Technical note

A theoretical description of viscous flow along a flat plate

R. C. Hastings
(retired)
formerly Royal Aircraft Establishment, UK

1.0 INTRODUCTION

Theoretical quantification of viscous effects in fluid flows is difficult, even if turbulence is absent, except when it is legitimate to simplify the Navier-Stokes equations in some way; for example by invoking the boundary-layer approximation[1,2,3] in appropriate cases of interacting viscous and inviscid flow. The technical importance of viscous effects was thought sufficient incentive to re-examine a very simple flow configuration — namely plane, uniform and steady flow of an incompressible, viscous fluid toward a vanishingly-thin flat plate aligned with the undisturbed stream — in search of fresh insights into the general theory for viscous-inviscid interactions.

The strategy was to exploit the analogy between vorticity transport in a viscous fluid and heat conduction in a moving solid[2,4]. The key to doing so was the realization that, if the perturbation of the undisturbed flow by the plate might be represented as the sum of a series of successive approximations, then the stream function of the viscous part of the flow field — not merely the vorticity which resulted from its existence — might be expressible at every stage as the solution of an analogous heat conduction problem.

The perturbation was found to consist of continuous distributions of disturbances which it is convenient to call drag poles and mixing eddies. Drag poles, which are concentrated sources of drag from Oseen theory[2,3,5,6], are placed on the plate itself. Mixing eddies, whose stream functions are suitably-signed source functions from the theory of heat conduction in a moving solid, are placed in the flow surrounding the plate; they modify the flow induced by the drag poles to account for the non-linear aspects of the flow dynamics.

2.0 ANALYSIS

2.1 Formulation of problem

In a Cartesian frame of reference, let a uniform stream, having static pressure p_0 and velocity u_0 parallel to the x-axis, flow past an infinitely-thin flat plate with its leading edge at the origin and its trailing edge at c on the x-axis. Denote the density and kinematic viscosity of the fluid by constants ρ_0 and ν_0 respectively. Local values of velocity components and static pressure are u, v (parallel respectively to x, y) and p.

The equation of continuity of mass is satisfied by the introduction of a stream function such that:

$$u = \partial\psi/\partial y,\ v = -\partial\psi/\partial x$$

Vorticy, ω, is defined by

$$\omega = \partial v/\partial x - \partial u/\partial y = -(\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2)$$

Dimensionless working variables are defined as follows, all except the last three in terms of physical quantities already introduced:

$$\begin{array}{l} X = u_0 x/\nu_0,\ Y = u_0 y/\nu_0,\ U = u/u_0,\ V = \nu/u_0 \\ R_{ec} = u_0 c/\nu_0,\ C_p = 2(p - p_0)/p_0 u_0^2,\ \Psi = \psi/\nu_0 \\ \Omega = \nu_0\omega/u_0^2,\ \mathrm{M} = m/\nu_0,\ S,\ \sigma \end{array} \quad \ldots (1)$$

Paper No. 2985. Manuscript received 16 June 2005, revised version received 30 August 2005, accepted 26 September 2005.

https://doi.org/10.1017/S0001924000000920 Published online by Cambridge University Press

The dimensionless chord has been denoted by R_{ec} because it, like any length in the XY plane, may be regarded as a Reynolds number. The quantities, M and S, are respectively drag-pole strength for dimensional strength m and mixing-eddy density, the term density signifying strength per unit area of the XY plane. Finally σ, whose significance is clear from Equation (12) below, will be called a seeding density, density again denoting a strength per unit area as it did for S.

It follows that the relationships between dimensionless stream function, velocity components and vorticity become:

$$U = \partial\Psi/\partial Y,\ V = -\partial\Psi/\partial X$$

and

$$\Omega = \partial V/\partial X - \partial U/\partial Y = -\nabla^2\Omega$$

if $\nabla^2 = \partial^2/\partial X^2 + \partial^2/\partial X^2$

Conventional differentiations and combinations of the Navier-Stokes equations for momentum transport in the X and Y directions lead to: the vorticity transport equation

$$U\partial\Omega/\partial X + V\partial\Omega/\partial Y = \nabla^2\Omega, \quad \ldots (2)$$

and an equation defining the static pressure field in terms of the velocity field

$$\nabla^2 C_p = -\nabla^2(U^2 + V^2) + 2\partial(V\Omega)/\partial X - 2\partial(U\Omega)/\partial Y \quad \ldots (3)$$

The boundary conditions for the flat plate problem in this dimensionless space are:

for $X \to -\infty$ and all Y

also for $Y \to \pm\infty$ and all X

$$\begin{aligned} U &\to 1 \\ V &\to 0 \\ C_p &\to 0 \end{aligned} \quad \ldots (4)$$

and for $Y = 0$ and $0 < X < R_{ec}$

$$U = V = 0. \quad \ldots (5)$$

It follows that the X axis is a streamline which it is convenient to take as a datum for ψ by setting:

$$\Psi = 0 \text{ on } Y = 0 \quad \ldots (6)$$

Now divide Ψ into three parts representing firstly the undisturbed stream in accord with Equations (4) and (6); then potential-flow, Ψ_p, and viscous, Ψ_v, parts of the perturbation:

$$\Psi = Y + \Psi_p + \Psi_v \quad \ldots (7)$$

By definition,

$$\nabla^2\Psi_p = 0, \quad \ldots (8)$$

and it is assumed that it will be useful to define S by:

$$\partial\Psi_v/\partial X - \nabla^2\Psi_v = S \quad \ldots (9)$$

It follows by differentiation of Equation (9) that:

$$\partial\Omega/\partial X - \nabla^2\Omega = -\nabla^2 S \quad \ldots (10)$$

Equations (10) and (2) are compatible only if:

$$\nabla^2 S = \sigma \quad \ldots (11)$$

where

$$\sigma = (U-1)\,\partial\Omega/\partial X + V\partial\Omega/\partial Y \quad \ldots (12)$$

In principle the above formulation of the problem allows the flow field to be determined, provided that there is a solution of Equation (11) which enables Equation (9) to be solved in a way compatible with the boundary conditions for the flow. As will now be shown, this can be done if the stream function may be expressed as a series of successive approximations.

2.2 Development of solution

Let Ψ be expanded in a series of approximations whose orders are indicated by subscripts.

Thus: $\Psi = Y + \sum_{r=1}^{\infty}\Psi_r$. $\quad \ldots (13)$

Similarly put:

$$\begin{aligned} U &= 1 + \sum_{r=1}^{\infty} U_r\ , \quad V = \sum_{r=1}^{\infty} V_r\ , \quad \Omega = \sum_{r=1}^{\infty}\Omega_r \\ S &= \sum_{r=2}^{\infty} S_r\ , \quad \sigma = \sum_{r=2}^{\infty}\sigma_r \end{aligned} \quad \ldots (14)$$

Finally, split Ψ_r into a potential part of Ψ_{rp} and, instead of the single Ψ_v of Equation (7), two viscous parts, Ψ_{ra} and Ψ_{rb},

where for all r

$$\nabla^2\Psi_{rp} = 0, \quad \ldots (15)$$

$$\Psi_{ra}/\partial X - \nabla^2\Psi_{ra} = 0 \quad \ldots (16a)$$

and for $r \geq 2$

$$\Psi_{rb}/\partial X - \nabla^2\Psi_{rb} = S_r \quad \ldots (16b)$$

Differentiation of Equations 16a) and (16b) leads to

$$\partial\Omega_r/\partial X - \nabla^2\Omega_r = 0 \text{ when } \Omega_r = -\nabla^2\Psi_{ra} \quad \ldots (17a)$$

and

$$\partial\Omega_r/\partial X - \nabla^2\Omega_r = -\nabla^2 S_r \text{ when } \Omega_r = -\nabla^2\Psi_{rb} \quad \ldots (17b)$$

On the other hand, substituting the Series (14) in Equation (2) produces the sequence of equations:

$$\partial\Omega_1/\partial X - \nabla^2\Omega_1 = 0, \text{ for } r = 1 \quad \ldots (18a)$$

and

$$\partial\Omega_r/\partial X - \nabla^2\Omega_r = -\sigma_r, \text{ if } r > 1 \quad \ldots (18b)$$

$$\text{where } \sigma_r = \sum_{q=1}^{r-1}\left(U_q\partial\Omega_{r-q}/\partial X + V_q\partial\Omega_{r-q}/\partial Y\right) \quad \ldots (19)$$

Comparison of Equations (17b) and (18b) shows that a pre-requisite to the determination of Ψ_{rb} is the determination of S_r from

$$\nabla^2 S_r = \sigma_r \quad \ldots (20)$$

Equation (20), unlike Equation (11) which it supersedes in this

https://doi.org/10.1017/S0001924000000920 Published online by Cambridge University Press

context, is a soluble Poisson equation for any r because σ_r is, by Equation (19), always composed of quantities obtainable from approximations of lower order.

The solution for the velocity field can now be completed. A first approximation is given by Oseen's linearised theory of viscous flow, represented here by Equations (15) and (16a). The appropriate solution $\Psi_1 = (\Psi_{1p} + \Psi_{1a})$ for the perturbation of the uniform stream consists of a continuous distribution of disturbances, which will be called drag poles, along the plate. This solution is capable of satisfying the boundary conditions of the problem and it is convenient to use it to meet the condition (5) that $U = V = 0$ on the plate. It also yields quantities needed to synthesise σ_2 and S_2 from Equations (19) and (20) for $r = 2$, firstly to provide the basis for a second approximation and secondly to contribute to all higher-order approximations for σ and S.

The heart of the second approximation, and of all higher-order approximations since they are formed in the same way, is a continuous distribution of the disturbances which will be called mixing eddies. There are eddies wherever S is non-zero (essentially in the boundary layers of the plate and, if its chord is finite, its wake) and those below the plate are the images of those above. The imaging pairs of eddies induce velocities along the plate and so drag poles have to be included in each approximation to restore the zero-velocity boundary condition at the plate and consequently to modify its drag.

Details of drag poles, mixing eddies and their relation to heat-conduction theory are given in the Appendix. The choice of names for the disturbances is intended to emphasise their main physical effects: drag poles supply drag forces while mixing eddies redistribute velocities wherever this is necessary to secure momentum balance in the viscous-flow region.

The complete solution, incorporating expressions from Equations (A1), (A2), (A4) and (A5) from the Appendix, may be written:

$$U = 1 + \sum_{r=1}^{r\max} \int_0^{R_{ec}} M_r'(t) U_m(X,Y,t)\,\mathrm{d}t + \sum_{r=2}^{r\max} \iint S_r(t_1,t_2) U_s(X,Y,t_1,t_2)\,\mathrm{d}t_1\mathrm{d}t_2 \qquad \ldots (21)$$

$$V = \sum_{r=1}^{r\max} \int_0^{R_{ec}} M_r'(t) V_m(X,Y,t)\,\mathrm{d}t + \sum_{r=2}^{r\max} \iint S_r(t_1,t_2) V_s(X,Y,t_1,t_2)\,\mathrm{d}t_1\mathrm{d}t_2 \qquad \ldots (22)$$

in which $M'_r(t)$ is the rth approximation to the drag-pole strength per unit length at t, and the local eddy density $S_r(t_1,t_2)$ is obtained from a preliminary solution of Equation (20) using σ_r from Equation (19)

$$S_r(X,Y) = (1/4\pi) \iint \sigma_r(t_1,t_2) \ln [(X-t_2)^2 + (Y-t_2)^2]\,\mathrm{d}t_1\mathrm{d}t_2 \qquad \ldots (23)$$

The double integrals in Equations (21) to (23) are to be evaluated over the parts of the XY plane where S and σ are non-zero. The solution is expressed in the form of velocity components, rather than a stream function, because there is no convenient form for a drag pole's stream function[3,6,8].

The summations in Equations (21) and (22), as well as in Equation (24) below, have been written as extending to a notional maximum value of r — namely r_{max}. One reason for this is that it is not clear whether or not the series of approximations converges. If the series converges the theory is exact and r_{max} may be taken to be infinite. A second reason for introducing r_{max} is that whether or not the theory is exact, solutions are likely to be computed by summing a finite number of approximations.

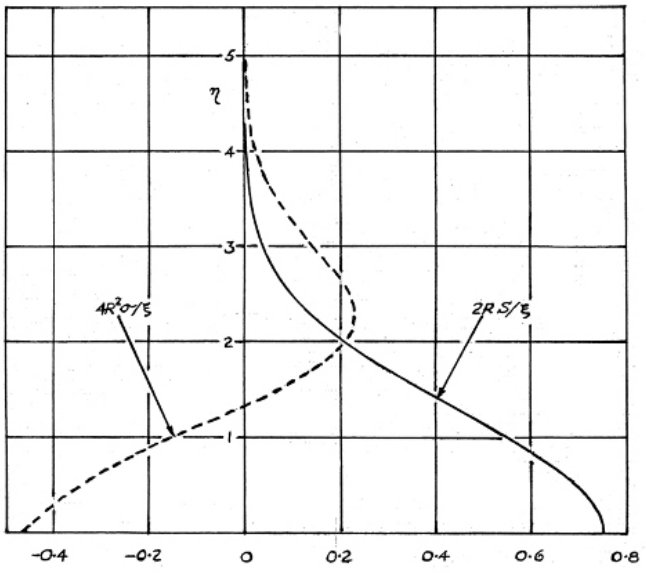


Figure 1. Mixing-eddy parameters σ and S for the semi-infinite plate at high Reyolds number (Blasius flow).

It is noted here for completeness (a) that once the flow field has been calculated, the associated static pressure distribution may be obtained by solving Equations (3) as a Poisson equation, and (b) that the drag coefficient of a finite plate follows, via Equation (A3), as:

$$C_D = (2/R_{ec}) \sum_{r=1}^{r\max} \int_0^{R_{ec}} M_r'(t)\,\mathrm{d}t\ . \qquad \ldots (24)$$

3.0 DISCUSSION

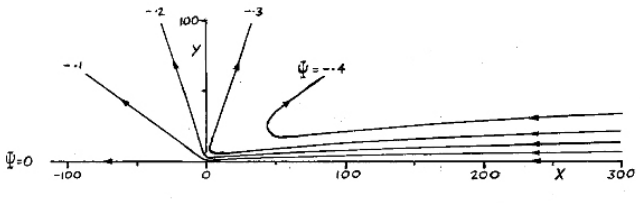
Implementation of the theory just outlined is beyond the scope of this note: however it is presumed that this will take the form of computerized flow models in which the continuous disturbance distributions leading to Equations (21) and (22) will be replaced by populations of discrete, finite-strength disturbances. It is useful, therefore, to consider both the likely characteristics of the parameters σ and S, which determine the strengths of the disturbance flow fields, and also the extent of the region influenced by a single disturbance.

Boundary-layer theory offers an insight into the behaviour of σ and S because, except when chord Reynolds numbers are low (and mixing eddies not very important), the flow along much of the chord of a plate will resemble Blasius boundary-layer flow[3,6] closely enough to provide guidance on the magnitudes and spatial distributions to be expected. The Blasius solution, couched in Kaplun's optimal parabolic variables[3,6,7] may be considered to be an exact solution from which σ and S may be evaluated: S from Equation (9) and σ either by differentiating S or from Equation (12). Dimensionless Kaplun variables are $\xi = \pm(R + X)^{1/2}$, where the signs refer to Y respectively $\geq 0 +$ or $\leq 0 -$, and $\eta = +(R - X)^{1/2}$. It can easily be shown that $2RS/\xi$ and $4R^2\sigma/\xi$ are functions of η only and take the forms shown in Fig. 1.

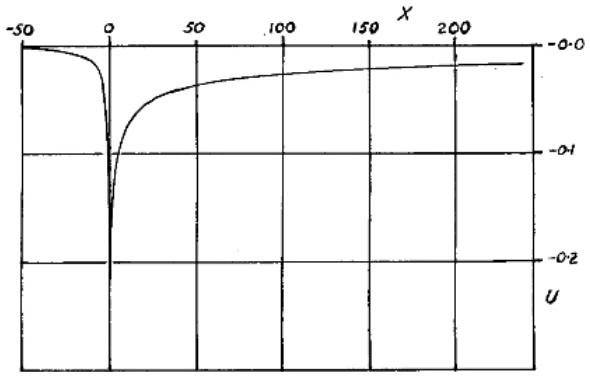
As may be seen from Figs 2 and 3, the region where either a drag pole or a mixing eddy exerts significant influence is very small in Reynolds number terms. For this reason it is expected that basic computational modules in flow models will not be individual disturbances but groups of them.

The upper half of the streamline pattern for a unit drag pole at $X = Y = 0$ is sketched in Fig. 2(a). This dimensionless pattern is essentially universal: only the Ψ-values assigned to streamlines would change

https://doi.org/10.1017/S0001924000000920 Published online by Cambridge University Press



(a) Upper half of the streamline pattern fairly near the pole.



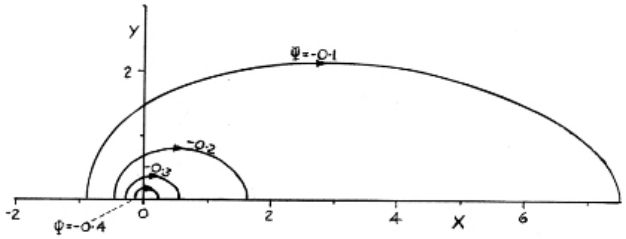
(b) Distribution of U along the X-axis.

Figure 2. Dimensionless flow properties for an isolated unit drag pole at the point (0,0).

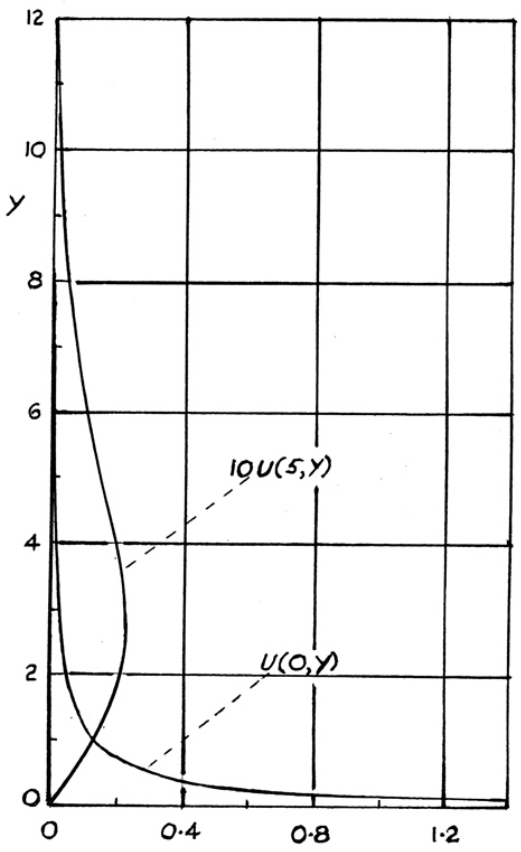
proportionately if the pole-strength were changed. For $X < 0$, and everywhere at large $|Y|$, the flow is predominately the outflow from the potential-flow source; in contrast, when $X > 0$ and $|Y|$ is somewhat smaller, viscous inflow dominates. The distribution of velocity along the X-axis in Fig. 2(b) emphasises how the really strong influence of the pole is confined to a very small region (Reynolds number based on distance from the pole well below unity) surrounding the pole. The lengthy region of weak viscous influence to the right of the pole will, because the pole can exist only in the presence of a left-to-right stream, always combine with that stream to form a wake.

Figure 3(a) is similar to Fig. 2(a) in showing a dimensionless flow pattern, but this time the upper halves of selected streamlines near a positive, unit, mixing eddy at $X = Y = 0$ are illustrated. It should be noted that the scales of the patterns in Figs 2(a) and 3(a) differ greatly although both cover the same range of stream function. This is because the eddy, being a purely viscous circulating flow which vanishes fairly quickly as R increases, is much the more compact of the two types of disturbance. The upper halves of U-velocity profiles at $X = 0$ and $X = 5$ are shown in Fig. 3(b). It should be noted that velocities would be reversed in the lower halves of the profiles and also that velocities at $X = 5$ have been multiplied by 10 for clarity.

Overall, the flow field method which is described here is believed to possess certain advantages. Firstly it offers simplicity because it avoids the direct solution of the vorticity transport equation as a quasi-linear, fourth-order, partial differential equation for the stream function of the flow. That equation is replaced by a sequence of pairs of linear, second-order equations, whose formal solutions can be written down immediately. Secondly it is possible to visualise how, at least in principle, the solution to the flat-plate problem might be generalised to make it apply to other configurations and, perhaps, time-dependent flows. Such extensions may be possible because the basic disturbances, here called drag poles and mixing eddies, from



(a) Upper half of the streamline pattern close to the (positive) eddy.



(b) Upper halves of profiles of U in the Y-direction at $X = 0$ and $X = 5$.

Figure 3. Dimensionless flow properties for an isolated unit mixing eddy at the point (0,0).

which the flat-plate solution is synthesised, are merely particular simplifications of functions which are, at their most general, both three-dimensional and time-dependent.

It is not known whether the proposed series solution for the flow field converges; nor is it known whether it is important that it should. Regarding the latter consideration, Van Dyke[6] has emphasised that convergence is often unimportant in perturbation solutions of physical problems. It seems likely that the utility of the theory will have to be assessed empirically by computing solutions for particular cases.

https://doi.org/10.1017/S0001924000000920 Published online by Cambridge University Press

4.0 CONCLUSIONS

- An analytical method has been described for determining the perturbation of a steady, uniform stream of incompressible, viscous fluid by the presence of an infinitely-thin, flat plate aligned with the undisturbed flow.
- The perturbation consists of a series of approximations; each except the first, and thus the perturbation as a whole, consists of continuous distributions of two types of disturbances. These have been named drag poles and mixing eddies.
- Implementation of the theory will presumably require modelling of the perturbation by populations comprising groupings of finite-strength poles and eddies. The first approximation consists of drag poles only.
- Extension of the present method to cover more general problems, including those involving three-dimensional and time-dependent flow, may be possible.

REFERENCES

1. PRANDTL, L. The mechanics of viscous fluids, *Aerodynamic Theory* 1935, DURAND, W.F. (ED). Vol 3, Springer, Berlin; also 1963, Dover, New York.
2. GOLDSTEIN, S. (ED). *Modern Developments in Fluid Dynamics*, 1938, Clarendon Press, Oxford; also 1965, Dover, New York.
3. ROSENHEAD, L. (ED). *Laminar Boundary Layers*, 1963, Clarendon Press, Oxford.
4. CARSLAW, H.S. and JAEGER, J.C. *The Conduction of Heat in Solids*, 1959, 2nd edition, Clarendon Press, Oxford.
5. OSEEN, C.W. *Hydrodynamik*, 1927, Akad Verlag, Leipzig.
6. VAN DYKE, M.D. *Perturbation Methods in Fluid Mechanics*, 1975, Annotated edition, Parabolic Press, Stanford.
7. KAPLUN, S. The rôle of co-ordinate systems in boundary-layer theory, Z. agnew, *Math Phys*, March 1954, **5**, pp 111-135.
8. BAIRSTOW, L., CAVE, B.M. and LANG, E.D. The resistance of a cylinder moving in a viscous fluid, *Phil Trans A*, 1923, **223**, pp 383-432.
9. FILON, L.N.G. The forces on a cylinder in a stream of viscous fluid, *Proc Roy Soc A*, 1926, **113**, pp 7-27.

A.1 APPENDIX

A.1.1 Drag poles, mixing eddies and heat conduction theory

The drag-pole solution of Oseen theory (sometimes called an Oseenlet[3,6]), at the point $(t,0)$ and having unit strength, has the following closed-form velocity components[3,6]:

$$U_m(X, Y, t) = ((1/2\pi)\{X_m / R^2_m - 0{\cdot}5\exp(X_m/2)\,[K_0(R_m/2) + (X_m / R_m)\,K_1(R_m/2)]\} \quad \ldots (A1)$$

$$V_m(X, Y, t) = (1/2\pi)\{(Y / R^2_m - 0{\cdot}5\exp(X_m/2)[(Y / R_m)\,K_1(R_m/2))]\} \quad \ldots (A2)$$

in which the first term of each comes from a potential-flow source $\theta_m/2\pi$ and where K_0 and K_1 are Bessel functions in standard notation,

while $X_m = X - t$, $R_m = (X^2_m + Y^2)^{1/2}$ and $\theta_m = \mathrm{Tan}^{-1}(Y/X_m)$.

The drag force per unit length normal to the XY plane, generated by a source whose strength is M, in a uniform stream whose velocity is u_0 , can be shown to be[9]:

$$D = \rho_0 \nu_0 u_0 M \quad \ldots (A3)$$

If the dependent variables in Equations (16a) and (16b) had been temperatures instead of stream functions, each of these equations would have been a dimensionless form of that governing conduction of heat in a solid moving at constant speed parallel to the X-axis[4]. The non-zero right side of Equation (16b) would represent a heat-source strength per unit area in the solid; consequently the solution of Equation (16b) would consist of two-dimensional point heat-source functions having the form $(1/2\pi)\exp(X/2)\,K_0(R/2)$, distributed over the XY plane. The same point function, interpreted as a stream function, generates an anticlockwise swirling flow which is defined here to be a negative mixing eddy. The convention that clockwise swirl be positive suits the analysis of Section 2: as may be seen from Figs 2 and 3, positive eddies, so defined, placed above the plate would assist the drag poles on the plate to oppose the oncoming stream and meet the condition that $U = 0$ on the plate.

It follows that a positive mixing eddy, at the point $(t_1, t_2)$and having unit strength, has the following velocity components:

$$U_s(X, Y, t_1, t_2) = (1/4\pi)\exp(X_s / 2)\,[(Y_s / R_s)\,K_1(R_s / 2)] \quad \ldots (A4)$$

$$V_s(X, Y, t_1, t_2) = (1/4\pi)\exp(X_s / 2)\,[K_0(R_s / 2) - (X_s / R_s)\,K_1(R_s / 2)] \quad \ldots (A5)$$

if $X_s = X - t_1$, $Y_s = Y - t_2$ and $R_s = (X^2_s + Y^2_s)^{1/2}$.

https://doi.org/10.1017/S0001924000000920 Published online by Cambridge University Press