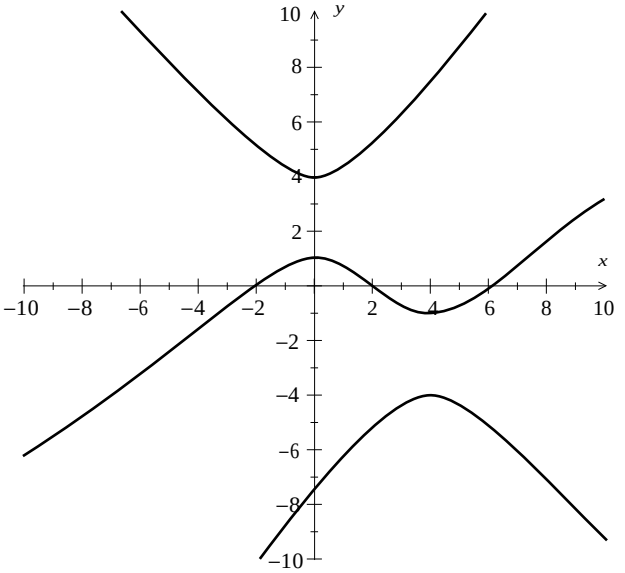


FIGURE 2

References

1. C. Foster, Symmetrical cubics, *Math. Gaz.* **90,** (March 2012) pp. 109-112.
2. Massachusetts Institute of Technology Open Courseware: http://ocw.mit.edu/courses/mathematics/

   Undergraduate 18.704 Seminar in Algebra and Number Theory: Rational Points on Elliptic Curves.

doi: 10.1017/mag.2014.14

SAMANTHA CAESAR
Keene State College, Keene, NH, USA
e-mail: *samantha.trudie.10@gmail.com*

99.02 Series involving fixed-column decimal representations

Introduction

As an end-of-term piece of homework, I set the students in one of my senior classes the task of devising their own question, along with its solution, related to at least one of the mathematical topics we had covered in the previous few months. My intention was to give the students the opportunity to tackle each other's questions. One that immediately caught my eye went, with a little bit of tidying up, as follows:

> Let a_0, a_1, a_2, a_3, … be the increasing and infinite sequence of all positive integers such that when each of these numbers is written in decimal notation, there is an 8 in the hundreds column. Work out the sum of the infinite series
>
> $$\frac{a_0}{2^0} + \frac{a_1}{2^1} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \ \ldots\ .$$

https://doi.org/10.1017/mag.2014.15 Published online by Cambridge University Press

Unfortunately, despite a valiant attempt, the student who came up with this question failed to provide a correct solution; it seems he had slightly underestimated its difficulty.

It occurred to me though that the evaluation of the above sum would make an interesting application of generating functions. Furthermore, this approach would readily allow us to obtain a solution to the corresponding generalised problem in which the increasing and infinite sequence of positive integers is such that when each of these numbers is written in decimal notation, there is an m in the nth column. It is shown here how these results may be derived.

A generating function

Let $\mathcal{S}$ denote the increasing and infinite sequence of all positive integers such that when each of these numbers is written in decimal notation, there is an 8 in the hundreds column, giving

$$\mathcal{S} = (800,\ 801,\ 802, \ldots,\ 899,\ 1800,\ 1801,\ 1802,\ \ldots,\ 1899,\ 2800, \ldots).$$

We require the sum of the series

$$\sum_{k=0}^{\infty} \frac{a_k}{2^k},$$

where $a_0 = 800$, $a_1 = 801$, $a_2 = 802$, …, $a_{99} = 899$, $a_{100} = 1800$, $a_{101} = 1801$, …, and will evaluate this via $F(x)$, which is given by

$$\begin{aligned} F(x) &= 800 + 801x + 802x^2 + \ldots + 899x^{99} + 1800x^{100} + 1801x^{101} \ldots \\ &= \sum_{k=0}^{\infty} a_k x^k, \end{aligned}$$

the ordinary generating function for $\mathcal{S}$, noting that its radius of convergence is 1.

First, we rewrite $F(x)$ as

$$\begin{aligned} F(x) = 1 &+ 2x + 3x^2 + 4x^3 + \ldots \\ &+ 799\left(1 + x + x^2 + \ldots + x^{99}\right) \\ &+ 1699\left(x^{100} + x^{101} + x^{102} + \ldots + x^{199}\right) \\ &+ 2599\left(x^{200} + x^{201} + x^{202} + \ldots + x^{299}\right) \\ &+ \ldots \\ = 1 + 2x + 3x^2 + 4x^3 + \ldots &+ \sum_{k=1}^{\infty}\left[(900k - 101)\sum_{j=0}^{99} x^{100(k-1)+j}\right]. \end{aligned}$$

Next, on using the formula for the sum to infinity of a geometric progression [1], we have

$$x + x^2 + x^3 + x^4 + \ldots = \frac{x}{1 - x}.$$

https://doi.org/10.1017/mag.2014.15 Published online by Cambridge University Press

Since also

$$\frac{d}{dx}\left(x + x^2 + x^3 + x^4 + \ldots\right) = 1 + 2x + 3x^2 + 4x^3 + \ldots,$$

it follows that

$$1 + 2x + 3x^2 + 4x^3 + \ldots = \frac{d}{dx}\left(\frac{x}{1 - x}\right) = \frac{1}{(1 - x)^2}. \qquad (1)$$

Then, on using the formula for the sum of the terms of a finite geometric progression [1, 2], we have

$$\sum_{j=0}^{99} x^{100(k-1)+j} = x^{100(k-1)}\left(\frac{1 - x^{100}}{1 - x}\right). \qquad (2)$$

In a similar manner to the way in which (1) was derived, it may be shown that

$$\sum_{k=1}^{\infty} kx^{100(k-1)} = \frac{1}{(1 - x^{100})^2}. \qquad (3)$$

On combining (2) and (3) we obtain, after some simplification,

$$\sum_{k=1}^{\infty}\left[(900k - 101)\sum_{k=1}^{\infty} x^{100(k-1)+j}\right] = \frac{799 + 101x^{100}}{(1 - x)(1 - x^{100})},$$

which leads to the result

$$\begin{aligned} F(x) &= \frac{1}{(1 - x)^2} + \frac{799 + 101x^{100}}{(1 - x)(1 - x^{100})} \\ &= \frac{800 - 799x + 100x^{100} - 101x^{101}}{(1 - x)^2(1 - x^{100})}. \end{aligned}$$

The answer to the student's question is, with the help of computer algebra,

$$\begin{aligned} F\left(\tfrac{1}{2}\right) &= \frac{800 - 799\left(\tfrac{1}{2}\right) + 100\left(\tfrac{1}{2}\right)^{100} - 101\left(\tfrac{1}{2}\right)^{101}}{\left(1 - \tfrac{1}{2}\right)^2\left(1 - \left(\tfrac{1}{2}\right)^{100}\right)} \\ &= \frac{2707701682087498001596958046683 4}{1690200800304305868662270940 5} \\ &= 1602.00000000000000000000000001419\ldots \end{aligned}$$

Notice that although the answer is non-integral, it is very close to 1602. It is worth pondering over this briefly, and deciding whether or not this will be the case for other sums involving fixed-column decimal representations of integers.

https://doi.org/10.1017/mag.2014.15 Published online by Cambridge University Press

A generalisation

Finally, let the integers m and n satisfy $0 \leqslant m \leqslant 9$ and $n \geqslant 1$, respectively. We generalise $F(x)$ here to $F_{m,n}(x)$, the generating function for the increasing and infinite sequence of all positive integers such that when each of these numbers is written in decimal notation, there is an m in the nth column, noting that we adopt here the convention that the right-most column is termed the 0th column (or units column), the next column is termed the 1st column (the tens column), and so on. It is relatively straightforward to show that this generating function is given by

$$F_{m,n}(x) = \frac{10^n m - (10^n m - 1)x + 10^n(9-m)x^{10^n} - (10^n(9-m)+1)x^{10^n+1}}{(1-x)^2(1-x^{10^n})}.$$

This could of course be generalised further by considering numbers written in bases other than decimal.

References

1. A. J. Sadler and D. W. S. Thorning, *Understanding Pure Mathematics*, Oxford University Press, 1987.
2. D. E. Knuth, *The Art of Computer Programming*, Volume 1, Addison-Wesley, 1968.

doi: 10.1017/mag.2014.15

MARTIN GRIFFITHS

Dept. of Mathematical Sciences, University of Essex, Colchester CO4 3SQ

99.03 Quick proofs that certain sums of fractions are not integers

We illustrate our method with the series of reciprocal primes $\sum \frac{1}{p_r}$ where $p_1, p_2, \ldots$ denote the primes in ascending order, so that $2 = p_1 < p_2 < \ldots$. The first few partial sums $\frac{1}{2}, \frac{1}{2}+\frac{1}{3} = \frac{5}{6}, \frac{1}{2}+\frac{1}{3}+\frac{1}{5} = \frac{31}{30}$ support the inductive hypothesis that, for $n \geqslant 1$, $\sum_{r=1}^{n} \frac{1}{p_r} = \frac{O_n}{E_n}$ where O_n is odd and E_n is even. Notice that we do not need to assume that $\frac{O_n}{E_n}$ is in its lowest form: any such representation suffices to deduce immediately (by contradiction) that $\sum_{r=1}^{n} \frac{1}{p_r}$ is not an integer. The inductive step is painless: if $\sum_{r=1}^{N} \frac{1}{p_r} = \frac{O_N}{E_N}$, then $\sum_{r=1}^{N+1} \frac{1}{p_r} = \frac{O_N}{E_N} + \frac{1}{p_{N+1}} = \frac{p_{N+1}O_N + E_N}{p_{N+1}E_N}$ which is of the required 'Odd/Even' form since p_{N+1} is odd.

Our other examples below may also be set out in the same way but, following the helpful suggestion of the referee, we can clarify and shorten the exposition by focusing on the structural properties of the types of rational numbers involved. To that end, let A denote the set of rational

https://doi.org/10.1017/mag.2014.15 Published online by Cambridge University Press