We can rewrite (1) as

$$\cos^2(\theta + \alpha) - \cos^2\theta = \cos^2\alpha$$

which, using the double-angle formula, gives

$$\tfrac{1}{2}\left[\cos(2\theta + 2\alpha) + 1 - \cos 2\theta - 1\right] = \cos^2\alpha$$

so that

$$-\sin(2\theta + \alpha)\sin\alpha = \cos^2\alpha$$

For fixed α, there are solutions for θ provided that $|\sin\alpha| \geqslant \cos^2\alpha$ or $\sin^2\alpha + |\sin\alpha| - 1 \geqslant 0$. This solves to give $|\sin\alpha| \geqslant \dfrac{1}{\phi}$ where $\phi = \dfrac{\sqrt{5}+1}{2}$ is the golden ratio: hence our title! The extreme case $\alpha = \sin^{-1}\dfrac{1}{\phi} \approx 38.2°$ corresponds to the smallest angle in the right-angled triangle with sides in the ratio $1 : \sqrt{\phi} : \phi$. This is sometimes known as the Kepler triangle in the literature and Kepler's construction of it is shown in Figure 1. With the diameter of the semicircle divided in golden ratio, all three of the right-angled triangles shown are Kepler triangles.

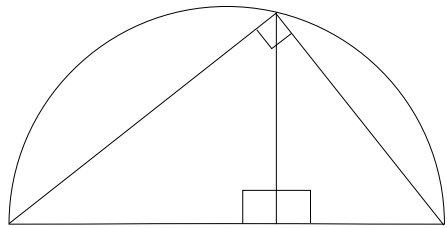


FIGURE 1

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101.05 Integration by symmetry

Introduction

In this article we present ideas that make integration understandable on the basis of elementary geometrical insights. We show how symmetry considerations can help to calculate areas under the graphs of power functions.

Integration can be seen as the calculation of the area under graphs of functions. Knowing how to calculate areas of rectangles or other polygons is not sufficient in order to handle curved shapes like areas under polynomial functions. But sometimes elementary geometrical transformations may give the figures in the problem a new shape which may be more easily accessible, sometimes even by elementary tools. Here we present an approach, where both rotational and reflection symmetry play the role of the catalyst which makes the task of integration surmountable.

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Power functions

In this Note we study the integral of power functions. We start with the parabola $y = x^2$.

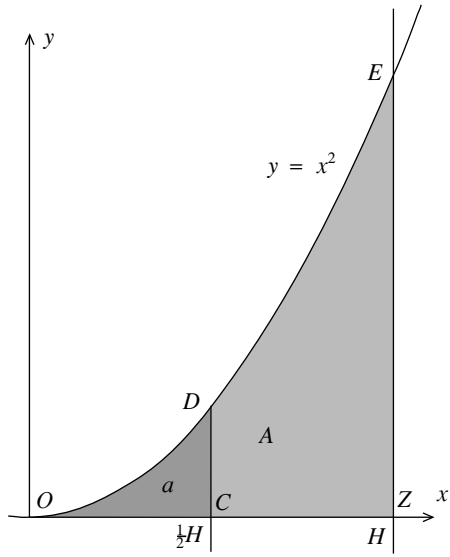


FIGURE 1

We are interested in the area A under the function over the interval $[0, H]$. The 'corners' of the area are denoted by OZE as in Figure 1. We start with the smaller area a over the interval $[0, \frac{1}{2}H]$ and try to find a connection with the area A over the whole interval $[0, H]$. First we dilate the small area a in the x-direction by multiplying every x-value of any point in the area by the factor 2 and get $\bar{a} = 2a$ for the new area.

Now we dilate the new area $\bar{a}$ along the y-direction using the magnifying factor 4. The new function $h(x)$ satisfies $h(2x) = 4x^2 = (2x)^2$ that is $h(z) = z^2$. Thus the new function coincides with the old one $f(x) = x^2$. Therefore $A = 4\bar{a} = 8a$ and we have found a connection between the areas over the intervals $[0, \frac{1}{2}H]$ and $[0, H]$.

Now symmetry enters the scene. We reflect the graph $y = x^2$ about the line $x = \frac{1}{2}H$ and get $y = (H - x)^2$. The difference area G between $y = x^2$ and $y = (H - x)^2$ over the interval $[\frac{1}{2}H, H]$ is easy to calculate since the difference function $x^2 - (H - x)^2 = 2Hx - H^2$ is linear. Now $y = 2Hx - H^2$ is the equation of the straight line CE, so G equals the area of the triangle CZE. Thus $G = \dfrac{1}{2} \times \dfrac{H}{2} \times H^2 = \dfrac{H^3}{4}$.

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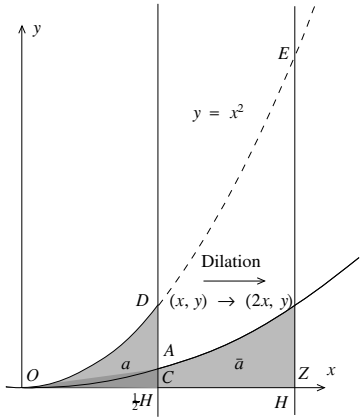


FIGURE 2

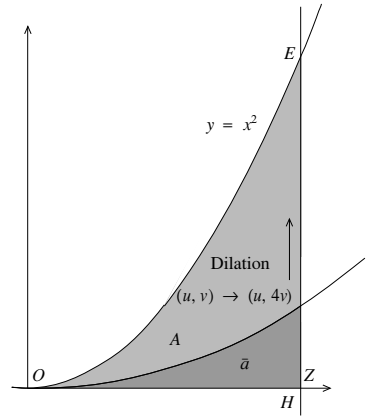


FIGURE 3

Now, by symmetry, we see from Figure 4 that A consists of three smaller shapes, the first one having area G and two remaining shapes each having area a. Thus we arrive at $A = G + 2a$. Using what we know about G and a we get $A = \frac{1}{4}H^3 + \frac{1}{4}A$ or $A = \frac{1}{3}H^3$. This is the result we get from integral calculus.

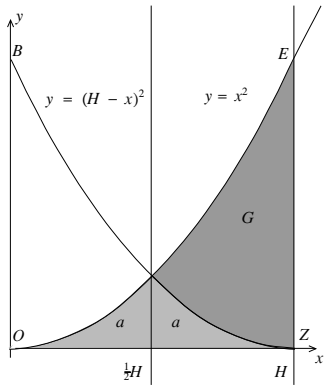


FIGURE 4

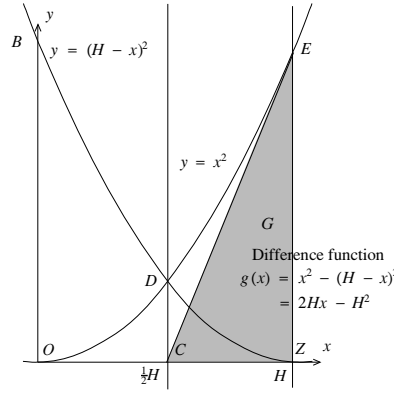


FIGURE 5

Now we move to the cubic. This time we use a different trick or let's say a different symmetry. We divide the desired area at $x = \frac{1}{2}H$ into two 'halves' and rotate the right one through 180° about C and glue it to the first part from below.

Of course, the area of the figure is unchanged by this process. Nevertheless, something dramatic happens. If we want to find the area of the second shape in the picture below (Figure 6) we have to study the 'difference' function $x^3 - (x - H)^3$. The surprise is that this function is no longer cubic but quadratic

$$x^3 - (x - H)^3 = H^3 - 3H^2x + 3Hx^2.$$

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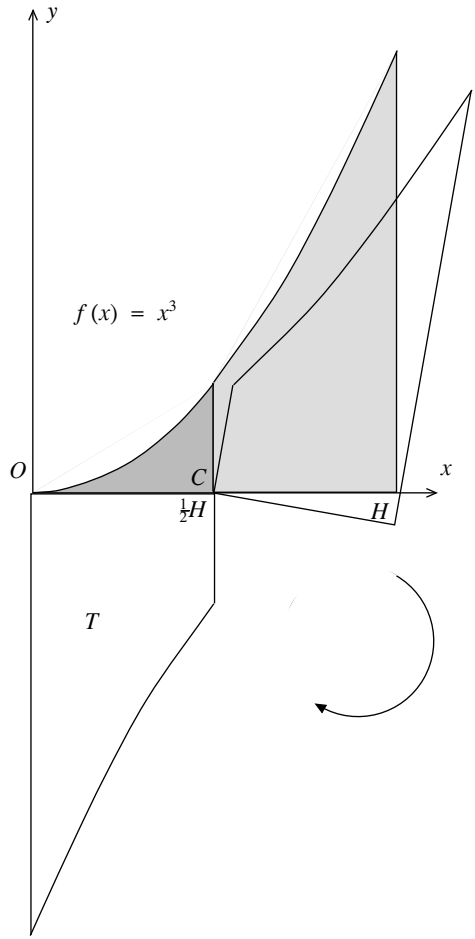


FIGURE 6

We have already learned how to find the area under quadratic functions and for the area T of the new shape over the interval $\left[0, \frac{1}{2}H\right]$ we get the following

$$T = H^3\left(\frac{H}{2}\right) - 3H^2 \times \frac{1}{2}\left(\frac{H}{2}\right)^2 + 3H \times \frac{1}{3}\left(\frac{H}{2}\right)^3 = \frac{H^4}{4}.$$

Again this result coincides with the result from integral calculus.

In the case of the power function $y = x^4$ we return to the method for the quadratic case. Dilating the area over $\left[0, \frac{1}{2}H\right]$ by the factor 2 in x-direction and then dilating the new area in y-direction by the factor $2^4 = 16$ yields $A = 32a$. After reflecting $f(x) = x^4$ we have to study the difference function

$$x^4 - (H - x)^4 = -H^4 + 4H^3x - 6H^2x^2 + 4Hx^3.$$

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This function again is of lower degree than the original quartic and we can calculate the difference area G using integration rules that we derived earlier in the article. We get

$$G = -H^4\left(H - \frac{H}{2}\right) + 4H^3\left(\frac{H^2 - \left(\frac{H}{2}\right)^2}{2}\right) - 6H^2\left(\frac{H^3 - \left(\frac{H}{2}\right)^3}{3}\right) + 4H\left(\frac{H^4 - \left(\frac{H}{2}\right)^4}{4}\right) = \frac{3H^5}{16}.$$

By reflection symmetry, we find again $A = G + 2a$. Using what we know about G and a we get this time $A = \frac{3H^5}{16} + \frac{A}{16}$ or $A = \frac{H^5}{5}$ as we expected from integral calculus.

Now we can go on and find new formulae for increasing powers using the different methods for odd and even powers. For even powers we have to use the argument using dilations and reflection symmetry whilst for odd powers we use the rotation trick and thus reduce the degree of the power function by one.

In the two articles [1] and [2] Rainer Kaenders presents an approach where affine transformations play the role of symmetry in this article. Also there a new road to integration is opened. For another approach to integration see Kirfel [3].

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101.06 Finding some integrals using an interesting self-similar substitution

After recently completing a section of work on techniques of integration I was asked by one of my students if the following integral could be found using a substitution other than a trigonometric (or hyperbolic) substitution:

$$\int \frac{dx}{(1 - x)\sqrt{1 - x^2}}.$$

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