106.48 On the first Fermat point for the triangle

When outwardly facing equilateral triangles are attached to the sides of a triangle ABC, the joins AA', BB', CC' meet at the first Fermat point P as shown in Figure 1. If the angles A, B, C are less than $\frac{2}{3}\pi$ then P is an internal point such that

$$\angle BPC = \angle CPA = \angle APB = \tfrac{2}{3}\pi \text{ and } QA + QB + QC > PA + PB + PC$$

for any other point Q (see [1]).

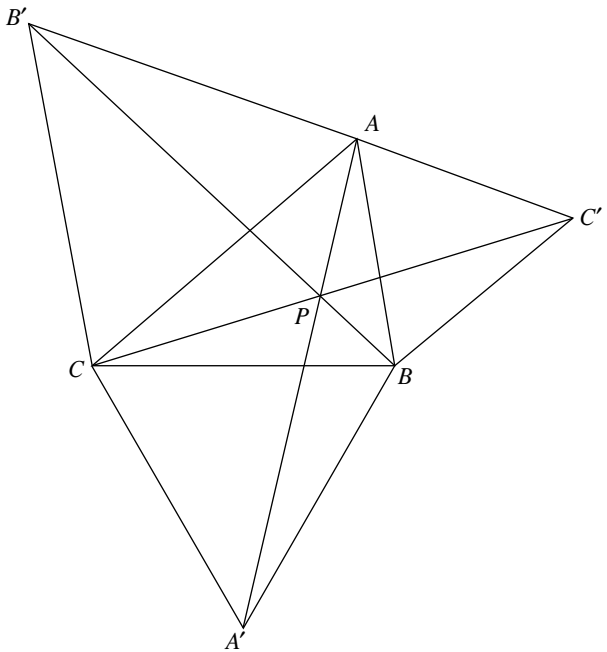


FIGURE 1

In this Note we will derive a formula for $PA + PB + PC$ ($P \in ABC$).

Preamble

By applying the Cosine Rule to triangles PBC, PCA, PAB, we have

$$a^2 = PB^2 + PC^2 + PB{\cdot}PC,\ b^2 = PC^2 + PA^2 + PC{\cdot}PA,$$

$$c^2 = PA^2 + PB^2 + PA{\cdot}PB,$$

which sum to

$$a^2 + b^2 + c^2 = 2\left(PA^2 + PB^2 + PC^2\right) + PB{\cdot}PC + PC{\cdot}PA + PA{\cdot}PB. \qquad (1)$$

Then (with triangle area Δ and circumradius R)

$$\Delta = \Delta PBC + \Delta PCA + \Delta PAB$$

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yields

$$\Delta = \frac{\sqrt{3}}{4}\left(PB{\cdot}PC + PC{\cdot}PA + PA{\cdot}PB\right) = \frac{abc}{4R}. \tag{2}$$

Hence (from (1, 2) and using the Sine Rule) we have

$$\begin{aligned}(PA+PB+PC)^2 &= PA^2+PB^2+PC^2+2\left(PB{\cdot}PC+PC{\cdot}PA+PA{\cdot}PB\right)\\ &= \tfrac{1}{2}\left(a^2+b^2+c^2\right)+\tfrac{3}{2}\left(PB{\cdot}PC+PC{\cdot}PA+PA{\cdot}PB\right)\\ &= \frac{1}{2}\left(a^2+b^2+c^2\right)+\frac{1}{2}\frac{\sqrt{3}abc}{R}\\ &= 2R^2\left(\sin^2A+\sin^2B+\sin^2C\right)+4\sqrt{3}R^2\sin A\sin B\sin C. \end{aligned}\tag{3}$$

Cotangent methodology

Let $\alpha = \cot A$, $\beta = \cot B$, $\gamma = \cot C$ $(\beta\gamma + \gamma\alpha + \alpha\beta = 1)$, $\Sigma = \alpha + \beta + \gamma$ $(\Sigma \geqslant \sqrt{3})$ and $\Pi = \alpha\beta\gamma$ $\left(\Pi \leqslant \frac{1}{3\sqrt{3}}\right)$. Then

$$\begin{aligned}\frac{1}{\sin A\sin B\sin C} &= \sqrt{(1+\alpha^2)(1+\beta^2)(1+\gamma^2)}\\ &= (\beta+\gamma)(\gamma+\alpha)(\alpha+\beta) = \Sigma-\Pi. \end{aligned}\tag{4}$$

Also,

$$\begin{aligned}\sin^2A+\sin^2B+\sin^2C &= \frac{1}{1+\alpha^2}+\frac{1}{1+\beta^2}+\frac{1}{1+\gamma^2}\\ &= \frac{2(\alpha+\beta+\gamma)}{(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)} = \frac{2\Sigma}{\Sigma-\Pi}. \end{aligned}\tag{5}$$

So using (4, 5) in (3) leads to

$$(PA+PB+PC)^2 = 4R^2\frac{\Sigma+\sqrt{3}}{\Sigma-\Pi},$$

that is,

$$PA+PB+PC = 2R\sqrt{\frac{\Sigma+\sqrt{3}}{\Sigma-\Pi}}. \tag{6}$$

Now since $\Sigma \geqslant \sqrt{3}$ and $\Pi \leqslant \frac{1}{3\sqrt{3}}$, we have

$$\frac{\Sigma+\sqrt{3}}{\Sigma-\Pi} = \frac{1+\sqrt{3}/\Sigma}{1-\Pi/\Sigma} \leqslant \frac{9}{8}$$

and

$$PA+PB+PC \leqslant 3R,$$

where the upper bound $3R$ is achieved only in the equilateral case when $\Sigma = \sqrt{3}$ and $\Pi = \frac{1}{3\sqrt{3}}$.

For the half-equilateral triangle (angles $\frac{1}{2}\pi$, $\frac{1}{3}\pi$, $\frac{1}{6}\pi$) we have $PA+PB+PC = \sqrt{7}\,R$, and $\left(1+\sqrt{3}\right)R$ for the half-square (angles $\frac{1}{2}\pi$, $\frac{1}{4}\pi$, $\frac{1}{4}\pi$).

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Concluding remark

Whereas $QA = QB + QC \geqslant PA + PB + PC$ for any point Q when P is an internal point, areal coordinates may be used to show that $QA^2 + QB^2 + QC^2 \geqslant GA^2 + GB^2 + GC^2$ for the centroid G.

Reference

1. G. Leversha, *The geometry of the triangle*, UKMT (2013) pp.144-147.

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106.49 An area problem using barycentric coordinates

In [1], the following question was raised: Given any triangle, show how to construct three Cevians such that the four triangles *ANE*, *BLF*, *CMD* and *LMN* in Figure 1 have equal areas. Answer: The three Cevians divide each side in the ratio $\Phi : 1$, where Φ is the golden ratio. In [1] and [2], the answer was derived by applying Ceva's theorem and by a Cartesian coordinate argument. In this Note, we give a more natural argument using barycentric coordinates and prove a more general statement (Theorem 1).

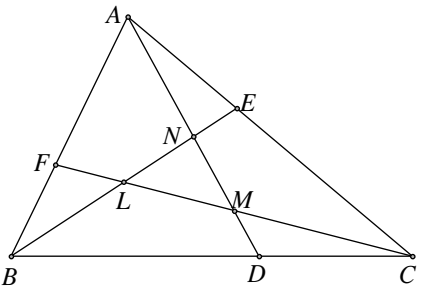


FIGURE 1

In absolute areal (barycentric) coordinates, with respect to the basic triangle ABC, we get $A = (1, 0, 0)$, $B\,(0, 1, 0)$ and $C = (0, 0, 1)$, and the points D, E and F on the lines BC, CA and AB are $D = (0, d, d')$, $E = (e', 0, e)$ and $F = (f, f', 0)$, where $d + d' = 1$, $e + e' = 1$ and $f + f' = 1$. These points are on the sides of the triangle ABC if, and only if, $0 < d, e, f < 1$. Let us consider the points

$$\begin{aligned} L &= \left(\frac{e'f}{1 - ef'}, \frac{e'f'}{1 - ef'}, \frac{ef}{1 - ef'}\right), \\ M &= \left(\frac{fd}{1 - fd'}, \frac{f'd}{1 - fd'}, \frac{f'd'}{1 - fd'}\right), \\ N &= \left(\frac{d'e'}{1 - de'}, \frac{de}{1 - de'}, \frac{d'e}{1 - de'}\right). \end{aligned} \tag{1}$$

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