CALCULATION OF HYDRODYNAMIC FORCES FOR UNSTEADY STOKES FLOWS BY SINGULARITY INTEGRAL EQUATIONS BASED ON FUNDAMENTAL SOLUTIONS

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ABSTRACT

The attractive feature of the singularity method for steady Stokes flows is that the hydrodynamic forces acting on the particle can be calculated by the total strength of distributed singularities. For unsteady Stokes flows, however we have to derive hydrodynamic forces acting on a solid body in terms of the strengths of both unsteady Stokeslets as well as unsteady potential dipoles if mass and force sources are both taken into consideration. Since the hydrodynamic force formulation results in a Volterra integral equation of the first kind, and the strengths are numerically approximated by means of the Lubich convolution quadrature method (CQM) in this study. As far as the numerical solutions of time-domain integral formulations of the unsteady Stokes equations are concerned, this paper requires only the Laplace-domain instead of the time- domain fundamental solutions of the governing equations. The stability and accuracy of the proposed method are verified through some well selected numerical examples. In total we include two examples presenting the accuracy of Lubich CQM, and another two examples for calculating general hydrodynamic forces of a sphere in oscillating or non-oscillating unsteady Stokes flows. It is concluded that this study is able to extend the unsteady Stokes flow theory to more general transient motions instead to limit to the oscillating flow assumption.

Keywords: Fundamental solutions, Stokeslets, Dipole, Volterra integral equation, Convolution quadrature method, Singularity methods.

1. INTRODUCTION

In most engineering applications such as rheological properties of colloidal systems, microorganisms, sedimentation and nanotechnology, and so on, the approximate analysis of hydromechanics at Stokes or low-Reynolds-number flows is based on the assumption of quasi-steady approach. If we let $\mathrm{Re} \to 0$, the inertia term is completely neglected, and the time variable is implicit in these equations. The steady Stokes equations are then reduced to the followings:

$$\mathbf{0} = -\nabla p + \mu \nabla^2 \mathbf{u} \quad \nabla \bullet \mathbf{u} = 0 \tag{1}$$

From analytical points of view, only for several simple specific geometries in translating, rotating and shearing flows, or parabolic flow pasting the regular solid body, solutions of exact steady Stokes flows were able to be obtained. In general the analytical methods include but not limit to the conventional harmonic functions method with the choice of a suitable coordinate system; the well-known classical methods and the singularity method as explored by Chwang and Wu [1]. In particular, the singularity methods provide the possibility to address problems in which the traditional analytic methods are either very hard or unable to solve. The basic idea is to simulate the flow in terms of discrete or continuous distributions of fundamental solutions of Stokes equations to satisfy the appropriate boundary conditions at the surface of the solid body. Moreover finding the hydrodynamic forces of arbitrary shape of solid body by the singularity methods is proved to be feasible. For examples, Dabros [2] distributed fundamental solutions of Stokes equations inside the particle, not particle surface, to estimate hydrodynamic forces and torques. Zhou and Pozrikidis [3] developed adaptive singularity method which moves the position of singular points inside the particle, and investigated the accuracy and performance of solutions. Young *et al.* [4] also developed a numerical scheme based on the method of fundamental solutions for the steady Stokes equations. They tested lid-driven flows

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in a square cavity and a cubic cavity to show that the scheme is computationally efficient than the finite element method in dealing with irregular domain. However the aforementioned collocation scheme is only focused on steady Stokes flows. The objective of this paper is motivated by two major concerns: (1) to study fluid unsteady inertia term $\partial\mathbf{u}/\partial t$ in more general motion instead of limiting to the oscillating motion and (2) to investigate the important properties of the singularity methods for steady and unsteady Stokes flows by theoretical and computational approaches.

The first motivation arises from the fact that although quasi-steady approach can simplify physical problems. For example, Feng and Joseph [5] studied the accuracy of using the quasi-steady scheme in tracking unsteady particle motions. However they found out significant discrepancy exists in the presence or absence of fluid unsteady term. Numerical results also showed vivid fact that the quasi-steady approach is never able to approximate truly unsteady model appropriately, since the unsteady inertia term $\partial\mathbf{u}/\partial t$ plays an important role in modeling the general motion of the particles. Therefore, for general unsteady Stokes flows, we will establish a strategy to calculate the flow field and hydrodynamic forces acting on particles by considering the unsteady inertial force in this paper. We use analytic space-time fundamental solutions of the unsteady Stokes equations. Namely, the flow due to an impulsive point force $\mathbf{F}\delta(\mathbf{x})\delta(t)$ is computed. A time-space domain formulation is expressed based on the unsteady fundamental solutions of Stokes equations with appropriate boundary conditions. This results in convolution integral equations, or the Volterra integral equations of the first kind, for the strength of unsteady Stokeslets. In general, the numerical solution of a convolution integral equation is a well-studied problem, and a large variety of numerical methods have been developed to rapidly and accurately obtain approximations. Collocation methods (Brunner [6]), Sinc collocation methods (Muhammad *et al*. [7]), convolution quadrature method (CQM) (Lubich and Schadle [8]), and Galerkin methods (Bedivan and Fix [9]) are available in the literature. In this study we will employ Lubich CQM for the time discretization to solve the convolution integral equation in the target object. Lubich CQM is solved numerically by using a quadrature formula. Their weights are determined by the Laplace transform of the kernel function and a linear multistep method. The numerical results show good stability and accuracy. However the drawback of CQM is also sensitive to the shape of Talbot contour, which is a deformation of the Bromwich contour.

The second motivation of this paper is to investigate the important properties of the singularity methods for steady and unsteady Stokes flows. We follow the vital rule that hydrodynamic forces acting on a particle in a viscous fluid are relative to the total strength of distributed Stokeslets as studied by Dabros [2]. For any shape of a solid body in an oscillatory flow, we also can obtain the hydrodynamic force formula as presented by Pozrikidis [10]:

$$\mathbf{F} = -8\pi\mathbf{a} + \lambda^2 V_c\, \mathbf{U}(t) \tag{2}$$

in which $\mathbf{a}$ is the total strength of the distributed Stokeslets, $\mathbf{U}(t)$ the particle velocity, V_c the volume of the particle and λ is a dimensionless parameter given by $\lambda^2 = -i\omega d^2/v$ and ω is the oscillation frequency by assuming $\mathbf{u}$ and p are under oscillatory motions. However, the above formula is not applicable to an arbitrary time-dependent motion with general non-oscillating velocity $\mathbf{U}(t)$. In present paper, unsteady fundamental solutions and Laplace transform will be used to find the hydrodynamic forces for any general non-oscillating velocity field. This paper extends and generalizes the unsteady results of Pozrikidis [10] from oscillatory flow to a general arbitrary motion.

The organization of the paper is as follows. In Section 2, we will present the fundamental solutions of unsteady Stokes flow due to a concentrated point force and is called an unsteady Stokeslet. The total forces exerted by unsteady Stokeslets for arbitrary particle shape is obtained. In Section 3, a review of Lubich CQM, the methodology for solving the convolution integral equation is presented. In Section 4, we provide four examples using CQM works. Two examples are that the behavior of the CQM are analyzed and tested numerically; include their main properties and accuracy. Another two examples, space-time Volterra integral equation and the hydrodynamic forces will be calculated for a translating sphere by CQM. Our conclusions are summarized in Section 5.

2. STRENGTH THEOREM

The fundamental solutions of unsteady Stokes flow represent the solutions due to a concentrated point force $\boldsymbol{\alpha}$, with a singularity at the point $\mathbf{x_0}$ and at time $t = 0$. It is noted that the fluid is treated as incompressible so that there is no mass source in the continuity equation. Under such assumptions, the governing equations of unsteady Stokes flow are then written as:

$$\nabla \bullet \mathbf{u} = 0 \tag{3}$$

$$\rho\frac{\partial \mathbf{u}}{\partial t} = -\nabla p + \mu\nabla^2\mathbf{u} + \boldsymbol{\alpha}\delta(\mathbf{x}-\mathbf{x_0})\delta(t) \tag{4}$$

where $\mathbf{u} = (u, v, w)$ is the velocity vector; ρ the constant density; p the pressure; μ the constant dynamic viscosity; $\hat{\mathbf{x}} = \mathbf{x} - \mathbf{x_0}$, and $r = |\hat{\mathbf{x}}|$ an Euclidean norm; $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ a constant vector and stands for the strength of the point force locating at $\mathbf{x_0}$ in a three- dimensional Euclidean space, respectively. The fundamental solutions of unsteady Stokes flow in the Cartesian coordinates are obtained by the techniques of integral transforms (see Banerjee [11]; Tsai *et al.* [12]) and are listed as follows:

$$u_i^S(\mathbf{x}, t) = \frac{1}{4\pi\rho} G_{ij}^S(\hat{\mathbf{x}}, t)\, \alpha_j \tag{5}$$

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$$G_{ij}^{S}(\hat{\mathbf{x}},t) = \frac{1}{\sqrt{\pi\nu t}}\left(\frac{r^2\delta_{ij}-\hat{x}_i\hat{x}_j}{2\nu t r^2}-\frac{2}{r^2}\delta_{ij}+\frac{3(r^2\delta_{ij}-\hat{x}_i\hat{x}_j)}{r^4}\right)\exp(-\lambda^2) - \frac{\mathrm{erf}(\lambda)(r^2\delta_{ij}-3\hat{x}_i\hat{x}_j)}{r^5} \tag{6}$$

where $\lambda = r/\sqrt{4\nu t}$ and $G_{ij}^{S}(\hat{\mathbf{x}},t)$ is a free-space Green's function, also called a fundamental solution or an unsteady Stokeslet. The induced pressure is

$$p^{S}(\mathbf{x},t) = \frac{\delta(t)}{4\pi}\frac{\hat{x}_i}{r^3}\alpha_i \tag{7}$$

In principle, the flow fields can be expressed as the convolution integral in time using the fundamental solutions in Eq. (5). Thus the induced velocity and pressure are given as follows:

$$u_i(x,t) = \frac{1}{4\pi\rho}\int_0^t G_{ij}^{S}(\hat{x},t-\tau)\,\alpha_j(\tau)\,d\tau \tag{8}$$

And

$$p(\mathbf{x},t) = \frac{1}{4\pi}\frac{\hat{x}_i}{r^3}\alpha_i(t) \tag{9}$$

Now, making use of Eq. (8), and considering the problem that a particle translates with the prescribed boundary condition $\mathbf{U}(t)$ as shown in Fig. 1. The flow field can be approximated by superposition from all unsteady Stokeslets. The velocity field is evaluated as

$$u_i(\mathbf{x},t) = \frac{1}{4\pi\rho}\sum_{k=1}^{N}\int_0^t G_{ij}^{S}(\mathbf{x}-\mathbf{x}_\mathbf{o}^k,t-\tau)\,\alpha_j^k(\tau)\,d\tau \tag{10}$$

where N is the number of Stokeslet points; $\mathbf{x}_\mathbf{o}^k$ is the position of the kth Stokeslet point, $\alpha_j^k(t)$ is the jth coordinate component of strength for the kth Stokeslet point collocated in the interior of particle. The method of using Eq. (10) is called the method of fundamental solutions (MFS), the singularity method (SM), the fundamental collocation method (FCM) or the charge simulation method (CSM) in the literature. For illustration Tsai *et al.* [12] have used the MFS to solve the 2D and 3D unsteady Stokes flow problems. The basic idea and feature of the singularity method is to distribute proper strengths of Stokeslets to satisfy the boundary conditions of regular or irregular solid bodies. For any given velocity $u_i(\mathbf{x}, t) = \mathbf{U}(t)$ at the boundary, Eq. (10) is a Volterra integral equation of the first kind. In principle, by taking the Laplace transform of Eq. (10) with respect to t, we can obtain the strengths of unsteady Stokeslets. The singularity method has been considered to be an efficient numerical and analytical method for solving both steady and unsteady Stokes flows.

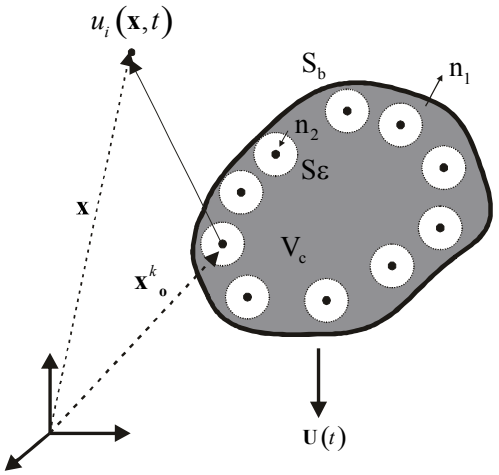


Fig. 1 Schematic diagram of the nodes distribution of Stokeslets

It should be mentioned here that for a steady flow, specifically, the attractive feature of the singularity method is that the force acting on the body is relative to the total strength of the distributed singularities as given by Dabros [2]. The surrounding fluid exerts a force on the particle like:

$$\mathbf{F} = -\sum_{k=1}^{N}\boldsymbol{\alpha}^k \tag{11}$$

The feature can be called a strength theorem. For unsteady Stokes flow, the strength theorem may be not yet validated. Next, we will address the strength theorem. If the strength of unsteady Stokeslets has known parameters, it can represent the flow field. Thus the flow field can be simulated by the superposition of the fundamental solutions. The momentum equation with finite point forces is expressed by:

$$\rho\frac{\partial u}{\partial t} = -\nabla p + \mu\nabla^2 u + \sum_{k=1}^{N}\alpha^k\,\delta(x-x_o^k)\,\delta(t) \tag{12}$$

The fluid force exerted on the particle is $\iint_{Sb}(-p+\boldsymbol{\tau})\bullet\mathbf{n}_1\,d\mathrm{A}$ and $\boldsymbol{\tau}$ is the viscous stress tensor, V_c the interior of volume, $\mathbf{n}_1$ is the unit outward normal at Sb and $\mathbf{n}_2$ is the inward unit at Sε. For the definitions shown in Fig. 1, because the interior of volume V_c has no singular points $\mathbf{x}_\mathbf{o}^k$ existing, we use the divergence theorem and obtain

$$\iint_{Sb}(-p+\boldsymbol{\tau})\bullet\mathbf{n}_1 d\mathrm{A} + \iint_{S\varepsilon}(-p+\boldsymbol{\tau})\bullet\mathbf{n}_2 d\mathrm{A} = \iiint_{Vc}(-\nabla p+\nabla\bullet\boldsymbol{\tau})\,d\mathrm{V} = \iiint_{Vc}(-\nabla p+\mu\nabla^2\mathbf{u})\,d\mathrm{V} \tag{13}$$

Applying Eq. (4), the Equation is

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$$\iint_{Sb}(-p+\boldsymbol{\tau})\bullet\mathbf{n}_1 dA+\iint_{S\varepsilon}(-p+\boldsymbol{\tau})\bullet\mathbf{n}_2 dA$$
$$=\iiint_{Vc}\left(-\sum_{k=1}^{N}\boldsymbol{\alpha}^k\delta(\mathbf{x}-\mathbf{x}_\mathbf{o}^k)\,\delta(t)+\rho\frac{\partial\mathbf{u}}{\partial t}\right)dV \tag{14}$$

Because singular points do not exist in V_c, Eq. (14) is further deduced to the following form:

$$\iint_{Sb}(-p+\boldsymbol{\tau})\bullet\mathbf{n}_1 dA=\iiint_{Vc}\rho\frac{\partial\mathbf{u}}{\partial t}dV-\iint_{S\varepsilon}(-p+\boldsymbol{\tau})\bullet\mathbf{n}_2 dA \tag{15}$$

The surrounding fluid exerts a force on the control surface Sε of sphere with radius ε centered at the point force $\boldsymbol{\alpha}$ due to an unsteady Stokeslet is given by the integral

$$\mathbf{F}_\mathbf{S}(t)=-\frac{1}{3}\boldsymbol{\alpha}(t)+\int_0^t\frac{\boldsymbol{\alpha}(t)}{\sqrt{\pi\nu(t-\tau)}}\left(-\frac{\varepsilon^3}{6\nu(t-\tau)^2}\right)e^{\frac{-\varepsilon^2}{4\nu(t-\tau)}}d\tau \tag{16}$$

Taking the Laplace transform of Eq. (16), we obtain:

$$\mathbf{F}_\mathbf{S}(p)=-\frac{1}{3}\boldsymbol{\alpha}(p)+\frac{1}{\sqrt{\pi\nu}}\left(-\frac{\varepsilon^3}{6\nu}\right)\boldsymbol{\alpha}(p)$$
$$\cdot\left(8\cdot\frac{\sqrt{\pi}}{2}\frac{\nu^{\frac{3}{2}}}{\varepsilon^3}+4\cdot\frac{\nu\sqrt{p\pi}}{\varepsilon^2}\right)e^{-\sqrt{\frac{\varepsilon^2 p}{\nu}}} \tag{17}$$

If we let $\varepsilon\to 0$ and then by taking the inverse Laplace transform of Eq. (17), we therefore obtain

$$\mathbf{F}_\mathbf{S}(t)=-\boldsymbol{\alpha}(t) \tag{18}$$

By referring to Eq. (15) and $\mathbf{n}_2$ is the inward unit at Sε, hence the second term of Eq. (15) is given by:

$$\lim_{\varepsilon\to 0}\iint_{S\varepsilon}(-p+\boldsymbol{\tau})\bullet\mathbf{n}_2\,dA=\sum_{k=1}^{N}\boldsymbol{\alpha}^k(t) \tag{19}$$

Hence the surrounding fluid exerts the force acting on the particle as:

$$\iint_{Sb}(-p+\boldsymbol{\tau})\bullet\mathbf{n}_1\,dA=-\sum_{k=1}^{N}\boldsymbol{\alpha}^k(t)+\iiint_{Vc}\rho\frac{\partial\mathbf{u}}{\partial t}\,dV \tag{20}$$

The volume integral of Eq. (20) means the rate of change of momentum of fluid with volume of particle is V_c; and $\mathbf{u}$ is in the interior of the particle and is equal to $\mathbf{U}(t)$, the particle moving velocity. The surrounding fluid exerts a force on the particle as

$$\mathbf{F}=-\sum_{k=1}^{N}\boldsymbol{\alpha}^k(t)+V_c\cdot\rho\frac{\mathrm{d}\mathbf{U}(t)}{\mathrm{dt}} \tag{21}$$

where V_c is the volume of the particle. This result agrees with Pozrikidis [10] (in Eq. (2) by some manipulation) that in the original notation the force on a particle with oscillating Stokeslets as a function of frequency ω. Moreover, if both unsteady Stokeslet and unsteady potential dipole are taken into consideration, we will have to add the inertia force due to the unsteady term. That is, if we consider both the mass and force sources, we can simply obtain the hydrodynamic forces of a sphere with radius a, from the following formula by adding the dipole into Eq. (21) to get the following formula (see also Eqs. (34) and (35) below for more descriptions):

$$\mathbf{F}=-\boldsymbol{\alpha}(t)-\frac{\rho}{3}\frac{d}{dt}\left(\mathbf{d}(t)\right)+\frac{4}{3}\pi a^3\rho\frac{\mathrm{d}\mathbf{U}(t)}{\mathrm{dt}} \tag{22}$$

$\boldsymbol{\alpha}(t)$ and $\mathbf{d}(t)$ are the strengths of unsteady Stokeslet and unsteady potential dipole, respectively.

3. CONVOLUTION QUADRATURE METHOD

To find the hydrodynamic forces acting on particles is essential to the study of mass transfer, rheology and micro-organism locomotion. Making use of Eq. (21), Eq. (10) and appropriating boundary conditions to be fulfilled, we can obtain the hydrodynamic forces. A fast and stable algorithm for computing convolution integral was proposed by Lubich and Schadle [8]. The convolution integrals in Eq. (10) can be expressed

$$u_i(\mathbf{x},t)$$
$$=\frac{1}{4\pi\rho}\sum_{k=1}^{N}\left(\int_{t-\Delta t}^{t}G_{ij}^S(\mathbf{x}-\mathbf{x}_\mathbf{o}^k,t-\tau)\,\alpha_j^k(\tau)\,d\tau\right.$$
$$\left.+\int_0^{t-\Delta t}G_{ij}^S(\mathbf{x}-\mathbf{x}_\mathbf{o}^k,t-\tau)\,\alpha_j^k(\tau)\,d\tau\right) \tag{23}$$

Thus in the RHS of Eq. (23), the first integral is calculated as follows:

$$\int_{t-\Delta t}^{t}G_{ij}^S(\mathbf{x}-\mathbf{x}_\mathbf{o}^k,t-\tau)\alpha_j^k(\tau)\,d\tau$$
$$\approx\ \Phi_1\ \alpha_j^k(t-\Delta t)+\Phi_2\frac{\alpha_j^k(t)-\alpha_j^k(t-\Delta t)}{\Delta t} \tag{24}$$

where

$$\Phi_1\approx\sum_{h=-M}^{M}\omega_h\ \mathrm{I}G_{ij}^S(\mathbf{x}-\mathbf{x}_\mathbf{o}^k,\lambda_h)/\lambda_h\exp(\Delta t\lambda_h) \tag{25}$$

$$\Phi_2\approx\sum_{h=-M}^{M}\omega_h\ \mathrm{I}G_{ij}^S(\mathbf{x}-\mathbf{x}_\mathbf{o}^k,\lambda_h)/\lambda_h^2\exp(\Delta t\lambda_h) \tag{26}$$

The kernel $\mathrm{I}G_{ij}^S(\mathbf{x}-\mathbf{x}_\mathbf{o}^k,\mathrm{s})$ is the Laplace transform of $G_{ij}^S(\mathbf{x}-\mathbf{x}_\mathbf{o}^k,t)$, the weights ω_h and quadrature points λ_h correspond to a Talbot contour Γ, 2M + 1 is the number of quadrature points on Γ. Since direct numerical integration along Bromwich contour in calculating inverse Laplace transform is difficult, we employ the Talbot contour, a special contour integration Γ to numerically solve the inversion problem of the Laplace transforms. The Talbot contour Γ is illustrated in Fig. 2, the Talbot contour is given by

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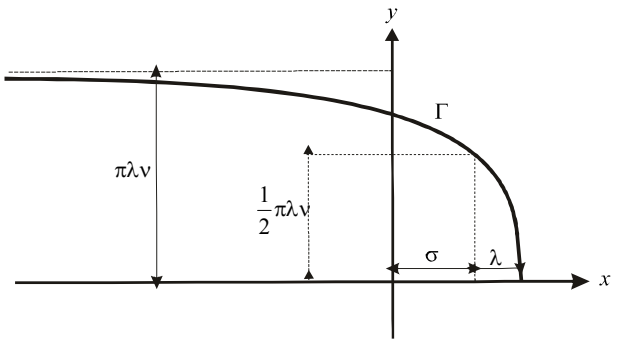


Fig. 2 Illustration of Talbot contour

$$\Gamma \mapsto \gamma(\theta) = \sigma + \lambda S_{\nu}(\theta) = \sigma + \lambda(\theta \cot\theta + i\nu\theta)$$
$$\theta \in (-\pi, \pi) \tag{27}$$

The real parameters $\lambda(>0)$, $\nu(>0)$ and σ determine the geometry of curve. The second integral in Eq. (23) is

$$\int_0^{t-\Delta t} G_{ij}^S(\mathbf{x}-\mathbf{x}_{\mathbf{0}}^k, t-\tau)\,\alpha_j^k(\tau)\,d\tau$$
$$\approx \sum_{\ell=1}^{L}\sum_{h=-M}^{M} w_h^{(\ell)}\ \mathrm{I}G_{ij}^S(\mathbf{x}-\mathbf{x}_{\mathbf{0}j}^k, \lambda_h)\exp(t-\tau_{\ell-1})\,\lambda_h^{(\ell)} y(\tau_{\ell-1}, \tau_\ell, \lambda_h^{(\ell)}) \tag{28}$$

Because the shape of Talbot contour is fundamentally important to obtain good results and the numerical overflow problem may happen for larger t, so the choices of $w_h^{(\ell)}$ and $\lambda_h^{(\ell)}$ corresponding to the differential Talbot contour Γ_1 in every interval of time by Lubich and Schadle [8] must be carefully undertaken. The $[0, t-\Delta t]$ time interval is split into a sequence of L number time intervals and $I_\ell = [\tau_{\ell-1}, \tau_\ell]$, $\tau_0 = t - \Delta t$, and $\tau_L = 0$. Another variables $y(\tau_{\ell-1}, \tau_\ell, \lambda_h^{(\ell)})$ is the solution of initial value problem at time $\tau_{\ell-1}$

$$\frac{dy}{dt} = \lambda_h^{(\ell)} y + \alpha_j^k(\mathrm{t}) \quad y(\tau_\ell) = 0 \tag{29}$$

For given velocity of the particle $u_i(\mathbf{x}, t) = \mathbf{U}(t)$ in Eq. (23), the no-slip condition leads to the following Volterra integral of the first kind for the strength α_j^k

$$\mathbf{U}(t) = \frac{1}{4\pi\rho}\sum_{k=1}^{N}\int_0^t G_{ij}^S(\mathbf{x}-\mathbf{x}_{\mathbf{0}}^k, t-\tau)\,\alpha_j^k(\tau)\,d\tau \tag{30}$$

By applying Eqs. (26), (28), (29) and (30), the strength α_j^k can be solved. This explains the method of fundamental solutions or singularity method in the literature. Furthermore we can substitute α_j^k into Eq. (21) to obtain the hydrodynamic drag on the particle.

4. NUMERICAL EXAMPLES

In this section, a Volterra integral equation of the first kind with a weakly singular kernel function and another Volterra equation of the second kind are considered. And the accuracy of Lubich CQM is presented. We also give another two examples to illustrate the application of strength theorem with Lubich CQM for unsteady Stokes flows. Strength theorem with Lubich CQM is used to calculate the strength of unsteady singularities and also to obtain the hydrodynamic forces acting on the particle for unsteady Stokes flows. Numerical results are compared with some exact solutions.

4.1 A Volterra Integral Equation of the First Kind. We Consider a Volterra Integral Equation with a Weakly Singular Kernel Function for $y(t)$.

$$\int_0^t K(t, \tau)\, y(\tau)\, d\tau = g(t) \tag{31}$$

The kernel $K(t, \tau)$ and $g(t)$ is given by

$$K(t, \tau) = \frac{1}{\sqrt{\pi(t-\tau)}}, \quad g(t) = 2\sqrt{\frac{t}{\pi}}$$

The Laplace transform is $K(s) = 1/\sqrt{s}$ and exact solution is $y(t) = 1.0$. In the numerical example we use Lubich CQM to accurately obtain approximations to $y(t)$. The Talbot contour is shown in Fig. 2, the choices of parameters were found to give good results with $\lambda = 8.0/t$, $\nu = 0.6$, $\sigma = 0.0$, the base B is an integer and B = 2 was found to be a good choice in our numerical experiments, time step $\Delta t = 0.001$ and the number of quadrature points on Talbot contours is N = 15.

Figure 3 shows the comparison of the exact solution and the numerical solution of Lubich CQM and absolute errors $|\, y_n(t) - y(t_n)\,|$ versus time. The approximation error is about 10^{-6} to 10^{-8} and the accuracy is very good.

4.2 A Volterra Integral Equation of the Second Kind. We Consider Now a Volterra Integral Equation of the Second Kind.

$$y(t) + \int_0^t K(t, \tau)\, y(\tau)\, d\tau = g(t) \tag{32}$$

The kernel function $K(t, \tau)$ and $g(t)$ are given by

$$K(t, \tau) = \cos(t-\tau), \quad g(t) = \sin(t)$$

The exact solution is $y(t) = 2\sqrt{3}/3\ \sin(\sqrt{3}/2\, t)\, e^{-t/2}$ and the Laplace transform is $K(s) = s/(s^2+1)$. In this numerical example we use Lubich CQM to accurately obtain approximations to $y(t)$. The choices of parameters were found to be $\lambda = 8.0/t$, $\nu = 0.6$, $\sigma = 0.0$, the base B = 2, time step $\Delta t = 0.001$ and the number of quadrature points on Talbot contours is N = 15. Figure 4 shows the comparison of the exact solution and the numerical solution of Lubich CQM and absolute errors $|\, y_n(t) - y(t_n)\,|$ versus time. The approximation error is about 10^{-5} to 10^{-8} and the method performed quite well.

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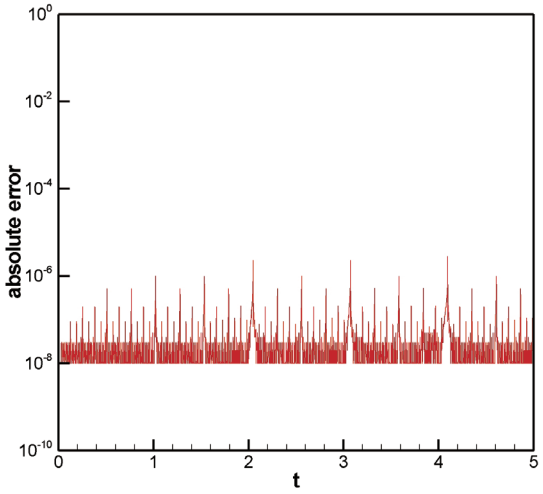


Fig. 3 Lubich CQM absolute error versus time for N = 15 and B = 2 for case 4.1

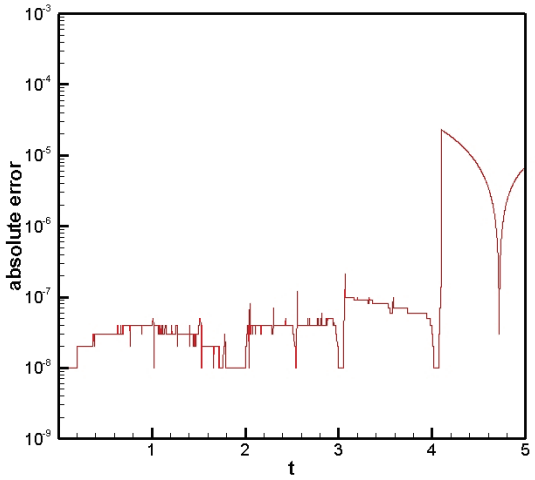


Fig. 4 Lubich CQM absolute error versus time for N = 15 and B = 2 for case 4.2

4.3 A Sphere Moving in Viscous Flow

To the best of our knowledge, there exist only a few solutions of the forces and torques (couples) acting on bodies obtained only for special geometries in unsteady Stokes flows. For example, due to the pioneering work of Stokes [13] the hydrodynamic forces of a translating sphere were obtained with the function of the frequency of oscillation ω. The inverse Fourier integral was used to derive the famous formula of the drag force F acting on a sphere in a time- dependent motion with velocity $\mathbf{U}(t)$ as given by Basset [14]:

$$\mathrm{F} = -6\pi\mu a\mathrm{U}(t) - \frac{2}{3}\pi\rho a^3\frac{\mathrm{dU}}{\mathrm{dt}} - 6\pi\rho a^2\sqrt{\frac{\nu}{\pi}}\int_0^t \frac{d\mathrm{U}}{d\tau}\frac{1}{\sqrt{t-\tau}}\,d\tau \tag{33}$$

In this famous equation the first term is the quasi-steady Stokes drag, the second is the added mass force and the last term is so-called the Basset force which stands for the growth of the time-dependent boundary layer at the body surface. By the singularity method given by Chwang & Wu [1], in the above problem we can employ an unsteady Stokeslet, an unsteady potential dipole with poles at the center of the sphere, to construct the solutions. The flow fields can be expressed as the convolution integral in time using the fundamental solutions. Thus the induced velocity is described by

$$u_i(\mathbf{x},t) = \frac{1}{4\pi\rho}\int_0^t G_{ij}^S(\hat{\mathbf{x}},t-\tau)\,\alpha_j(\tau)d\tau + \frac{1}{4\pi}G^{\mathrm{PD}}{}_{\mathrm{ij}}(\hat{\mathbf{x}})\,d_j(t) \tag{34}$$

Where the first term is an unsteady Stokeslet and the second term represents an unsteady potential dipole. Here, $\alpha_j(t)$, $d_j(t)$ are the strengths of the unsteady singularities, respectively. The fundamental solutions $G_{ij}^S(\hat{\mathbf{x}},t)$ can be found in Eq. (6). An unsteady potential dipole is shown as

$$G_{ij}^{\mathrm{PD}}(\hat{\mathbf{x}}) = -\frac{\delta_{ij}}{r^3} + 3\frac{\hat{x}_i\,\hat{x}_j}{r^5} \tag{35}$$

Without loss of generality, we consider a sphere of radius a centered at the origin which translates with velocity U(t) in the z-direction, where U(t) is not a necessary time period but arbitrary and is differentiable and continuous. For the reason of simplicity and comparison with Eq. (33) purposes, there are two simple, but typical examples to be calculated.

4.3(a) An oscillatory motion

It is assumed that the isolated sphere moves with oscillatory translational motion and the boundary conditions are then

$$\begin{aligned}&\mathbf{u} = \mathrm{U}(t)\,\mathbf{e}_z = \mathrm{U}_o\sin(\omega t)\,\mathbf{e}_z \quad \text{at} \quad r = a\\ &\mathbf{u} = 0 \quad \text{at} \quad r\to\infty\end{aligned} \tag{36}$$

4.3(b) A non-oscillatory or general motion

The isolated sphere moves with arbitrary translational motion and the boundary conditions are assumed to be then

$$\begin{aligned}&\mathbf{u} = \mathrm{U}(t)\,\mathbf{e}_z = \mathrm{U}_o\mathrm{erf}(k\mathrm{t})\,\mathbf{e}_z \quad \text{at} \quad r = a\\ &\mathbf{u} = 0 \quad \text{at} \quad r\to\infty\end{aligned} \tag{37}$$

where U_o is velocity amplitude, ω is angular frequency, erf(t) is Error function, k is the constant, and $\mathbf{e}_z$ denotes a unit vector in the z-direction and the initial condition for $t = 0$ is

$$\mathbf{u} = \mathrm{U} = 0 \tag{38}$$

In 4.3(a) oscillatory motion case, the exact Basset integral in Eq. (33) is given by

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$$\int_0^t \frac{dU}{d\tau}\frac{1}{\sqrt{t-\tau}}\,d\tau = U_o\,\omega\int_0^t \frac{\cos(\omega t)}{\sqrt{t-\tau}}\,d\tau$$
$$= U_o\sqrt{2\pi\omega}\left(\cos(\omega t)\,Fresc\left(\sqrt{\frac{2\omega t}{\pi}}\right)\right.$$
$$\left.+\sin(\omega t)\,Fress\left(\sqrt{\frac{2\omega t}{\pi}}\right)\right) \qquad (39)$$

Here Fresc(t) and Fress(t) denote the Fresnel cosine and sine functions, respectively. However, in (b) general motion case, the exact Basset integral only can be solved with numerical scheme.

Hence all the variables are dimensional, in our numerical experiments silicone oil is used with large viscosity ($\nu = 0.01\text{m}^2/\text{s}$, $\rho = 980.0\text{kg}^3/\text{m}^3$), the radius of sphere is $r = 0.006$m, the amplitude $U_o = 0.01$m/s, angular frequency $\omega = \pi$ rad/s, $k = 3.0$. In order to obtain $\alpha_j(t)$, $d_j(t)$ in Eq. (34), we use Lubich CQM and collocation method. The choices of Talbot contour parameters were found to give good results with the base B = 2, $\lambda = 16.0/(2B^{(\ell)}\Delta t)$, ν = 0.6, σ = 0.0, time step $\Delta t = 0.01$. The integration is carried out from $t = 0$ until $t = 2$, and the number of quadrature points on Talbot contours is N = 101. The surface of sphere is divided into n = 12 boundary points. There are three components of velocity, so that there are $3n$ = 36 boundary equations. Since there are two coefficients $\alpha_j(t)$ and $d_j(t)$ for a singular point at the center of the sphere, we need 6 unknown coefficients, as $3n = 36 > 6$. Therefore, singular value decomposition (SVD) is used to obtain $\alpha_j(t)$, $d_j(t)$. In (a) oscillatory motion case, Fig. 5 shows the strengths of unsteady Stokeslet $\alpha_3(t)$ and the dipole $d_3(t)$ as denoted by solid line and dashed line, respectively. The strong oscillating behaviors of physical characteristics of the strengths of unsteady Stokeslet and dipole are vividly illustrated in Fig. 5. The dipole strength $d_3(t)$ is far less than the Stokeslet $\alpha_3(t)$, the ratio of $d_3(t)$ / $\alpha_3(t)$ is about equal to the order of 10^{-7}, and the phase difference is π. The relative error between the numerical results defined by Eq. (22) and exact solution in Eq. (33) is shown as Fig. 6. Here, it can be seen that the accuracy of numerical scheme is stable and accurate except for the maximum relative error is found to be the time when the hydrodynamic force approaches zero at $t = 0$, $t = 1$, $t = 2$. The reason is likely to be still difficult to calculate accurately the force when the value approximates zero with numerical method. The relative error is very small in the time of our computation coverage.

In 4.3(b) non-oscillatory or general motion cases, Fig. 7 shows the strengths of unsteady Stokeslet $\alpha_3(t)$ and the dipole $d_3(t)$ which are denoted by solid line and dashed line, respectively. The strong decaying behaviors of physical characteristics of the strengths of unsteady Stokeslet and dipole are vividly illustrated in Fig. 7. The relative error between the numerical results and

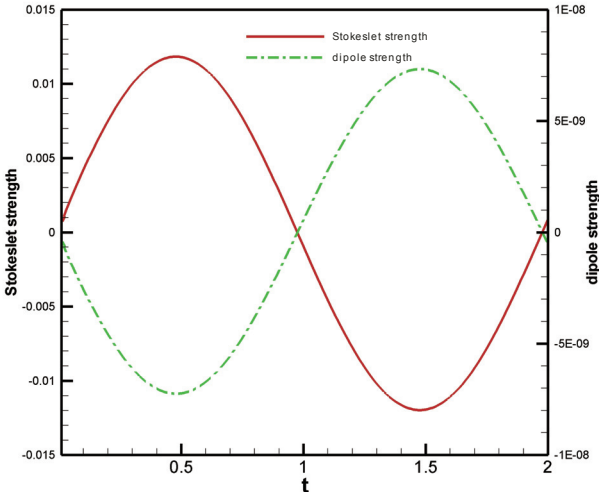


Fig. 5 The strengths of unsteady Stokeslet $\alpha_3(t)$ and the dipole $d_3(t)$ as denoted by solid and dashed lines for case 4.3(a)

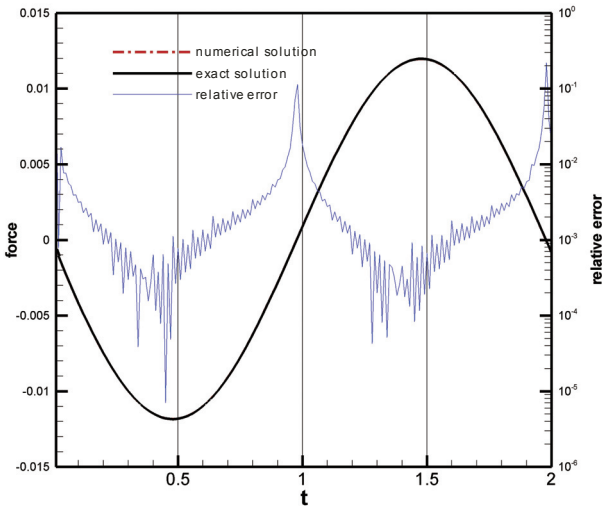


Fig. 6 The numerical scheme is stable and accurate except for the maximum relative error is found to be the time when the hydrodynamic force approaches zero at $t = 0$, 1, 2 for case 4.3(a)

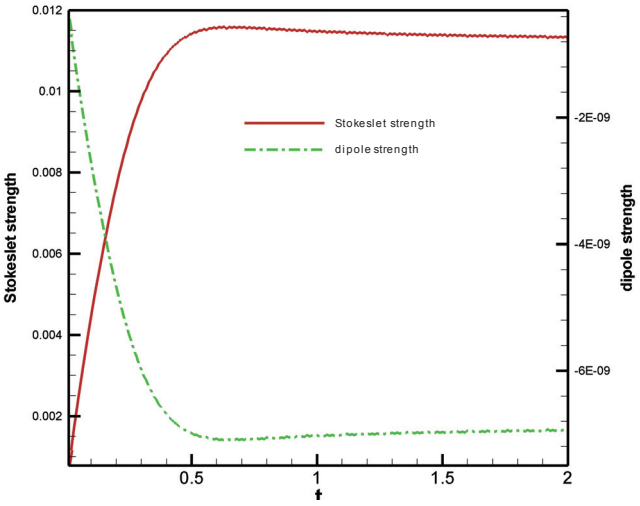


Fig. 7 The strengths of unsteady Stokeslet $\alpha_3(t)$ and the dipole $d_3(t)$ as denoted by solid and dashed lines for case 4.3(b)

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exact solutions in Eq. (33) is shown as Fig. 8. Here, it also can be seen that the accuracy of numerical scheme is still stable and accurate except that the maximum relative error is found to be the time when the sphere starts suddenly. The reason is likely to be still difficult to calculate accurately the force when the value approximates zero with numerical scheme. The relative error is very small in the time of our computational coverage. The novel method of the formula in Eq. (22) combining with Lubich CQM might be considered as excellent and efficient for calculating the hydrodynamic forces in unsteady Stokes flows.

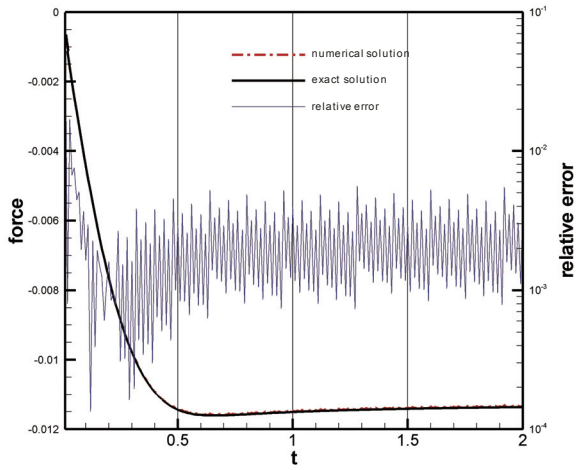


Fig. 8 The numerical scheme is stable and accurate except for the maximum relative error is found to be the time that the sphere starts suddenly for case 4.3(b)

5. CONCLUSIONS

The singularity method is a classical method to be applied to obtain exact solutions of Stokes flows. More importantly, it provides the possibility of solving problems that the traditional analytical methods cannot solve. If the singularities are adaptively chosen, namely by combing proper singularities, we are able to simulate the Stokes flow fields, and find hydrodynamic forces and flow fields. It also provides the alternative using numerical method to solve more complicated unsteady Stokes equations. In theory, if the strength of singularities is known, by referring to Eq. (5) as well as Eq. (7), we can describe the velocity fields induced by arbitrary motion of solid particles, including the pressure. In particular, the hydrodynamic force is related to the total strength of the distributed singularities in unsteady Stokes problems. A novel technique for the analysis of hydrodynamic force problems with Lubich CQM coupling fundamental solutions has been presented in this work. The main advantage of this approach is that it requires only Laplace transform of fundamental solutions without worrying about the time discretization to obtain stable and accurate results. Therefore, the proposed formulation may be used as a powerful simulation tool for Stokes or low-Reynolds number flow systems.

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