HYPERELLIPTIC THREEFOLDS

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It is a classical result (5) that the algebraic surfaces which admit a continuous group of birational self-transformations, or automorphisms, and which are neither rational nor scrollar, belong to one or other of two families: (i) the elliptic surfaces; these possess an elliptic group of ∞^1 automorphisms; (ii) the hyperelliptic surfaces of rank 1 (which, for brevity, we shall call hyperelliptic); these possess a completely transitive permutable group of ∞^2 automorphisms. The chief properties of the two classes of surface, in so far as they are required in the present work, are described in § 1.

The algebraic threefolds analogous to such surfaces fall into one or other of three categories:

(1) *Elliptic threefolds*; these admit a continuous group of ∞^1 automorphisms whose trajectories form a *congruence* (∞^2 system of index 1) of birationally equivalent elliptic curves.

(2) *Hyperelliptic threefolds*; these possess a continuous group of ∞^2 automorphisms whose trajectories form a pencil (rational or irrational) of hyperelliptic surfaces.

(3) *Abelian* (*Picard*) *threefolds* (of genus 3), which admit a completely transitive permutable continuous group of ∞^3 automorphisms.

It is an interesting fact that all these threefolds have (absolute) linear genus unity and also arithmetic genus unity; they therefore belong to the extensive class of threefolds of linear genus unity which has been surveyed in (9). The Abelian threefolds, like the hyperelliptic surfaces, are particular examples of Abelian varieties V_p of genus p, and many of their properties, obtained by transcendental methods, may be inferred from the general theory of those varieties (2, 7, 8, 11). The elliptic threefolds have been studied from the geometrical point of view in (10); and the present work is devoted to a similar study of the hyperelliptic type.

We first show that any hyperelliptic threefold V contains, not only the pencil $\{C\}$ of hyperelliptic surfaces already mentioned, but also a congruence Γ of birationally equivalent curves $\mathscr{C}$. These meet the generic surface C in d ($\geqslant 1$) points; the number d, called the determinant of V, plays an essential part in the theory. The curves of Γ cut on C an involution I_d of order d, which is without united points, and whose image is a surface C^* which is properly or improperly hyperelliptic. We may thus represent V on a d-ple threefold V^* which is the product of C^* and a curve $\mathscr{C}^*$ whose genus is that of the pencil $\{C\}$; the coincidence surface of the representation consists of a number of algebraically isolated surfaces, each suitably counted, which are submultiples of C; thus in general V is endowed with divisor σ greater than unity.

From the mapping of V on V^* we deduce equivalences for the canonical and pluri-

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canonical systems of V, and so calculate the geometric genus P_g and the plurigenera P_i. These results are then applied to some interesting special cases; in particular, a calculation of P_i ($i = 2, 3, 4, 6$) leads to the theorem: *in order that V should contain a congruence of rational curves, it is necessary and sufficient that $P_{12} = 0$.* This result, which expresses a well-defined geometrical property by the vanishing of an invariant, is the first of its kind in the theory of threefolds. We may also note the result that, in the case where the curves $\mathscr{C}$ are rational, the congruence Γ always possesses a rational pencil of unisecant surfaces, from which it follows that V is then birationally equivalent to a point-cone whose generators map the curves $\mathscr{C}$.

Finally, we consider the analytical representation of V; in order to attack the general problem, it would first be necessary to set up projective models for all the involutions without coincidences which a given hyperelliptic surface can carry. This question has not so far been considered; we have, however, obtained representations in the case where these involutions are cyclic, referring briefly to some particular non-cyclic cases in § 9.

1. *Elliptic and hyperelliptic surfaces.* We begin by recalling the main facts concerning the elliptic and hyperelliptic surfaces, the theory of which is expounded in (5) and (6).

An elliptic surface contains a pencil, of some genus $\rho \geqslant 0$, of elliptic curves $\mathscr{C}$, and an elliptic pencil of curves $\mathscr{K}$, of genus $\pi \geqslant 1$; the pencil $\{\mathscr{C}\}$ will in general contain a certain number of multiple elliptic members but neither $\{\mathscr{C}\}$ nor $\{\mathscr{K}\}$ possesses nodal curves. Conversely, it may be proved that any surface containing two such pencils is elliptic; it is convenient to include in this scheme the case $\pi = 0$, corresponding to the elliptic scrollar surfaces, which are then called improperly elliptic. In the classification of the elliptic surfaces the number $[\mathscr{C}\mathscr{K}] = d$, called the determinant, plays an important part.

The hyperelliptic surfaces (of rank† 1) with general moduli are of two types: the Jacobi surface, which is birationally equivalent to the surface representing the pairs of points of a curve of genus 2; and the Picard surface of divisor δ (> 1), which is mapped on the Jacobi surface by an involution of order δ, free from united points. These surfaces possess a pure canonical curve of order zero; hence, on any such surface which is free from exceptional curves, the virtual grade n and virtual genus π of any curve system satisfy the relation $n = 2\pi - 2$.

The elliptic surfaces, on the other hand, have a canonical curve (effective or virtual) compounded of the pencil $\{\mathscr{C}\}$. The types $\rho = 0, \pi = 1$, which possess a virtual canonical curve of order zero, have a natural parametric representation (of rank greater than unity) by means of hyperelliptic functions; we shall therefore call them *improperly hyperelliptic.* It may be shown that all the elliptic and hyperelliptic surfaces are characterized by the two properties of having absolute linear genus $p^{(1)} = 1$, and arithmetic genus $p_a = -1$; and it may further be proved that the first property is a consequence of the second.

† A hyperelliptic surface is of rank 1 if we can choose parameters (u, v) such that there is a (1, 1) unexceptional correspondence between the points of the surface and the values of (u, v) which are distinct, modulo the periods.

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Any hyperelliptic surface F of general moduli is invariant under two types of automorphism; in terms of the customary parametric representation of F by means of hyperelliptic functions of two variables (u, v), these are expressed by the equations

$$u = u' + a, \quad v = v' + b, \tag{1}$$

$$u = -u' + a, \quad v = -v' + b, \tag{2}$$

where a and b are arbitrary constants. The transformations (1), which will be termed of the first kind, obviously form a continuous two-parameter transitive permutable group; and it is important to observe that any involution generated by such transformations is free from united points. Instead the transformations (2), of the second kind, are involutory, and any involutions which they generate must possess united points. It may be shown that any involution without united points which is carried by F is hyperelliptic, i.e. is mapped by a hyperelliptic surface of rank 1.

If we allow F to have particular moduli, then the above results may be subject to modification. To begin with, while the general hyperelliptic surface, free from exceptional curves, has base number unity, there exist *k-ply singular surfaces* ($k = 1, 2, 3$) of Jacobi or Picard type, whose base number is $k+1$. Thus, in particular, while a general hyperelliptic surface contains no elliptic curves, if the surface is specialized so as to contain one such curve, then it must contain two pencils, both elliptic, of these curves; the pencils are unisecant if, and only if, the surface is of Jacobi type. Again, F may possess other automorphisms than those described above, which are termed singular transformations; and finally, in the case where F contains elliptic curves, it can carry involutions, free from united points, which are mapped by improper hyperelliptic surfaces; such involutions are all cyclic (1).

2. *Hyperelliptic threefolds.* We consider a non-singular threefold V, which admits a continuous permutable group $\mathcal{G}$ of ∞^2 automorphisms. The trajectories of this group must form a pencil $\{C\}$, of some genus $\rho \geqslant 0$, of surfaces C, each of which is invariant under $\mathcal{G}$; and we assume that the generic surface C is irreducible.

We shall suppose that $\mathcal{G}$ operates transitively, without exceptions, on C; this hypothesis, with the previous assumption of permutability, serves to exclude various types of threefold which have already been investigated. It follows then that C must be a hyperelliptic surface; and, unless the contrary is stated, we shall assume that C has general moduli. We shall also assume that C represents biunivocally and unexceptionally the transformations of $\mathcal{G}$, so that the surface is non-singular and free from exceptional curves. It will appear from § 3 that, as a hypothesis of generality, we may suppose that every irreducible member of the pencil $\{C\}$ is non-singular; however, it will also appear that we may expect $\{C\}$ to contain a certain number of reducible members, each consisting of an irreducible surface counted with a certain multiplicity. Evidently the pencil $\{C\}$ is free from base points.

We first show that V *contains a hyperelliptic congruence of birationally equivalent curves.* For, consider any irreducible non-singular curve $\mathcal{K}$, not lying on any surface C, and meeting the generic C in a certain number of distinct points. The transforms of $\mathcal{K}$ under $\mathcal{G}$ form an irreducible ∞^2 system $\{\mathcal{K}\}$ of irreducible non-singular curves, no two of which coincide and which, since they cannot lie on a surface, have an index $\nu \geqslant 1$. If $\nu = 1$, these generate the congruence Γ in question. Supposing, however,

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that $\nu > 1$, consider the set of ν curves $\mathscr{K}$ issuing from the same point P of V; since the curves $\mathscr{K}$ cut on C an involution which is free from united points, the image of the system $\{\mathscr{K}\}$ is itself a hyperelliptic surface C^* (§ 1), in general birationally distinct from C. On C^* the sets of ν curves are mapped by an ∞^3 series of sets of ν distinct points, given by values (u_i, v_i) of the parameters which specify a point of the surface $(i = 1, 2, \ldots, \nu)$. Let P be a general point of V, corresponding to the set $(\bar{u}_i, \bar{v}_i)$ of points on C^*; and consider the sets which satisfy the equivalences

$$\Sigma u_i \equiv \Sigma \bar{u}_i, \quad \Sigma v_i \equiv \Sigma \bar{v}_i. \tag{A}$$

The latter impose at most two conditions on the sets in question; in fact, we shall see that they impose precisely two. Suppose, first, that this is the case; then the locus of P such that the corresponding set of ν points on C^* satisfies (A) is a non-singular curve; and the aggregate of such curves either form, or are compounded of, a congruence Γ. Evidently Γ is without base points and singular points (through which pass ∞^1 curves of the system); the curves $\mathscr{C}$ belonging to Γ are all birationally equivalent, of genus $\pi \geqslant 0$, and cut the generic C in sets of an involution I_d of a certain order $d \geqslant 1$, which is without united points and which is properly hyperelliptic.

Next suppose, if possible, that there are ∞^2 sets of ν points (u_i, v_i) equivalent to the given set $(\bar{u}_i, \bar{v}_i)$; then, reasoning as before, we see that the locus of P, subject to (A), is a surface, and that the aggregate of such surfaces either form, or are compounded of, a pencil without base points. This pencil will cut C in a pencil of curves, also without base points; but, for general values of the moduli, C cannot contain such a pencil.†

Lastly, if all the ∞^3 sets are equivalent, we may readily show that all the curves of $\{\mathscr{K}\}$ are equivalent; this means that, for every pair (u_1, v_1), (u_2, v_2) of points of C^*, we have $u_1 \equiv u_2$, $v_1 \equiv v_2$. In conclusion, then, the only tenable hypothesis is the first, that is, the congruence Γ exists in all cases.

Conversely, it may be shown that any threefold V which contains a pencil $\{C\}$ and a congruence Γ of the types just described is necessarily hyperelliptic; the proof, based on the properties‡ of transformations of the first kind, is exactly similar to that of the analogous result in the theory of the elliptic surfaces (5).

In all that follows, the number $[\mathscr{C}C] = d$, which we call the *determinant* of V, is of capital importance. On any given curve $\mathscr{C}$, the pencil $\{C\}$ cuts an involution γ_d of order d and genus ρ, without fixed points. We observe that *each curve $\mathscr{C}$ is invariant under a permutable group $\mathscr{G}_d$ of automorphisms.* This group consists of those transformations of $\mathscr{G}$ which leave each $\mathscr{C}$ invariant.

The above result, which plays an essential part in the classification of the hyperelliptic threefolds, places no restriction on the genus π of $\mathscr{C}$ but it means that, for $\pi > 2$, $\mathscr{C}$ cannot be of general moduli and also that, for $\pi \geqslant 1$, d can assume only a finite number of values; at present these have been completely determined in the cases $\pi = 1, 2, 3$ alone (see § 9).

† If, however, C contains elliptic curves, this case may arise; V is then a Picard threefold containing three elliptic pencils of Picard surfaces, each with its associated congruence of elliptic curves.

‡ Precisely, there exists a unique automorphism of V which leaves each surface C invariant and which makes correspond any two chosen points on a given surface C.

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3. *Representation on a threefold of determinant unity.* Consider first the case $d = 1$; since the pencil $\{C\}$ is unisecant to each curve $\mathscr{C}$, it follows that $\pi = \rho$, and that the congruence Γ is birationally equivalent to C. We may thus map V on the product $\mathscr{C} \times C$ of a curve $\mathscr{C}$ and a surface C; and, by the assumptions made in § 2, this representation has no exceptional elements.

In order to obtain a representation in the case $d > 1$, we first construct the threefold $V^* = \mathscr{C}^* \times C^*$, where $\mathscr{C}^*$ is a curve of genus ρ, birationally equivalent to the pencil $\{C\}$, and C^* is a surface which maps unexceptionally the congruence Γ; then the generic point of V^* is made to correspond to the set of d points in which the generic surface C meets the generic curve $\mathscr{C}$. We thus obtain a mapping of V on the d-ple threefold V^*.

We have now to determine the branch surface of the representation. As we have seen, the involution I_d cut by Γ on the generic C has no coincidences: hence, either there is no branch surface at all or else the branch surface consists of a certain number of irreducible components, each suitably counted, belonging to the pencil $\{\bar{C}^*\}$ on V^* formed by the surfaces $\bar{C}^*$, each of which maps the product of a fixed point of $\mathscr{C}^*$ and a variable point of C^*; we may, without risk of confusion, denote the generic surface of this pencil by C^*. Let C_s^* denote a typical $(s-1)$-fold component of the branch surface; then this maps a surface C_s on V which is an $(s-1)$-fold component of the coincidence surface and which, counted s times, constitutes a member of the pencil $\{C\}$. The surface C_s is itself invariant under $\mathscr{G}$; if, in conformity with the previous hypothesis of generality, we assume that it is non-singular, we have the result that every curve $\mathscr{C}$ of Γ cuts C_s in d/s distinct points, i.e. Γ cuts on the surface an involution of order d/s, without united points, and also that C_s is mapped unexceptionally on the d/s-ple surface C_s^*, the representation being without branch points.

It is clear that, with the assumptions we have introduced, the representation of V on V^* possesses no other singular features than those just described: with an obvious notation, we have on V a certain number N ($\geqslant 0$) of s_i-fold members C_{s_i} ($i = 1, 2, \ldots, N$) of the pencil $\{C\}$; each surface C_{s_i} is irreducible and non-singular, and each number s_i ($1 < s_i \leqslant d$) is a divisor of d.

The curves of Γ set up a $(d/s, 1)$ correspondence between the points of the surface C_s and the curves of Γ; and since this correspondence has no coincidences, and Γ is hyperelliptic, it follows that C_s is likewise hyperelliptic.

We observe that each surface C_s may be supposed to be algebraically isolated; for if it were variable in a pencil (rational or irrational), V would contain a pencil $\{C_s\}$ of hyperelliptic surfaces which are d'-secant to Γ, where $d' < d$, and which we could substitute for the pencil $\{C\}$ (see § 7).

This remark has an interesting consequence: in the case where two at least of the characters s_i are equal—say $s_j = s_k = s$—we have on V two isolated surfaces C_{sj}, C_{sk} such that $sC_{sj} \equiv sC_{sk} \equiv C$. It follows that V *is endowed with divisor* $\sigma > 1$.

An important relation between the characters s_i is obtained by considering the coincidences in the involution γ_d cut by $\{C\}$ on any curve $\mathscr{C}$; since $\mathscr{C}$ meets each surface C_s in d/s distinct points, we have

$$2(\pi - 1) - 2d(\rho - 1) = d \sum_1^N \frac{s_i - 1}{s_i}. \tag{3}$$

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We can also obtain from the present representation *a lower limit to the superficial irregularity of* V; this is the (constant) irregularity of the generic surface on V, i.e. any non-singular surface which is variable in an ∞^2 system with irreducible free characteristic curve. Consider first the case $d = 1$; here it is known (12) that the irregularity q_2^* in question is the sum of the genus of $\mathscr{C}^*$ and the irregularity of C^*. Since the generic surface on V is in $(d, 1)$ correspondence with a generic surface on V^*, its irregularity satisfies the inequality $q_2 \geqslant q_2^*$ (1); thus in all cases we have $q_2 \geqslant \rho + 2$.

4. *The canonical and pluricanonical systems.* We now suppose that the threefold V^* is free from exceptional surfaces; in the present state of the theory it is not known whether such an assumption is restrictive (9). Consequently C^* must be free from exceptional curves of the first species, since any such curve would give rise to an exceptional surface on V^*.

With this hypothesis, a virtual pure canonical surface K^* of V^* is given by the equivalence $K^* \equiv (2\rho - 2)\, C^*$, where the $2\rho - 2$ surfaces C^* in question form a canonical set in $\{C^*\}$. Now, on V, a virtual pure canonical surface K is equivalent to the sum of the transform of K^* and the coincidence surface; thus

$$K \equiv \Sigma(s-1)\, C_s + (2\rho - 2)\, C,$$

where $sC_s \equiv C$ $(s = s_1, s_2, \ldots, s_N)$. Hence *the virtual canonical system* $|K|$ *of* V *is compounded of the pencil* $\{C\}$; *and the variable part of the surface* K *consists of a canonical set of surfaces in* $\{C\}$.

Also, since each surface $(s-1)\, C_s$ is algebraically isolated, it follows that *the geometric genus* P_g *of* V *is equal to* ρ.

We proceed to calculate the plurigenera P_i, i.e. the dimensions, each augmented by unity, of the systems $|iK|$ $(i = 2, 3, \ldots)$. A virtual i-canonical surface iK is given by

$$iK \equiv \Sigma i(s-1)\, C_s + i(2\rho - 2)\, C,$$

and consists of a certain number λ of surfaces C, together with fixed components hC_s, where h is some positive number less than s. Using the equivalence $sC_s \equiv C$, we have

$$iK \equiv \{iN + 2i(\rho - 1)\}\, C - \Sigma i C_s \equiv \lambda C + \Sigma h C_s.$$

It follows from this that the number $h + i$ must be some positive multiple k of s; and since $0 < h < s$, k must be the greatest integer contained in $(i + s - 1)/s$. We thus find that, with the usual notation,

$$\lambda = i(N + 2\rho - 2) - \Sigma\left[\frac{i+s-1}{s}\right].$$

Since the set λC of surfaces in $\{C\}$ is non-special whenever $\rho > 1$, it follows that, provided $\lambda > 0$, $P_i - 1 = \lambda - \rho$. Hence, when $\rho > 1$,

$$P_i = 1 + i(N + 2\rho - 2) - \rho - \Sigma\left[\frac{i+s-1}{s}\right]. \tag{4}$$

When, however, $\rho = 1$, $N = 0$, we have instead $P_i = 1$; and when $\rho = 0$, we must take P_i to be zero whenever the expression on the right-hand side of (4) is negative.

We next obtain the numerical invariants of V. Now, since the system $|K|$ is compounded of a pencil without base points, its virtual grade Ω_0 is zero; thus, in virtue of Pannelli's equations, $2\Omega_1 - 2 = 3\Omega_0$, $\omega = 4\Omega_0 + 1$,

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its virtual curvilinear genus Ω_1 and its absolute linear genus ω are both unity. Again, since all the components of a surface K (effective or virtual) have arithmetic genus -1, and no two components intersect, the arithmetic genus Ω_2 of K is equal to -1; hence, from Severi's relation, $2P_a = \Omega_0 - \Omega_1 + \Omega_2 + 4$, we deduce that $P_a = 1$. Thus *the absolute linear genus and arithmetic genus of V are each equal to unity.*

It has been remarked in § 1 that all the elliptic and hyperelliptic surfaces have (absolute) linear genus 1 and arithmetic genus -1; similarly, all the elliptic, hyperelliptic and Abelian threefolds have linear genus 1 and arithmetic genus 1. This property has already been established for the elliptic threefolds (10); for the Abelian threefolds it follows from the fact that these have a pure canonical surface of order zero, and that the virtual characters of such a surface are $\Omega_0 = 0$, $\Omega_1 = 1$, $\Omega_2 = -1$, $\omega = 1$. The result holds also in the case where C has special moduli (§ 6).

However, while the elliptic and hyperelliptic surfaces are characterized by the stated values of their invariants, an analogous result does not hold for threefolds; thus the completely regular threefolds which possess a canonical surface of order zero also have linear genus and arithmetic genus unity; and there may possibly be other classes of threefold with the same numerical invariants.

5. *Some particular cases.* We shall now examine some of the simplest and most interesting of the various possibilities, beginning with the case $P_g = 0$.

When $P_g\ (= \rho) = 0$, the multiple model V^* is the product of a hyperelliptic surface C^* and a rational curve; we may therefore take V^* to be a cone (or cylinder) with base C^*. In the special case where $d = 1$, and where C is a Jacobi surface, V belongs to the class of quasi-Abelian varieties, which admit a parametric representation by means of degenerate Abelian functions (13); and when C is a Picard surface of divisor δ, V may be mapped by an involution I_δ, without coincidences, on a quasi-Abelian threefold.

In the case where $\rho = 0, d = 1$, we obviously have $\pi = 0$, and also $P_i = 0\ (i = 1, 2, \ldots)$. In the case where $d > 1$, the analysis of the equations (3) and (4) is identical with that given by Enriques (5) for the elliptic surfaces of geometric genus zero, so that we need only summarize his results. As already remarked, each of the characters s must be a divisor of d; in the present case, by considering the representation on a d-ple cylinder V^*, it may further be proved that each must also divide the least common multiple of the others; and a Diophantine analysis† shows that, *in all cases where $\pi > 0$, at least one of the plurigenera P_4, P_6 is greater than zero.*

Of the types for which $P_g = 0$, those with $\pi = 1$ are specially interesting, since they possess virtual canonical systems of order zero. For, suppose that the system $|iK|$ were effective and of order greater than zero; then it would have to cut null curves on the surfaces of $\{C\}$; on the other hand, any surface A belonging to Γ contains a pencil of elliptic curves $\mathscr{C}$ and therefore, if it possesses an effective canonical system, this must be compounded of curves $\mathscr{C}$. Thus the system $|A'|$ adjoint to A likewise belongs to Γ, and so also the surface $iK = i(A' - A)$.

The hyperelliptic threefolds with $P_g = 0, \pi = 1$, bear the same relation to the Abelian threefolds as the improper hyperelliptic surfaces do to the hyperelliptic, and might

† For this, only the first property of the numbers s is required.

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therefore be called improperly Abelian. We note that a complete analysis of the arithmetically possible cases cannot be obtained from equation (3) since, when $\pi = 1$, the character d cancels out of the equation; for the elliptic surfaces, Enriques (5) has obtained the corresponding results from geometrical reasoning, and these can be applied to threefolds.

Suppose now that $P_g = 1$, so that $\rho = 1$. If $d = 1$, and C is a Jacobi surface, we have a subspecies of Jacobi threefold which maps the triads of points obtained by selecting one point from a given elliptic curve and a pair from a given curve of genus 2. If, instead, C is a Picard surface of divisor δ, we obtain a subspecies of Picard threefold which can be mapped on the Jacobi threefold. It will be noticed that these two varieties are not only special cases of Abelian threefolds but also of elliptic threefolds, for each contains a congruence of birationally equivalent elliptic curves and an elliptic pencil of (hyperelliptic) surfaces.

In the case where $d > 1$, we have from (3) and (4) the relations

$$2(\pi - 1) = d\Sigma \frac{s-1}{s}, \tag{5}$$

$$P_i = iN - \Sigma\left[\frac{i+s-1}{s}\right]. \tag{6}$$

Here, if $N = 0$, $P_i = 1$ $(i = 1, 2, \ldots)$; this is the case where V has a canonical surface of order zero. If, as we shall assume, $N > 0$, V possesses a canonical surface of order greater than zero. We shall now see what effect this assumption has on the values of the plurigenera.

From (6), we have $P_3 = 3N - \Sigma\left[\frac{s+2}{s}\right]$; and since, in all cases, $s \geqslant 2$, it follows that $P_3 \geqslant N$, whence $P_3 \geqslant 2$, except possibly when $N = 1$. The exceptional case can actually occur, as, for example, when $d = 4$, $s = 2$.

Again, from (6), we have $P_4 = 4N - \Sigma\left[\frac{s+3}{s}\right]$; and since $\left[\frac{s+3}{s}\right] \leqslant 2$, it follows that, in all cases, $P_4 \geqslant 2$. Hence, *if* $P_g = 1$, *and* V *possesses a canonical surface of order greater than zero, then* $P_3 \geqslant 2$, *except possibly when* $N = 1$, $s = 2$; *and in all cases,* $P_4 \geqslant 2$.

6. *The case where* Γ *is improperly hyperelliptic.* Until now we have assumed that the surface C has general moduli; as remarked in § 1, if this condition is not fulfilled, the involution I_d cut on C by Γ may be improperly hyperelliptic†, in which case V^* is the product of a curve $\mathscr{C}^*$ and an elliptic surface C^* of the type that possesses a virtual canonical curve of order zero. These surfaces have been studied by Bagnera and De Franchis (1) and also by Enriques (5), who have shown that they may be classified into seven families. Each such surface C^* contains a rational pencil $|\mathscr{L}^*|$ of elliptic curves, and an elliptic pencil $\{\mathscr{K}^*\}$ of elliptic curves, and depends for its further specification on the determinant $[\mathscr{K}^*\mathscr{L}^*]$ as well as on the characters t of the t-fold elliptic curves $\mathscr{L}_t^*$ of the pencil $|\mathscr{L}^*|$. The corresponding surface C contains two (elliptic) pencils of elliptic curves which are images of $|\mathscr{L}^*|$ and $\{\mathscr{K}^*\}$ respectively.

† The existence of the congruence Γ in this case is established as in § 2, although the parametric representation of C^* is different; this derives from the fact (§ 1) that C^* may be mapped by a cyclic involution, without coincidences, on a hyperelliptic surface.

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Precisely as in § 4, we see that, on C^*, a virtual pure canonical curve $\mathscr{X}^*$ is given by the equivalence

$$\mathscr{X}^* \equiv \Sigma(t-1)\,\mathscr{L}_t^* - 2\mathscr{L}^*.$$

Assuming, then, that V^* is free from exceptional surfaces, we have the result (12) that, on V^*, a virtual pure canonical surface K^* is given by

$$K^* \equiv \{\Sigma(t-1)\,\mathscr{L}_t^* - 2\mathscr{L}^*\} \times \mathscr{C}^* + 2(\rho-1)\,C^*.$$

Now each of the elliptic surfaces $L^* = \mathscr{L}^* \times \mathscr{C}^*$, $L_t^* = \mathscr{L}_t^* \times \mathscr{C}^*$, meets any surface C^* of the pencil $\{C^*\}$ in an elliptic curve; hence the surface L corresponding to L^* is mapped on a d-ple elliptic surface, with branch curve consisting entirely of elliptic curves of a pencil, and so is itself elliptic. The surfaces L form a rational pencil $|L|$ on V, containing as t-fold members the surfaces L_t, images of L_t^*, which are likewise elliptic. It follows that, on V, we have

$$K \equiv \Sigma(t-1)\,L_t - 2L + \Sigma(s-1)\,C_s + 2(\rho-1)\,C, \tag{7}$$

where, as in § 4, C_s denotes an s-fold component of the coincidence surface on V.

Here the expression $\Sigma(t-1)\,L_t - 2L$ represents a virtual surface of order zero; hence the numerical invariants of $|K|$, calculated from (7), are the same as the corresponding characters in the general case considered in § 4. Moreover, the genera P_g and P_i, which are found by calculating freedoms of effective systems, will be given by the formulae of § 4. The difference between this case and the preceding is that here the superficial irregularity of V satisfies the inequality $q_2 \geqslant \rho + 1$; this follows from the fact that C^* now has irregularity 1.

7. *Necessary and sufficient condition that* $\pi = 0$. We shall now prove that *in order that* $\pi = 0$, *it is necessary and sufficient that* $P_{12} = 0$.

For, if $\pi = 0$, all the surfaces belonging to Γ are rational or scrollar, so that V contains infinitely many linear systems of surfaces whose canonical and pluricanonical systems are all virtual. Hence $P_g = P_i = 0$, and thus, in particular, $P_{12} = 0$.

Next, if $P_{12} = 0$, it follows that $P_g = P_4 = P_6 = 0$; now, by the results of § 5, whenever $\pi > 0$, at least one of the characters P_4, P_6 must be greater than zero; hence we conclude that $\pi = 0$.

We note the corollary that, if V contains a congruence of rational curves distinct from Γ, then the curves of Γ must themselves be rational; for this hypothesis implies that $P_{12} = 0$.

The hyperelliptic threefolds for which $\pi = 0$ are in a sense analogous to the scrollar surfaces for, as we shall now show, when $\pi = 0$, the congruence Γ always possesses unisecant surfaces: *if* $\pi = 0$, V *is birationally equivalent to a point-cone projecting a surface* C^*.

We may assume that $d > 1$, since, when $d = 1$, there is nothing to prove. Substituting $\rho = 0$ in (4), we find that the virtual dimension of $|2K|$ is $N-4$; and since we must have $P_2 = 0$, it follows that $N \leqslant 3$. Putting $\rho = 0, \pi = 0$ in (3), we see that the characters s_i are to be determined from the relation

$$\Sigma 1/s_i = N - 2 + 2/d. \tag{8}$$

Each number s_i must be a divisor of d and, further, each must divide the least common multiple of the others (§ 5).

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We now find that, when $N = 1$, (8) has no solutions; that, when $N = 2$, there is one solution $s_1 = s_2 = d$; while for $N = 3$, there is the sole possibility $s_1 = s_2 = s_3 = 2, d = 4$.

Consider first the case $N = 2$; we then have two surfaces C_1, C_2, say, each unisecant to Γ, and such that $2C_1 \equiv 2C_2 \equiv C$. Hence C_1 and C_2 must belong to a pencil of unisecant surfaces; for otherwise any surface A (necessarily scrollar) generated by the curves of Γ would be cut by C_1 and C_2 in two disequivalent curves $\mathscr{A}_1$, $\mathscr{A}_2$ such that $2\mathscr{A}_1 \equiv 2\mathscr{A}_2$, whence A would have divisor σ greater than unity; whereas every scrollar surface has divisor unity. Moreover, the pencil to which $\mathscr{A}_1$ and $\mathscr{A}_2$ belong must be rational, and hence the pencil containing C_1 and C_2 is rational.

Consider next the case $N = 3$; here we have three surfaces C_i $(i = 1, 2, 3)$ each bisecant to Γ, and hence, by the previous reasoning, V must contain a rational pencil of such surfaces. Considering the series g_2^1 cut by these surfaces on the generic $\mathscr{C}$, we see that the pencil must contain two surfaces unisecant to Γ; and again applying the previous reasoning, we deduce the existence of a rational pencil of unisecants. Thus in all cases V may be mapped on the product of a rational curve C^* and a surface birationally equivalent to C^*.

This result is in contrast with the known fact that a *rational* congruence of rational curves need not possess unisecant surfaces.

8. *Analytical representations in the cyclic case.* The problem of constructing the threefold V when the requisite characters of C and Γ are assigned depends, in the first instance, upon that of obtaining a mapping for C on the d-ple surface C^*; this representation, as we know, must be free from branch points. The general problem involved here has not yet been considered; if, however, we restrict the question to the case where dC^* is cyclic, the solution in general terms is readily obtainable.

To begin with, it is clear that there exist cyclic involutions I_d $(d \geqslant 2)$ without coincidences on any given hyperelliptic surface; in fact we obtain such an involution from the transformation (1) by choosing the constants a and b so that $ad \equiv 0$, $bd \equiv 0$, modulo the relevant periods. Supposing, then, that C^* is situated in S_3 and endowed with ordinary singularities, and that the equation of C^*, in non-homogeneous coordinates, is $F(x, y, z) = 0$, it follows that the surface C is representable by the pair of equations

$$v^d = G(x, y, z), \quad F(x, y, z) = 0,$$

where $G(x, y, z)$ is a suitably chosen polynomial. If we interpret (x, y, z, v) as non-homogeneous coordinates in space S_4, we thus exhibit C as the complete intersection of two primals in that space. We have to secure that the representation of C on C^* should be without branch points, and also that C should be irreducible. To satisfy the first condition, we suppose that the equation $G(x, y, z) = 0$ represents a d-adjoint surface Φ of C^* whose order is some multiple of d—in which case there will be no branch points at infinity—and which has d-point intersection with C^* at all common points (if any) outside the double curve of C^*. To satisfy the second, we require that the locus of such common points should not be the curve of s-point intersection with C^* for some s-adjoint surface (not necessarily irreducible), where s is any divisor of d. By what has been said above, the existence of such a surface Φ, of appropriately high order, is assured.

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Consider first the case where $P_g = 0$, i.e. in which the surfaces C form a rational pencil $|C|$. Let (x, y, z, t) denote non-homogeneous coordinates in S_4, and take V^* to be the cylinder, with base C^*, whose equation is $F(x, y, z) = 0$. Suppose, further, that the surfaces of the pencil $|C^*|$ are cut by the primes $t =$ constant, and that the surfaces C_s^* are given by $t = a_i$ $(i = 1, 2, \ldots, N)$; then it follows that V is represented by the pair of equations

$$v^d = \prod_1^N (t - a_i)^{r_i} G(x, y, z), \quad F(x, y, z) = 0, \tag{9}$$

where r_i are positive integers less than d, satisfying certain conditions. Interpreting (x, y, z, t, v) as non-homogeneous coordinates in S_5, we thereby exhibit V as the complete intersection of two primals.

The conditions to be imposed on the exponents r_i are precisely the same as those in the analogous representation of the elliptic surfaces of geometric genus zero (5); we therefore quote the results: (i) In order that there should be no branch points at infinity on the generators of V^*, we require that $\Sigma r_i \equiv 0 \pmod{d}$. (ii) Each exponent r_i must be chosen so that d/s_i is the highest common factor of d and r_i. (iii) As previously remarked (§ 5), each of the numbers s_i must divide the least common multiple of the others.

Consider next the case where $P_g\ (= \rho) > 0$; and suppose that C and C^* are specified as before. Let $\mathscr{C}^*$ be the plane curve whose equation is $f(t, u) = 0$, and let $\mathscr{C}$ be mapped on the cyclic d-ple curve $\mathscr{C}^*$ by means of the equation $v^d = Ag(t, u)$, where A is a constant and $g(t, u)$ a polynomial chosen in the manner to be described; then the equations

$$v^d = g(t, u)\, G(x, y, z), \quad F(x, y, z) = 0, \quad f(t, u) = 0 \tag{10}$$

map V on the d-ple threefold V^* which represents the pairs of points of C^* and $\mathscr{C}^*$; they exhibit V as the complete intersection of three primals of a space S_6 in which (x, y, z, t, u, v) are non-homogeneous coordinates.

The conditions that the curve $g(t, u) = 0$ must satisfy are similar to those stated above. (i) In order that there should be no branchings at infinity, we must suppose that the order of g is a multiple of d. (ii) The curve g must have r_i-point intersections with the curve f at N simple points of f $(1 \leqslant r_i < d)$, each number r_i being such that the highest common factor of d and r_i is d/s_i. (iii) The residual intersections (if any) of g and f must be at simple points of the two curves, and each must be a d-point intersection. (iv) Finally, in order that the curve $\mathscr{C}$ so represented should be irreducible, we have to impose a condition on the sets of points specified in (ii) and (iii). Thus, the curve g must not reduce to a curve counted d times; and, more generally, g must not be of the form $g = h^{d/\delta}$, where δ is some divisor of d; nor, when g is not of this form, must the set of intersections of f and g be equivalent to the set of intersections of f and such a curve h.

9. *Notes and examples.* (1) As remarked in § 2, the classification of V depends in the first instance on the determination of all possible groups $\mathscr{G}_d$. The cases $\pi = 1, 2$ have been dealt with by Enriques (5) and Bolza (1, 6) respectively. With regard to the general problem, we may note that, for present purposes, it does not suffice merely to obtain the possible groups of automorphisms—as has actually been done for $\pi = 4, 5$ (3)—but

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it is also necessary to examine all subgroups of these; this has been accomplished for the case $\pi = 3$ (4).

(2) The representation given in § 8 for the cyclic threefolds obviously applies also to the case where C^* is improperly hyperelliptic; in fact it is for this case that the representation is perfectly general, and that examples are immediately available. Thus, with the notation of § 8, suppose that $F(x, y, z) = 0$ denotes the equation of an improperly hyperelliptic surface of order μ which possesses an effective m-canonical curve of order zero, but no effective pluricanonical curve of lower index; then, as first observed by Enriques and Severi (6), if $G(x, y, z) = 0$ is the equation of the unique m-adjoint of order $m(\mu - 4)$, the equations $v^m = G(x, y, z)$, $F(x, y, z) = 0$ represent an irreducible hyperelliptic surface, necessarily of particular moduli, and containing two elliptic pencils of elliptic curves. Substituting these functions F and G in (9) or (10), we obtain threefolds of the required type.

(3) As an illustration of a case which is in general non-cyclic, suppose that $\mathscr{C}$ is hyperelliptic of genus $\pi > 1$; then the group $\mathscr{G}_d$ of automorphisms must leave invariant the unique g_2^1 of which the canonical series of $\mathscr{C}$ is compounded, and it must also leave invariant the set of $2(\pi + 1)$ double points of this g_2^1. (This is the principle on which rests the determination of the possible groups $\mathscr{G}_d$ in the case $\pi = 2$.)

The aggregate of series g_2^1 on the ∞^2 curves of Γ (which we assume to be properly hyperelliptic) serve to define an involution I_2 on V. Since every curve $\mathscr{C}$ is non-singular, no two of the double points in any g_2^1 can coincide, and so the coincidence surface B of I_2 must meet every curve $\mathscr{C}$ in $2(\pi + 1)$ distinct points. There are now two possibilities: (a) The generic surface C is united in I_2; in this case we must have $\rho = 0$, for the series cut by $\{C\}$ on $\mathscr{C}$ is then compounded of the g_2^1. (b) The generic C is not united in I_2; then the locus of conjugates in I_2 of the points of C is a second surface of $\{C\}$.

In either case B consists of a number of surfaces C, together with the surfaces C_s; for B is in $(2\pi + 2, 1)$ correspondence with the curves of the hyperelliptic congruence Γ, and the correspondence is free from united points; hence B is either hyperelliptic or consists of a number of hyperelliptic surfaces without common points. Further, the generic C cannot meet B; whence the result.

By means of I_2 we can map V on a double threefold V' which contains a congruence Γ' of rational curves $\mathscr{C}'$, images of $\mathscr{C}$; it follows from § 7 that we may take V' to be a point-cone (or cylinder). The branch surface B' consists of a number of surfaces of the pencil $\{C'\}$ corresponding to $\{C\}$, together with the surfaces C'_s, images of C_s.

(4) Case (b) is the simplest illustration of a *polygonal* pencil $\{C\}$, i.e. a pencil whose image curve contains a g_n^1 $(n \geqslant 2)$. In this case we may map V on an n-ple threefold V^* which contains a rational pencil $|C^*|$ of surfaces, the generic member of which represents n distinct surfaces of $\{C\}$. The branch surface on V^* consists of surfaces belonging to $|C^*|$.

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