0 \in C^\alpha(M)$ there is a unique mild solution u of (3.3) in $C^{\alpha,2\alpha}([0,T]\times M)\cap C^{1+\alpha/2,2+\alpha}((0,T]\times M)$ such that

$$||u||_{C^{\alpha,2\alpha}} \leqslant C(||u_0||_{C^{2\alpha}} + ||f||_{C^{\alpha/2,\alpha}}),$$

where C is depending only on $\max_{i,j=1} ||a_{ij}||_{C^\alpha}$, $\max_{i=1} ||b_i||_{C^\alpha}$, and $||c||_{C^\alpha}$.

Proof of Theorem 3.1 We shall start to rewrite equation (3.2) in local coordinates ξ on the reference hypersurface Σ with $g = g_\Sigma$. For the principal term in (3.2), define

$$h(u,Du,D^2u) := \frac{\sqrt{(u+1)^2+|\nabla u|^2}}{(u+1)^2}\operatorname{div}\left(\frac{\nabla u}{\sqrt{(u+1)^2+|\nabla u|^2}}\right) - \frac{n}{u+1}.$$

Set

$$\mathscr{G}(u_0) = G\big(|\tilde{\nabla}(e^{\tau\tilde{\Delta}}H_0)|^2\big),$$

and put the mapping

$$\mathscr{F} : u \mapsto \mathscr{F}(u) := -K\left[\int_{\mathscr{R}} \chi(\sigma[u]\,\sigma[u_0])\,dx\right]^{-\frac{1}{n+1}} \frac{\sqrt{(u+1)^2+|\nabla u|^2}}{u+1}\beta^\epsilon(u-u_0).$$

Then we obtain the following evolution equation, corresponding to (3.2)

$$\partial_t u = \mathscr{G}(u_0)h(u,Du,D^2u) + \mathscr{F}(u). \tag{3.4}$$

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We shall look for the solution to (3.4) by applying the contraction mapping principle in the following space:

$$Y(T) = \{u \in C^{1+\alpha/2,2+\alpha}([0,T]\times\Sigma)\,;\, u(0) = u_0,$$
$$||u_0||_{C^{2+\alpha}} \leqslant \varepsilon_0,\ ||u-u_0||_{C^{1+\alpha/2,2+\alpha}} \leqslant R,\ u \geqslant -2/3\}.$$

Note that $(Y(T), d)$ is a complete metric space with the distance $d(u,v) = ||u-v||_{C^{1+\alpha/2,2+\alpha}}$.

Suppose that $\mathscr{A}_0$ is the linearization of $h = h(u,p,q)$ at the point (u_0, Du_0, D^2u_0), i.e.,

$$\mathscr{A}_0 = \sum_{i,j=1}^{n} h_{q_{ij}}(u_0, Du_0, D^2u_0)D_{ij} + \sum_{i=1}^{n} h_{p_i}(u_0, Du_0, D^2u_0)D_i + h_u(u_0, Du_0, D^2u_0).$$

Now we define the operator Γ by $\Gamma(u) = w$, where w is the solution of the linear problem

$$\begin{cases} \partial_t w = \mathscr{G}(u_0).\mathscr{A}_0 w + \mathscr{G}(u_0)h(u, Du, D^2u) + \mathscr{F}(u) - \mathscr{G}(u_0).\mathscr{A}_0 u, \quad 0 \leqslant t \leqslant T, \\ w(0) = u_0. \end{cases} \tag{3.5}$$

Here, let us see that Γ is well-defined on $Y(T)$. Concerning to the coefficient $\mathscr{G}(u_0)$, $e^{\tau\tilde{\Delta}}H_0 =: v(\tau,\xi)$ is the solution of the Cauchy problem of the linear heat equation:

$$\partial_\tau v = \tilde{\Delta} v, \quad v(0) = H_0,$$

and let $\tilde{a}_{ij}(u_0)$ and $\tilde{b}_i(u_0)$ be the coefficients determined by

$$\begin{aligned} \tilde{\Delta} v &= \frac{1}{(u_0+1)^2}\left(\Delta_\Sigma v - \frac{u_{0,i}u_{0,j}}{(u_0+1)^2 + |\nabla u_0|^2} v_{,ij} - \Gamma^k_{ij}(u_0)v_{,k}\right) \\ &=: \sum_{i,j=1}^{n} \tilde{a}_{ij}(u_0)D_{ij}v + \sum_{i=1}^{n} \tilde{b}_i(u_0)D_i v. \end{aligned}$$

One see that $\tilde{a}_{ij}(u_0)$ satisfies a uniformly elliptic condition, and $\tilde{a}_{ij}(u_0)$, $\tilde{b}_{ij}(u_0)$ and the mean curvature H_0 are contained in $C^\alpha(\Sigma)$. Hence, by Theorem 3.2 there is a unique mild solution $v(\tau)$ in $C^{2+\alpha}(\Sigma)$, and for some $C = C(||u_0||_{C^{2+\alpha}}) > 0$,

$$||\tilde{\nabla}(e^{\tau\tilde{\Delta}}H_0)||_{C^{1+\alpha}} \leqslant C.$$

Then we have $\mathscr{G}(u_0) \in C^\alpha(\Sigma)$ such that

$$||\mathscr{G}(u_0)||_{1+C^\alpha} \leqslant C$$

for $C = C(||u_0||_{C^{2+\alpha}}) > 0$, and there is $\nu = \nu(||u_0||_{C^{2+\alpha}}) > 0$ such that

$$\mathscr{G}(u_0) \geqslant \nu > 0.$$

By virtue of Theorem 3.2, for every $u \in Y(T)$, there is a unique solution $w = \Gamma(u)$ to problem (3.5), belonging to $C^{1+\alpha/2,2+\alpha}$.

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We first show that if T is small enough (depending on R), Γ is a contraction, that is, for every $u, v \in Y(T)$,

$$||\Gamma(u) - \Gamma(v)||_{C^{1+\alpha/2,2+\alpha}} \leqslant C(R)T^{\alpha/2}||u - v||_{C^{1+\alpha/2,2+\alpha}}.$$

Let $w = \Gamma(u) - \Gamma(v)$. w satisfies

$$\begin{cases} \partial_t w = \mathscr{G}(u_0)\mathscr{A}_0 w + \psi, & 0 \leqslant t \leqslant T, \\ w(0) = 0, \end{cases} \tag{3.6}$$

where we set

$$\begin{cases} \psi(t,\xi) = \mathscr{G}(u_0)(\xi)\big(\tilde{h}[u(t,\xi)] - \tilde{h}[v(t,\xi)] - \mathscr{A}_0(u(t,\xi) - v(t,\xi))\big) + \mathscr{F}(u)(t,\xi) - \mathscr{F}(v)(t,\xi), \\ \tilde{h}[u(t,\xi)] = h(u(t,\xi), Du(t,\xi), D^2u(t,\xi)). \end{cases}$$

By Theorem 3.2, there is a $C > 0$ such that

$$||w||_{C^{1+\alpha/2,2+\alpha}} \leqslant C||\psi||_{C^{\alpha/2,\alpha}}.$$

So, we must estimate $||\psi||_{C^{\alpha/2,\alpha}}$. Set $\psi = \mathscr{G}(u_0)\psi_1 + \psi_2$, that is,

$$\psi_1 = \tilde{h}[u] - \tilde{h}[v] - \mathscr{A}_0(u - v), \quad \psi_2 = \mathscr{F}(u) - \mathscr{F}(v).$$

To estimate $||\psi_1||_{C^{\alpha/2,\alpha}}$, we rewrite ψ_1 as

$$\begin{aligned} \psi_1 = &\int_0^1 (\tilde{h}_u[\theta u + (1-\theta)v] - \tilde{h}_u[u_0])(u - v)\, d\theta \\ &+ \int_0^1 \sum_{i=1}^n (\tilde{h}_{p_i}[\theta u + (1-\theta)v] - \tilde{h}_{p_i}[u_0])D_i(u - v)\, d\theta \\ &+ \int_0^1 \sum_{i,j=1}^n (\tilde{h}_{q_{ij}}[\theta u + (1-\theta)v] - \tilde{h}_{q_{ij}}[u_0])D_{ij}(u - v)\, d\theta. \end{aligned}$$

One can see straightforwardly that $\tilde{h} = h(u,p,q)$ is smooth and the first derivatives are globally Lipschitz continuous on $[-2/3,\infty) \times \mathbb{R}^n \times \mathbb{R}^{n^2}$, i.e. for some $L > 0$,

$$\begin{aligned} |h_u(u,p,q) - h_u(u',p',q')| + |h_{p_i}(u,p,q) - h_u p_i(u',p',q')| + |h_{q_{ij}}(u,p,q) - h_{q_{ij}}(u',p',q')| \\ \leqslant L(|u - u'| + |p - p'| + |q - q'|). \end{aligned}$$

Then, the way of the C^α-estimate in Lunardi [9, Theorem 8.5.4] is available for $||\psi_1||_{C^{\alpha/2,\alpha}}$. Hence we obtain

$$||\psi_1||_{C^{\alpha/2,\alpha}} \leqslant C T^{\alpha/2}||u - v||_{C^{1+\alpha/2,2+\alpha}}. \tag{3.7}$$

To estimate $||\psi_2||_{C^{\alpha/2,\alpha}}$, we let

$$\tilde{f}[u] = f(u, Du) := \frac{\sqrt{(u+1)^2 + |\nabla u|^2}}{u+1}.$$

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The function $\tilde{f} = f(u,p)$ is smooth on $[-2/3,\infty)\times\mathbb{R}^n$, on which there is a $L'>0$, independent of u,p such that

$$|f_u(u,p)-f_u(u',p')|+\sum_{i=1}^{n}|f_{p_i}(u,p)-f_{p_i}(u',p')|\leqslant L'(|u-u'|+|p-p'|).$$

Moreover, we set the mapping

$$\mathscr{Q}(u):=-K\beta^{\varepsilon}(u-u_0)\left[\int_{\mathscr{R}}\chi(\sigma[u]\,\sigma[u_0])\,dx\right]^{-\frac{1}{n+1}}.$$

That is, $\mathscr{F}(u)=\mathscr{Q}(u)\tilde{f}[u]$. Note that and $\mathscr{Q}$ is Fréchet differentiable from $C^{1+\alpha/2,2+\alpha}$ into itself, whence we denote the derivative at u by $\mathscr{Q}_u[u]$. Then we obtain the following formula

$$\mathscr{Q}_u[u]\phi=\mathscr{Q}_1[u]\int_{X_\phi}\frac{\sigma[u_0]}{\tilde{f}[\phi]}dA_\phi+\mathscr{Q}_2[u]\phi,$$

for any $\phi\in C^{1+/\alpha/2,2+\alpha}$, where

$$\mathscr{Q}_1(u)=\frac{K\beta^{\varepsilon}(u-u_0)}{n+1}\left[\int_{\mathscr{R}}\chi(\sigma[u]\,\sigma[u_0])\,dx\right]^{-\frac{n+2}{n+1}},$$

$$\mathscr{Q}_2(u)=-K\beta^{\varepsilon}_u(u-u_0)\left[\int_{\mathscr{R}}\chi(\sigma[u]\,\sigma[u_0])\,dx\right]^{-\frac{1}{n+1}},$$

and X_ϕ corresponds to the hypersurface with a graph ϕ over Σ, dA_ϕ is the area element on X_θ. Similarly, by computing the Fréchet derivative of $\mathscr{Q}_u[u]u$ straightforwardly, one can see that there is a $L''>0$, depending on u_0 such that

$$\left|\mathscr{Q}_1[u]\int_{X_u}\frac{\sigma[u_0]}{\tilde{f}[u]}dA_u-\mathscr{Q}_1[v]\int_{X_v}\frac{\sigma[u_0]}{\tilde{f}[v]}dA_v\right|+|\mathscr{Q}_2[u]-\mathscr{Q}_2[v]|\leqslant L''|u-v| \tag{3.8}$$

for any $u,v\in C^{1+/\alpha/2,2+\alpha}$ with $u,v\geqslant -2/3$. Now, we have

$$\begin{aligned}\psi_2=\mathscr{F}(u)-\mathscr{F}(v)&=\int_0^1\frac{\partial}{\partial\theta}\mathscr{F}[\theta u+(1-\theta)v]d\theta\\&=\int_0^1\mathscr{Q}_1(\theta u+(1-\theta)v)\left(\int_{X_{\theta u+(1-\theta)v}}\frac{\sigma[u_0]}{\tilde{f}[\theta u+(1-\theta)v]}dA_{\theta u+(1-\theta)v}\right)\tilde{f}[\theta u+(1-\theta)v](u-v)\,d\theta\\&\quad+\int_0^1\mathscr{Q}_2(\theta u+(1-\theta)v)\left(\tilde{f}_u[\theta u+(1-\theta)v](u-v)+\sum_{i=1}^{n}\tilde{f}_{p_i}[\theta u+(1-\theta)v]D_i(u-v)\right)d\theta.\end{aligned}$$

Thanks to (3.8) and the Lipschitz continuity of $\tilde{f}$, similarly as in the case of $||\psi_1||_{C^{\alpha/2,\alpha}}$, we find a $C>0$, depending on R and $||u_0||_{C^{1+\alpha/2,2+\alpha}}$ such that

$$||\psi_2||_{C^{\alpha/2,\alpha}}\leqslant CT^{(1+\alpha)/2}||u-v||_{C^{1+\alpha/2,2+\alpha}}. \tag{3.9}$$

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Hence, there is a $C > 0$ such that

$$||w||_{C^{1+\alpha/2,2+\alpha}} \leqslant C T^{\alpha/2} ||u - v||_{C^{1+\alpha/2,2+\alpha}},$$

and, if T is small enough, Γ is a 1/2-contraction:

$$||\Gamma(u) - \Gamma(v)||_{C^{1+\alpha/2,2+\alpha}} \leqslant 1/2 ||u - v||_{C^{1+\alpha/2,2+\alpha}}.$$

On the other hand, we shall see that Γ is a mapping $Y(T)$ into itself, if R is suitably large. For every $u_0 \in C^{2+\alpha}(\Sigma)$ with $u_0 \geqslant -2/3$,

$$||\Gamma(u) - u_0||_{C^{1+\alpha/2,2+\alpha}} \leqslant R/2 + ||\Gamma(u_0) - u_0||_{C^{1+\alpha/2,2+\alpha}}.$$

Let $v = \Gamma(u_0) - u_0$, and then v satisfies

$$\begin{cases} \partial_t v = \mathscr{G}(u_0)\mathscr{A}_0 v + \mathscr{G}(v)h(u_0, Du_0, D^2u_0) + \mathscr{F}(u_0) - \mathscr{G}(u_0)\mathscr{A}_0 u_0, & 0 \leqslant t \leqslant T, \\ v(0) = 0. \end{cases}$$

By Theorem 3.2, there is a $C > 0$, depending on $||u_0||_{C^{1+\alpha/2,2+\alpha}}$ such that

$$||v||_{C^{1+\alpha/2,2+\alpha}} \leqslant C(||\mathscr{G}(v)||_{C^{\alpha/2,\alpha}} ||h(u_0, Du_0, D^2u_0)||_{C^{\alpha/2,\alpha}} + ||\mathscr{F}(u_0)||).$$

Since $v(0) = 0$, similarly arguing, there is a $C > 0$ (depending on $||v||_{C^{1+\alpha/2,2+\alpha}}$ and $\sup_{s \geqslant 0} |G'(s)|$) such that

$$||\mathscr{G}(v)||_{C^{\alpha/2,\alpha}} \leqslant C T.$$

Once again, by choosing T small enough (depending on $||u_0||_{C^{1+\alpha/2,2+\alpha}}$), then we have

$$||v||_{C^{1+\alpha/2,2+\alpha}} \leqslant C||\mathscr{F}(u_0)|| =: C',$$

where C' is independent of T and R. Thus, we obtain

$$||\Gamma(u) - u_0||_{C^{1+\alpha/2,2+\alpha}} \leqslant R/2 + C'.$$

We find a small $T > 0$ again such that $\Gamma(u) \geqslant -2/3$ on $[0, T] \times \Sigma$ from the continuity. Then it follows that Γ is a contraction mapping on $Y(T)$ for R is so large ($R > 2C'$), and hence, Γ has a unique fixed point in $Y(T)$.

As for uniqueness, let u, v be solutions with the same initial data u_0. Then $w = u - v$ satisfies

$$\begin{cases} \partial_t w = \mathscr{G}(u_0) \left(\sum_{i,j=1}^{n} a_{ij} D_{ij} w + \sum_{i=1}^{n} b_i D_i w + cw \right), & 0 \leqslant t \leqslant T, \\ w(0) = 0, \end{cases}$$

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where

$$a_{ij} = \int_0^1 \sum_{i,j=1}^{n} \tilde{h}_{q_{ij}}[\theta u + (1-\theta)v]\, d\theta,$$

$$b_i = \int_0^1 \sum_{i=1}^{n} (\tilde{h}_{p_i} + \tilde{f}_{p_i})[\theta u + (1-\theta)v]\, d\theta,$$

and

$$c = \int_0^1 \left(\tilde{h}_u + \int_{X_\theta} \frac{\sigma[u_0]}{\tilde{f}[\theta u + (1-\theta)v]}\, dA_\theta \mathcal{Q}_1 \tilde{f} + \mathcal{Q}_2 \tilde{f}_u \right) [\theta u + (1-\theta)v] d\theta.$$

Since a_{ij} satisfies the uniformly elliptic condition on $Y(T)$, the standard contradiction argument of the maximum principle yields $w \equiv 0$, i.e., uniqueness holds. Finally, we get the C^∞-regularity of $u(t)$ for $t > 0$ from the bootstrap argument. Hence, we complete the proof of Theorem 3.1. □

We should notice that especially if $G \equiv 1$, equation (3.2) satisfies the comparison principle, from which it follows that if X_0 is so large, (where the meaning of "large" is determined by not only X_0 but also K and d), the one-point shrinking phenomenon does not occur in contrast with the usual mean curvature flow ($K = 0$).

Theorem 3.3 *Let $G \equiv 1$. Suppose that there is a solution to $f(R) = 0$ as in (2.4), and let us denote the minimal one by $\underline{R}$. If X_0 contains in a sphere of radius $\underline{R}$, then the solution $X(t)$ remains surrounding the sphere.*

4 Experimental results

In this last section, we shall show numerical experiments in the case when 2-D image processing. For simplicity, we put $G \equiv 1$ and $\epsilon = d$ in (2.3), whence $\beta^d(r)$ is a step function.

Suppose that X_0 is a quite complicated or noisy initial hypersurface. Then we must take regularization width d in (2.3) small enough for the consistency. However, this restriction is not so pleased for the purpose of denoising, and moreover the computation of $\mathrm{dist}(x, X_0)$ is very hard in practice. Thus, in order to avoid these unfavorable difficulties let us replace by the another (considerably rough) approximation:

$$\sigma^\epsilon_{X_0}(X) = \beta^\epsilon(\sigma_{X_0}(X)|X - X_0|). \tag{4.1}$$

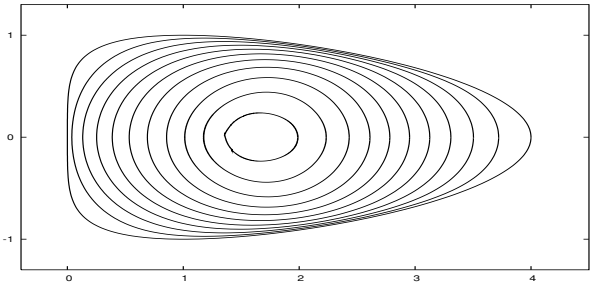


FIG. 1. Evolution when $K = 0$ (curve shortening flow).

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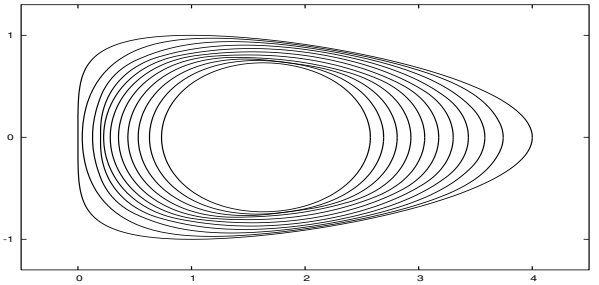


FIG. 2. Evolution by (2.2) with $K = 1$ and $\epsilon = 0.2$.

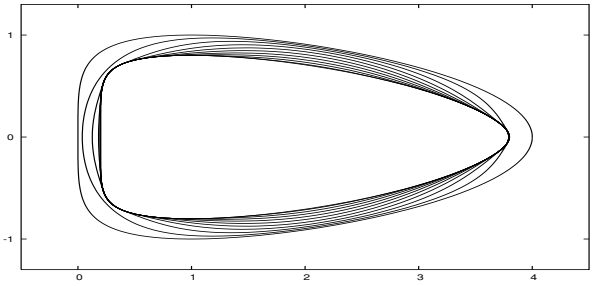


FIG. 3. Evolution by (2.2) with $K = 10$ and $\epsilon = 0.2$.

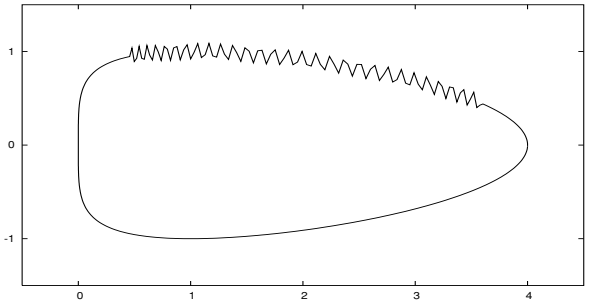


FIG. 4. Original image curve with an anisotropic noise.

For comparison we exhibit numerical simulations by varying parameter $K > 0$ and the vanishing case $K = 0$, that is, the so-called curve shortening flow for the same original image and time-increment.

The method of the discretization for our mean curvature flow model (2.2) with specific regularization (4.1) should be precisely manifested, but the details will be discussed elsewhere.

In Figure 3, it can be seen that the evolved curve by (2.2) does not shrink (cf. Theorem 3.3). In the region such that the distance from the initial curve is less than or equal to $\epsilon = 0.2$, the second integral term on the RHS of (2.2) does not work, that is, the curve shortening flow. On the other hand, on the exterior region the repulsive

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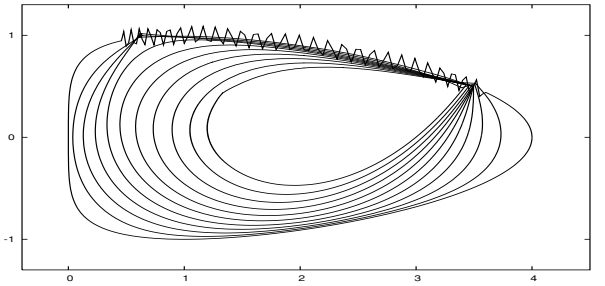


FIG. 5. Evolution by (2.2) when $K = 0$.

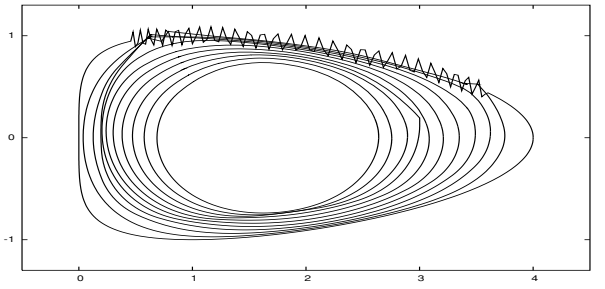


FIG. 6. Evolution by (2.2) with $K = 1$ and $\epsilon = 0.2$.

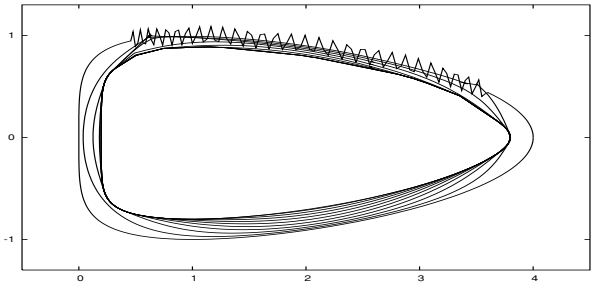


FIG. 7. Evolution by (2.2) with $K = 10$ and $\epsilon = 0.2$.

effect on account of the term clearly appears since K is large enough, and thus the curve asymptotically converges to the inner curve (probably stable equilibrium).

In Figure 5, the moving speed of the upper side is slower relative to the lower side due to the noise. As a result, a cusp on the upper right appears.

5 Conclusion

The surface energy $E(X; X_0)$ in (1.4) associated to a given initial surface X_0 was introduced, and then we obtain a mean curvature evolution equation with nonlocal nonlinearities as a the negative gradient flow. It have shown that the Cauchy problem is locally wellposed on a family of closed, embedded hypersurface of class $C^{2+\alpha}$. Furthermore, we perform the

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numerical simulations, which show that the repulsive effect of the second term of RHS in (2.2) plays an important role in preserving its geometric shape features while smoothing tiny ones under parameters K and ϵ chosen suitably.

Acknowledgements

The author was supported by Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists.

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