French mathematician Georges Dostor [3, p. 73]. As a corollary we have that the area of a parallelogram with sides of lengths a and b is given by

$$K = \tfrac{1}{4}\sqrt{(p+q+2a)(p+q-2a)(p+q+2b)(p+q-2b)}$$

where p and q are the lengths of the diagonals [3, p.72].

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100.32 Linear quadrisection and the Kiepert hyperbola

For a general scalene triangle ABC, the Cartesian coordinate system provided by the asymptotes of the associated Kiepert hyperbola has one obvious advantage, namely, that a point $P\left(x, \frac{1}{x}\right)$ is instantly recognised as lying on this curve ([1]).

In this Note we will locate two significant points lying on different branches of the hyperbola and also encounter three examples of linear quadrisection. However, we opt just to outline some of the routine coordinate geometry.

In our Cartesian framework, the vertices $A\left(a, \frac{1}{a}\right)$, $B\left(b, \frac{1}{b}\right)$, $C\left(c, \frac{1}{c}\right)$, centroid $G\left(g, \frac{1}{g}\right)$ and orthocentre $H\left(h, \frac{1}{h}\right)$ lie on the curve $xy = 1$. Thus $3g = a + b + c$, $\frac{3}{g} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ with $abch = -1$ for rectangularity. The origin $O(0, 0)$ is the Kiepert centre and g, h and two of the parameters a, b, c may be taken as positive.

We will need the following result.

Lemma: We have

$$\begin{aligned}(b+c)(c+a)(a+b) &= (3g-a)(3g-b)(3g-c)\\ &= 27g^3 - 9(a+b+c)g^2 + 3(bc+ca+ab)g - abc\\ &= \left(3\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)g - 1\right)abc\\ &= 8abc = -\frac{8}{h}.\end{aligned}$$

The triangle $A'B'C'$

The tangents to any circumscribing rectangular hyperbola at the vertices A, B, C are given by the equations $a^2y + x = 2a$, $b^2y + x = 2b$, $c^2y + x = 2c$, respectively, and these may be shown to meet in pairs at the vertices of the triangle $A'B'C'$ with coordinates

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$$A'\left(\frac{2bc}{b+c}, \frac{2}{b+c}\right), \qquad B'\left(\frac{2ca}{c+a}, \frac{2}{c+a}\right), \qquad C'\left(\frac{2ab}{a+b}, \frac{2}{a+b}\right).$$

Now the lemma may be used to show that the centroid of the triangle $A'B'C'$ is given by

$$G'\left(\frac{g}{4} - \frac{3}{4g^2h}, \frac{1}{4g} - \frac{3g^2h}{4}\right),$$

where the point $P\left(-\frac{1}{g^2h}, -g^2h\right)$ lies on the negative branch of the Kiepert hyperbola.

Clearly G' is the point of quadrisection of the chord GP nearer to P.

The centroid partition

We next consider the circumcentres of the three triangles BCG, CAG, ABG partitioned by the centroid G. These may be shown to be given respectively by

$$O_A\left(2g - \frac{a(g+h)}{2g}, \frac{2}{g} - \frac{g+h}{2ah}\right),$$
$$O_B\left(2g - \frac{b(g+h)}{2g}, \frac{2}{g} - \frac{g+h}{2bh}\right),$$
$$O_C\left(2g - \frac{c(g+h)}{2g}, \frac{2}{g} - \frac{g+h}{2ch}\right).$$

Another routine calculation leads to the coordinates for the circumcentre O^* (say) of the triangle $O_AO_BO_C$, namely

$$O^*\left(\frac{3g}{2} - \frac{3h}{4} + \frac{g^2}{4h}, \frac{3}{2g} - \frac{3}{4h} + \frac{h}{4g^2}\right).$$

The above coordinates for O^* reveal the point $Q\left(\frac{g^2}{h}, \frac{h}{g^2}\right)$ on the positive branch of the hyperbola where G lies between H and Q. Moreover, O^* is the point of quadrisection of QU nearer to $U\left(2g - h, \frac{2}{g} - \frac{1}{h}\right)$ which point is clearly the reflection of H in G on the Euler line.

The van Lamoen circle

The circumcentre of the six triangles partitioned by the medians of a triangle ABC are concyclic which fact was discovered by van Lamoen in 2000. Then (as explained in [2, Chap. 24]) this circle is a Tucker circle of the triangle $O_AO_BO_C$ whose centre O^{**} (say) lies on the Brocard diameter K^*O^* of this triangle where the Lemoine point K^* is the centroid G.

Finally, the reader is invited to use synthetic methods to establish our third quadrisection which is given by

$$[O^{**}] = \frac{3}{4}[G] + \frac{1}{4}[O^*].$$

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100.33 A surprising relationship between Pell-Lucas, Pell, square and centred square numbers

For a positive integer $k \geqslant 3$, the nth term of the kth polygonal number, $F(n; k)$, and the nth term of the kth centred polygonal number, $C(n; k)$, are given by

$$F(n; k) = \frac{(k-2)n^2 + (4-k)n}{2},$$

$$C(n; k) = \frac{kn^2 - kn + 2}{2},$$

respectively. It was shown in [1] that the numbers that are both k-gonal and centred k-gonal are given by

$$u(n;k) = \frac{k}{16(k-2)}\left\{\frac{-2k^2+18k-32}{k} + \left[k-1+\sqrt{k(k-2)}\right]^{2n+1} + \left[k-1-\sqrt{k(k-2)}\right]^{2n+1}\right\}$$

for $n \geqslant 0$ and $k \geqslant 3$. The Pell numbers are defined recursively by $P_0 = 0$, $P_1 = 1$ and $P_{n+2} = 2P_{n+1} + P_n$ for all $n \geqslant 0$. The Pell-Lucas numbers are defined recursively by $Q_0 = 2$, $Q_1 = 2$ and $Q_{n+2} = 2Q_{n+1} + Q_n$ for all $n \geqslant 0$. The purpose of this Note is to show an unexpected relationship between Pell-Lucas numbers, Pell numbers and $u(n; 4)$ – the square numbers that are centred square numbers.

The Binet formula for Q_n is

$$Q_n = r^n + s^n$$

for $n \geqslant 0$ where $r = 1 + \sqrt{2}$ and $s = 1 - \sqrt{2}$. Since $r^2 = 3 + 2\sqrt{2}$, $s^2 = 3 - 2\sqrt{2}$ and $rs = -1$,

$$\begin{aligned} u(n; 4) &= \frac{1}{8}\left[2 + \left(3 + 2\sqrt{2}\right)^{2n+1} + \left(3 - 2\sqrt{2}\right)^{2n+1}\right] \\ &= \frac{1}{8}\left[2 + \left(r^2\right)^{2n+1} + \left(s^2\right)^{2n+1}\right] \qquad (1) \\ &= \frac{1}{8}\left(2 + Q_{4n+2}\right). \end{aligned}$$

Since $Q_m^2 - Q_{2m} = 2(-1)^m$ and $Q_m^2 - 8P_m^2 = 4(-1)^m$, (1) can be written as

$$u(n; 4) = \tfrac{1}{8}\left(4 + Q_{2n+1}^2\right) = P_{2n+1}^2.$$

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